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On the Dynamics of a Modified van der Pol–Duffing Oscillator

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Abstract: The 3-dimensional modified van der Pol–Duffing oscillator has been studied by several authors. We complete its study, first characterizing its zero-Hopf equilibria and then its zero-Hopf bifurcations—i.e., we provide sufficient conditions for the existence of three, two or one periodic solutions, bifurcating from the zero-Hopf equilibrium localized at the origin of coordinates. Recall that an equilibrium point of a 3-dimensional differential system whose eigenvalues are zero and a pair of purely imaginary eigenvalues is a zero-Hopf equilibrium. Finally, we determine the dynamics of this system near infinity, i.e., we control the orbits that escape to or come from the infinity.

Keywords: van der Pol–Duffing oscillator; zero-Hopf; dynamics at infinity

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1. Introduction and Statement of the Main Results

In the papers [1,2], a modified van der Pol–Duffing oscillator circuit was studied, given by the three-dimensional differential system

$$
\begin{aligned}
\frac{dx}{dt} &= \dot{x} = -m(x^3 - \mu x - y + \alpha), \\
\frac{dy}{dt} &= \dot{y} = x - y - z, \\
\frac{dz}{dt} &= \dot{z} = \beta y - \gamma z,
\end{aligned}
\tag{1}
$$

where $\alpha, \beta, m, \mu, \gamma \in \mathbb{R}$ are parameters and $m \neq 0$; otherwise, all the planes $x =$ constant are invariant and the differential system on these planes are linear. Note that if $\gamma = 0$, the circuit reduces to the well-known van der Pol–Duffing oscillator circuit, which is equivalent to Chua's autonomous circuit but with a cubic nonlinear element. The authors of [1,2] observed that for convenient parameter values of $m, \mu, \alpha, \beta, \gamma$, system (1) exhibits a double-scroll chaotic attractor.

Several authors have studied the van der Pol–Duffing oscillator circuit and its various generalizations; see, for instance, [1–10] and the references cited therein. More precisely, in the paper [3], some bifurcations of the van der Pol–Duffing oscillator are studied, but not from the 3-dimensional modified van der Pol–Duffing oscillator. In reference [4], some applications to secure communications of the synchronized modified van der Pol–Duffing oscillators with offset terms are performed. Papers [1,2,5] studied the adaptive synchronization of two chaotic systems, wherein one of these systems is the modified van der Pol–Duffing oscillator. In reference [6], the authors analyze the response of a

Duffing–van der Pol oscillator under delayed feedback control. In the paper [7], the authors studied the existence of strange attractors and how to control the chaos in a Duffing–van der Pol oscillator with two external periodic forces. In reference [8], the dynamics and some hidden attractors of a general autonomous van der Pol–Duffing oscillator are analyzed. The transmission of signals by synchronization in a chaotic van der Pol–Duffing oscillator is studied in the paper [9]. Finally, in reference [10], the authors work with the hyperchaos and its bifurcations in a driven van der Pol–Duffing oscillator circuit.

For $\alpha = 0$, an important property of the modified van der Pol–Duffing system (1) is its symmetry under the change in coordinates $(x, y, z) \to (-x, -y, -z)$. So, if $(x(t), y(t), z(t))$ is a solution of the system, then $(-x(t), -y(t), -z(t))$ is also a solution. Moreover, system (1) with $\alpha = 0$, $m \neq 0$, and $(\beta\mu + \mu\gamma + \gamma)(\beta + \gamma) > 0$ exhibits three equilibria: the origin and the two equilibria,

$$E^{\pm} = \left(\pm\frac{\sqrt{\beta\mu + \mu\gamma + \gamma}}{\sqrt{\beta + \gamma}}, \pm\frac{\gamma\sqrt{\beta\mu + \mu\gamma + \gamma}}{(\beta + \gamma)^{3/2}}, \pm\frac{\beta\sqrt{\beta\mu + \mu\gamma + \gamma}}{(\beta + \gamma)^{3/2}} \right),$$

choosing all the plus, or all the minus. When $m \neq 0$ and $(\beta\mu + \mu\gamma + \gamma)(\beta + \gamma) \leq 0$, the system has only one equilibrium point at the origin. If $m = 0$, system (1) admits a straight line of equilibrium points.

Here, a *zero-Hopf equilibrium* in a three-dimensional autonomous differential system refers to an equilibrium point characterized by a zero eigenvalue and a pair of purely imaginary eigenvalues. A *zero-Hopf bifurcation* takes place when some periodic orbits bifurcate from a zero-Hopf equilibrium moving the parameters of the diffeential system. Further details regarding zero-Hopf bifurcations can be found in [11–15].

In the following propositions, we describe the zero-Hopf equilibria of the modified van der Pol–Duffing system (1) for $\alpha = 0$.

Proposition 1. *For $\alpha = 0$, there are two 2-parameter families of the modified van der Pol–Duffing system (1), for which the origin of coordinates is a zero-Hopf equilibrium point:*

(i) $m = \frac{1+\gamma}{\mu}$, $\beta = -\gamma\frac{1+\mu}{\mu}$, *and* $\mu(\gamma^2\mu + 2\gamma(\mu + 1) + \mu + 1) < 0$.

(ii) $m = 0$, $\gamma = -1$, $\mu \in \mathbb{R}$, *and* $\beta > 1$.

Proposition 2. *For $\alpha = 0$, there are no parameter families of the modified van der Pol–Duffing system (1) for which the equilibria $E^{\pm}$ are zero-Hopf.*

Using the averaging theory (see Section 2.1 for more details on the averaging theory), we investigate when the modified van der Pol–Duffing system (1), having a zero-Hopf equilibrium point at the origin of coordinates, has a zero-Hopf bifurcation producing some limit cycles. Recall that when in a neighborhood of a periodic orbit there are no other periodic orbits, such a periodic orbit is a *limit cycle*.

Theorem 1. *If we set*

$$\begin{aligned}
&\beta = -\gamma_0\frac{1 + \mu_0}{\mu_0} + \beta_1\varepsilon + \beta_2\varepsilon^2, \quad m = \frac{1 + \gamma_0}{\mu_0} + m_1\varepsilon + m_2\varepsilon^2, \\
&\mu = \mu_0 + \mu_1\varepsilon + \mu_2\varepsilon^2, \quad \gamma = \gamma_0 + \gamma_1\varepsilon + \gamma_2\varepsilon^2, \\
&\alpha = 0, \quad \text{and} \quad \mu_0(\gamma_0^2\mu_0 + 2\gamma_0(\mu_0 + 1) + \mu_0 + 1) < 0,
\end{aligned}$$

where β_1, β_2, γ_1, γ_2, μ_1, μ_2, m_1, and $m_2 \in \mathbb{R}$, and ε is a sufficiently small parameter. Then, the modified van der Pol–Duffing system (1) with $\varepsilon = 0$ has a zero-Hopf bifurcation at the *equilibrium point located at the origin of coordinates. For a sufficiently small* $\varepsilon \neq 0$*, the following statements hold:*

(*a*) *If* $\frac{CD-AF}{AE-BD} > 0$, $\frac{BF-CE}{AE-BD} > 0$, *and* $\frac{C}{A} < 0$, *three periodic orbits bifurcate from the origin. See Figure 1.*

(*b*) *If* $\frac{CD-AF}{AE-BD} > 0$, $\frac{BF-CE}{AE-BD} > 0$, *and* $\frac{C}{A} > 0$, *two periodic orbits bifurcate from the origin.*

(*c*) *If* $\frac{CD-AF}{AE-BD} < 0$ *or* $\frac{BF-CE}{AE-BD} < 0$, *and* $\frac{C}{A} < 0$, *one periodic orbit bifurcates from the origin.*

The values of the constants A, B, C, D, E, *and* F *are defined in Appendix A.*

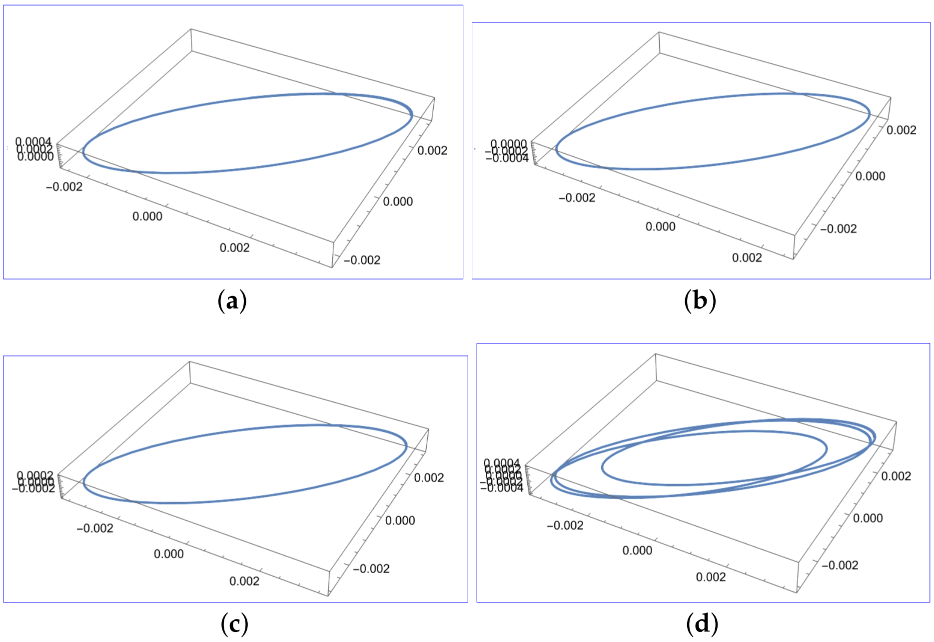


Figure 1. The three limit cycles of the modified van der Pol–Duffing system (1) under the assumptions of statement (**a**) of Theorem 1 for the values of the parameters $\gamma_0 = 1/4$, $\gamma_1 = \gamma_2 = 0$, $\mu_0 = -3/4$, $\mu_1 = -1$, $\mu_2 = 0$, $\beta_1 = \beta_2 = 0$, $m_1 = 0$, $m_2 = -1/2$ and $\varepsilon = 1/500$. (**a**) The limit cycle obtained from the initial conditions (r_1, z_1) of the proof of Theorem 1. (**b**) Idem from the initial conditions (r_2, z_2). (**c**) Idem from the initial conditions (r_3, z_3). (**d**) The three limit cycles together. Moreover, applying statement (c) of Theorem 1 to these three periodic orbits, we determine that they are unstable.

For polynomial differential systems like the modified van der Pol–Duffing system (1), the dynamics in a neighborhood of infinity can be studied using the Poincaré compactification. In general terms, this compactification consists of identifying the entire $\mathbb{R}^3$ with the interior of the closed ball of $\mathbb{R}^3$ of radius one, centered at the origin of coordinates. Then, the boundary of this ball, the 2-dimensional sphere $\mathbb{S}^2$, can be identified with the infinity of $\mathbb{R}^3$, because in the space $\mathbb{R}^3$, we can escape to or come from infinity in as many directions as there are points on the 2-dimensional sphere $\mathbb{S}^2$. See Section 2 for further details on the Poincaré compactification.

In the forthcoming result, utilizing Poincaré compactification, we characterize the dynamics of the modified van der Pol–Duffing system (1) in a neighborhood of infinity.

Theorem 2. *For* $\alpha, \beta, \gamma, \mu \in \mathbb{R}$, *and* $m \neq 0$, *the phase portrait of the modified van der Pol–Duffing system (1) on the sphere of infinity is topologically equivalent to the one shown in Figure 2. In particular, for* $m < 0$, *there exist two stable star nodes and a circle of equilibria. At each of these equilibria, two orbits start if* $m < 0$, *or end if* $m > 0$. *See the phase portrait at infinity in Figure 2 for* $m < 0$; *and for* $m > 0$, *it is the same phase portrait but with the orientation of all orbits reversed.*

Looking at Theorem 2 and at the dynamics of the orbits on the sphere of the infinity in Figure 2, it follows that when the parameter $m < 0$ in the modified van der Pol–Duffing oscillator, all the orbits that escape at infinity outside the plane $x = 0$ end at infinity at the endpoints of the x-axis. While when the parameter $m > 0$, all the orbits coming from the infinity outside the plane $x = 0$ start at infinity at the endpoints of the x-axis.

In Section 2, we introduce the basic definitions and necessary results for proving Theorems 1 and 2. Section 3 is dedicated to proving Propositions 1 and 2. In Section 4, we prove Theorem 1, and in Section 5, we prove Theorem 2.

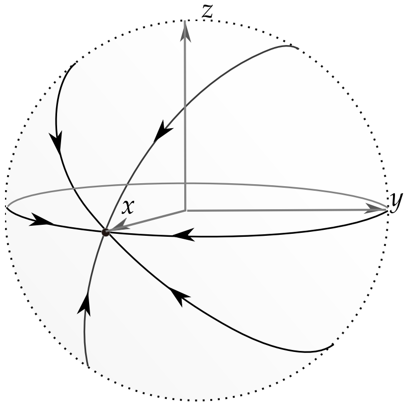


Figure 2. Phase portrait of the modified van der Pol–Duffing system (1) on the Poincaré sphere for $m < 0$.

2. Preliminaries

2.1. The Averaging Theory of First and Second Order

Consider the non-autonomous differential system

$$\dot{\mathbf{x}} = \varepsilon \mathbf{F}_1(t,\mathbf{x}) + \varepsilon^2 \mathbf{F}_2(t,\mathbf{x}) + \varepsilon^3 \mathbf{F_3}(t,\mathbf{x},\varepsilon), \quad (t,\mathbf{x},\varepsilon) \in \mathbb{R} \times \Omega \times [-\varepsilon_0, \varepsilon_0], \tag{2}$$

where Ω is an open bounded subset of $\mathbb{R}^n$. We assume that $\mathbf{F}_i$ for $i = 1,2,3$ are T-periodic in the variable t and sufficiently smooth functions. The averaged functions of first and second order of system (2) are

$$\begin{aligned} \mathbf{g}_1(\mathbf{x}) &= \int_0^T \mathbf{F}_1(t,\mathbf{x})\, dt, \\ \mathbf{g}_2(\mathbf{x}) &= \int_0^T \mathbf{F}_2(t,\mathbf{x}) + \frac{\partial \mathbf{F}_1}{\partial \mathbf{x}}(t,\mathbf{x}) \left(\int_0^t \mathbf{F}_1(s,\mathbf{x})\, ds \right) dt, \end{aligned} \tag{3}$$

respectively. The following result shows that under suitable hypotheses, the zeros of the averaged functions $\mathbf{g}_i$ for $i = 1,2$ will provide T-periodic solutions of system (2).

Theorem 3. *Assume that $\mathbf{g}_i$, for $i \in \{1,2\}$, is the first non-vanishing averaging function of system (2). Then, the following two statements hold:*

(i) *If $\mathbf{g}_1(\mathbf{x}) \not\equiv 0$ and there exists $\mathbf{x}^* \in \Omega$ such that $\mathbf{g}_1(\mathbf{x}^*) = 0$ and the determinant of the Jacobian matrix $Det(D\mathbf{g}_1(\mathbf{x}^*)) \neq 0$, then for a sufficiently small $|\varepsilon| \neq 0$, there exists a T-periodic solution $\mathbf{x}(t,\varepsilon)$ of system (2) such that $\mathbf{x}(0,0) = \mathbf{x}^*$.*

(ii) *If $\mathbf{g}_1(\mathbf{x}) \equiv 0$ and $\mathbf{g}_2(\mathbf{x}) \not\equiv 0$, and there exists $\mathbf{x}^* \in \Omega$ such that $\mathbf{g}_2(\mathbf{x}^*) = 0$ and the determinant of the Jacobian matrix $Det(D\mathbf{g}_2(\mathbf{x}^*)) \neq 0$, then for a sufficiently small $|\varepsilon| \neq 0$, there exists a T-periodic solution $\mathbf{x}(t,\varepsilon)$ of system (2) such that $\mathbf{x}(0,0) = \mathbf{x}^*$.*

(iii) *Under the assumptions of statement (i) (respectively, (ii)) if the real part of all eigenvalues of the Jacobian matrix $D\mathbf{g}_1(\mathbf{x}^*)$ (respectively, $D\mathbf{g}_2(\mathbf{x}^*)$) is negative, then the periodic solution*

$\mathbf{x}(t,\varepsilon)$ *is locally stable. If the real part of some eigenvalue of the Jacobian matrix* $D\mathbf{g}_1(\mathbf{x}^*)$ *(respectively,* $D\mathbf{g}_2(\mathbf{x}^*)$*) is positive, then the periodic solution* $\mathbf{x}(t,\varepsilon)$ *is unstable.*

For a proof of Theorem 3, see Theorem A of [16] and Theorems 11.5 and 11.6 of [17].

2.2. Poincaré Compactification in $\mathbb{R}^3$

Now, we present some results on the Poincaré compactification in $\mathbb{R}^3$, which will be crucial for proving Theorem 2. Thus, in this subsection, we summarize the formulas concerning the Poincaré compactification for a cubic polynomial differential system in $\mathbb{R}^3$. The Poincaré compactification for polynomial systems in $\mathbb{R}^2$ was introduced by Poincaré for studying the global dynamics of polynomial differential systems, including the behavior of the unbounded solutions. Later on, this compactification was extended to $\mathbb{R}^n$ for $n > 2$ in [18].

Here, we consider the polynomial differential system

$$\dot{x} = P_1(x,y,z), \qquad \dot{y} = P_2(x,y,z), \qquad \dot{z} = P_3(x,y,z), \tag{4}$$

or its vector field $X = (P_1, P_2, P_3)$, where the maximum of the degrees of the polynomials P_i for $i = 1,2,3$ is three. This polynomial system can be extended to an analytic system on the unit closed ball centered at the origin of coordinates, referred to as the *Poincaré ball*, with its interior diffeomorphic to $\mathbb{R}^3$, while its boundary, the 2-dimensional sphere $\mathbb{S}^2$, is identified as the infinity of $\mathbb{R}^3$. As usual, we consider six open local charts on $\mathbb{S}^2$: $U_1 = \{(x,y,z)|x > 0\}, U_2 = \{(x,y,z)|y > 0\}, U_3 = \{(x,y,z)|z > 0\}, V_1 = \{(x,y,z)|x < 0\}$, $V_2 = \{(x,y,z)|y < 0\}$, and $V_3 = \{(x,y,z)|z < 0\}$.

The flow of the Poincaré compactification of the polynomial differential system (4) in the local chart U_1 is determined by the differential system:

$$\begin{aligned}
\dot{u} &= w^3\left(-uP_1\left(\frac{1}{w},\frac{u}{w},\frac{v}{w}\right) + P_2\left(\frac{1}{w},\frac{u}{w},\frac{v}{w}\right)\right),\\
\dot{y} &= w^3\left(-vP_1\left(\frac{1}{w},\frac{u}{w},\frac{v}{w}\right) + P_3\left(\frac{1}{w},\frac{u}{w},\frac{v}{w}\right)\right),\\
\dot{z} &= -w^4P_1\left(\frac{1}{w},\frac{u}{w},\frac{v}{w}\right).
\end{aligned}$$

When $w = 0$, we have the points of the infinite sphere in all the local charts.

The flow of Poincaré compactification of system (4) in the local chart U_2 is governed by the following differential system:

$$\begin{aligned}
\dot{u} &= w^3\left(-uP_2\left(\frac{u}{w},\frac{1}{w},\frac{v}{w}\right) + P_1\left(\frac{u}{w},\frac{1}{w},\frac{v}{w}\right)\right),\\
\dot{v} &= w^3\left(-vP_2\left(\frac{u}{w},\frac{1}{w},\frac{v}{w}\right) + P_3\left(\frac{u}{w},\frac{1}{w},\frac{v}{w}\right)\right),\\
\dot{w} &= -w^4P_2\left(\frac{u}{w},\frac{1}{w},\frac{v}{w}\right).
\end{aligned}$$

The flow of Poincaré compactification of system (4) on U_3 is governed by the following differential system:

$$\begin{aligned}
\dot{u} &= w^3\left(-uP_3\left(\frac{u}{w},\frac{v}{w},\frac{1}{w}\right)+P_1\left(\frac{u}{w},\frac{v}{w},\frac{1}{w}\right)\right),\\
\dot{v} &= w^3\left(-vP_3\left(\frac{u}{w},\frac{v}{w},\frac{1}{w}\right)+P_2\left(\frac{u}{w},\frac{v}{w},\frac{1}{w}\right)\right),\\
\dot{w} &= -w^4P_3\left(\frac{u}{w},\frac{v}{w},\frac{1}{w}\right),
\end{aligned}$$

The expression for the extended differential system in the local chart V_i, for $i = 1,2,3$, is the same compared to that in the local chart U_i.

3. Proofs of Propositions 1 and 2

Proof of Proposition 1. For $\alpha = 0$, the characteristic polynomial of the Jacobian matrix of the modified van der Pol–Duffing system (1) evaluated at the origin is

$$p(x) = -\lambda^3 + \lambda^2(\mu m - 1 - \gamma) + \lambda(m\mu(\gamma+1) + m - \beta - \gamma) + (m(\mu(\beta+\gamma)+\gamma)).$$

Imposing that $p(x) = -\lambda(\lambda^2+\omega^2)$, we obtain the following relations:

1. $m = 0$, $\gamma = -1$, and $\omega^2 = \beta - 1$;
2. $\mu \neq 0$, $\beta = -\frac{\gamma(\mu+1)}{\mu}$, $m = \frac{\gamma+1}{\mu}$, and $\omega^2 = -\frac{\gamma^2\mu+2\gamma(\mu+1)+\mu+1}{\mu}$.

So, the proposition follows. □

Proof of Proposition 2. Assume $\alpha = 0$. Due to the invariance of the differential system (1) under the symmetry $(x,y,z) \to (-x,-y,-z)$, it is sufficient to prove the proposition for the equilibrium point E^+. The characteristic polynomial of the Jacobian matrix of the modified van der Pol–Duffing system (1) at E^+ is

$$\begin{aligned}
p(x) = -\lambda^3 + \lambda^2&\left(\gamma\left(\frac{3m}{\beta+\gamma}+1\right)+2\mu m+1\right)\\
&+\lambda\left(\frac{m(-2(\gamma+1)\mu(\beta+\gamma)+\beta-\gamma(3\gamma+2))}{\beta+\gamma}-\beta-\gamma\right)\\
&-2m(\gamma+\mu(\beta+\gamma)).
\end{aligned}$$

The proposition follows by showing that the polynomial $p(x)$ cannot be written as the polynomial $-\lambda(\lambda^2+\omega^2)$ when the equilibrium E^+ exists. Indeed, in order that $p(x) = -\lambda(\lambda^2+\omega^2)$, the following system must have some solution:

$$\begin{aligned}
&\gamma\left(\frac{3m}{\beta+\gamma}+1\right)+2\mu m+1 = 0,\\
&\frac{m(-2(\gamma+1)\mu(\beta+\gamma)+\beta-\gamma(3\gamma+2))}{\beta+\gamma}-\beta-\gamma = \omega^2,\\
&2m(\gamma+\mu(\beta+\gamma)) = 0.
\end{aligned} \tag{5}$$

Note that $\gamma + \mu(\beta+\gamma) \neq 0$; otherwise, the equilibrium point E^* would be the equilibrium $(0,0,0)$, which was already studied in Proposition 1. Then, from the third equation of system (5), we obtain that $m = 0$, but in the definition of the modified van der Pol–Duffing oscillator, $m \neq 0$; otherwise, the oscillator is reduced to trivial linear differential systems on the invariant planes $x =$ constant. This completes the proof of the proposition. □

4. Proof of Theorem 1

Under the assumptions of Theorem 1, system (1) becomes

$$\begin{aligned} \dot{x} &= \frac{(\gamma_0 + \mu_0\varepsilon(m_1 + m_2\varepsilon) + 1)\left(-x^3 + x(\mu_0 + \varepsilon(\mu_1 + \mu_2\varepsilon)) + y\right)}{\mu_0} \\ \dot{y} &= x - y - z \\ \dot{z} &= y\varepsilon(\beta_1 + \beta_2\varepsilon) - \frac{\gamma_0(\mu_0 + 1)y}{\mu_0} - z(\gamma_0 + \varepsilon(\gamma_1 + \gamma_2\varepsilon)). \end{aligned} \tag{6}$$

One of the main difficulties for applying the averaging theory to the differential system (6) is to write this system into the normal form (2) for applying the averaging theory, i.e., we need a small parameter in the differential system and also that the differential system be periodic in the independent variable. At the moment, this is not the case in the differential system (6). The small parameter ε is introduced with the following rescaling equation.

Upon rescaling the system by taking $(x, y, z) = \varepsilon(X, Y, Z)$, we obtain

$$\begin{aligned} \dot{X} &= \frac{(\gamma_0 + 1)(\mu_0 X + Y)}{\mu_0} + \varepsilon\left(m_1(\mu_0 X + Y) + \frac{(\gamma_0 + 1)\mu_1 X}{\mu_0}\right) + \\ &\quad \varepsilon^2\left(\frac{X(\gamma_0\mu_2 + \mu_2 + \mu_0\mu_1 m_1 + \mu_0^2 m_2) + \mu_0 m_2 Y - (\gamma_0 + 1)X^3}{\mu_0}\right) + \mathcal{O}\left(\varepsilon^3\right), \\ \dot{Y} &= X - Y - Z, \\ \dot{Z} &= -\frac{\gamma_0(\mu_0(Y + Z) + Y)}{\mu_0} + \varepsilon(\beta_1 Y - \gamma_1 Z) + \varepsilon^2(\beta_2 Y - \gamma_2 Z) + \mathcal{O}\left(\varepsilon^3\right). \end{aligned} \tag{7}$$

The subsequent computations will be easier if the linear part of the differential system at the origin of coordinates is written into its real normal Jordan form.

We use the following linear change of variables in order to have the linear part of system (7) when $\varepsilon = 0$ in its real Jordan normal form:

$$\begin{aligned} X &= \frac{(\gamma_0 + 1)\sqrt{-\mu_0((\gamma_0 + 1)^2\mu_0 + 2\gamma_0 + 1)}\overline{X} - (\gamma_0 + 1)(\gamma_0\mu_0 + \mu_0 + 1)\overline{Y} + \overline{Z}}{\mu_0 + 1}, \\ Y &= \frac{\sqrt{-\mu_0((\gamma_0 + 1)^2\mu_0 + 2\gamma_0 + 1)}\overline{X} - \mu_0(\gamma_0\overline{Y} + \overline{Z})}{\mu_0 + 1}, \\ Z &= \gamma_0\overline{Y} + \overline{Z}. \end{aligned} \tag{8}$$

In these new variables, system (7) becomes

$$\begin{aligned} \dot{\overline{X}} &= -\frac{\sqrt{(\gamma_0 + 1)^2\mu_0 + 2\gamma_0 + 1}}{\sqrt{-\mu_0}}\overline{Y} + \varepsilon G_1^1(\overline{X}, \overline{Y}, \overline{Z}) + \varepsilon^2 G_1^2(\overline{X}, \overline{Y}, \overline{Z}) + \mathcal{O}\left(\varepsilon^3\right), \\ \dot{\overline{Y}} &= \frac{\sqrt{(\gamma_0 + 1)^2\mu_0 + 2\gamma_0 + 1}}{\sqrt{-\mu_0}}\overline{X} + \varepsilon G_2^1(\overline{X}, \overline{Y}, \overline{Z}) + \varepsilon^2 G_2^2(\overline{X}, \overline{Y}, \overline{Z}) + \mathcal{O}\left(\varepsilon^3\right), \\ \dot{\overline{Z}} &= \varepsilon G_3^1(\overline{X}, \overline{Y}, \overline{Z}) + \varepsilon^2 G_3^2(\overline{X}, \overline{Y}, \overline{Z}) + \mathcal{O}\left(\varepsilon^3\right), \end{aligned} \tag{9}$$

where

$$G_1^1(\overline{X}, \overline{Y}, \overline{Z}) = \frac{\beta_1\mu_0\overline{X}\sqrt{(\gamma_0 + 1)^2\mu_0 + 2\gamma_0 + 1} + \sqrt{-\mu_0}(\mu_0(\beta_1 + \gamma_1) + \gamma_1)(\gamma_0\overline{Y} + \overline{Z})}{(\mu_0 + 1)\sqrt{(\gamma_0 + 1)^2\mu_0 + 2\gamma_0 + 1}},$$

$$
\begin{aligned}
G_2^1(\overline{X},\overline{Y},\overline{Z}) = \ & \frac{1}{\mu_0(\mu_0+1)((\gamma_0+1)^2\mu_0+2\gamma_0+1)} \\
& \Big(-\sqrt{-\mu_0}\,\overline{X}\sqrt{(\gamma_0+1)^2\mu_0+2\gamma_0+1} \\
& \times((\gamma_0+1)^2(\mu_0+1)\mu_1+\mu_0(\gamma_0\mu_0+\mu_0+1)(-\beta_1+\mu_0 m_1+m_1)) \\
& +\mu_0\overline{Y}(\gamma_0^2\mu_0(\mu_0(-\beta_1+\mu_0 m_1+m_1)) \\
& -\gamma_0(\mu_0+1)(\gamma_1(\mu_0+1)+\mu_0(\beta_1-2(\mu_0+1)m_1))+\mu_0(\mu_0+1)^2 m_1) \\
& +(\gamma_0+1)^2(\mu_0+1)\mu_1\overline{Y}(\gamma_0\mu_0+\mu_0+1) \\
& -\mu_0\overline{Z}(\gamma_0\mu_0+\mu_0+1)(\mu_0(\beta_1+\gamma_1)+\gamma_1) \\
& -(\gamma_0+1)(\mu_0+1)\mu_1\overline{Z}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
G_3^1(\overline{X},\overline{Y},\overline{Z}) = \ & \frac{1}{\sqrt{-\mu_0}\mu_0((\gamma_0+1)^2\mu_0+2\gamma_0+1)^{3/2}} - \sqrt{-\mu_0((\gamma_0+1)^2\mu_0+2\gamma_0+1)} \\
& (-\sqrt{-\mu_0}((\gamma_0+1)\overline{Z}(\mu_0(\mu_0(\beta_1+\gamma_1)+\gamma_1)-\gamma_0\mu_1)+ \\
& \overline{Y}\gamma_0((\gamma_0+1)\gamma_1\mu_0(\mu_0+1)+\mu_0^2(m_1+\beta_1+2\gamma_0 m_1+\beta_1\gamma_0+(\gamma_0+1)^2\mu_0 m_1) \\
& +(\gamma_0+1)^2\mu_1(\gamma_0\mu_0+\mu_0+1)))-\mu_0\overline{X}((\gamma_0+1)^2\mu_0+2\gamma_0+1) \\
& \gamma_0(\gamma_0+1)^2\mu_1+(\gamma_0+1)\mu_0(\beta_1+\gamma_0\mu_0 m_1)+\gamma_0\mu_0 m_1\Big).
\end{aligned}
$$

The expressions for $G_1^2(\overline{X},\overline{Y},\overline{Z})$, $G_2^2(\overline{X},\overline{Y},\overline{Z})$, and $G_3^2(\overline{X},\overline{Y},\overline{Z})$ are omitted due to their length.

Now, we write the differential system (9) in the cylindrical coordinates $(\overline{X},\overline{Y},\overline{Z}) = (r\cos\theta, r\sin\theta, z)$. And taking θ as the new time of the system, system (9) becomes the following non-autonomous and periodic differential system with period 2π:

$$
\begin{aligned}
\frac{dr}{d\theta} &= F_1^1(\theta,r,z)\varepsilon + F_1^2(\theta,r,z)\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
\frac{dz}{d\theta} &= F_2^1(\theta,r,z)\varepsilon + F_2^2(\theta,r,z)\varepsilon^2 + \mathcal{O}(\varepsilon^3),
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
F_1^1(\theta,r,z) &= \frac{\sqrt{-\mu_0}(\tilde{G}_1^1(\theta,r,z)\cos(\theta)+\tilde{G}_2^1(\theta,r,z)\sin(\theta))}{\sqrt{(\gamma_0+1)^2\mu_0+2\gamma_0+1}}, \\
F_1^2(\theta,r,z) &= \frac{\tilde{G}_3^1(\theta,r,z)\sqrt{-\mu_0}}{\sqrt{(\gamma_0+1)^2\mu_0+2\gamma_0+1}}, \\
F_2^1(\theta,r,z) &= \frac{1}{r((\gamma_0+1)^2\mu_0+2\gamma_0+1)}\Big(\tilde{G}_2^1(\theta,r,z)^2\mu_0\cos(\theta)\sin(\theta)+ \\
& \sqrt{-\mu_0}r\sqrt{\gamma_0^2\mu_0+2\gamma_0(\mu_0+1)+\mu_0+1}\sin(\theta)\tilde{G}_2^2(\theta,r,w)+ \\
& -\sin(\theta)\tilde{G}_1^1(\theta,r,z)^2\mu_0\cos(\theta)+\tilde{G}_2^1(\theta,r,z)\tilde{G}_1^1(\theta,r,z)\mu_0\cos(2\theta) \\
& +\tilde{G}_1^2(\theta,r,z)\sqrt{-\mu_0}r\sqrt{\gamma_0^2\mu_0+2\gamma_0(\mu_0+1)+\mu_0+1}\cos(\theta)\Big), \\
F_2^2(\theta,r,z) &= \frac{1}{r((\gamma_0+1)^2\mu_0+2\gamma_0+1)}(\tilde{G}_2^1(\theta,r,z)\tilde{G}_3^1(\theta,r,z)\mu_0\cos(\theta) \\
& +\tilde{G}_3^2(\theta,r,z)((\gamma_0+1)^2\mu_0+2\gamma_0+1)\sqrt{-\mu_0}r \\
& -\tilde{G}_1^1(\theta,r,z)\tilde{G}_3^1(\theta,r,z)\mu_0\sin(\theta)),
\end{aligned}
$$

where $\tilde{G}_i^j(\theta,r,z) = G_i^j(r\cos(\theta), r\sin(\theta), z)$ for $i = 1,2$ and $j = 1,2,3$.

Now, the differential system (10) is written in the normal form to apply the averaging theory described in Section 2.1, where $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r,z)$, $\mathbf{F}_1(t,\mathbf{x}) = \mathbf{F}_1(\theta,r,z) = (F_1^1(\theta,r,z), F_1^2(\theta,r,z))$ and $\mathbf{F}_2(t,\mathbf{x}) = \mathbf{F}_2(\theta,r,z) = (F_2^1(\theta,r,z), F_2^2(\theta,r,z))$.

From (3), the first-order averaging function of system (10) is

$$\mathbf{g}_1(r,z) = \int_0^{2\pi} \mathbf{F}_1(\theta,r,z)d\theta = 2\pi\begin{pmatrix} g_{11}(r,z) \\ g_{12}(r,z) \end{pmatrix},$$

where

$$\begin{aligned} g_{11}(r,z) = {} & \frac{r}{2\sqrt{-\mu_0}((\gamma_0+1)^2\mu_0+2\gamma_0+1)^{3/2}}\Big(-(\gamma_0+1)^2\mu_1(\gamma_0\mu_0+\mu_0+1) \\ & +\mu_0(\gamma_0\gamma_1-\mu_0((\gamma_0+1)(\beta_1-\gamma_0\gamma_1)+2\gamma_0 m_1+m_1) \\ & -(\gamma_0+1)^2\mu_0^2 m_1)\Big), \\ g_{12}(r,z) = {} & \frac{z(\gamma_0+1)(\mu_0(\mu_0(\beta_1+\gamma_1)+\gamma_1)-\gamma_0\mu_1)}{\sqrt{-\mu_0}((\gamma_0+1)^2\mu_0+2\gamma_0+1)^{3/2}}. \end{aligned}$$

Depending on the values of the parameters, the solutions of $\mathbf{g}_1(r,z)=0$ in $(0,\infty)\times\mathbb{R}$ may either not exist, or be nonisolated. As a result, the first-order averaging theory does not provide any information on the possible periodic orbits that can bifurcate from the zero-Hopf equilibrium. To apply the second-order averaging theory, it is necessary that the first-order averaging function $\mathbf{g}_1(r,z)$ be identically zero. This condition can be achieved by setting $\beta_1 = (\gamma_0\mu_1-\gamma_1\mu_0^2-\gamma_1\mu_0)/(\mu_0^2)$ and $m_1 = (-\gamma_0\mu_1+\gamma_1\mu_0-\mu_1)/(\mu_0^2)$. As outlined in Section 2.2, the averaging function of the second-order $\mathbf{g}_2(r,z)$ is

$$\begin{aligned} \mathbf{g}_2(r,z) &= \int_0^{2\pi}\Big(D_{(r,z)}F_1(\theta,r,z)\int_0^{\theta}F_1(s,r,z)ds+F_2(\theta,r,z)\Big)d\theta \\ &= \begin{pmatrix} \dfrac{r(Ar^2+Bz^2+C)}{4(-\mu_0)^{3/2}(\mu_0+1)^2(\gamma_0^2\mu_0+2\gamma_0(\mu_0+1)+\mu_0+1)^{3/2}} \\ \dfrac{z(Dr^2+Ez^2+F)}{(-\mu_0)^{5/2}(\mu_0+1)^2(\gamma_0^2\mu_0+2\gamma_0(\mu_0+1)+\mu_0+1)^{3/2}} \end{pmatrix}, \end{aligned}$$

where the constants A, B, C, D, E, and F are defined in Appendix A. Therefore, the system $\mathbf{g}_2(r,z)=0$ has isolated simple solutions $(0,\infty)\times\mathbb{R}$ if, and only if, $\dfrac{CD-AF}{AE-BD}>0$ and $\dfrac{BF-CE}{AE-BD}>0$, or $\dfrac{C}{A}<0$.

For $\dfrac{C}{A}<0$, $\dfrac{CD-AF}{AE-BD}>0$, and $\dfrac{BF-CE}{AE-BD}>0$, system $\mathbf{g}_2(r,z)=0$ has three isolated solutions in $(0,\infty)\times\mathbb{R}$. These solutions are

$$(r_1,z_1) = \left(\sqrt{\frac{BF-CE}{AE-BD}}, \sqrt{\frac{CD-AF}{AE-BD}}\right),$$

$$(r_2,z_2) = \left(\sqrt{\frac{BF-CE}{AE-BD}}, -\sqrt{\frac{CD-AF}{AE-BD}}\right),$$

and $(r_3,z_3) = \left(\sqrt{-\dfrac{C}{A}},0\right)$. The Jacobian determinant of $\mathbf{g}_2(r,z)$ at these solutions are

$$\mathrm{Det}\left(\frac{\partial\mathbf{g}_2}{\partial(r,z)}\right)\Big|_{(r,z)=(r_i,z_i)} = \frac{(CD-AF)(CE-BF)}{(BC-AE)\mu_0^4(\mu_0+1)^4((\gamma_0+1)^2\mu_0+2\gamma_0+1)^3} \neq 0,$$

for $i = 1, 2$ and

$$\mathrm{Det}\left(\frac{\partial \mathbf{g}_2}{\partial(r,z)}\right)\bigg|_{(r,z)=(r_3,z_3)} = \frac{C(CD-AF)}{2A\mu_0^4(\mu_0+1)^4((\gamma_0+1)^2\mu_0+2\gamma_0+1)^3} \neq 0.$$

Hence, Theorem 3 guarantees that for a sufficiently small $|\varepsilon| > 0$, there exists a periodic solution of system (10), corresponding to the point (r_i, z_i) of the form $(r(\theta,\varepsilon), z(\theta,\varepsilon))$ such that $(r_i(0,\varepsilon), z_i(0,\varepsilon)) \to (r_i, z_i)$ as $\varepsilon \to 0$. Thus, system (9) has the periodic solution:

$$(u_i(t,\varepsilon), v_i(t,\varepsilon), w_i(t,\varepsilon)) = (r_i(t,\varepsilon)\cos\theta(t,\varepsilon), r_i(t,\varepsilon)\sin\theta(t,\varepsilon), w_i(t,\varepsilon))$$

for sufficiently small $|\varepsilon| > 0$, with

$$(u_i(0,\varepsilon), v_i(0,\varepsilon), w_i(0,\varepsilon)) \to (r_i, 0, w_i),$$

as $\varepsilon \to 0$. Therefore, from the linear change of variables (8), system (7) has the periodic solution $(X_i(t,\varepsilon), Y_i(t,\varepsilon), Z_i(t,\varepsilon))$ for sufficiently small $|\varepsilon| > 0$ such that

$$(X_i(0,\varepsilon), Y_i(0,\varepsilon), Z_i(0,\varepsilon)) \to (X_0, Y_0, Z_0),$$

as $\varepsilon \to 0$, where

$$\begin{aligned} X_0 &= \frac{(\gamma_0+1)\sqrt{-\mu_0}\sqrt{(\gamma_0+1)^2\mu_0+2\gamma_0+1}\,r_i + w_i}{\mu_0+1}, \\ Y_0 &= \frac{\sqrt{-\mu_0}\sqrt{(\gamma_0+1)^2\mu_0+2\gamma_0+1}\,r_i - \mu_0 w_i}{\mu_0+1}, \\ Z_0 &= w_i. \end{aligned}$$

Consequently, system (6) has a periodic solution

$$(x_i(t,\varepsilon), y_i(t,\varepsilon), z_i(t,\varepsilon)) = (\varepsilon X_i(t,\varepsilon), \varepsilon Y_i(t,\varepsilon), \varepsilon Z_i(t,\varepsilon))$$

for sufficiently small $|\varepsilon| > 0$ such that $(x_i(0,\varepsilon), y_i(0,\varepsilon), z_i(0,\varepsilon)) \to (0,0,0)$ as $\varepsilon \to 0$. Therefore, this periodic solution bifurcates from the zero-Hopf equilibrium point localized at the origin of coordinates when $\varepsilon = 0$. In summary, the modified van der Pol–Duffing system (1) has three periodic orbits bifurcating from the origin. This completes the proof of statement (a).

For $\frac{CD-AF}{AE-BD} > 0$, $\frac{BF-CE}{AE-BD} > 0$, and $\frac{C}{A} > 0$, system $\mathbf{g}_2(r,z) = 0$ has two isolated solutions in $(0,\infty) \times \mathbb{R}$. Applying the same arguments used in the proof of statement (a) when we repeat through the changes in coordinates, the modified van der Pol–Duffing system (1) has two periodic orbits bifurcating from the origin. This completes the proof of statement (b).

For $\frac{CD-AF}{AE-BD} < 0$ or $\frac{BF-CE}{AE-BD} < 0$, and $\frac{C}{A} < 0$, the system $\mathbf{g}_2(r,z) = 0$ has one isolated solution in $(0,\infty) \times \mathbb{R}$. Applying the same arguments used in the proof of statement (a) when we repeat through the changes in coordinates, the modified van der Pol–Duffing system (1) has one periodic orbit bifurcating from the origin. This completes the proof of statement (c).

5. Proof of Theorem 2

Using the notation of Section 2.2, it follows that for $m \neq 0$, the expression of the Poincaré compactification of the modified van der Pol–Duffing system (1) in the local chart U_1 is

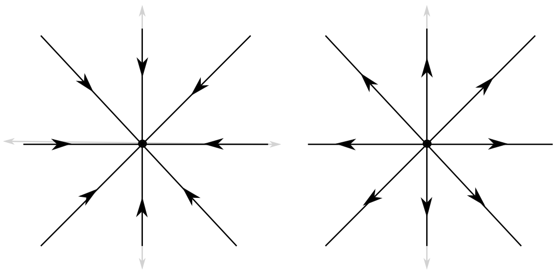
$$\begin{aligned} \dot{u} &= mu + w^2 - (1 + m\mu)uw^2 - vw^2 - mu^2w^2, \\ \dot{v} &= mv + (\beta uw^2 - \gamma + m\mu)vw^2 - muvw^2, \\ \dot{w} &= mw(1 - \mu w^2 - uw^2). \end{aligned} \tag{11}$$

System (11) admits a unique infinite singular point, $p_0 = (0, 0, 0)$, such that the Jacobian matrix of the differential system (11) has the eigenvalue m with multiplicity 3. Hence, p_0 is an unstable star node if $m > 0$, and a stable star node if $m < 0$. The phase portraits of system (11) restricted to infinity are shown in Figure 3.

Now, we consider the Poincaré compactification of system (1) in the local chart U_2:

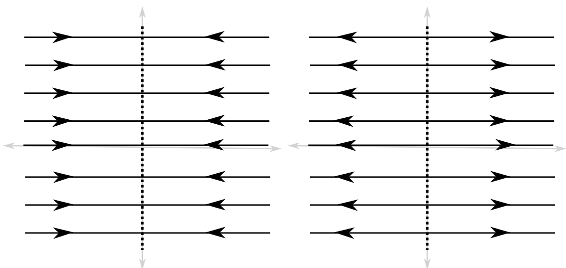
$$\begin{aligned} \dot{u} &= mw^2 - mu^3 + (1 + m\mu)uw^2 - u^2w^2 + uvw^2, \\ \dot{v} &= (\beta + (1 - \gamma)v - uv + v^2)w^2, \\ \dot{w} &= (1 - u + v)w^3. \end{aligned} \tag{12}$$

System (12) has the straight line $(0, v, 0)$ for all v of equilibrium points. Analyzing together systems (11) and (12) on $w = 0$, we obtain the phase portraits shown in Figure 4.



(a) $m < 0$ **(b)** $m > 0$

Figure 3. Local phase portrait of system (11) at the infinite singular point $(0, 0, 0)$ in the local chart U_1.



(a) $m > 0$ **(b)** $m < 0$

Figure 4. Local phase portrait of system (12) in a neighborhood of the straight line $v = w = 0$ filled with infinite singularities in the local chart U_2.

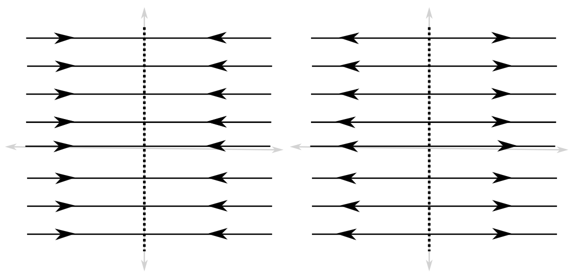
In the local chart U_3, system (1) becomes

$$\begin{aligned} \dot{u} &= -mu^3 + (\gamma + m\mu)uw^2 + mvw^2 - \beta uvw^2, \\ \dot{v} &= (-1 + u - (1 - \gamma)v - v^2\beta)w^2, \\ \dot{w} &= (\gamma - \beta v)w^3. \end{aligned} \tag{13}$$

In this chart, we only need to study if the origin of coordinates is an equilibrium point because all the other infinite equilibrium points have been studied in the local charts U_1 ad U_2, and the origin is an equilibrium on the circle of infinite equilibria (see Figure 5).

The flow of the Poincaré compactification in the local chart V_i, for $i = 1, 2, 3$, is the same as the flow in the respective local chart U_i.

Using the information about the infinite singularities in the local charts U_i and V_i, $i = 1, 2, 3$, we obtain the phase portrait of the modified van der Pol–Duffing system (1) on the sphere $\mathbb{S}^2$ of the infinity for $m \neq 0$. This completes the proof of Theorem 2.



(**a**) $m > 0$ (**b**) $m < 0$

Figure 5. Local phase portrait of system (13) at the infinite singular point $(0, 0, 0)$ in the local chart U_3.

6. Conclusions

In previous works on the modified van der Pol–Duffing oscillator, its zero-Hopf equilibria, zero-Hopf bifurcations, and how its orbits escape to or come from infinity were not studied. These three objects have thus been classified in this paper.

In Propositions 1 and 2, we have proven that only the equilibrium point localized at the origin of coordinates of the modified van der Pol–Duffing oscillator can be a zero-Hopf equilibrium for convenient values of the parameters of the system.

In Theorem 1, we have classified the zero-Hopf bifurcations from the zero-Hopf equilibrium at the origin of coordinates of the modified van der Pol–Duffing oscillator. Thus, we have characterized when three, two or one periodic orbits bifurcate from the zero-Hopf equilibrium. The tool used for obtaining these results on the zero-Hopf bifurcations is the averaging theory for studying periodic orbits.

Finally, though Theorem 2, we have studied the behavior of the orbits in a neighborhood of the infinity, using the Poincaré compactification.

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Data Availability Statement: This paper has no data.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Appendix A

$$
\begin{aligned}
A = \ & -3\pi(\gamma_0+1)^4\mu_0(\mu_0+1)(\gamma_0\mu_0+\mu_0+1),\\
B = \ & -12\pi(\gamma_0+1)^2\mu_0(\gamma_0\mu_0+\mu_0+1),\\
C = \ & 4\pi(\mu_0+1)^2\big(\mu_0^3 m_2+\beta_2\mu_0^3+\mu_0^4 m_2+\gamma_1\mu_0^2\mu_1-\mu_1^2-\mu_0\mu_1^2+\\
& \mu_0\mu_2+\mu_0^2\mu_2+\gamma_0^3\mu_0(\mu_0\mu_2-\mu_1^2)+\gamma_0^2(-\gamma_2\mu_0^3+\\
& -\mu_1(-\gamma_1\mu_0^2+3\mu_0\mu_1+\mu_1)+\mu_0^4+(3\mu_0+1)\mu_0\mu_2)+\\
& \gamma_0(\beta_2\mu_0^3-\gamma_2\mu_0^2(\mu_0+1)+2\mu_0^3(\mu_0+1)m_2+\gamma_1\mu_0\mu_1+\\
& 2\gamma_1\mu_0^2\mu_1-3\mu_0\mu_1^2+(3\mu_0+2)\mu_0\mu_2-2\mu_1^2)\big),\\
D = \ & 3\pi\gamma_0(\gamma_0+1)^3\mu_0^2(\mu_0+1),\\
E = \ & 2\pi\gamma_0(\gamma_0+1)\mu_0^2,\\
F = \ & 2\pi\mu_0(\gamma_0+1)(\mu_0+1)^2\big(\beta_2\mu_0^3-\gamma_0\mu_0\mu_2+\gamma_0\mu_1^2-\gamma_1\mu_0\mu_1+\gamma_2\mu_0^3+\gamma_2\mu_0^2\big).
\end{aligned}
$$

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