

# ON THE LIMIT CYCLES BIFURCATING FROM THE PERIODIC ORBITS OF THE HAMILTONIAN SYSTEM WITH HAMILTONIAN

$$H = (x^{2n} + y^{2n})/(2n)$$

GERARDO H. ANACONA<sup>1</sup>, JAUME LLIBRE<sup>2</sup> AND BRUNO FREITAS<sup>1</sup>

**ABSTRACT.** This article concerns with the weak 16–th Hilbert problem. More precisely, we consider the centers  $\dot{x} = -y^{2n-1}$ ,  $\dot{y} = x^{2n-1}$  and we perturb them by all homogeneous polynomial of degree  $2n - 1$  for  $n = 2, 3, 4, 5, 6, 7, 8$ . We prove that the maximum number of limit cycles that can bifurcate from the periodic orbits of these centers for  $n = 2, 3, 4, 5, 6, 7, 8$ , under the mentioned perturbations and using the averaging theory of first order are 1, 4, 3, 9, 5, 6, 7, respectively.

## 1. INTRODUCTION

The second part of the 16th Hilbert’s problem asks for the maximum number  $H(n)$  of limit cycles and their distribution for all planar polynomial differential systems of a given degree. For more details on the 16th Hilbert’s problem see [6, 8, 9], and the references quoted therein. The problem on the number  $H(n)$  remains open, even for  $n = 2$ , and a general result about the distribution of limit cycles in planar polynomial differential systems can be found in [10].

A weak form of the 16th Hilbert’s problem, known now as the weak 16th Hilbert’s problem was proposed by Arnold [1], asking for the maximum number  $Z(m, n)$  of isolated zeros of the Abelian integrals of all polynomial 1-form of degree  $n$  over algebraic ovals of degree  $m$ , for more information on the weak 16th Hilbert’s problem see [4, 7, 9], and the hundreds of references quoted therein. Of course  $Z(m, n) \leq H(n)$ . But the weak 16th Hilbert’s problem is also extremely hard to study. In fact the weak 16th Hilbert’s problem is the problem of studying the maximum number of limit cycles that can bifurcate from the periodic orbits of a polynomial differential center having a rational first integral of degree  $m$  when it is perturbed inside the class of all polynomial differential systems of degree  $n$ .

Our goal is to provide lower bounds for the maximum number of limit cycles that can bifurcate from the periodic orbits of the center

$$(1) \quad \dot{x} = -y^{2n-1}, \quad \dot{y} = x^{2n-1},$$

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of degree  $2n - 1$  when it is perturbed by all homogeneous polynomials of degree  $2n - 1$ , i.e.

$$(2) \quad \dot{x} = -y^{2n-1} + \varepsilon \left( \sum_{i+j=0}^{2n-1} a_{i,j} x^i y^j \right), \quad \dot{y} = x^{2n-1} + \varepsilon \left( \sum_{i+j=0}^{2n-1} b_{i,j} x^i y^j \right),$$

where  $\varepsilon$  a small parameter. As usual in the differential systems (1) and (2) the dot denotes derivative with respect to the time.

System (1) is a Hamiltonian system with Hamiltonian  $H = (x^{2n} + y^{2n})/(2n)$ .

The centers (1) are global, i.e. the periodic orbits surrounding the center  $(0, 0)$  filled the whole plane  $\mathbb{R}^2$  with the exception of the point  $(0, 0)$ .

When we perturb the center (1) by all homogeneous polynomial of degree  $2n - 1$  we are studying a particular case of the weak Hilbert problem.

Our main result is the following.

**Theorem 1.** *For  $|\varepsilon| \neq 0$  sufficiently small and using averaging of first order the following statements hold.*

- (a) *For  $n = 2$  there are systems (2) having at least 1 limit cycles bifurcating from the periodic orbits of the center (1).*
- (b) *For  $n = 3$  there are systems (2) having at least 4 limit cycles bifurcating from the periodic orbits of the center (1).*
- (c) *For  $n = 4$  there are systems (2) having at least 3 limit cycles bifurcating from the periodic orbits of the center (1).*
- (d) *For  $n = 5$  there are systems (2) having at least 2 limit cycles bifurcating from the periodic orbits of the center (1).*
- (e) *For  $n = 6$  there are systems (2) having at least 5 limit cycles bifurcating from the periodic orbits of the center (1).*
- (f) *For  $n = 7$  there are systems (2) having at least 6 limit cycles bifurcating from the periodic orbits of the center (1).*
- (g) *For  $n = 8$  there are systems (2) having at least 7 limit cycles bifurcating from the periodic orbits of the center (1).*

In sections 2 and 3 we prove statements (a) and (b) of Theorem 1, respectively. Since the proofs of the other statements are very similar, for those statements in section 4 we only provide the averaging function of first order and study their simple zeros.

In view of Theorem 1 it remains open the following question: *For  $n \geq 4$  the maximum number of limit cycles of the differential system (2) that can be obtained using the averaging theory described in the appendix is  $n - 1$ ?*

## 2. PROOF OF STATEMENT (a) OF THEOREM 1

We write the differential system (1) in polar coordinates  $(r, \theta)$ , where  $x = r \cos \theta$  and  $y = r \sin \theta$ , and we get

$$(3) \quad \dot{r} = r^{2n-1}(\sin \theta \cos^{2n-1} \theta - \cos \theta \sin^{2n-1} \theta), \quad \dot{\theta} = r^{2n-2}(\sin^{2n} \theta + \cos^{2n} \theta).$$

Taking the variable  $\theta$  as the new independent variable or time, the differential system (3) becomes the differential equation

$$(4) \quad \frac{dr}{d\theta} = \frac{r(\sin \theta \cos^{2n-1} \theta - \cos \theta \sin^{2n-1} \theta)}{\sin^{2n} \theta + \cos^{2n} \theta}.$$

An easy computation shows that the periodic solutions  $r(\theta, r_0)$  surrounding the center  $r = 0$  such that  $r(\theta, r_0) = r_0$  are

$$(5) \quad r(\theta, r_0) = r_0 (\sin^{2n} \theta + \cos^{2n} \theta)^{-\frac{1}{2n}}.$$

In polar coordinates the differential system (2) with  $n = 2$  writes

$$\begin{aligned} \dot{r} = & \epsilon (\cos \theta (a_0 + r(a_2 + b_1) \sin \theta + r^2(a_5 + b_4) \sin^2 \theta) + r^2 \cos^3 \theta (a_3 + r(a_7 + b_6) \sin \theta) \\ & + r^3(a_9 + b_8) \cos \theta \sin^3 \theta + r \cos^2 \theta (a_1 + r(a_4 + b_3) \sin \theta + r^2(a_8 + b_7) \sin^2 \theta) \\ & + a_6 r^3 \cos^4 \theta + \sin \theta (b_0 + b_2 r \sin \theta + b_5 r^2 \sin^2 \theta + b_9 r^3 \sin^3 \theta)) + \frac{1}{4} r^3 \sin 4\theta, \\ \dot{\theta} = & \frac{1}{4} r^2 (\cos 4\theta + 3) + \frac{\epsilon}{r} [-\sin \theta (a_0 + a_2 r \sin \theta + a_5 r^2 \sin^2 \theta + a_9 r^3 \sin^3 \theta) \\ & + \cos \theta (-r(a_1 - b_2) \sin \theta - r^2(a_4 - b_5) \sin^2 \theta - (r^3(a_8 - b_9) \sin^3 \theta) + b_0) \\ & + r \cos^2 \theta (-r(a_3 - b_4) \sin \theta - (r^2(a_7 - b_8) \sin^2 \theta) + b_1) \\ & - r^2 \cos^3 \theta (r(a_6 - b_7) \sin \theta - b_3) + b_6 r^3 \cos^4 \theta]. \end{aligned}$$

Again taking the variable  $\theta$  as the new independent variable, we obtain the equivalent differential equation

$$\begin{aligned} r' = & \frac{r \sin 4\theta}{\cos 4\theta + 3} + \epsilon \left[ \frac{1}{r^2(\cos 4\theta + 3)} (4(r^2 \cos^3(\theta)(a_3 + r(a_7 + b_6) \sin(\theta)) \right. \\ & + r \cos^2 \theta (a_1 + r(a_4 + b_3) \sin \theta + r^2(a_8 + b_7) \sin^2 \theta) \\ & + \cos \theta (a_0 + r(a_2 + b_1) \sin \theta + r^2(a_5 + b_4) \sin^2 \theta + r^3(a_9 + b_8) \sin^3 \theta) \\ & + a_6 r^3 \cos^4 \theta + \sin \theta (b_0 + b_2 r \sin \theta + b_5 r^2 \sin^2 \theta + b_9 r^3 \sin^3 \theta)) \\ (6) \quad & - \frac{1}{r^2(\cos 4\theta + 3)^2} [4 \sin 4\theta (-\sin \theta (a_0 + a_2 r \sin \theta + a_5 r^2 \sin^2 \theta + a_9 r^3 \sin^3 \theta) \\ & + \cos \theta (-r(a_1 - b_2) \sin \theta - r^2(a_4 - b_5) \sin^2 \theta - (r^3(a_8 - b_9) \sin^3(\theta)) + b_0) \\ & + r \cos^2 \theta (-r(a_3 - b_4) \sin \theta - (r^2(a_7 - b_8) \sin^2 \theta) + b_1) \\ & - r^2 \cos^3 \theta (r(a_6 - b_7) \sin \theta - b_3) + b_6 r^3 \cos^4 \theta)] \\ & = F_0(\theta, r) + \epsilon F_1(\theta, r) + \mathcal{O}(\epsilon^2). \end{aligned}$$

Here the prime denotes derivative with respect to the variable  $\theta$ . The differential equation (6) is in the normal form (13) for applying the averaging theory of first order described in the Appendix, where  $\mathbf{x} = r$ ,  $t = \theta$  and  $T = 2\pi$ . Moreover, the differential equation (6) satisfies all the assumptions of Theorem 2 of the appendix.

The periodic solution  $r(\theta, r_0)$  given in (5) of the unperturbed equation (4) for  $n = 2$  is

$$r(\theta, r_0) = \frac{r_0}{\sqrt[4]{\sin^4 \theta + \cos^4 \theta}}.$$

All the periodic solutions surrounding the center of system (1) with  $n = 2$  intersect the positive  $x$ -axis  $\{(x, y) : x > 0, y = 0\}$ . So, without loss of generality, we can assume that the initial condition  $r_0$  of all these periodic solutions belong to the interval  $(0, k)$ , with  $k$  big enough, and consequently we can apply Theorem 2 of the appendix in the bounded interval  $[\varepsilon, k]$  with  $\varepsilon > 0$  sufficiently small.

The variational differential equation (15) of the appendix for the differential equation (6) with  $\varepsilon = 0$  in every of its periodic solutions are

$$\frac{dM}{d\theta} = \frac{\sin 4\theta}{\cos 4\theta + 3} M.$$

Its solution with  $M(0) = 1$  is

$$M(\theta) = \frac{\sqrt{2}}{\sqrt[4]{\cos 4\theta + 3}}.$$

Now we can compute the averaged function (16) of the appendix, that for our differential equation (6) is

$$\begin{aligned} \mathbf{F}(r_0) &= \frac{1}{\sqrt{2}} \int_0^{2\pi} \sqrt[4]{\cos 4\theta + 3} \left[ \frac{1}{r^2(\cos 4\theta + 3)} (4(r^2 \cos^3(\theta)(a_3 + r(a_7 + b_6) \sin(\theta)) \right. \\ &\quad + r \cos^2 \theta (a_1 + r(a_4 + b_3) \sin \theta + r^2(a_8 + b_7) \sin^2 \theta) \\ &\quad + \cos \theta (a_0 + r(a_2 + b_1) \sin \theta + r^2(a_5 + b_4) \sin^2 \theta + r^3(a_9 + b_8) \sin^3 \theta) \\ &\quad + a_6 r^3 \cos^4 \theta + \sin \theta (b_0 + b_2 r \sin \theta + b_5 r^2 \sin^2 \theta + b_9 r^3 \sin^3 \theta)) \\ &\quad - \frac{1}{r^2(\cos 4\theta + 3)^2} [4 \sin 4\theta (-\sin \theta (a_0 + a_2 r \sin \theta + a_5 r^2 \sin^2 \theta + a_9 r^3 \sin^3 \theta) \\ &\quad + \cos \theta (-r(a_1 - b_2) \sin \theta - r^2(a_4 - b_5) \sin^2 \theta - (r^3(a_8 - b_9) \sin^3(\theta)) + b_0) \\ &\quad + r \cos^2 \theta (-r(a_3 - b_4) \sin \theta - (r^2(a_7 - b_8) \sin^2 \theta) + b_1) \\ &\quad - r^2 \cos^3 \theta (r(a_6 - b_7) \sin \theta - b_3) + b_6 r^3 \cos^4 \theta)] \Big] d\theta \\ &= 2K \left( \frac{1}{2} \right) (a_1 + b_2) + \frac{1}{2\sqrt{2}} [\pi(3a_6 + a_8 + b_7 + 3b_9)] r_0^2 \end{aligned}$$

The positive zero of  $\mathbf{F}(r_0) = 0$  is

$$r_0 = 2\sqrt[4]{2} \sqrt{\frac{K(\frac{1}{2})}{\pi}} \sqrt{-\frac{a_1 + b_2}{3a_6 + a_8 + b_7 + 3b_9}},$$

if  $(a_1 + b_2)(3a_6 + a_8 + b_7 + 3b_9) < 0$ . Hence, by Theorem 2 of the appendix, we get at most one limit cycle. This completes the proof of statement (a) of Theorem 1. Here the function  $K(s)$  is the complete elliptic integral of the first kind, and  $K(1/2) = 0.590170299508048..$

### 3. PROOF OF STATEMENT (b) OF THEOREM 1

In polar coordinates the differential system (2) with  $n = 3$  becomes

$$\begin{aligned}
 (7) \quad \dot{r} = & \frac{1}{4}r^5 \sin 4\theta + \epsilon \left[ \cos \theta (a_0 + r^4(a_{14} + b_{13}) \sin^4 \theta + r(a_2 + b_1) \sin \theta + r^5(a_{20} + b_{19}) \sin^5 \theta \right. \\
 & + r^2(a_5 + b_4) \sin^2 \theta + r^3(a_9 + b_8) \sin^3 \theta) + r \cos^2 \theta (a_1 + r^3(a_{13} + b_{12}) \sin^3 \theta + a_{15}r^5 \cos^6 \theta \\
 & + r^4(a_{19} + b_{18}) \sin^4 \theta + r(a_4 + b_3) \sin \theta + r^2(a_8 + b_7) \sin^2 \theta) + r^4 \cos^5 \theta (a_{10} + r(a_{16} + b_{15}) \sin \theta) \\
 & + r^3 \cos^4 \theta (r(a_{11} + b_{10}) \sin \theta + r^2(a_{17} + b_{16}) \sin^2 \theta + a_6) + r^2 \cos^3 \theta (r^2(a_{12} + b_{11}) \sin^2 \theta \\
 & + r^3(a_{18} + b_{17}) \sin^3 \theta + a_3 + r(a_7 + b_6) \sin \theta) + \sin \theta (b_0 + b_{14}r^4 \sin^4 \theta + b_2r \sin \theta + b_{20}r^5 \sin^5 \theta \\
 & \left. + b_5r^2 \sin^2 \theta + b_9r^3 \sin^3 \theta) \right] \\
 \dot{\theta} = & r^4 (\sin^6 \theta + \cos^6 \theta) + \frac{\epsilon}{r} \left[ -\sin \theta [a_0 + a_{14}r^4 \sin^4 \theta + a_2r \sin \theta + a_{20}r^5 \sin^5 \theta + a_5r^2 \sin^2 \theta \right. \\
 & + a_9r^3 \sin^3 \theta] + \cos \theta [-r(a_1 - b_2) \sin \theta - r^4(a_{13} - b_{14}) \sin^4 \theta - (r^5(a_{19} - b_{20}) \sin^5 \theta) \\
 & - r^2(a_4 - b_5) \sin^2 \theta - r^3(a_8 - b_9) \sin^3 \theta + b_0] + r^3 \cos^4 \theta [-r(a_{10} - b_{11}) \sin \theta \\
 & - (r^2(a_{16} - b_{17}) \sin^2 \theta) + b_6] + r^2 \cos^3 \theta [-r^2(a_{11} - b_{12}) \sin^2 \theta - (r^3(a_{17} - b_{18}) \sin^3 \theta) \\
 & - r(a_6 - b_7) \sin \theta + b_3] + r \cos^2 \theta [-r^3(a_{12} - b_{13}) \sin^3 \theta - (r^4(a_{18} - b_{19}) \sin^4 \theta) \\
 & \left. - r(a_3 - b_4) \sin \theta) - r^2(a_7 - b_8) \sin^2 \theta + b_1] - r^4 \cos^5 \theta (r(a_{15} - b_{16}) \sin \theta - b_{10}) + b_{15}r^5 \cos^6 \theta \right]
 \end{aligned}$$

Taking as independent variable the variable  $\theta$ , we obtain the equivalent differential equation

$$\begin{aligned}
 (8) \quad r' = & \frac{r \sin 4\theta}{4 (\sin^6 \theta \cos^6 \theta)} + \epsilon \left[ \frac{1}{r^4 (\sin^6 \theta + \cos^6 \theta)} \left[ \cos \theta [a_0 + r^4(a_{14} + b_{13}) \sin^4 \theta + r(a_2 + b_1) \sin \theta \right. \right. \\
 & + r^5(a_{20} + b_{19}) \sin^5 \theta + r^2(a_5 + b_4) \sin^2 \theta + r^3(a_9 + b_8) \sin^3 \theta] + r \cos^2 \theta [a_1 \\
 & + r^3(a_{13} + b_{12}) \sin^3 \theta + r^4(a_{19} + b_{18}) \sin^4 \theta + r(a_4 + b_3) \sin \theta + r^2(a_8 + b_7) \sin^2 \theta] \\
 & + r^4 \cos^5 \theta (a_{10} + r(a_{16} + b_{15}) \sin \theta) + r^3 \cos^4 \theta [r(a_{11} + b_{10}) \sin \theta + r^2(a_{17} + b_{16}) \sin^2 \theta + a_6] \\
 & + r^2 \cos^3 \theta [r^2(a_{12} + b_{11}) \sin^2 \theta + r^3(a_{18} + b_{17}) \sin^3 \theta + a_3 + r(a_7 + b_6) \sin \theta] + a_{15}r^5 \cos^6 \theta \\
 & \left. + \sin \theta [b_0 + b_{14}r^4 \sin^4 \theta + b_2r \sin \theta + b_{20}r^5 \sin^5 \theta + b_5r^2 \sin^2 \theta + b_9r^3 \sin^3 \theta] \right] \\
 & - \frac{\sin 4\theta}{4r^4 (\sin^6 \theta + \cos^6 \theta)^2} \left[ -\sin \theta [a_0 + a_{14}r^4 \sin^4 \theta + a_2r \sin \theta + a_{20}r^5 \sin^5 \theta + a_5r^2 \sin^2 \theta \right. \\
 & + a_9r^3 \sin^3 \theta] + \cos \theta [-r(a_1 - b_2) \sin \theta - r^4(a_{13} - b_{14}) \sin^4 \theta - (r^5(a_{19} - b_{20}) \sin^5 \theta) \\
 & \left. - r^2(a_4 - b_5) \sin^2 \theta - r^3(a_8 - b_9) \sin^3 \theta + b_0] + r^3 \cos^4 \theta [-r(a_{10} - b_{11}) \sin \theta \right.
 \end{aligned}$$

$$\begin{aligned}
& - (r^2(a_{16} - b_{17}) \sin^2 \theta + b_6] + r^2 \cos^3 \theta [-r^2(a_{11} - b_{12}) \sin^2 \theta - (r^3(a_{17} - b_{18}) \sin^3 \theta) \\
& - r(a_6 - b_7) \sin \theta + b_3] r \cos^2 \theta [-r^3(a_{12} - b_{13}) \sin^3 \theta - (r^4(a_{18} - b_{19}) \sin^4 \theta) - r(a_3 - b_4) \\
& \cdot \sin \theta - r^2(a_7 - b_8) \sin^2 \theta + b_1] - r^4 \cos^5 \theta (r(a_{15} - b_{16}) \sin \theta - b_{10}) + b_{15} r^5 \cos^6 \theta] \\
& = F_0(\theta, r) + \varepsilon F_1(\theta, r) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Again this differential equation is in the normal form (13) for applying the averaging theory described in the appendix.

The periodic solution  $r(\theta, r_0)$  given in (5) of the unperturbed equation (4) for  $n = 3$  is

$$r(\theta, r_0) = \frac{r_0}{\sqrt[6]{\sin^6 \theta + \cos^6 \theta}}.$$

The variational differential equation (15) of the appendix for the differential equation (8) with  $\varepsilon = 0$  in every of its periodic solutions is

$$\frac{dM}{d\theta} = \frac{\sin 4\theta}{4(\sin^6 \theta + \cos^6 \theta)} M$$

Its solution satisfying  $M(0) = 1$  is

$$M(\theta) = \frac{\sqrt{2}}{\sqrt[6]{3 \cos 4\theta + 5}};$$

The averaged function (16) of the appendix for the differential equation (8) is

$$\begin{aligned}
\mathbf{F}(r_0) = & \int_0^{2\pi} \frac{\sqrt[6]{3 \cos 4\theta + 5}}{\sqrt{2}} \left[ \frac{1}{r^4 (\sin^6 \theta + \cos^6 \theta)} [\cos \theta [a_0 + r^4(a_{14} + b_{13}) \sin^4 \theta \right. \\
& + r(a_2 + b_1) \sin \theta r^5(a_{20} + b_{19}) \sin^5 \theta + r^2(a_5 + b_4) \sin^2 \theta \\
& + r^3(a_9 + b_8) \sin^3 \theta] + r \cos^2 \theta [a_1 + r^3(a_{13} + b_{12}) \sin^3 \theta + r^4(a_{19} + b_{18}) \sin^4 \theta \\
& + r(a_4 + b_3) \sin \theta + r^2(a_8 + b_7) \sin^2 \theta] + r^4 \cos^5 \theta (a_{10} + r(a_{16} + b_{15}) \sin \theta) \\
& + r^3 \cos^4 \theta [a_6 + r(a_{11} + b_{10}) \sin \theta + r^2(a_{17} + b_{16}) \sin^2 \theta] + r^2 \cos^3 \theta \\
& \cdot [r^2(a_{12} + b_{11}) \sin^2 \theta + r^3(a_{18} + b_{17}) \sin^3 \theta + a_3 r(a_7 + b_6) \sin \theta] \\
& + a_{15} r^5 \cos^6 \theta + \sin \theta [b_0 + b_2 r \sin \theta + b_{14} r^4 \sin^4 \theta + b_{20} r^5 \sin^5 \theta + b_5 r^2 \sin^2 \theta] \\
& + b_9 r^3 \sin^3 \theta] - \frac{\sin 4\theta}{4r^4 (\sin^6 \theta + \cos^6 \theta)^2} [-\sin \theta [a_0 + a_{14} r^4 \sin^4 \theta + a_2 r \sin \theta \\
& + a_{20} r^5 \sin^5 \theta + a_5 r^2 \sin^2 \theta + a_9 r^3 \sin^3 \theta] + \cos \theta [-r(a_1 - b_2) \sin \theta \\
& - r^4(a_{13} - b_{14}) \sin^4 \theta - r^5(a_{19} - b_{20}) \sin^5 \theta - r^2(a_4 - b_5) \sin^2 \theta \\
& - r^3(a_8 - b_9) \sin^3 \theta + b_0] + r^3 \cos^4 \theta [-r(a_{10} - b_{11}) \sin \theta - r^2(a_{16} - b_{17})
\end{aligned}$$

$$\begin{aligned}
& \cdot \sin^2 \theta + b_6] + r^2 \cos^3 \theta [-r^2(a_{11} - b_{12}) \sin^2 \theta - (r^3(a_{17} - b_{18}) \sin^3 \theta) \\
& - r(a_6 - b_7) \sin \theta + b_3] r \cos^2 \theta [-r^3(a_{12} - b_{13}) \sin^3 \theta - (r^4(a_{18} - b_{19}) \sin^4 \theta) \\
& - r(a_3 - b_4) \sin \theta - r^2(a_7 - b_8) \sin^2 \theta + b_1] - r^4 \cos^5 \theta (r(a_{15} - b_{16}) \sin \theta - b_{10}) \\
& + b_{15} r^5 \cos^6 \theta] d\theta \\
& = Ar_0^5 + Br_0^3 + Cr_0^2 + Dr_0 + E,
\end{aligned}$$

where

$$\begin{aligned}
A &= 3.49066a_{15} + 1.0472a_{17} + 0.698132a_{19} + 0.698132b_{16} + 1.0472b_{18} + 3.49066b_{20}, \\
B &= 3.64298a_6 + 1.21433a_8 + 1.21433b_7 + 3.64298b_9, \\
C &= -2.22945b_3, \\
D &= 3.85524a_1 + 3.85524b_2, \\
E &= -2.39718b_0.
\end{aligned}$$

From Theorem 2 and the Descartes Theorem, it follows that the polynomial  $\mathbf{F}(r_0)$  given in (9) can have four positive simple roots. Therefore, by Theorem 2 the differential equation (8) can have four limit cycles. This completes the proof of statement (b) of Theorem 1.

#### 4. STATEMENTS (c) – (f) OF THEOREM 1

Following the steps of the proofs of statements (a) and (b) of Theorem 1 we obtain the following averaged function for the differential system (2) with  $n = 4$

$$(10) \quad \mathbf{F}(r_0) = Ar_0^7 + Br_0^5 + Cr_0^3 + Dr_0.$$

Where

$$\begin{aligned}
A &= 3.5916a_{28} + 1.06264a_{30} + 0.637582a_{32} + 0.513086a_{34} + 0.513086b_{29} + 0.637582b_{31}, \\
& + 1.06264b_{33} + 3.5916b_{35}, \\
B &= 3.6748a_{15} + 1.14634a_{17} + 0.73496a_{19} + 0.73496b_{16} + 1.14634b_{18} + 3.6748b_{20}, \\
C &= 3.77839a_6 + 1.25946a_8 + 1.25946b_7 + 3.77839b_9, \\
D &= 3.91384a_1 + 3.91384b_2.
\end{aligned}$$

Again from Theorem 2 and the Descartes Theorem, it follows that the polynomial  $\mathbf{F}(r_0)$  given in (10) can have three positive simple roots. Therefore, by Theorem 2 the differential system (2) with  $n = 4$  can have three limit cycles. This completes the proof of statement (c) of Theorem 1.

For  $n = 5$  the averaged function is

$$(11) \quad \mathbf{F}_0(r_0) = Ar_0^9 + Br_0^5 + Cr_0$$

where

$$\begin{aligned}
A &= 1.26163 \times 10^6 (-0.000488281a_{51} - 0.0000610352a_{53} - 0.0000610352b_{46} \\
&\quad - 0.000488281b_{48}), \\
B &= 1.26163 \times 10^6 (0.00488281a_{15} + 0.000732422a_{19} + 0.000732422b_{16} + \\
&\quad 0.00488281b_{20}), \\
C &= 1.26163 \times 10^6 (0.000488281a_1 + 0.000488281b_2).
\end{aligned}$$

By Theorem 2 the differential system (2) with  $n = 5$  can have two limit cycles. This completes the proof of statement (d) of Theorem 1.

For  $n = 6$  the averaged function is

$$(12) \quad \mathbf{F}_0(r_0) = A\mathbf{r}_0^{11} + B\mathbf{r}_0^9 + C\mathbf{r}_0^7 + D\mathbf{r}_0^5 + E\mathbf{r}_0^3 + F\mathbf{r}_0,$$

where

$$\begin{aligned}
A &= 1.11072a_{68} + 0.632414a_{70} + 0.451724a_{72} + 0.37024a_{74} + 0.337172b_{67} \\
&\quad + 0.37024b_{69} + 0.451724b_{71} + 0.632414b_{73} + 1.11072b_{75} + 3.70889b_{77}, \\
B &= 33.0314a_{47} + 103.111a_{49} + 146.977a_{51} + 112.981a_{53} + 112.981b_{46} \\
&\quad + 146.977b_{48} + 103.111b_{50} + 33.0314b_{52} + 34.4483b_{54}, \\
C &= -235.417a_{28} + 421.636a_{30} + 540.751a_{32} + 241.276a_{34} + 241.276b_{29} \\
&\quad + 540.751b_{31} + 421.636b_{33} - 235.417b_{35}, \\
D &= 1666.01a_{15} + 4513.83a_{17} + 5216.59a_{19} + 5216.59b_{16} + 4513.83b_{18} \\
&\quad + 1666.01b_{20}, \\
E &= 445233.a_6 + 199383.a_8 + 199383.b_7 + 445233.b_9, \\
F &= -8.83872 \times 10^6 a_1 - 8.83872 \times 10^6 b_2.
\end{aligned}$$

By Theorem 2 the differential system (2) with  $n = 6$  can have five limit cycles. This completes the proof of statement (e) of Theorem 1.

For  $n = 7$  the averaged function is

$$\mathbf{F}_0(r_0) = A\mathbf{r}_0^{13} + B\mathbf{r}_0^{11} + C\mathbf{r}_0^9 + D\mathbf{r}_0^7 + E\mathbf{r}_0^5 + F\mathbf{r}_0^3 + G\mathbf{r}_0,$$

where

$$\begin{aligned}
A &= 0.310323a_{101} + 0.289897a_{103} + 1.13414a_{93} + 0.642949a_{95} + 0.450968a_{97} \\
&\quad + 0.357764a_{99} + 0.642949b_{100} + 1.13414b_{102} + 3.74941b_{104} + 0.310323b_{49} \\
&\quad + 0.289897b_{92} + 0.357764b_{96} + 0.450968b_{98}, \\
B &= 1.15979a_{68} + 0.668748a_{70} + 0.478117a_{72} + 0.387684a_{74} + 0.344957b_{67} \\
&\quad + 0.387684b_{69} + 0.478117b_{71} + 0.668748b_{73} + 1.15979b_{75} + 3.77612b_{77}, \\
C &= 0.106875a_{46} + 1.18898a_{47} + 0.0511989a_{48} + 0.698827a_{49} + 0.132849a_{50} \\
&\quad + 0.510671a_{51} + 0.424775a_{53} + 0.424775b_{46} + 0.510671b_{48} - 0.132849b_{49} \\
&\quad + 0.698827b_{50} - 0.0511989b_{51} + 1.18898b_{52} + 3.80587b_{54}, \\
D &= 3.83938a_{28} + 0.106906a_{29} + 1.22268a_{30} + 0.111814a_{31} + 0.734545a_{32} \\
&\quad + 0.11994a_{33} + 0.550644a_{34} + 0.136414a_{35} - 0.136414b_{28} + 0.550644b_{29} \\
&\quad - 0.11994b_{30} + 0.734545b_{31} - 0.111814b_{32} + 1.22268b_{33} - 0.106906b_{34} \\
&\quad + 3.83938b_{35}, \\
E &= 3.87765a_{15} + 1.26227a_{17} + 0.777952a_{19} + 0.777952b_{16} + 1.26227b_{18} + 3.87765b_{20}), \\
F &= 3.9221a_6 + 1.30985a_8 + 1.30985b_7 + 3.9221b_9, \\
G &= 3.97487a_1 + 3.97487b_2.
\end{aligned}$$

By Theorem 2 the differential system (2) with  $n = 7$  can have six limit cycles. This completes the proof of statement (f) of Theorem 1.

For  $n = 8$  the averaged function is

$$\mathbf{F}_0(r_0) = A\mathbf{r}_0^{14} + B\mathbf{r}_0^{12} + C\mathbf{r}_0^{10} + D\mathbf{r}_0^8 + E\mathbf{r}_0^6 + F\mathbf{r}_0^4 + G\mathbf{r}_0^2 + H,$$

where

$$\begin{aligned}
A &= 0.649406a_{124} + 0.450442a_{126} + 0.350343a_{128} + 0.295184a_{130} + 0.265065a_{132} \\
&\quad + 0.251614a_{134} + 0.251614b_{121} + 0.265065b_{123} + 0.295184b_{125} + 0.350343b_{127} \\
&\quad + 0.450442b_{129} + 0.649406b_{131} + 1.14861b_{133} - 2.6223810^{12}b_{135}), \\
B &= 0.318561a_{101} + 0.291883a_{103} + 1.16806a_{93} + 0.668726a_{95} + 0.470317a_{97} \\
&\quad + 0.371514a_{99} + 0.668726b_{100} + 1.16806b_{102} + 3.79447b_{104} + 0.318561b_{49} \\
&\quad + 0.291883b_{92} + 0.371514b_{96} + 0.470317b_{98}, \\
C &= 1.18971a_{68} + 0.690633a_{70} + 0.493309a_{72} + 0.396571a_{74} + 0.346973b_{67} \\
&\quad + 0.396571b_{69} + 0.493309b_{71} + 0.690633b_{73} + 1.18971b_{75} + 3.8167b_{77}, \\
D &= 1.21409a_{47} + 0.715787a_{49} + 0.520323a_{51} + 0.426808a_{53} + 0.426808b_{46} \\
&\quad + 0.520323b_{48} + 0.715787b_{50} + 1.21409b_{52} + 3.84127b_{54}, \\
E &= 3.86869a_{28} + 1.24185a_{30} + 0.745109a_{32} + 0.552671a_{34} + 0.552671b_{29} \\
&\quad + 0.745109b_{31} + 1.24185b_{33} + 3.86869b_{35}, \\
F &= 3.89967a_{15} + 1.27394a_{17} + 0.779933a_{19} + 0.779933b_{16} + 1.27394b_{18} + 3.89967b_{20}, \\
G &= 3.93514a_6 + 1.31171a_8 + 1.31171b_7 + 3.93514b_9, \\
H &= 3.97651a_1 + 3.97651b_2.
\end{aligned}$$

By Theorem 2 the differential system (2) with  $n = 8$  can have seven limit cycles. This completes the proof of statement (f) of Theorem 1.

## 5. APPENDIX: AVERAGING THEORY AND DESCARTES THEOREM

**5.1. Averaging theory.** In this subsection we present the basic results from the averaging theory that we need for proving the main results of this paper.

We consider the problem of the bifurcation of  $T$ -periodic orbits from differential systems of the form

$$(13) \quad \mathbf{x}' = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \mathcal{O}(\varepsilon^2),$$

with  $\varepsilon = 0$  to  $\varepsilon \neq 0$  sufficiently small. Here the functions  $F_0, F_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$  and  $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$  are  $C^2$  functions,  $T$ -periodic in the first variable, and  $\Omega$  is an open subset of  $\mathbb{R}^n$ . The main assumption is that the unperturbed system

$$(14) \quad \mathbf{x}' = F_0(t, \mathbf{x}),$$

has a sub-manifold of dimension  $n$  of periodic orbits. A solution of this problem is given using the averaging theory.

Let  $\mathbf{x}(t, \mathbf{z})$  be the solution of system (14) such that  $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$ . We write the linearization of the unperturbed system along the periodic solution  $\mathbf{x}(t, \mathbf{z})$  as

$$(15) \quad \mathbf{y}' = D_{\mathbf{x}}F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}.$$

Let  $M_{\mathbf{z}}(t)$  be the fundamental matrix of the linear differential system (15) such that  $M_{\mathbf{z}}(0)$  is the identity matrix.

We assume that there exists an open set  $V$  whose closure  $Cl(V)$  is contained in  $\Omega$  such that for each  $\mathbf{z} \in Cl(V)$ , the solution  $\mathbf{x}(t, \mathbf{z})$  is  $T$ -periodic. So the set of periodic orbits with initial conditions in  $Cl(V)$  is isochronous for the system (14); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of  $T$ -periodic orbits from the periodic orbits  $\mathbf{x}(t, \mathbf{z})$  with  $\mathbf{z} \in Cl(V)$  is given in the following result.

**Theorem 2.** *We assume that there exists an open and bounded set  $V$  with  $Cl(V) \subset \Omega$  such that for each  $\mathbf{z} \in Cl(V)$ , the solution  $\mathbf{x}(t, \mathbf{z})$  is  $T$ -periodic, then consider the averaged function  $\mathbf{F} : Cl(V) \rightarrow \mathbb{R}^n$  defined by*

$$(16) \quad \mathbf{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z})) dt.$$

*If there exists  $p \in V$  with  $F(p) = 0$  and  $\det(\frac{d\mathbf{F}}{d\mathbf{z}}(p)) \neq 0$ , then there exists a  $T$ -periodic solution  $\mathbf{x}(t, \varepsilon)$  of system (13) such that  $\mathbf{x}(0, \varepsilon) \rightarrow p$  when  $\varepsilon \rightarrow 0$ .*

Theorem 3 is due to Malkin [11] and Roseau [12], a shorter proof is in [3].

**5.2. Descartes Theorem.** Since the previous averaging theory reduces the problem of finding limit cycles of the differential system (13) in finding simple zeros of the averaged function  $\mathbf{F}_0(\mathbf{z})$ . In our problem the averaged function  $\mathbf{F}_0(\mathbf{r})$  only depends on the radial variable  $r$  of the polar coordinates, and in order to study its simple we shall use the next result, a proof of it can be found in [2].

**Theorem 3. (Descartes Theorem)** *Consider the real polynomial  $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \dots + a_{i_r}x^{i_r}$  with  $0 \leq i_1 < i_2 < \dots < i_r$  and  $a_{i_j} \neq 0$  real constants for  $j \in \{0, 1, 2, \dots, r\}$ . When  $a_{i_j}a_{i_{j+1}} < 0$ , we say that in  $a_{i_j}$  and  $a_{i_{j+1}}$  there is a variation of sign. If the number of variations of signs is  $m$ , then  $p(x)$  has at most  $m$  positive real roots. Moreover, it is always possible to choose the coefficients of  $p(x)$  in such a way that  $p(x)$  has exactly  $r - 1$  positive real roots.*

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<sup>1</sup> DEPARTAMENTO DE MATEMÁTICA, IME–UFG, CAMPUS SAMAMBAIA, CP 131, CEP 74001–970, GOIÂNIA, GOIÁS, BRAZIL

*Email address:* `gerardohomero@discente.ufg.br`

*Email address:* `freitasm@ufg.br`

<sup>2</sup> DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BEL-LATERRA, BARCELONA, CATALONIA, SPAIN

*Email address:* `jllibre@mat.uab.cat`