

ON NESTED CENTRAL CONFIGURATIONS OF THE $3n$ BODY PROBLEM

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ABSTRACT. In this work we consider the existence of $(3, n)$ -crowns in the classical Newtonian $3n$ -body problem, which are central configurations formed by three groups of n bodies with the same mass within each group, located at the vertices of three concentric regular polygons. We consider the case with dihedral symmetry, called nested $(3, n)$ -crowns, where the vertices of the polygons are aligned. We characterize the set of admissible radii for the polygons for which nested $(3, n)$ -crowns exist. We conclude with numerical evidences that suggest uniqueness for each set of three masses.

1. INTRODUCTION

The classical Newtonian N -body problem concerns about N punctual bodies with positives masses interacting between themselves through the gravitational attraction according to Newton’s gravitational law [14]. Since for $N \geq 3$ the knowledge of explicit solutions it is difficult, the study of particular solutions has been always a challenge. In the literature, we find Euler’s work [8], where the author construct a solution for the 3-body problem in which the bodies are collinear at each instant and the dynamics of the solution is given by a periodic rotation. Also Lagrange in [10] has constructed a solution for the 3-body problem, but in Lagrange’s solution the bodies remain at the vertices of an equilateral triangle at each instant of the motion. Such kinds of solutions were called homographic by Wintner in [19], where the initial form of the configuration is preserved in time evolution. In other words, the motion is given by dilations and rotations centered at the center of mass of the system.

In order to construct homographic solutions, the initial positions must satisfy special conditions: the acceleration vector on each body must be proportional to its position vector, with the same proportionality constant, see [13] and references therein. Such initial conditions are called central configurations. Their relevance reveals in many different aspects: the knowledge of a central configuration allows to compute a homographic solution, there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum, and simultaneous collisions of N bodies are performed in central configurations. For a deeper discussion of introduction and description of main properties about central configurations we refer the reader [18], [16] or [13].

Considering bodies with masses $\mu_i > 0$, $i = 1, 2, \dots, N$ and positions $\mathbf{r}_i \in \mathbb{R}^2$, $i = 1, 2, \dots, N$, the algebraic equations to compute central configurations are

$$(1) \quad \frac{\partial U}{\partial \mathbf{r}_i} + \lambda \mu_i \mathbf{r}_i = 0, \quad i = 1, \dots, N,$$

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where λ is mentioned above as proportionality constant,

$$U = \sum_{i < j} \frac{\mu_i \mu_j}{r_{ij}},$$

and r_{ij} is the Euclidean distance between $\mathbf{r}_i$ and $\mathbf{r}_j$. The equations (1) are well defined considering $r_{ij} \neq 0$, that is avoiding collisions.

The study of central configurations formed by groups of polygons (also called rings) goes back from Maxwell's work in 1859, see, for example [13] and the references given there. A regular polygon with equal masses at the vertices forms a central configuration, see [9]. Reversely, if the configuration of $N > 3$ bodies is a regular polygon, then Perko and Walter proved that the configuration is central if and only if the masses at the vertices are equal, see for example [15], and the references given there.

Several works have studied central configurations formed by κ groups of n bodies. Moeckel and Simó in [12], consider spatial configurations where two homothetic regular n -gons, $n \geq 2$, lie in two parallel planes with center on a common vertical axis. Considering equal masses on each polygon and unitary radius for one of it, they study bifurcations from planar (when the two planes are coincident) to spatial central configurations, proving that these bifurcations occur just for $n > 472$. The authors also prove that when the two planes are coincident, for each fixed ratio of masses there are exactly two possible central configurations. Earlier Elmabsout in [2] show the existence of two homothetic polygons central configurations, later Zhang and Xie, in [22] study the necessary conditions for the masses for the same problem. Without assuming equality in the masses, Zhang and Zhou, in [23], prove that a central configuration given by two concentric and coplanar regular n -gons rotated each other by an angle equal 0 or π/n satisfy that the masses in each polygon must be equal. Also, Yu and Zhang in [20] consider the case of two regular n -gons, and assuming equal masses in each polygon they have proved that the configuration is central if the polygons are rotated an angle of 0 or π/n .

Several authors refer to a planar central configuration formed by κ groups of n bodies in concentric n -gons as nested if all the vertices of the polygons are aligned and twisted if some polygons are rotated by an angle of π/n . Llibre and Mello in [11] have studied the twisted case for $\kappa = 3$ and $n = 2$ and $n = 3$, and the case $\kappa = 4$ and $n = 2$. Corbera, Delgado and Llibre in [5] using an implicit function theorem argument have proved the existence for the general case of nested crowns with arbitrary κ and n . In [24] Zhao and Chen have studied the existence in the case $\kappa = p + g$ with p aligned and g twisted n -gons. The nomenclature used in this work was introduced by Barrabés and Cors in [4], where the authors have studied the twisted case for $\kappa = 2$ proving the existence for an arbitrary n , and a complete description for $n = 3$ and $n = 4$. See also, [3], [6], [7] and [17] for other examples.

The aim of this work concerns about the inverse problem for central configurations in the $3n$ -body problem. We characterize the region of admissible radii for the so called nested $(3, n)$ -crowns. In Section 2 we state the geometric problem precisely, then using a suitable coordinates we get the functions to be studied in order to guarantee positivity of the masses, and we state the Main Theorem. In Section 3 we list and derive some technical lemmas useful for the proof of the main result which is placed in Section 4. In the last Section, we provide several numerical evidences to conjecture the uniqueness of the nested $(3, n)$ -crowns given a set of ordered masses.

2. STATEMENT OF THE PROBLEM AND MAIN RESULT

We start with the definitions and notations used through the paper, and present the equations of the problem. Although we will focus on the planar $3n$ -body problem, we

present the equations for the planar κn body problem. The details of the derivation of the equations can be found in [4], so we will omit them.

Consider the planar Newtonian κn -body problem consisting in κ groups of n bodies such that all bodies in the j -th group have equal mass m_j , $j = 1, \dots, \kappa$. Let $\mathbf{q}_{ji} \in \mathbb{R}^2$, $j = 1, \dots, \kappa$, $i = 1, \dots, n$, be the position of each body in a reference frame where the center of mass is at the origin of coordinates. Our aim is to study the existence of central configurations of the κn -body problem where the bodies within each group form a regular polygon. In [4] the concept of *crown* has been introduced.

Definition 1. [4] *A central configuration formed by κ groups of n bodies in a regular n -gon such that all the masses of the same group are equal is called a crown of κ rings or a (κ, n) -crown.*

We denote by $\mathbf{q}_j = \mathbf{q}_{j1}$ the *leader* of each group. As the bodies $\mathbf{q}_{ji}$, $i = 1, \dots, n$ are in a regular n -gon, once the position of the leader is known, all the others bodies in the same n -gon or ring are fixed. Introducing complex notation:

$$(2) \quad \mathbf{q}_j = a_j e^{i\varpi_j}, \quad \mathbf{q}_{ji} = \mathbf{q}_j e^{i2\pi(i-1)/n}, \quad i = 2, \dots, n, \quad j = 1, \dots, \kappa,$$

where ϖ_j and a_j are, respectively, the polar angle of the leader and the radius of the j -th polygon. Due to the rotational symmetry of the configuration, it is clear that it is enough to consider $\varpi_j \in (-\pi/n, \pi/n]$, $j = 1, \dots, \kappa$.

Introducing the notation (2) into the equations for central configuration (1) we get a system with $2\kappa n$ equations, although due to the symmetries of the problem only 2κ of them are independent. Therefore, it is enough to consider the equations related to the leaders

$$\frac{\partial U}{\partial \mathbf{q}_j} + \lambda m_j \mathbf{q}_j = 0, \quad j = 1, \dots, \kappa,$$

which using (2), can be rewritten as

$$(3) \quad \sum_{\substack{l=1 \\ l \neq j}}^{\kappa} m_l \sum_{k=1}^n \frac{a_l e^{i(\varpi_l - \varpi_j + 2\pi k/n)} - a_j}{(a_l^2 + a_j^2 - 2a_l a_j \cos(\varpi_l - \varpi_j + 2\pi k/n))^{3/2}} + \frac{m_j}{a_j^2} \sum_{k=1}^{n-1} \frac{e^{i2\pi k/n} - 1}{(2 - 2\cos(2\pi k/n))^{3/2}} + \lambda a_j = 0, \quad j = 1, \dots, \kappa.$$

Notice that the equations depend on the difference $\varpi_l - \varpi_j$, which represents how much one ring is rotated with respect to the other. Furthermore, equations (3) imply that both their real and imaginary parts must be null. Also notice that there is no order established beforehand on the sequence of a_j , $j = 1, \dots, \kappa$.

Definition 2. [4] *A (κ, n) -crown is called nested if $\varpi_l - \varpi_j = 0$ for all $l, j = 1, \dots, \kappa$.*

The case of $(2, n)$ -crowns has been extensively studied by several authors. Elmabsout [2] proved in 1991 the existence of central configurations for the Newtonian case with two polygons, followed by further contributions from Elmabsout and Bang in [1]. Zhang and Zhou in [23] prove that the masses within each group must be equal. Yu and Zhang [20] prove that either $\varpi_1 - \varpi_2 = 0$ (nested crown) or $\varpi_1 - \varpi_2 = \pi/n$. Also, Yu and Zhang [21] wonder if the two regular n -gons can have different number of bodies, and the answer was negative. Moeckel and Simó [12] prove that for every mass ratio, there are exactly two nested $(2, n)$ -crowns. Finally, Barrabés and Cors [4] prove that there exist one $(2, n)$ -crown when $\varpi_1 - \varpi_2 = \pi/n$ for any value of the mass ratio.

In the case of (κ, n) -crowns with $\kappa > 2$, there are several partial results for specific values of κ and n (see the Introduction).

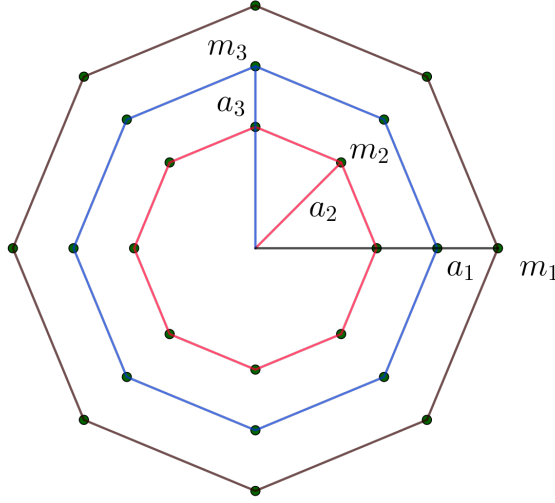


FIGURE 1. Representation of a nested $(3,8)$ -crown with $a_1 = 1$, $m_1 = 1$, $a_2 = 1/2$, $m_2 \simeq 0.059246$, $a_3 = 3/4$, $m_3 \simeq 0.937317$.

In this work we focus on the study of the existence of nested $(3,n)$ -crowns, see Figure 1. Specifically we consider the inverse problem, that is, we describe the set of admissible radii a_j , $j = 1, 2, 3$ for which there exist nested $(3,n)$ -crowns.

Notice that, in the nested case $\varpi_1 = \varpi_2 = \varpi_3 = 0$, the imaginary part of equations (3) are trivially satisfied. We eliminate λ using the first equation, so system (3) reduce to two equations linear respect to the masses. Without loss of generality, we set $0 < a_2 < a_3 < a_1 = 1$ and $m_1 = 1$. Note that with these assumptions we already avoid collisions. Thus in order to find a $(3,n)$ -nested crown we need to solve the following system of equations (see [4] for more details)

$$(4) \quad \begin{aligned} f(a_2)m_2 + \Omega(a_2, a_3)m_3 &= g(a_2), \\ \Omega(a_3, a_2)m_2 + f(a_3)m_3 &= g(a_3), \end{aligned}$$

where

$$(5) \quad f(x) = \frac{S_n}{x^2} - xF(x),$$

$$(6) \quad g(x) = S_n x - \frac{1}{x^2} F\left(\frac{1}{x}\right) = \frac{1}{x} f\left(\frac{1}{x}\right),$$

$$(7) \quad \Omega(a_2, a_3) = \frac{1}{a_2^2} F\left(\frac{a_3}{a_2}\right) - a_2 F(a_3),$$

with

$$(8) \quad F(x) = \sum_{k=1}^n \frac{1 - x \cos\left(\frac{2k\pi}{n}\right)}{[1 + x^2 - 2x \cos\left(\frac{2k\pi}{n}\right)]^{3/2}},$$

and

$$(9) \quad S_n = \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{\sin(k\pi/n)}.$$

It is straightforward that system (4) can be solved explicitly for m_2 and m_3 :

$$(10) \quad m_2 = \frac{N_2(a_2, a_3)}{D(a_2, a_3)}, \quad m_3 = \frac{N_3(a_2, a_3)}{D(a_2, a_3)},$$

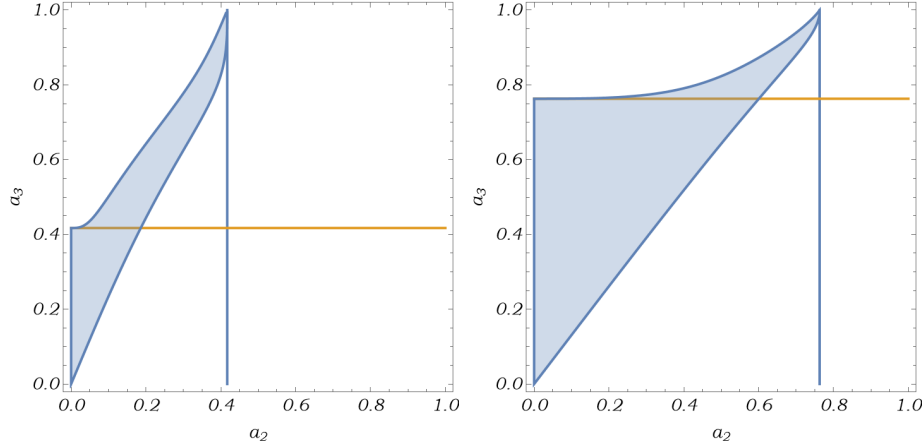


FIGURE 2. Region R of admissible values of (a_2, a_3) for nested $(3,2)$ -crowns (left) and nested $(3,230)$ -crowns. The horizontal line (in yellow) represents $a_3 = x_\#$, while the vertical line (in blue) represents $a_2 = x_\#$. In left where $x_\#(2) \simeq 0.417221$, and right $x_\#(230) \simeq 0.763037$. See Main Theorem and Lemma 3.

where

$$(11) \quad N_2(a_2, a_3) = \Omega(a_2, a_3)g(a_3) - f(a_3)g(a_2),$$

$$(12) \quad N_3(a_2, a_3) = \Omega(a_3, a_2)g(a_2) - f(a_2)g(a_3),$$

$$(13) \quad D(a_2, a_3) = \Omega(a_2, a_3)\Omega(a_3, a_2) - f(a_2)f(a_3),$$

provided that $D(a_2, a_3) \neq 0$.

The main result characterize the set of admissible values for the radii, that is, the values a_2 and a_3 in the (a_2, a_3) plane such that the masses m_2 and m_3 given in (10) are positive. Notice that without loss of generality, we can consider $(a_2, a_3) \in T$ where

$$(14) \quad T = \{(a_2, a_3); 0 < a_2 < a_3 < 1\}.$$

Due to the symmetry of the functions involved ($N_3(a_2, a_3) = N_2(a_3, a_2)$ and $D(a_2, a_3) = D(a_3, a_2)$), if there exist a nested $(3, n)$ -crown with $(a_2, a_3) \in T$ and (m_2, m_3) , then there also exist a nested $(3, n)$ -crown with (a_3, a_2) and (m_3, m_2) . That is, if (a_2, a_3) is admissible, so it is (a_3, a_2) .

Main Theorem. *For each $n \in \mathbb{N}$, $n \geq 2$, there exists $x_\#(n) \in (0, 1)$ such that the set of admissible values $(a_2, a_3) \in T$ for which there exists nested $(3, n)$ -crowns consists in a simply connected region $R \subset T$ limited by the segment between $(0, 0)$ and $(0, x_\#(n))$, a graph connecting $(0, x_\#(n))$ and $(x_\#(n), 1)$, and a graph connecting $(0, 0)$ and $(x_\#(n), 1)$.*

In Figure 2 we show the region R for two values of n . We will prove that $\lim_{n \rightarrow \infty} x_\#(n) = 1$, consequently the admissible region R tends to the whole region T as the number of the bodies in each ring grows.

3. TECHNICAL LEMMAS

Clearly, all the functions involved in the problem depend on a natural number $n \geq 2$, which is the number of the bodies in each regular n -gon. Therefore, in what follows n is a fixed value.

The following Lemma comes from Propositions 6 and 7 of [1] due to Bang and Elmabsout. See also Moeckel and Simó [12].

Lemma 1. [Bang and Elmabsout (2003)] For any $n > 1$, the function

$$G(x) = \sum_{k=1}^n \frac{1}{(1 + x^2 - 2x \cos(2k\pi/n))^{1/2}}.$$

satisfy the following properties:

- (i) $G(1/x) = xG(x)$,
- (ii) $G'(x) < 0$ for $x > 1$,
- (iii) for $x \in [0, 1)$ all the derivatives of $G(x)$ are positive.

The following Lemmas collect several properties of the functions F , f and Ω .

Lemma 2. Let F be the function given in (8). Then:

- (i) For $x \in (0, 1)$, $F(x) > 0$ and $F'(x) > 0$.
- (ii) For $x > 1$, $F(x) < 0$ and $F'(x) > 0$.

Proof. Consider the function G given in Lemma 1.

For $x \in (0, 1)$ is clear from the definition that $F(x) > 0$. Moreover, using Lemma 1

$$(15) \quad F(x) = (xG(x))'$$

and $F'(x) = 2G'(x) + xG''(x) > 0$ using Lemma 1 (iii).

For $x > 1$, using the definitions of functions F and G it is a straightforward computation that

$$(16) \quad F\left(\frac{1}{x}\right) = -x^2 G'(x).$$

Then, using also Lemma 1 we have that

$$(17) \quad G'\left(\frac{1}{x}\right) = -x^2 F(x) > 0.$$

Thus, $F(x) < 0$ for $x > 1$. Moreover, considering again (16) in the following form

$$G'(1/x) = -x^2 F(x).$$

and differentiating it, we obtain

$$x^4 F'(x) = G''(1/x) - 2x^3 F(x) > 0$$

for $x > 1$. This concludes the proof. $\square$

Lemma 3. Let f be the function given in (5). Then:

- (i) For $x > 1$, $f(x) > 0$.
- (ii) For $x \in (0, 1)$, there exists a unique $x_{\sharp} = x_{\sharp}(n) \in (0, 1)$ such that $f(x_{\sharp}) = 0$, $f(x) > 0$ if $x \in (0, x_{\sharp})$ and $f(x) < 0$ if $x \in (x_{\sharp}, 1)$.
- (iii) For $x \in (0, 1)$, $f'(x) < 0$.

Moreover,

$$\lim_{n \rightarrow \infty} x_{\sharp} = 1.$$

Proof. From Lemma 2, first statement follows. For the second and third statement, using (15) it follows that $f(x) = S_n x^{-2} - x(xG(x))'$, so differentiating we have

$$f'(x) = -2S_n x^{-3} - (xG(x))' - x(xG(x))'' < 0,$$

because G and all its derivatives are positive if $x \in (0, 1)$, see Lemma 1. Also,

$$-\infty = \lim_{x \rightarrow 1^-} f(x) < f(x) < \lim_{x \rightarrow 0^+} f(x) = +\infty.$$

Therefore, there exists a unique $x_{\sharp} \in (0, 1)$ such that $f(x_{\sharp}) = 0$.

Finally, $x_{\sharp}$ is the solution of

$$S_n = x^3 F(x),$$

in the interval $(0, 1)$. In this interval F is an increasing function with $\lim_{x \rightarrow 1} F(x) = +\infty$ and $\lim_{n \rightarrow \infty} S_n = +\infty$. Therefore, we have that $x_{\sharp} \rightarrow 1$. This concludes the proof. $\square$

Lemma 4. *Let F be the function given in (8). The function $\varphi(x) = x^3 F(x)$ satisfies $\varphi'(x) > 0$ for all $x \neq 1$.*

Proof. For $x \in (0, 1)$, by Lemma 2 it is clear that $\varphi'(x) > 0$.

For $x > 1$, using (17)

$$\varphi(x) = x^3 F(x) = x^3 \left(-\frac{1}{x^2} G' \left(\frac{1}{x} \right) \right) = -x G' \left(\frac{1}{x} \right).$$

Differentiating twice we have

$$(18) \quad \varphi'(x) = -G' \left(\frac{1}{x} \right) + \frac{1}{x} G'' \left(\frac{1}{x} \right),$$

$$(19) \quad \varphi''(x) = \frac{1}{x^2} G'' \left(\frac{1}{x} \right) - \frac{1}{x^2} G'' \left(\frac{1}{x} \right) - \frac{1}{x^3} G''' \left(\frac{1}{x} \right) = -\frac{1}{x^3} G''' \left(\frac{1}{x} \right).$$

It follows by Lemma 1, that $\varphi''(x) < 0$, that is, the function $\varphi'(x)$ is decreasing. Moreover, using (18), Lemma 1 item (iii), and the fact that $\lim_{x \rightarrow 0} G'(x) = 0$, we have that

$$\lim_{x \rightarrow +\infty} \varphi'(x) = \lim_{x \rightarrow +\infty} \left[-G' \left(\frac{1}{x} \right) + \frac{1}{x} G'' \left(\frac{1}{x} \right) \right] \geq 0.$$

Therefore, $\varphi'(x) > 0$ for $x > 1$. This concludes the proof. $\square$

Lemma 5. *Let Ω be the function given in (7). Then:*

- (i) *For $x > y > 1$, $\Omega(x, y) > 0$.*
- (ii) *For $0 < x < y < 1$, $\Omega(x, y) < 0$ and $\Omega(y, x) > 0$.*

Proof. Recall that from its definition

$$\Omega(x, y) = \frac{1}{x^2} F \left(\frac{y}{x} \right) - x F(y).$$

For $x > y > 1$, he have that $0 < \frac{y}{x} < 1$, so using Lemma 2 it follows that $\Omega(x, y) > 0$.

For $0 < x < y < 1$, he have that $\frac{y}{x} > 1$, again using Lemma 2 we have that $\Omega(x, y) < 0$. Finally, introducing the change $x = \frac{1}{t}$ and $y = \frac{s}{t}$, we have that $\frac{1}{t} < \frac{s}{t} < 1$ and

$$\Omega(y, x) = st^2 \left[\varphi \left(\frac{1}{s} \right) - \varphi \left(\frac{1}{t} \right) \right] > 0$$

using Lemma 4. $\square$

4. PROOF OF THE MAIN THEOREM

We fix a value of n and in what follows $x_{\sharp} = x_{\sharp}(n)$ is given by Lemma 3. We divide the region T (14) in two subregions:

$$(20) \quad T_1 = \{(a_2, a_3) \in T; 0 < a_2 < x_{\sharp}\}, \quad T_2 = T \setminus T_1.$$

Consider also the region

$$(21) \quad \tilde{R} = \{(a_2, a_3) \in T; 0 < a_2 \leq x_{\sharp} \cdot a_3 \leq x_{\sharp}\} \subset T_1.$$

See Figure 3.

First we prove that there are no admissible values in T_2 .

Theorem 1. *There are not nested $(3, n)$ -crowns for any $(a_2, a_3) \in T_2$.*

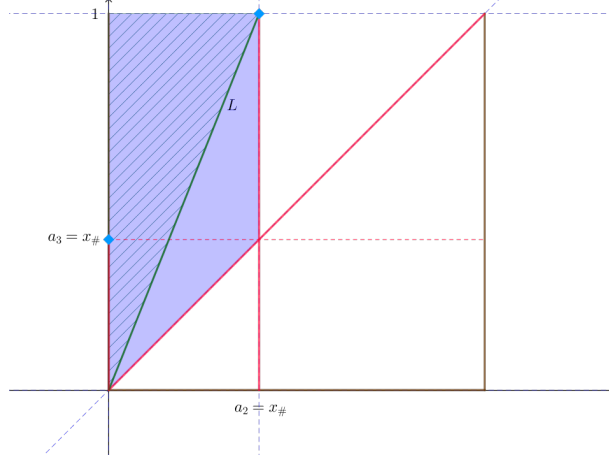


FIGURE 3. Regions T_1 (20) (in blue) and $\tilde{R}$ (21) (scratched). The line L in green corresponds to $a_3 = a_2/x_\#$.

Proof. For $(a_2, a_3) \in T$, by Lemma 5 we have that $\Omega(a_2, a_3) < 0$ and $\Omega(a_3, a_2) > 0$. For $(a_2, a_3) \in T_2$, $x_\# < a_2, a_3$ and by Lemma 3 (ii), follows that $f(a_2) < 0$ and $f(a_3) < 0$. Therefore, the denominator (13) satisfies

$$D(a_2, a_3) = \Omega(a_2, a_3)\Omega(a_3, a_2) - f(a_2)f(a_3) < 0.$$

Now, consider the numerator (12). Using (6) and Lemma 3 we have that $g(a_2) > 0$ and $g(a_3) > 0$. Therefore, $N_3(a_2, a_3) > 0$, and from (10), the mass $m_3(a_2, a_3) < 0$. $\square$

Using Theorem 1, it follows that the possible existence of nested $(3, n)$ -crowns can only occur in the region T_1 . See Figure 3.

Next, we study the sign of each factor involved in the expressions (10). Using 6, we rewrite the numerator of m_2, m_3 (11), (12) as

$$(22) \quad N_2(a_2, a_3) = \frac{1}{a_3}\Omega(a_2, a_3)f\left(\frac{1}{a_3}\right) - \frac{1}{a_2}f(a_3)f\left(\frac{1}{a_2}\right),$$

$$(23) \quad N_3(a_2, a_3) = \frac{1}{a_2}\Omega(a_3, a_2)f\left(\frac{1}{a_2}\right) - \frac{1}{a_3}f(a_2)f\left(\frac{1}{a_3}\right),$$

where f and Ω are given in (5) and (7).

Proposition 1. *Let $N_3(a_2, a_3)$ be the function given in (23) for $(a_2, a_3) \in T$ (14), $x_\#$ given in Lemma 3 and the region $\tilde{R}$ (21). Then:*

- (1) $\lim_{(a_2, a_3) \rightarrow (a, a)} N_3(a_2, a_3) = +\infty$, for any $a \in (0, 1)$,
- (2) $N_3(x_\#, a_3) > 0$ for $x_\# < a_3 < 1$, and $N_3(x_\#, a_3) = 0$,
- (3) $N_3(a_2, a_3) < 0$ for all $(a_2, a_3) \in \tilde{R}$,
- (4) $\frac{\partial N_3}{\partial a_3}(a_2, a_3) < 0$ for $x_\# a_3 \leq a_2 < x_\#$.

Proof. First claim is consequence of the fact that $\lim_{x \rightarrow 1^-} F(x) = +\infty$. Second claim is straightforward from Lemmas 3 and 5 for $a_3 < 1$, and the fact that $\Omega(1, x_\#) = 0$.

For the third claim, we write N_3 in terms of the function F :

$$\begin{aligned}
N_3(a_2, a_3) &= \frac{1}{a_2} \Omega(a_3, a_2) f\left(\frac{1}{a_2}\right) - \frac{1}{a_3} f(a_2) f\left(\frac{1}{a_3}\right) \\
&= \frac{1}{a_2} \left(S_n a_2^2 - \frac{1}{a_2} F\left(\frac{1}{a_2}\right) \right) \left(\frac{1}{a_3^2} F\left(\frac{a_2}{a_3}\right) - a_3 F(a_2) \right) \\
&\quad - \frac{1}{a_3} \left(S_n a_3^2 - \frac{1}{a_3} F\left(\frac{1}{a_3}\right) \right) \left(\frac{S_n}{a_2^2} - a_2 F(a_2) \right) \\
&= S_n \frac{a_2}{a_3^2} F\left(\frac{a_2}{a_3}\right) - \frac{1}{a_2^2 a_3^2} F\left(\frac{1}{a_2}\right) F\left(\frac{a_2}{a_3}\right) + \frac{a_3}{a_2^2} F\left(\frac{1}{a_2}\right) F(a_2) \\
&\quad - S_n^2 \frac{a_3}{a_2^2} + S_n \frac{1}{a_2^2 a_3^2} F\left(\frac{1}{a_3}\right) - \frac{a_2}{a_3^2} F\left(\frac{1}{a_3}\right) F(a_2).
\end{aligned}$$

On one hand, the 1st and 4th terms:

$$S_n \frac{a_2}{a_3^2} F\left(\frac{a_2}{a_3}\right) - S_n^2 \frac{a_3}{a_2^2} = -\frac{S_n}{a_3} f\left(\frac{a_2}{a_3}\right).$$

On the other hand, the second term can be written as follows:

$$\begin{aligned}
\frac{1}{a_2^2 a_3^2} F\left(\frac{1}{a_2}\right) F\left(\frac{a_2}{a_3}\right) &= \frac{1}{a_3^2 a_3} F\left(\frac{1}{a_2}\right) \left(\frac{a_2}{a_3} F\left(\frac{a_2}{a_3}\right) - S_n \frac{a_3^2}{a_2^2} + S_n \frac{a_3^2}{a_2^2} \right) \\
&= -\frac{1}{a_3^2 a_3} F\left(\frac{1}{a_2}\right) f\left(\frac{a_2}{a_3}\right) + S_n \frac{a_3}{a_2^5} F\left(\frac{1}{a_2}\right)
\end{aligned}$$

Introducing the above expressions in $N_3(a_2, a_3)$:

$$\begin{aligned}
N_3(a_2, a_3) &= -\frac{S_n}{a_3} f\left(\frac{a_2}{a_3}\right) + \frac{1}{a_2^3 a_3} F\left(\frac{1}{a_2}\right) f\left(\frac{a_2}{a_3}\right) - S_n \frac{a_3}{a_2^5} F\left(\frac{1}{a_2}\right) \\
&\quad + \frac{a_3}{a_2^2} F\left(\frac{1}{a_2}\right) F(a_2) + S_n \frac{1}{a_2^2 a_3^2} F\left(\frac{1}{a_3}\right) - \frac{a_2}{a_3^2} F\left(\frac{1}{a_3}\right) F(a_2) \\
&= f\left(\frac{a_2}{a_3}\right) \left(\frac{1}{a_3^2 a_3} F\left(\frac{1}{a_2}\right) - \frac{S_n}{a_3} \right) + \frac{a_3}{a_2^2} F\left(\frac{1}{a_2}\right) \left(a_2 F(a_2) - \frac{S_n}{a_2^2} \right) \\
&\quad + \frac{1}{a_3^2} F\left(\frac{1}{a_3}\right) \left(\frac{S_n}{a_2^2} - a_2 F(a_2) \right) \\
&= f\left(\frac{a_2}{a_3}\right) \left(\frac{1}{a_3^2 a_3} F\left(\frac{1}{a_2}\right) - \frac{S_n}{a_3} \right) + a_3 f(a_2) \left(\frac{1}{a_3^2} F\left(\frac{1}{a_3}\right) - \frac{1}{a_2^2} F\left(\frac{1}{a_2}\right) \right) \\
(24) \quad &= \frac{1}{a_3} f\left(\frac{a_2}{a_3}\right) \left(\frac{1}{a_3^2} F\left(\frac{1}{a_2}\right) - S_n \right) + a_3 f(a_2) \left(\varphi\left(\frac{1}{a_3}\right) - \varphi\left(\frac{1}{a_2}\right) \right),
\end{aligned}$$

where $\varphi(x) = x^3 F(x)$ is given in Lemma 4.

Then, using that $f(a_2/a_3) \geq 0$ and $f(a_2) > 0$ in $\tilde{R}$ (Lemma 3), $F(1/a_2) < 0$ (Lemma 2), and that the function φ is increasing (Lemma 4), the claim is proved.

For the last claim, we consider $x_\# a_3 \leq a_2 < x_\#$ and we derive the above expression for N_3 (24) with respect the second variable:

$$\begin{aligned}
\frac{\partial N_3}{\partial a_3}(a_2, a_3) &= -\frac{1}{a_3^2} f\left(\frac{a_2}{a_3}\right) \left(\frac{1}{a_2^2} F\left(\frac{1}{a_2}\right) - S_n \right) - \frac{a_2}{a_3^3} f'\left(\frac{a_2}{a_3}\right) \left(\frac{1}{a_2^2} F\left(\frac{1}{a_2}\right) - S_n \right) \\
&\quad + f(a_2) \left(\varphi\left(\frac{1}{a_3}\right) - \varphi\left(\frac{1}{a_2}\right) \right) - \frac{1}{a_3} f(a_2) \varphi'\left(\frac{1}{a_3}\right) < 0,
\end{aligned}$$

using again Lemmas 2, 3, and 4. This completes the proof. $\square$

The above results proves the following result.

Proposition 2. *The set $N_3(a_2, a_3) = 0$ is a graph for $(a_2, a_3) \in T_1 \setminus \tilde{R}$ connecting the points $(0, 0)$ and $(x_\sharp, 1)$.*

Proof. From Proposition 1 it is clear that $N_3(a_2, a_3) = 0$ only for $(a_2, a_3) \in T_1 \setminus \tilde{R}$, and that fixed a value $a_2 \in (0, 1)$ there exists a unique value for a_3 such that $N_3(a_2, a_3) = 0$. $\square$

Remark 1. *The above Proposition proves that $N_3(a_2, a_3) > 0$ for all $(a_2, a_3) \in T_1$ above the graph $N_3 = 0$ for $a_2 \in (0, x_\sharp)$.*

Proposition 3. *Let $N_2(a_2, a_3)$ be the function given in (22) and T as in (14). Then $N_2(a_2, a_3) < 0$ for all $(a_2, a_3) \in T$.*

Proof. We consider first the case $a_3 < x_\sharp$. For the points satisfying $0 < a_2 < a_3 \leq x_\sharp < 1$, by Lemma 5 we have that $\Omega(a_2, a_3) < 0$ and using Lemma 3 we have that $f(1/a_2) > 0$, $f(1/a_3) > 0$ and $f(a_3) \geq 0$. Then $N_2(a_2, a_3) < 0$.

Now we consider the case $x_\sharp < a_3 < 1$. Using (24) we have

$$\begin{aligned} N_2(a_2, a_3) &= N_3(a_3, a_2) \\ &= \frac{1}{a_2} f\left(\frac{a_3}{a_2}\right) \left(\frac{1}{a_3^3} F\left(\frac{1}{a_3}\right) - S_n \right) + a_2 f(a_3) \left(\varphi\left(\frac{1}{a_2}\right) - \varphi\left(\frac{1}{a_3}\right) \right). \end{aligned}$$

In this case, using Lemmas 2 and 3, $f(a_3/a_2) > 0$, $f(a_3) < 0$, and $F(1/a_3) < 0$, and using Lemma 4 we have that $N_2(a_2, a_3) < 0$. $\square$

Remark 2. *From Proposition 3, there exist nested $(3, n)$ -crown for $(a_2, a_3) \in T_1$, if and only if, $N_3(a_2, a_3) < 0$ and $D(a_2, a_3) < 0$.*

Proposition 4. *Let $D(a_2, a_3)$ be the function given in (13) for $(a_2, a_3) \in T$ (14) and $x_\sharp$ given in Lemma 3. Then:*

- (1) $D(a_2, a_3) < 0$ in $\{(a_2, a_3) \in T_1; a_3 \leq x_\sharp \text{ or } a_3 \leq a_2/x_\sharp\}$;
- (2) $\lim_{a_3 \rightarrow 1^-} D(x_\sharp, a_3) = 0$, $\lim_{a_2 \rightarrow 0^+} D(a_2, x_\sharp) = 0$;
- (3) for any $a_2 \in (0, x_\sharp)$, $\lim_{a_3 \rightarrow 1^-} D(a_2, a_3) > 0$;
- (4) $\frac{\partial D}{\partial a_3}(a_2, a_3) > 0$ for any $(a_2, a_3) \in \tilde{R}$.

Proof. To prove the first claim, we consider first $0 < a_2 < a_3 \leq x_\sharp$. In this region, using Lemma 5 we have that $\Omega(a_2, a_3) < 0$ and $\Omega(a_3, a_2) > 0$ and using Lemma 3 $f(a_2) > 0$, $f(a_3) > 0$, so $D(a_2, a_3) < 0$.

Now we consider $0 < a_2 < a_3 \leq a_2/x_\sharp < 1$. We rewrite the function as follows:

$$\begin{aligned} D(x, y) &= \left(\frac{1}{x^2} F\left(\frac{y}{x}\right) - xF(y) \right) \left(\frac{1}{y^2} F\left(\frac{x}{y}\right) - yF(x) \right) - \left(\frac{S_n}{x^2} - xF(x) \right) \left(\frac{S_n}{y^2} - yF(y) \right) \\ &= \frac{1}{x^2 y^2} F\left(\frac{y}{x}\right) F\left(\frac{x}{y}\right) - \frac{y}{x^2} F(x) F\left(\frac{y}{x}\right) - \frac{x}{y^2} F(y) F\left(\frac{x}{y}\right) - \frac{S_n^2}{x^2 y^2} \\ &\quad + S_n \frac{y}{x^2} F(y) + S_n \frac{x}{y^2} F(x). \end{aligned}$$

Then

$$\begin{aligned}
D(a_2, a_3) &= \frac{1}{a_2^2 a_3^2} F\left(\frac{a_3}{a_2}\right) F\left(\frac{a_2}{a_3}\right) - \frac{a_3}{a_2^2} F(a_2) F\left(\frac{a_3}{a_2}\right) - \frac{a_2}{a_3^2} F(a_3) F\left(\frac{a_2}{a_3}\right) \\
&\quad - \frac{S_n^2}{a_2^2 a_3^2} + \frac{S_n a_3}{a_2^2} F(a_3) + \frac{S_n a_2}{a_3^2} F(a_2) \\
&= a_2 a_3 F\left(\frac{a_3}{a_2}\right) \left(\frac{1}{a_2^3 a_3^3} F\left(\frac{a_2}{a_3}\right) - \frac{1}{a_2^3} F(a_2) \right) + \frac{1}{a_3} F(a_3) \left(S_n \frac{a_3^2}{a_2^2} - \frac{a_2}{a_3} F\left(\frac{a_2}{a_3}\right) \right) \\
&\quad + \frac{S_n}{a_3^2} \left(-\frac{S_n}{a_2^2} + a_2 F(a_2) \right) \\
&= a_2 a_3 F\left(\frac{a_3}{a_2}\right) \left(\frac{1}{a_2^3 a_3^3} F\left(\frac{a_2}{a_3}\right) - \frac{a_3^3}{a_2^2} F\left(\frac{a_2}{a_3}\right) + \frac{a_3^3}{a_2^2} F\left(\frac{a_2}{a_3}\right) - \frac{1}{a_2^3} F(a_2) \right) \\
&\quad + \frac{1}{a_3} F(a_3) f\left(\frac{a_2}{a_3}\right) - \frac{S_n}{a_3^2} f(a_2) \\
&= a_2 a_3 F\left(\frac{a_3}{a_2}\right) \left(\frac{1}{a_2^3 a_3^3} F\left(\frac{a_2}{a_3}\right) (1 - a_3^6) + \varphi\left(\frac{a_2}{a_3}\right) - \varphi(a_2) \right) \\
&\quad + \frac{1}{a_3} F(a_3) f\left(\frac{a_2}{a_3}\right) - \frac{S_n}{a_3^2} f(a_2) < 0
\end{aligned}$$

because by Lemma 2 $F(a_3/a_2) < 0$ and $F(a_2/a_3) > 0$, by Lemma 4 φ is an increasing function, and by Lemma 3, $f(a_2/a_3) \leq 0$ for $a_3 \leq a_2/x_\sharp$ and $f(a_2) > 0$.

The second claim is a straightforward calculation.

For the third claim, using the last expression for $D(a_2, a_3)$ we have that

$$\lim_{a_3 \rightarrow 1^-} D(a_2, a_3) = f(a_2) \lim_{a_3 \rightarrow 1^-} F(a_3) - S_n f(a_2) = +\infty.$$

Finally,

$$\begin{aligned}
\frac{\partial D}{\partial y}(x, y) &= -\frac{2}{x^2 y^3} F\left(\frac{y}{x}\right) F\left(\frac{x}{y}\right) + \frac{1}{x^3 y^2} F'\left(\frac{y}{x}\right) F\left(\frac{x}{y}\right) - \frac{1}{x y^4} F\left(\frac{y}{x}\right) F'\left(\frac{x}{y}\right) \\
&\quad - \frac{1}{x^2} F(x) F\left(\frac{y}{x}\right) - \frac{y}{x^3} F(x) F'\left(\frac{y}{x}\right) + \frac{2x}{y^3} F(y) F\left(\frac{x}{y}\right) - \frac{x}{y^2} F'(y) F\left(\frac{x}{y}\right) \\
&\quad + \frac{x^2}{y^4} F(y) F'\left(\frac{x}{y}\right) + \frac{2S_n^2}{x^2 y^3} + \frac{S_n}{x^2} F(y) + \frac{S_n y}{x^2} F'(y) - \frac{2S_n x}{y^3} F(x) \\
&= A + B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
A &= -\frac{2}{x^2 y^3} F\left(\frac{y}{x}\right) F\left(\frac{x}{y}\right) - \frac{1}{x y^4} F\left(\frac{y}{x}\right) F'\left(\frac{x}{y}\right) - \frac{1}{x^2} F(x) F\left(\frac{y}{x}\right) \\
&\quad + \frac{2x}{y^3} F(y) F\left(\frac{x}{y}\right) + \frac{x^2}{y^4} F(y) F'\left(\frac{x}{y}\right) + \frac{S_n}{x^2} F(y), \\
B &= \frac{1}{x^3 y^2} F'\left(\frac{y}{x}\right) F\left(\frac{x}{y}\right) - \frac{y}{x^3} F(x) F'\left(\frac{y}{x}\right) \\
&= \frac{1}{x^3} F'\left(\frac{y}{x}\right) \Omega(y, x), \\
C &= -\frac{x}{y^2} F'(y) F\left(\frac{x}{y}\right) + \frac{S_n y}{x^2} F'(y) = \frac{1}{y} F'(y) f\left(\frac{x}{y}\right), \\
E &= \frac{2S_n^2}{x^2 y^3} - \frac{2S_n x}{y^3} F(x) = \frac{2S_n}{y^3} f(x).
\end{aligned}$$

Recall that $F' > 0$ everywhere and for $0 < x < y < 1$ $F(y/x) < 0$ and $F(x/y) > 0$. Therefore, $A > 0$. From Lemma 5, $\Omega(y, x) > 0$, so $B > 0$. Finally, $(x, y) \in \tilde{R}$, so $f(x) > 0$ and $f(x/y) > 0$, so $C > 0$ and $E > 0$. The claim is proved. $\square$

The above proposition proves the following result.

Proposition 5. *The set $D(a_2, a_3) = 0$ is a graph for $(a_2, a_3) \in \tilde{R}$ connecting the points $(0, x_\#)$ and $(x_\#, 1)$.*

Therefore, using Propositions 2, 3 and 5 the Main Theorem is proved. We may note that the conclusion was possible using the line $a_3 = a_2/x_\#$ to control the sign of $D(a_2, a_3)$ and $N_3(a_2, a_3)$. In Figure 4, we show the regions involved for the case $n = 4$.

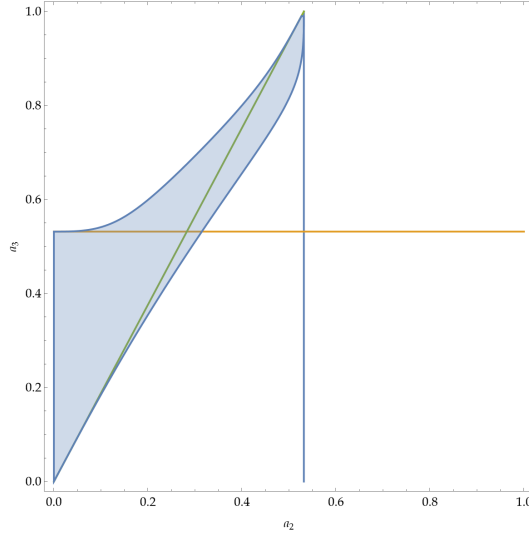


FIGURE 4. Representation of Theorem 2 for $n = 4$. The line L in green corresponds to $a_3 = a_2/x_\#$ the other straight lines are $a_2 = x_\#$ and $a_3 = x_\#$. The above blue curve represents $D(a_2, a_3) = 0$ and the below blue curve represents $N_3(a_2, a_3) = 0$. In the blue region we have positive masse.

5. DISCUSSION

We have proved that for any $(a_2, a_3) \in R$, the region limited by the graphs $N_3(a_2, a_3) = 0$ and $D(a_2, a_3) = 0$ in $a_2 \in (0, x_\#)$ (see Main Theorem), there exists a nested $(3, n)$ -crown. In this sense we have solved what is called the *invers problem*, that is, we have characterize the admissible radii. The direct problem consisting in proving the existence of nested $(3, n)$ -crown given a pair of positive masses, say (m_2, m_3) (recall $m_1 = 1$), is much more complicated in general.

In order to try to tackle this problem, we have studied numerically the behavior of the curves $m_2(a_2, a_3) = k$ and $m_3(a_2, a_3) = k$, for k any positive constant. We have observed that the region R is foliated by both sets of curves for all values of k . In particular we have observed that:

- The set $m_3(a_2, a_3) = k$ has only one component in R which is a graph connecting a point $(0, a^*(k))$ and the point $(x_\#, 1)$. Assuming this is true, is not difficult to see that $a^*(k) \in (0, x_\#)$ is the only solution of the equation

$$(25) \quad g(x) = kf(x),$$

where f and g are given in (5) and (6), and $a^*(k)$ is an increasing function of k .

- The set $m_2(a_2, a_3) = k$ has only one component in R which is a graph connecting the point $(0, x_\#)$ and a point on the curve $N_3(a_2, a_3) = 0$ whose first coordinate is also the solution of equation (25), $a_2 = a^*(k)$.

In Figures 5 and 6 we show some curves $m_3(a_2, a_3) = k$ and $m_2(a_2, a_3) = k$ for $n = 2$ and $n = 10$.

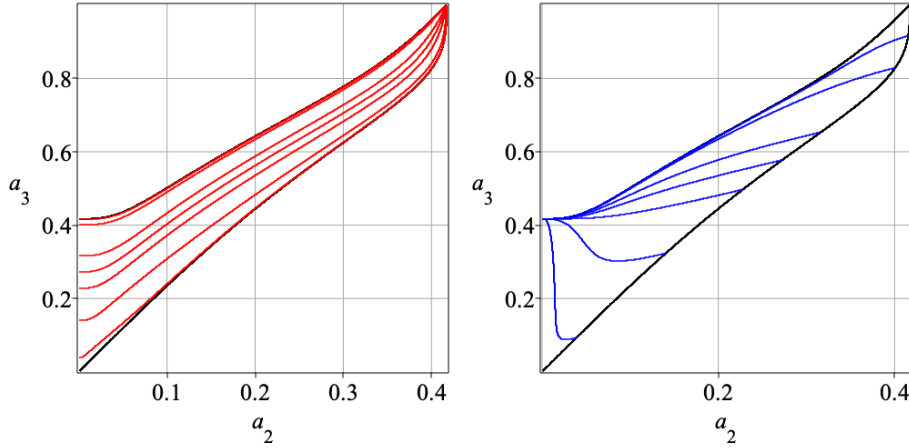


FIGURE 5. For $n = 2$, curves $D(a_2, a_3) = 0$, $N_3(a_2, a_3) = 0$ (in black) and $m_3(a_2, a_3) = k$ (left) and $m_2(a_2, a_3) = k$ (right) for $k = 0.001, 0.05, 0.25, 0.5, 1, 10, 100$.

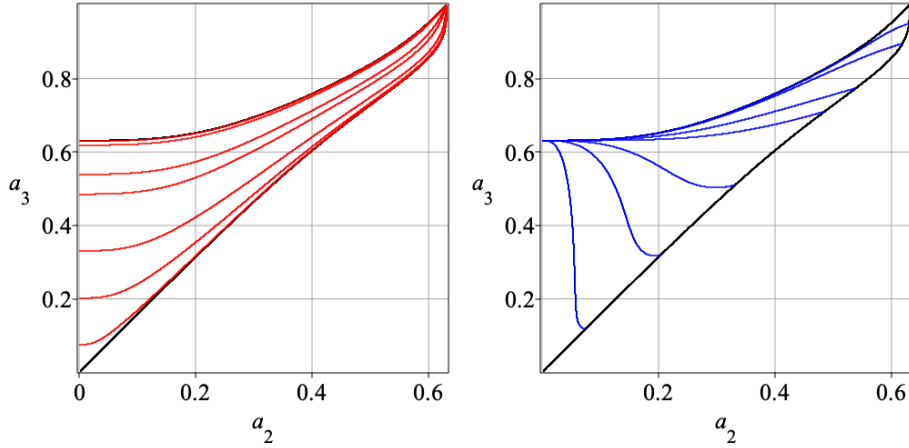


FIGURE 6. For $n = 10$, curves $D(a_2, a_3) = 0$, $N_3(a_2, a_3) = 0$ (in black) and $m_3(a_2, a_3) = k$ (left) and $m_2(a_2, a_3) = k$ (right) for $k = 0.001, 0.02, 0.1, 0.5, 1, 10, 100$.

Assuming that the behavior of the curves described above is true, then it is clear that the curves $m_3(a_2, a_3) = \text{constant}$ and $m_2(a_2, a_3) = \text{constant}$ intersect only once, for any positive constants. Thus, we propose the following conjecture.

Conjecture: For any pair (m_2, m_3) of positive masses, there exists a unique nested $(3, n)$ -crown.

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DATA AVAILABILITY

The authors declare that the data supporting the findings of this study are available within the paper.

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