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The dimension of planar elliptic measures arising from Lipschitz matrices in Reifenberg flat domains

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Abstract

In this paper we show that, given a planar Reifenberg flat domain with small constant and a divergence form operator associated to a real (not necessarily symmetric) uniformly elliptic matrix with Lipschitz coefficients, the Hausdorff dimension of its elliptic measure is at most 1. More precisely, we prove that there exists a subset of the boundary with full elliptic measure and with σ-finite one-dimensional Hausdorff measure. For Reifenberg flat domains, this result extends a previous work of Thomas H. Wolff for the harmonic measure.

Keywords Elliptic measure · Reifenberg flat domain

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1 Introduction and Main results

We study the dimension of planar elliptic measures in Reifenberg flat domains with small constant, assuming also Lipschitz continuity of the coefficients of the matrix. In fact, that regularity is only needed near the boundary.

Let $L_A u := -\operatorname{div}(A\nabla u)$ be a second order operator in divergence form associated to the $(n+1) \times (n+1)$ real matrix $A(\cdot) = (a_{ij}(\cdot))_{1\le i,j\le n+1}$ such that there exists $\lambda \geq 1$ with

$$\lambda^{-1}|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \text{ for all } \xi \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}, \tag{1.1a}$$

$$\langle A(x)\xi, \eta\rangle \leq \lambda|\xi||\eta| \text{ for all } \xi, \eta \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}. \tag{1.1b}$$

We say that a measurable matrix is uniformly elliptic with ellipticity constant $\lambda \geq 1$ if (1.1a) and (1.1b) are satisfied. The uniform ellipticity condition implies that the matrix has bounded coefficients.

Let $\Omega \subset \mathbb{R}^2$ be an open set and fix a point $p \in \Omega$. Consider the operator

$$\begin{aligned} T : C(\partial\Omega) &\to \quad \mathbb{R} \\ f \quad &\mapsto u_f^{L_A}(p), \end{aligned}$$

where $u_f^{L_A}$ is the L_A-harmonic extension of f in Ω via Perron's method. By the maximum principle for L_A-harmonic functions with uniformly elliptic matrices (see [18, p. 46] for example), the operator T is linear, bounded and positive. For bounded open sets $\Omega \subset \mathbb{R}^{n+1}$ with $n \geq 2$ we do the same construction.

The elliptic (L_A-harmonic) measure in Ω with pole $p \in \Omega$ is the unique Radon probability measure $\omega^p_{\Omega,A}$ (by the Riesz representation theorem) such that

$$u_f^{L_A}(p) = \int_{\partial\Omega} f(\xi)\, d\omega^p_{\Omega,A}(\xi) \text{ for every } f \in C(\partial\Omega).$$

Via the Perron method construction, although the characteristic function $\mathbf{1}_E$ of a set $E \subset \partial\Omega$ is not continuous in $\partial\Omega$, the function $x \mapsto \omega^x_{\Omega,A}(E)$ is L_A-harmonic in Ω. For a detailed construction of elliptic/harmonic measures, see [21, Section 11]. If the set Ω, the matrix A, or the pole $p \in \Omega$ is clear from the context, we will omit it in $\omega^p_{\Omega,A}$.

In this work, we study the Hausdorff dimension of elliptic measures arising from uniformly elliptic matrices. This is defined as

$$\dim_{\mathcal{H}} \omega^p_{\Omega,A} := \inf\{\dim_{\mathcal{H}} F : \omega^p_{\Omega,A}(F^c) = 0\}.$$

Naturally $\dim_{\mathcal{H}} \omega \leq \dim_{\mathcal{H}} \partial\Omega$. The Hausdorff dimension of the harmonic measure (i.e., with $\omega = \omega_{Id}$) has been studied by several authors, both in the plane and in higher dimensions, showing a different behavior in each case.

In the plane, Carleson proved $\dim_{\mathcal{H}} \omega \leq 1$ for "snowflake type" sets and $\dim_{\mathcal{H}} \omega < 1$ for some self similar "Cantor type" sets, both results in [9]. More precise results were obtained for simply connected domains by Makarov in [33] by showing $\dim_{\mathcal{H}} \omega = 1$. The upper bound $\dim_{\mathcal{H}} \omega \leq 1$ with no assumptions on the domain was shown by Jones and Wolff in [24], and later it was improved by Wolff in [44] by proving that there is a subset $F \subset \partial\Omega$ with full harmonic measure $\omega(F) = 1$ and σ-finite length.

In the same direction for higher dimensions, in $\mathbb{R}^{n+1}$ with $n \geq 2$, Bourgain proved in [7] that there exists a dimensional constant $b_n > 0$ such that $\dim_{\mathcal{H}} \omega \leq n + 1 - b_n$. By the results on the previous paragraph, we can take $b_1 = 1$ and this choice is optimal. For $n \geq 2$, Wolff constructed in [45] a domain $\Omega_n \subset \mathbb{R}^{n+1}$ such that $\dim_{\mathcal{H}} \omega_{\Omega_n} > n$, and in particular the Bourgain constant can't equal 1, i.e., $b_n < 1$. In a recent work [6], Badger and Genshaw refined the proof in [7] to find estimates on the Bourgain constant $b_n \in (0, 1)$.

From the results in the previous paragraphs we have that $\dim_{\mathcal{H}} \omega < n + 1$ (with some precise gap), but possibly $\dim_{\mathcal{H}} \omega = \dim_{\mathcal{H}} \partial\Omega$. In fact, the situation $\dim_{\mathcal{H}} \omega < \dim_{\mathcal{H}} \partial\Omega$ holds for many non-trivial domains. This phenomenon is frequently called the "dimension drop" for harmonic measure. The dimension drop is closely related to the results mentioned above. Indeed, similar techniques to the ones in [7, 24] are used in some of the following articles. The first work in this direction is due to Kaufman and Wu [29] for the planar $\log 4/\log 3$-dimensional Koch snowflake. Subsequently, Carleson [9] extended this result to self-similar Cantor sets in the plane. For this type of domains, see Makarov and Volberg [35], Batakis [5], and also [42, 43]. Jones and Wolff showed that the dimension drop happens for some uniform and disconnected planar domains, see Theorem 2.1 in [15, Section IX.2]. For IFS domains (iterated functions systems), see Urbański and Zdunik [41], and Batakis and Zdunik [8].

For general AD-regular domains of fractional dimension, one may ask if the dimension drop happens. It holds on uniformly "non-flat" AD-regular domains with codimension smaller than 1, as shown by Azzam in [4]. Recently, for $n \geq 1$ and $s \in [n - \frac{1}{2}, n)$, the third author showed in [40] that the dimension drop occurs for higher codimensional s-AD-regular subsets of C^1 n-dimensional manifolds in $\mathbb{R}^{n+1}$. In contrast, David, Jeznach and Julia proved in [12] that this last result may fail for s close enough to $n - 1$.

The situation is different for uniformly elliptic matrices. In [38] in the planar case and in [39] in higher dimensions, for every $\varepsilon > 0$, Sweezy constructed a domain $\Omega \subset \mathbb{R}^{n+1}$ and an elliptic operator in divergence form L_A whose associated elliptic measure $\omega_{\Omega,A}$ satisfies $\dim_{\mathcal{H}} \omega_{\Omega,A} \geq n+1-\varepsilon$. However, as ε becomes smaller, the ellipticity constant of the resulting matrix increases. Such planar domains and elliptic measures are constructed by the push forward under quasiconformal mappings, and the higher dimensional analog is deduced from the planar case.

Using a new approach in the planar case, David and Mayboroda in [14] constructed an elliptic operator $L = -\operatorname{div} a\nabla$, where a is a uniformly elliptic and continous scalar function on the complementary of the four corner Cantor set of dimension 1, whose elliptic measure ω_{aId} is proportional to the one-dimensional Hausdorff measure on the Cantor set. Operators of this form are the so-called "good elliptic operators". Following the same strategy, Perstneva in [36] constructed "good elliptic operators" on the complementary of the planar d-dimensional Koch-type snowflake with $1 < d < \log 4/\log 3$, whose elliptic measure is proportional to the d-dimension Hausdorff measure on the Koch snowflake.

After Sweezy's results, it is natural to ask which conditions on the matrix imply the analogous result of [24, 44] for the elliptic case. In this paper we study the metric properties of the elliptic measure when assuming regularity conditions on the matrix A. For other results in this line, see for example [37], about the rectifiability of the elliptic measure for Hölder matrices.

Suppose from now on that the domain is (δ, r_0)-Reifenberg flat and the coefficients of the matrix are also Lipschitz. Roughly speaking, a domain $\Omega \subset \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat if for every ball B centered at $\partial\Omega$ and with radius smaller than r_0, the δ-neighborhood of a line through the center of B contains $B \cap \partial\Omega$. See Definition 2.7 for the details. In this situation we show that the dimension of the elliptic measure in Reifenberg flat domains with small constant is at most 1. More precisely, for this type of domains we obtain the analogous result of [44, Theorem 1] in the following theorem.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a (δ, r_0)-Reifenberg flat domain, $p \in \Omega$, and A be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ. Suppose also that A is Lipschitz. Then there exists $\delta_0 = \delta_0(\lambda) > 0$ such that if $0 < \delta \leq \delta_0$ then there is a set $F \subset \partial\Omega$ satisfying $\omega_{\Omega,A}(F) = 1$ and with σ-finite one-dimensional Hausdorff measure. In particular* $\dim_{\mathcal{H}} \omega_{\Omega,A} \leq 1$.

Despite we are requiring to work with δ-Reifenberg flat domains with small enough constant δ, such sets can be constructed with Hausdorff dimension strictly larger than 1. For example, a suitable variant of the Koch snowflake can be constructed to be δ-Reifenberg flat.

It is well-known that the Hausdorff dimension of elliptic measures only depends on how the matrix is near the boundary and hence it is only necessary to assume the Lipschitz regularity around the boundary. In fact, in our proof we use the regularity only around the boundary, and so the theorem is still true assuming Lipschitz continuity in a small neighbourhood of $\partial\Omega$.

Similarly as in [44, Proof of Theorem 1], Theorem 1.1 follows from a more quantitative result involving a good covering of a subset of the boundary with big elliptic measure, see Main Lemma 3.1 for the precise statement.

One might think that this result could be obtained by the application of quasiconformal mappings. For symmetric matrices with determinant 1, the principal solution of the associated Beltrami equation, which depends only on the matrix, can act as a change of variables which inherits the extra regularity of the coefficients. Then, we obtain that the elliptic measure is the pushforward of the harmonic measure in the image domain. In the general case, we can obtain the elliptic measure as a pushforward of a harmonic measure using a quasiconformal change of variables which depends also on the Green function of the domain, as long as it satisfies the capacity density condition. See the article [16] by the first author. In this case, the extra dependence of the Green function does not allow us to obtain extra regularity estimates for the change of variables, and the σ-finiteness of length can not be attained.

A key point, and the main difficulty in this paper is to obtain the lower bound

$$\int_{\partial\widetilde{\Omega}} \log \frac{d\widetilde{\omega}^p}{d\sigma}(\xi)\, d\widetilde{\omega}^p(\xi) \geq -\text{const} > -\infty,$$

with a bound independent of the smoothness, where $\widetilde{\Omega}$ is the modified domain appearing in the proof of Lemma 3.3 and $\widetilde{\omega}^p$ the elliptic measure in $\widetilde{\Omega}$ with respect to the matrix A. This lower bound is known in the harmonic case for general smooth domains in the plane. This fact is the key point in the study of the dimension in the planar case, see for example [9, 24, 44]. Actually, the behavior of this integral is also crucial in the study of the dimension of harmonic measures in higher dimensions in [45].

The first occurrence (as far as we know) of the use of this lower bound in the study of the dimension of the harmonic measure is in [9]. For the proof see [24], and for further details see [11].

The previous lower bound is used in the proof of Lemma 3.3 to obtain a subset with big elliptic measure as it is done in [24] and [44], and it can be deduced from

$$\int_{\partial\widetilde{\Omega}} \log |S\nabla g_p^T(\xi)|^2\, d\widetilde{\omega}^p(\xi) \geq -\text{const} > -\infty, \tag{1.2}$$

where g_p^T is the Green function in $\widetilde{\Omega}$ with respect to the matrix A^T, and S is the square root matrix of the symmetric part $A_0 = (A + A^T)/2$, i.e., $S^T S = A_0$. In Section 7 we obtain also an upper bound for the integral above, see Lemma 7.1.

A fundamental tool to obtain the estimate (1.2) is the relation

$$|\nabla g_p^T(y)| \gtrsim \frac{g_p^T(y)}{\operatorname{dist}(y, \partial\widetilde{\Omega})}$$

near the boundary. In Reifenberg flat domains this estimate was obtained by Lewis, Lundström and Nyström in [30], see Lemma 2.13 below. The converse inequality is well-known for general domains and Hölder matrices and follows by Schauder estimates. Obtaining an inequality of this type for other domains may allow to apply the techniques exposed in this paper.

Very similar results to the one in Theorem 1.1 about the dimension are true for p-harmonic measures, i.e., when the associated operator is the p-Laplacian $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ for Reifenberg flat sets with small constant (see [32]), and simply connected sets in the plane (see [31]).

2 Preliminaries and definitions

2.1 Notation

- We use $c, C \geq 1$ to denote constants that may depend only on the dimension and the constants appearing in the hypotheses of the results, and whose values may change at each occurrence.
- We write $a \lesssim b$ if there exists a constant $C \geq 1$ such that $a \leq Cb$, and $a \approx b$ if $C^{-1}b \leq a \leq Cb$.
- If we want to stress the dependence of the constant on a parameter η, we write $a \lesssim_\eta b$ or $a \approx_\eta b$ meaning that $C = C(\eta) = C_\eta$.
- The ambient space is $\mathbb{R}^2$. However, some auxiliary results will be stated in general, i.e., in $\mathbb{R}^{n+1}$ for $n \geq 1$.
- The diameter of a set $E \subset \mathbb{R}^{n+1}$ is denoted by $\operatorname{diam} E := \sup_{x,y\in E} |x-y|$.
- We denote by $B_r(x)$ or $B(x,r)$ the open ball with center x and radius r, i.e., $B_r(x) = B(x,r) = \{y \in \mathbb{R}^{n+1} : |y-x| < r\}$. We denote $B_r := B_r(0)$.
- Given a ball B, we denote by r_B or $r(B)$ its radius, and by c_B or $c(B)$ its center.
- We say that a matrix A is Hölder continuous with exponent $\alpha \in (0,1]$ in a set U, or briefly $C^{0,\alpha}(U)$, if its coefficients are Hölder continuous with exponent α. That is, there exists a constant $C_\alpha > 0$ (called the Hölder seminorm) such that

$$\left|a_{ij}(x) - a_{ij}(y)\right| \leq C_\alpha |x-y|^\alpha \text{ for all } x, y \in U \text{ and } 1 \leq i, j \leq n+1.$$

For shortness we write C^α instead of $C^{0,\alpha}$ if $\alpha \in (0,1)$, and when $\alpha = 1$ we say "Lipschitz continuous". In this case we write C_L instead of C_1, i.e.,

$$\left|a_{ij}(x) - a_{ij}(y)\right| \leq C_L |x-y| \text{ for all } x, y \in U \text{ and } 1 \leq i, j \leq n+1.$$

- We say that a function f is κ-Lipschitz in U if $|f(x) - f(y)| \leq \kappa|x-y|$ for all $x, y \in U$.
- We denote the characteristic function of a set E by $\mathbf{1}_E$.
- Denote $\mathcal{D}(\mathbb{R}^{n+1})$ the standard dyadic grid. That is, $\mathcal{D}(\mathbb{R}^{n+1}) = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k(\mathbb{R}^{n+1})$ where $\mathcal{D}_k(\mathbb{R}^{n+1})$ is the collection of all cubes of the form

$$\{x \in \mathbb{R}^{n+1} : m_i 2^{-k} \leq x_i < (m_i+1)2^{-k} \text{ for } i = 1, \ldots, n+1\},$$

where $m_i \in \mathbb{Z}$.

- Given $t > 0$ and a set $E \subset \mathbb{R}^{n+1}$, we write $U_t(E) := \{x \in \mathbb{R}^{n+1} : \text{dist}(x, E) < t\}$ for the t-neighborhood E.

2.2 CDC, NTA and Reifenberg flat domains

In this subsection we introduce the capacity density condition (CDC), non-tangentially accessible domains (NTA) and Reifenberg flat domains, the main object of our study.

Definition 2.1 Let K be a compact subset of Ω. Its capacity is

$$\text{Cap}(K, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in C_0^\infty(\Omega) \text{ and } u \geq 1 \text{ on } K \right\}.$$

Just from the definition of capacity we obtain the following facts. For more properties see [21, 2.2. Theorem].

Lemma 2.2 *The set function* $E \mapsto \text{Cap}(E, \Omega)$*,* $E \subset \Omega$*, enjoys the following properties:*

(1) If $E_1 \subset E_2$*, then* $\text{Cap}(E_1, \Omega) \leq \text{Cap}(E_2, \Omega)$.
(2) If $\Omega_1 \subset \Omega_2$ *are open and* $E \subset \Omega_1$*, then* $\text{Cap}(E, \Omega_2) \leq \text{Cap}(E, \Omega_1)$.

Definition 2.3 (CDC domain. [21, (11.20), (2.13)]) A domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, satisfies the capacity density condition, CDC for short, if there exist constants $c_0, r_0 > 0$ such that

$$\text{Cap}\left(\overline{B(x_0, r)} \cap \Omega^c, B(x_0, 2r)\right) \geq c_0 r^{n-1},$$

for all $x_0 \in \partial\Omega$, $x_0 \neq \infty$ and $r \leq r_0$.

In order to define NTA domains we need to introduce some concepts. Given a domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and a fixed constant C, we define:

- *C-Whitney ball*: A ball $B(x, r) \subset \Omega$ is a C-Whitney ball in Ω if $C^{-1}r < \text{dist}(B(x, r), \partial\Omega) < Cr$.
- *C-Harnack chain*: For $p_1, p_2 \in \Omega$, a C-Harnack chain from p_1 to p_2 in Ω is a sequence of C-Whitney balls such that the first ball contains p_1, the last contains p_2, and such that consecutive balls intersect. The number of balls is called the length of the C-Harnack chain.

Consecutive balls in a C-Harnack chain must have comparable radius. Given a positive L_A-harmonic function in Ω, a C-Harnack chain between two points $p_1, p_2 \in \Omega$ allows us (via Harnack's inequality) to obtain $u(p_1) \approx u(p_2)$, where the constant involved depends on C and the length of the C-Harnack chain.

Definition 2.4 (NTA domain. [23, Section 3]) A domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is called non-tangentially accessible, NTA for short, if there exists constants $C > 1$ and $r_0 > 0$ such that:

(1) *Interior Corkscrew condition*: For any $\xi \in \partial\Omega$, $0 < r < r_0$, there exists a point $A_r(\xi) \in \Omega$ such that $|A_r(\xi) - \xi| < r$ and $\operatorname{dist}(A_r(\xi), \partial\Omega) > C^{-1}r$. The point $A_r(\xi) = A(\xi, r)$ is called the Corkscrew point of the point x at radius r.
(2) *Exterior Corkscrew condition*: $\overline{\Omega}^c$ satisfies the interior Corkscrew condition.
(3) *Harnack chain condition*: If $\varepsilon > 0$ and $p_1, p_2 \in \Omega$ satisfy that $p_1, p_2 \in \Omega \cap B(\xi, r_0/4)$ for some $\xi \in \partial\Omega$, $\operatorname{dist}(p_j, \partial\Omega) > \varepsilon$, and $|p_1 - p_2| < 2^k\varepsilon$, then there exists a Harnack chain from p_1 to p_2 of length Ck and such that the diameter of each ball is bounded below by $C^{-1} \min_{j=1,2} \operatorname{dist}(p_j, \partial\Omega)$.

Remark 2.5 A domain with the exterior Corkscrew condition satisfies the capacity density condition, see [21, Theorem 6.31]. In particular, NTA domains satisfy the capacity density condition.

Definition 2.6 (Hausdorff distance) Given nonempty sets A and B, we denote their Hausdorff distance $\operatorname{dist}_{\mathcal{H}}(A, B)$ by

$$\operatorname{dist}_{\mathcal{H}}(A, B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\right\}.$$

Definition 2.7 (Reifenberg flat domain) Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set, and let $0 < \delta < 1/2$ and $r_0 > 0$. We say that Ω is a (δ, r_0)-Reifenberg flat domain if it satisfies the following conditions:

(1) For every $x \in \partial\Omega$ and every $0 < r \leq r_0$, there exists a hyperplane $\mathcal{P}(x, r)$ containing x such that

$$\operatorname{dist}_{\mathcal{H}} (\partial\Omega \cap B(x, r), \mathcal{P}(x, r) \cap B(x, r)) \leq \delta r.$$

(2) For every $x \in \partial\Omega$ and every $0 < r \leq r_0$, one of the connected components of

$$B(x, r) \cap \left\{y \in \mathbb{R}^{n+1} : \operatorname{dist}(y, \mathcal{P}(x, r)) \geq 2\delta r\right\}$$

is contained in Ω and the other is contained in $\mathbb{R}^{n+1}\setminus\Omega$.

For small enough $\delta > 0$ we have that a (δ, r_0)-Reifenberg flat domain is also an NTA domain (see [28, Section 3]) and it satisfies the capacity density condition, see Remark 2.5.

2.3 Partial differential equations

We want to study the elliptic equation

$$L_A u(x) := -\operatorname{div}(A(\cdot)\nabla u(\cdot))(x) = 0, \tag{2.1}$$

which should be understood in the distributional sense. We simply write L instead of L_A when the matrix is clear from the context.

Definition 2.8 We say that a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a solution of (2.1), or L_A-harmonic, in an open set $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, if

$$\int \langle A(y)\nabla u(y), \nabla\varphi(y)\rangle\, dy = 0, \text{ for all } \varphi \in C_c^\infty(\Omega).$$

By the De Giorgi-Nash-Moser theorem a solution $u \in W^{1,2}_{\text{loc}}(\Omega)$ of (2.1) is locally Hölder continuous. If the matrix has Hölder regularity then the solution is locally $C^{1,\beta}$ for some $\beta \in (0, 1)$, see [22, Theorem 3.13]. Assuming Lipschitz regularity of the coefficients, L_A-harmonic functions enjoy more regularity. More precisely, weak solutions of $-\operatorname{div} A\nabla u = 0$ with A Lipschitz are twice weakly differentiable, and there is a "Caccioppoli type" inequality for second derivatives.

Theorem 2.9 *(See [18, Theorem 8.8]) Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $L_A u = 0$, see (2.1), in $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, where A is uniformly elliptic and Lipschitz continuous in Ω. Then for any subdomain $\Omega' \subset\subset \Omega$, we have $u \in W^{2,2}(\Omega')$ and*

$$\|u\|_{W^{2,2}(\Omega')} \leq C\|u\|_{W^{1,2}(\Omega)}, \tag{2.2}$$

where C depends on the dimension, the ellipticity constant and Lipschitz constant of A, and the value $\operatorname{dist}(\Omega', \partial\Omega)$.

If, in addition, the domain has boundary of class C^2, then L_A-harmonic functions are globally in $W^{2,2}$.

Theorem 2.10 *(See [18, Theorem 8.12]) Assume, in addition to the hypotheses of Theorem 2.9, that Ω is bounded with $\partial\Omega$ of class C^2 and that there exists a function $\varphi \in W^{2,2}(\Omega)$ for which $u - \varphi \in W^{1,2}_0(\Omega)$. Then we have also $u \in W^{2,2}(\Omega)$ and*

$$\|u\|_{W^{2,2}(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\varphi\|_{W^{2,2}(\Omega)}\right),$$

where C depends on the dimension, the ellipticity constant and Lipschitz constant of A, and $\partial\Omega$.

Remark 2.11 Let $U \subset \Omega$ an open subset. Assuming $A \in C^{0,1}(U)$, then any weak L_A-harmonic function $u \in W^{1,2}_{\text{loc}}(\Omega)$ (in fact $u \in W^{2,2}_{\text{loc}}(U)$ by Theorem 2.9) satisfies $\operatorname{div} A\nabla u = 0$ a.e. in U. Indeed, since the matrix A is differentiable a.e. (by Rademacher's theorem) and $u \in W^{2,2}_{\text{loc}}(U)$ (by Theorem 2.9), we have $\operatorname{div} A\nabla u \in L^1_{\text{loc}}(U)$. Moreover, since u is L_A-harmonic then $\int \operatorname{div} A\nabla u \cdot \psi = \int A\nabla u \nabla\psi = 0$ for any $\psi \in C_c^\infty(U) \subset C_c^\infty(\Omega)$. By the fundamental lemma of calculus of variations[1] we conclude $\operatorname{div} A\nabla u = 0$ a.e. in U.

The following theorem about the Hölder continuity of L_A-harmonic functions up to the boundary of regular enough domains will allow us to bound the elliptic measure on a specific domain (an annulus) by means of studying the Green function near the boundary.

[1] The fundamental lemma of calculus of variations: If U open set, $f \in L^1_{\text{loc}}(U)$ and $\int f\phi = 0$ for any $\phi \in C_c^\infty(U)$, then $f = 0$ a.e. in U.

Theorem 2.12 *(See [18, Corollary 8.36]) Let T be a (possibly empty) $C^{1,\alpha}$ boundary portion of a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and suppose $u \in W^{1,2}(\Omega)$ is a weak solution of (2.1) in Ω, where $A \in C^\alpha$ $(0 < \alpha < 1)$, such that $u = 0$ on T (in the sense of $W^{1,2}(\Omega)$). Then $u \in C^{1,\alpha}(\Omega \cup T)$, and for any $\Omega' \subset\subset \Omega \cup T$ we have*

$$\|u\|_{C^{1,\alpha}(\Omega')} := \|u\|_{1;\Omega'} + [u]_{1,\alpha;\Omega'} \leq C \sup_{\Omega} |u|,$$

where

$$\|u\|_{1;\Omega'} := \sup_{\Omega'} |u| + \sup_{\Omega'} |\nabla u|, \quad [u]_{1,\alpha;\Omega'} := \sup_{\substack{x,y \in \Omega' \\ x \neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\alpha},$$

for C depending on n, the value of $\operatorname{dist}(\Omega', \partial\Omega \backslash T)$, the $C^{1,\alpha}$ character of T, and the ellipticity constants and the Hölder norm of the matrix A.

Since bounded Lipschitz functions are also Hölder continuous for any exponent $\alpha \in (0,1)$, the previous theorem remains true for matrices with bounded Lipschitz coefficients.

2.3.1 Non-degeneracy of $|\nabla u|$ in Reifenberg flat domains with small constant

Despite the following result applies to more general matrices (see [30, Definition 1.1 and Lemma 3.35]), it is only stated in our setting for our purposes, i.e., for uniformly elliptic (1.1a)-(1.1b) and Hölder continuous matrices. In fact, the Hölder continuity is only used near the boundary.

Lemma 2.13 *([30, Lemma 3.35]) Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a (δ, r_0)-Reifenberg flat domain, $\xi \in \partial\Omega$, and $0 < r < \min\{r_0, 1\}$. Let $0 < \alpha < 1$ and $A \in C^\alpha(\{x : \operatorname{dist}(x, \partial\Omega) < 10r_0\})$ be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ and Hölder seminorm C_α. Suppose that u is a positive L_A-harmonic function in $\Omega \cap B(\xi, 4r)$, that is continuous in $\overline{\Omega} \cap B(\xi, 4r)$, and that $u = 0$ on $\partial\Omega \cap B(\xi, 4r)$. There exist $\hat{\delta} = \hat{\delta}(n, \lambda, C_\alpha, \alpha)$, $\gamma = \gamma(n, \lambda, C_\alpha, \alpha)$ and $\hat{c} = \hat{c}(n, \lambda, C_\alpha, \alpha)$ such that if $0 < \delta \leq \hat{\delta}$, then*

$$\gamma^{-1} \frac{u(y)}{\operatorname{dist}(y, \partial\Omega)} \leq |\nabla u(y)| \leq \gamma \frac{u(y)}{\operatorname{dist}(y, \partial\Omega)} \text{ whenever } y \in \Omega \cap B(\xi, r/\hat{c}).$$

Remark 2.14 Note that if the matrix has bounded Lipschitz coefficients, i.e., $\alpha = 1$, then the same result holds with constants depending on n, the ellipticity constant λ and the value $C_L \|A\|_{L^\infty(\mathbb{R}^{n+1})}$, where C_L is the Lipschitz seminorm of A. The value $C_L \|A\|_{L^\infty(\mathbb{R}^{n+1})}$ comes from the fact that bounded Lipschitz functions are Hölder continuous for any exponent. Indeed, a quick computation shows that the matrix A is Hölder continuous with exponent $1/2$ with Hölder seminorm $C_{1/2} := \left(2C_L \|A\|_{L^\infty(\mathbb{R}^{n+1})}\right)^{1/2}$.

The comparability in Lemma 2.13 will allow to bound the error terms in the study of the key term in (7.1).

2.4 The fundamental solution and the Green function

We denote by $\mathcal{E}_x^A(y)$ the fundamental solution with pole at x for L_A in $\mathbb{R}^{n+1}$, $n \geq 1$, so that $L_A \mathcal{E}_x^A(\cdot) = \delta_x$ in the distributional sense, where δ_x is the Dirac mass at the point $x \in \mathbb{R}^{n+1}$. We write $\mathcal{E}_x(y)$ when the matrix A is clear from the context. For a construction of the fundamental solution for real and uniformly elliptic matrices we refer to [20] for higher dimensions, $\mathbb{R}^{n+1}$ with $n \geq 2$, and [26, Appendix] for the planar case.

In higher dimensions the fundamental solution behaves "in many senses" as in the harmonic case, see [20] for more details, but in the plane the situation is more delicate due to the change of sign of $\mathcal{E}_0^{Id}(x) = \log|x|$ in $|x| = 1$, the fundamental solution of the Laplacian in the plane.

Here we collect the formal definition and some properties of the fundamental solution in the plane.

Definition 2.15 ([27, Definition 2.5]) A function $\mathcal{E}_x : \mathbb{R}^2 \to \mathbb{R}$ is called a fundamental solution for $L_A = \operatorname{div} A\nabla\cdot$ with pole at x if

(1) $\mathcal{E}_x \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x\}) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for all $p < 2$, and

$$\int_{\mathbb{R}^2} \langle A(z)\nabla\mathcal{E}_x(z), \nabla\varphi(z)\rangle\, dz = -\varphi(x), \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^2),$$

(2) $|\mathcal{E}_x(y)| = \mathcal{O}(\log|x-y|)$ as $|y| \to \infty$.

The following result controls the fundamental solution far from the pole similarly as the fundamental solution for the harmonic case.

Theorem 2.16 *([27, Theorem 2.6]) For each $x \in \mathbb{R}^2$ there exists a unique (modulo an additive constant) fundamental solution $\mathcal{E}_x$ for L_A with pole at x, and positive constants C_1, C_2, $R_1 < 1 < R_2$, which depend only on λ, such that*

$$\begin{aligned} C_1 \log(1/|x-y|) \leq -\mathcal{E}_x(y) &\leq C_2 \log(1/|x-y|) \text{ for } |x-y| < R_1, \text{ and} \\ C_1 \log(|x-y|) \leq \mathcal{E}_x(y) &\leq C_2 \log(|x-y|) \quad \text{ for } |x-y| > R_2. \end{aligned}$$

From the previous result and the maximum principle we obtain the following pointwise bound.

Corollary 2.17 $|\mathcal{E}_x(y)| \lesssim 1 + |\log|x-y||$ *for all $x, y \in \mathbb{R}^2$, where the constant depends on λ.*

Proof For $|x-y| < R_1$ and $|x-y| > R_2$, Theorem 2.16 gives $|\mathcal{E}_x(y)| \lesssim |\log|x-y|| \leq 1 + |\log|x-y||$. In $\mathcal{A} = \{y \in \mathbb{R}^2 : R_1 < |x-y| < R_2\}$ we have $L\mathcal{E}_x(y) = 0$, and hence by the maximum principle we obtain $C_2 \log R_1 \leq \mathcal{E}_x(\cdot) \leq C_2 \log R_2$ in the annulus $\mathcal{A}$. So $|\mathcal{E}_x(y)| \lesssim 1 \leq 1 + |\log|x-y||$. □

We have the following relation between the fundamental solutions of the operators with matrices A and A^T. The same holds in higher dimensions, see [20, (3.43)].

Lemma 2.18 *([27, Lemma 2.7]) Fix $x, y \in \mathbb{R}^2$. Let $\mathcal{E}_x$ be the fundamental solution for an elliptic operator L_A with pole at x, and $\mathcal{E}^T_y$ be the fundamental solution to the adjoint operator L_{A^T} with pole at y. Then $\mathcal{E}_x(y) = \mathcal{E}^T_y(x)$.*

Now we focus on Green's function. Given a bounded Wiener regular domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, and a uniformly elliptic matrix A, let $g_x = g_x^{\Omega,A}$ denote the Green function, in Ω with pole at $x \in \Omega$ with respect to the matrix A, constructed in [13, Theorem 2.12] in the planar case, and [19, Theorem 1.1] in higher dimensions. We denote $g^T_x = g_x^{\Omega,A^T}$ the Green function with respect to the matrix A^T. In particular, g_x satisfies

$$\int_\Omega A(z)\nabla g_x(z)\nabla\varphi(z)\,dz = \varphi(x), \text{ for all } \varphi \in C_c^\infty(\Omega), \tag{2.3}$$

and the following:

(1) $g_x(y) = g^T_y(x)$ for all $x, y \in \Omega$ and $x \neq y$. See [13, Theorem 2.12 (2.18)] in the plane, and [19, Theorem 1.3] in higher dimensions.
(2) For each $x \in \Omega$ and any $0 < r < \text{dist}(x, \partial\Omega)$, $g_x \in W^{1,2}(\Omega\setminus B_r(x))$. See [13, Theorem 2.12 (2.15)] in the plane, and [19, Theorem 1.1 (1.3)] in higher dimensions.

If the domain Ω has boundary of class C^2 and the matrix is Lipschitz continuous in a neighborhood $U_{2s}(\partial\Omega) = \{x \in \mathbb{R}^{n+1} : \text{dist}(x, \partial\Omega) < 2s\}$, then the Green function also satisfies:

(3) For each $x \in \Omega$ and any $0 < r < \min\{s, \text{dist}(x, \partial\Omega)\}$, $g_x \in W^{2,2}(\Omega \cap U_r(\partial\Omega))$. This is a consequence of (2) by Theorems 2.9 and 2.10.

Next we show that $L_{A^T} g^T_x = \omega^x_{\Omega,A} - \delta_x$ in the distributional sense. With this identity we can move from integrating on the boundary to the interior of the set.

Lemma 2.19 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded Wiener regular domain and $\varphi \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$. Then*

$$\int_{\partial\Omega} \varphi(\xi)\,d\omega^x_{\Omega,A}(\xi) - \varphi(x) = -\int_\Omega A^T(z)\nabla g^T_x(z)\nabla\varphi(z)\,dz, \textit{ for a.e. } x \in \Omega. \tag{2.4}$$

Sketch of proof In higher dimensions this is proved in [1, (2.6)]. Here we detail the differences in the planar case.

As Ω is bounded Wiener regular and $\varphi \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$, the L_A-harmonic function u solving the Dirichlet problem with boundary data $\varphi|_{\partial\Omega}$ can be taken to be in $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and $u - \varphi \in W_0^{1,2}(\Omega)$. Indeed, by the Lax-Milgram theorem in $W_0^{1,2}(\Omega)$ there is a unique function $v \in W_0^{1,2}(\Omega)$ with $\int_\Omega A\nabla v\nabla\vartheta = -\int_\Omega A\nabla\varphi\nabla\vartheta$ for any $\vartheta \in W_0^{1,2}(\Omega)$. Taking $u = v + \varphi$ it is clear that $L_A u = 0$, $u \in W^{1,2}(\Omega)$

and $u-\varphi \in W_0^{1,2}(\Omega)$. On the other hand, since Ω is bounded Wiener regular, the L_A-harmonic extension u of $\varphi|_{\partial\Omega}$ is continuous up to the boundary, see [21, Theorem 6.27]. Moreover, by the definition of elliptic measure and the uniqueness of solutions we have

$$u(x)=\int_{\partial\Omega}\varphi(\xi)\,d\omega^x_{\Omega,A}(\xi).$$

Write

$$\int_\Omega A^T\nabla g_x^T\nabla\varphi=\int_\Omega A^T\nabla g_x^T\nabla u+\int_\Omega A^T\nabla g_x^T\nabla(\varphi-u)=:\mathrm{I}+\mathrm{II}.$$

The same proof of [1, (2.10) and (2.12)] applies also in the plane to have that the left-hand side integral is absolutely convergent and $\mathrm{II}=\varphi(x)-u(x)$ for a.e. $x\in\Omega$, replacing the use of (2.8) and (2.9) in [1] by the fact that the Green function satisfies $\nabla g_z\in L^p(\Omega)$ for all $p\in[1,2)$ and $g_z\in W^{1,2}_{\mathrm{loc}}(\Omega\backslash\{z\})$, see [13, Remark 2.19 and (3.66)] and item 2 in p. 10 respectively. Using that $|g_x^T(z)|\lesssim|\log|z-x||$ when $|x-z|\le 2\varepsilon$ is small enough, see [13, (2.17)], in the planar case the term I^2_ε defined in [1, p. 10855] is controlled by

$$|\mathrm{I}^2_\varepsilon|\lesssim\frac{|\log\varepsilon|}{\varepsilon}\int_{B_{2\varepsilon}(x)}|\nabla u|\lesssim\varepsilon\,|\log\varepsilon|\,\mathcal{M}(\nabla u\mathbf{1}_\Omega)(x),$$

and as $\varepsilon\,|\log\varepsilon|\to 0$ as $\varepsilon\to 0$, the same proof there implies that $\mathrm{I}=0$ for a.e. $x\in\Omega$. Therefore, (2.4) holds also in the planar case. □

We will show below that from the equality (2.4) it follows that

$$g_y(x)=-\mathcal{E}_y(x)+\int_{\partial\Omega}\mathcal{E}_y(\xi)\,d\omega^x_{\Omega,A}(\xi),\ \text{ for all } x,y\in\Omega. \tag{2.5}$$

(Probably this is already known but we will show the full details in the plane for completeness). Recall that $x\mapsto\int_{\partial\Omega}\mathcal{E}_y(\xi)\,d\omega^x_{\Omega,A}(\xi)$ is the L_A-harmonic extension of $\mathcal{E}_y$ inside Ω. Assuming that $g_y(x)=0$ if $y\notin\Omega$, we have that (2.5) also holds in this case, since Ω is Wiener regular and therefore the Green function is continuous through the boundary. Moreover, by (2.5) and since Ω is bounded, the Green function also satisfies:

(4) For each $x\in\Omega$, $g_x(y)\ge 0$ for any $y\in\Omega\backslash\{x\}$.

This was already proved in higher dimension in [19, Theorem 1.1]. However, since the situation is more delicate for unbounded planar domains due to the logarithmic behaviour of the fundamental solution, we only consider bounded planar Wiener regular domains.

Proof of (2.5) in the planar case Let $0<\varepsilon\ll\min\{|x-y|,\mathrm{dist}(x,\partial\Omega),\mathrm{dist}(y,\partial\Omega)\}$, $\psi^y=\psi^y_\varepsilon\in C^\infty_c(B_{2\varepsilon}(y))$ such that $\psi^y_\varepsilon=1$ in $B_\varepsilon(y)$ and $|\nabla\phi^y_\varepsilon|\lesssim 1/\varepsilon$, and $\psi^x=\psi^x_\varepsilon$ defined analogously.

Applying (2.4) to $(1-\psi^y)\mathcal{E}_y \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and using that $1-\psi^y = 1$ in $\partial\Omega \cup \{x\}$, we have

$$\int_{\partial\Omega} \mathcal{E}_y(\xi)\, d\omega^x_{\Omega,A}(\xi) - \mathcal{E}_y(x) = -\int_\Omega A^T(z)\nabla g^T_x(z)\nabla((1-\psi^y)\mathcal{E}_y)(z)\, dz.$$

Write the right-hand side term as

$$\begin{aligned} &-\int_\Omega A^T\nabla g^T_x \nabla((1-\psi^y)\mathcal{E}_y)\, dz \\ &\quad= \int_{B_{2\varepsilon}(y)\setminus B_\varepsilon(y)} A^T\nabla g^T_x \nabla\psi^y \cdot \mathcal{E}_y\, dz + \int_{B_{2\varepsilon}(y)} A\nabla\mathcal{E}_y \nabla g^T_x \cdot \psi^y\, dz \\ &\qquad - \int_{B_{2\varepsilon}(x)\setminus B_\varepsilon(x)} A\nabla\mathcal{E}_y\nabla\psi^x \cdot g^T_x\, dz - \int_\Omega A\nabla\mathcal{E}_y \nabla((1-\psi^x)g^T_x)\, dz \\ &=: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon. \end{aligned}$$

Using that $|g^T_x(z)| \lesssim |\log|x-z||$ and $|\mathcal{E}_y(z)| \lesssim |\log|y-z||$ when $|x-z| \leq 2\varepsilon$ and $|y-z| \leq 2\varepsilon$ for small enough $\varepsilon > 0$, see [13, (2.17)] and Theorem 2.16, the terms I_ε and III_ε are bounded by

$$|\mathrm{I}_\varepsilon| + |\mathrm{III}_\varepsilon| \lesssim \varepsilon\, |\log\varepsilon| \left\{ \left(⨍_{B_{2\varepsilon}(y)} |\nabla g^T_x|^2\, dz \right)^{1/2} + \left(⨍_{B_{2\varepsilon}(x)} |\nabla \mathcal{E}_y|^2\, dz \right)^{1/2} \right\}. \tag{2.6}$$

For the bound of II_ε, since g^T_x is L_{A^T}-harmonic in $\Omega\setminus B_{10\varepsilon}(y)$, there exists $p = p(\lambda) > 2$ such that

$$\left(⨍_{B_{2\varepsilon}(y)} |\nabla g^T_x|^p\, dz \right)^{1/p} \lesssim_\lambda \left(⨍_{B_{4\varepsilon}(y)} |\nabla g^T_x|^2\, dz \right)^{1/2},$$

see [25, Lemma 1.1.12], and let $1 \leq q < 2$ be its Hölder exponent conjugate, i.e., $1/p + 1/q = 1$. By Hölder's inequality and the choice of $p > 2$, the term II_ε is controlled by

$$|\mathrm{II}_\varepsilon| = \left| \int_{B_{2\varepsilon}(y)} A\nabla\mathcal{E}_y\nabla g^T_x \cdot \psi^y\, dz \right| \lesssim \varepsilon^2 \left(⨍_{B_{2\varepsilon}(y)} |\nabla\mathcal{E}_y|^q\, dz \right)^{1/q} \left(⨍_{B_{4\varepsilon}(y)} |\nabla g^T_x|^2\, dz \right)^{1/2}.$$

By [10, Theorem 0.1] we have $\left(⨍_{B_{2\varepsilon}(y)} |\nabla\mathcal{E}_y|^q\, dz \right)^{1/q} \lesssim_\lambda 1/\varepsilon$, and so

$$|\mathrm{II}_\varepsilon| \lesssim \varepsilon \left(⨍_{B_{4\varepsilon}(x)} |\nabla g^T_x|^2\, dz \right)^{1/2}. \tag{2.7}$$

Since $g^T_x \in W^{1,2}_{\mathrm{loc}}(\Omega\setminus\{x\})$ and $\mathcal{E}_y \in W^{1,2}_{\mathrm{loc}}(\Omega\setminus\{y\})$, in particular $|\nabla g^T_x|^2 \in L^1_{\mathrm{loc}}(\Omega\setminus\{x\})$ and $|\nabla\mathcal{E}_y|^2 \in L^1_{\mathrm{loc}}(\Omega\setminus\{y\})$, by the Lebesgue differentiation theorem we have that

$\fint_{B_{4\varepsilon}(y)} |\nabla g_x^T|^2 \to |\nabla g_x^T(y)|^2$ for a.e. $y \in \Omega$ and $\fint_{B_{4\varepsilon}(x)} |\nabla \mathcal{E}_y|^2 \to |\nabla \mathcal{E}_y(x)|^2$ for a.e. $x \in \Omega$ respectively. That is, by (2.6) and (2.7) we have $|\mathrm{I}_\varepsilon| + |\mathrm{II}_\varepsilon| + |\mathrm{III}_\varepsilon| = 0$ a.e. $x, y \in \Omega$.

On the other hand, from the Dirac delta property of the fundamental solution, $(1-\psi^x)g_x^T \in W_0^{1,2}(\Omega)$ and the density of $C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^{n+1})$ in $W_0^{1,2}(\Omega)$, we obtain $\mathrm{IV}_\varepsilon = (1-\psi^x(y))g_x^T(y) = g_y(x)$, and (2.5) is proved for a.e. $x, y \in \Omega$. By continuity, it also holds for all $x, y \in \Omega$. □

To end this section, we see how the Green function is related to the density of the elliptic measure in smooth domains. Assume now A is Lipschitz continuous in an open neighborhood of $\partial\Omega$, say $U_s(\partial\Omega) = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, \partial\Omega) < s\}$. Under this assumption A is differentiable a.e. by Rademacher's theorem, and L_A-harmonic functions are in $W^{2,2}$ by Theorem 2.10.

Lemma 2.20 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded domain with smooth boundary (and hence Wiener regular) and $A \in C^{0,1}(U_s(\partial\Omega))$. The elliptic measure $\omega_{\Omega,A}^p$ can be written as*

$$d\omega_{\Omega,A}^p = -\langle A^T \nabla g_p^T, \nu\rangle \, d\sigma, \text{ for a.e. } p \in \Omega \backslash U_{2s}(\partial\Omega), \tag{2.8}$$

where ν is the unit outer normal to $\partial\Omega$ and σ is the surface measure on $\partial\Omega$.

Proof Let $\varphi \in C_c^\infty(U_s(\partial\Omega))$ and set $\phi_p(z) = g_p^T(z) + \mathcal{E}_p^T(z)$ for $z \in \mathbb{R}^{n+1}$, see (2.5) when $z \in \Omega$. Notice that $\phi_p(z) = -\mathcal{E}_p^T(z)$ in Ω^c, and hence ϕ_p is L_{A^T}-harmonic in Ω and $\overline{\Omega}^c$.

The claim follows since the right-hand side of (2.4) is

$$-\int_{\partial\Omega} \varphi(\xi)\langle A^T(\xi)\nabla g_p^T(\xi), \nu(\xi)\rangle \, d\sigma(\xi) - \varphi(p).$$

Indeed, since A is differentiable a.e. in $\operatorname{supp}\varphi$ (Rademacher's theorem), $g_p^T \in W^{2,2}(\operatorname{supp}\varphi)$ (Theorem 2.10) and $\varphi \in C_c^\infty(U_s(\partial\Omega))$, in particular $A^T\nabla g_p^T \cdot \varphi \in W^{1,2}(\operatorname{supp}\varphi)$. As the integration by parts formula holds for $W^{1,2}$ functions, then

$$\begin{aligned}
-\int_\Omega A^T \nabla g_p^T \nabla\varphi &= \int_\Omega \varphi \operatorname{div}\left(A^T\nabla g_p^T\right) - \int_\Omega \operatorname{div}\left(A^T \nabla g_p^T \cdot \varphi\right) \\
&= \int_\Omega \varphi \operatorname{div}\left(A^T\nabla g_p^T\right) - \int_{\partial\Omega} \varphi\langle A^T\nabla g_p^T, \nu\rangle \, d\sigma \\
&= \int_{\mathbb{R}^{n+1}} \varphi \operatorname{div}\left(A^T\nabla\phi_p\right) - \int_{\mathbb{R}^{n+1}} \varphi \operatorname{div}\left(A^T\nabla\mathcal{E}_p^T\right) \\
&\quad - \int_{\partial\Omega} \varphi\langle A^T\nabla g_p^T, \nu\rangle \, d\sigma \\
&= \int_{\mathbb{R}^{n+1}} \varphi \operatorname{div}\left(A^T\nabla\phi_p\right) - \varphi(p) - \int_{\partial\Omega} \varphi\langle A^T\nabla g_p^T, \nu\rangle \, d\sigma,
\end{aligned}$$

where in the last equality we used that φ has compact support and $\int \varphi \operatorname{div}\left(A^T \nabla \mathcal{E}_p^T\right) = \varphi(p)$ by the definition of the fundamental solution. Since ϕ_p is L_{A^T}-harmonic in $\mathbb{R}^{n+1} \setminus \partial\Omega$, $\mathcal{H}^{n+1}(\partial\Omega) = 0$ and $A \in C^{0,1}(\operatorname{supp}\varphi)$, we have that $\operatorname{div}\left(A^T \nabla \phi_p\right) = 0$ a.e. in $\operatorname{supp}\varphi$ by Remark 2.11, and the claim follows. □

3 Main Lemma and preliminary reductions

As in [44], Theorem 1.1 will follow from the following more quantitative result.

Main Lemma 3.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded (δ, r_0)-Reifenberg flat domain, a point $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$, and A be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ, and suppose also that A is κ-Lipschitz in $U_{r_0}(\partial\Omega) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial\Omega) < r_0\}$. For a given $0 < r \leq 1$ satisfying $r\kappa \|A\|_{L^\infty(\mathbb{R}^2)} \leq 1$, there exists $\delta_0 = \delta_0(\lambda) > 0$ such that for every $0 < \delta \leq \delta_0$ we have the following:*

For any $0 < \tau < 1$, sufficiently large M, and $\rho \in (0, r/M)$ there is a set $F \subset \partial\Omega$ such that $\omega^p_{\Omega,A}(F) \geq C^{-1}\tau$ and a countable covering $F \subset \bigcup_i B(z_i, r_i)$ where

(1) $\sum_i r_i \leq CM^\tau$,
(2) $\sum_{\{i : r_i > \rho\}} r_i \leq CM^{-1}$,

with universal constant C.

Remark 3.2 Given M sufficiently large to satisfy the conclusions of the lemma, the particular choice $\rho = r/(2M)$ yields that the number of balls $B(z_i, r_i)$ with $r_i > r/(2M)$ is universally bounded, that is, $\sum_{\{i : r_i > r/(2M)\}} 1 \leq 2C/r$.

By means of a linear deformation of the plane (see Section 3.1 below) and a rescaling, we see that it suffices to prove the following weaker lemma to obtain Main Lemma 3.1.

Lemma 3.3 *(Weak form of Main Lemma 3.1) Let $\Omega \subset \mathbb{R}^2$, $p \in \Omega$ and A as in Main Lemma 3.1. Suppose also that $A_0 = \frac{A+A^T}{2}$ is of the form $A_0 = R^T B R$ with $R \in C^{0,1}(U_{r_0}(\partial\Omega))$ a rotation, and $B \in C^{0,1}(U_{r_0}(\partial\Omega))$ diagonal. Then there exists $\delta_0 = \delta_0(\lambda, \kappa\|A\|_{L^\infty(\mathbb{R}^2)}) > 0$ such that for every $0 < \delta \leq \delta_0$ we have the following:*

For any $0 < \tau < 1$, sufficiently large M (how large depends on τ and on the constants in the hypothesis), and $\rho \in (0, 1/M)$ there is a set $F \subset \partial\Omega$ such that $\omega^p_{\Omega,A}(F) \geq C^{-1}\tau$ and a countable covering $F \subset \bigcup_i B(z_i, r_i)$ with

(1)* $\sum_i r_i \leq CM^\tau$,
(2)* $\sum_{\{i : r_i > \rho\}} r_i \leq CM^{-1}$,

with universal constant C.

We remark that this weaker form replaces the assumption on the parameter r by the additional assumption $A_0 = R^T B R$, and allows δ_0 to depend also on $\kappa\|A\|_{L^\infty}$, and ρ to be in $(0, 1/M)$. For the proof of Lemma 3.3, see Section 5.

3.1 The change of variables in Reduction 1

In this subsection we first collect some auxiliary results about changes of variables that will be useful to prove some technical lemmas, secondly we construct the precise linear deformation in the plane that allows the reduction from Main Lemma 3.1 to Lemma 3.3, and finally we see how it distorts planar Reifenberg flat domains.

3.1.1 Linear changes of variables

We will see how L_A-harmonic functions behave under linear changes of variables. See [2, Lemmas 3.8 and 3.9] for a detailed proof of the following two results.

Lemma 3.4 *Let $D \in \mathbb{R}^{(n+1)\times(n+1)}$ be a constant matrix with* $\det D \neq 0$, *$n \geq 0$. A function f is L_A-harmonic in Ω if and only if $\widetilde{f} = f \circ D$ is $L_{\widetilde{A}}$-harmonic in $D^{-1}(\Omega)$, where $\widetilde{A}(\cdot) = D^{-1}A(D\cdot)(D^{-1})^T$ and $D^{-1}(\Omega) = \{D^{-1}x : x \in \Omega\}$.*

By the definition of elliptic measure, the previous lemma implies the following relation of elliptic measures under a linear change of variables.

Corollary 3.5 *Let $D \in \mathbb{R}^{(n+1)\times(n+1)}$ be a constant matrix such that* $\det D \neq 0$, *$n \geq 0$, and let Ω be a Wiener regular domain. Let $\omega = \omega_{\Omega,A}$ be the elliptic measure in Ω with matrix A, and $\widetilde{\omega} = \omega_{D^{-1}(\Omega),\widetilde{A}}$ where $\widetilde{A}(\cdot) = D^{-1}A(D\cdot)\left(D^{-1}\right)^T$. Then $\omega^x(E) = \widetilde{\omega}^{D^{-1}x}(D^{-1}(E))$ for every $x \in \Omega$ and $E \subset \partial\Omega$.*

3.1.2 Lipschitz diagonalization of symmetric matrices in the plane

In the study of the integral (7.1) in Section 7 we will use that after a suitable linear change of variables D, the symmetric part of the matrix $\widetilde{A}$ in Corollary 3.5 diagonalizes in the form $R^T BR$, where R is a Lipschitz rotation and B is Lipschitz diagonal. In this subsection we see that we can always reduce to this case.

We need to follow this strategy because in general it is not true that Lipschitz elliptic symmetric matrices diagonalize in the aforementioned form, as we can see in the following example.

Example 3.6 Let A_1, A_2 be two constant symmetric matrices diagonalizing with different eigenvectors, and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a Lipschitz function with $f\mathbf{1}_{\{|x|<1\}} < 0$, $f\mathbf{1}_{\{|x|>1\}} > 0$ and $\|f\|_\infty \leq \varepsilon$ for small enough fixed constant $\varepsilon > 0$.

Set $A(x) := Id + f(x)\mathbf{1}_{\{|x|<1\}}(x)A_1 + f(x)\mathbf{1}_{\{|x|>1\}}(x)A_2$ and take $\varepsilon > 0$ small enough to ensure the ellipticity condition on the matrix A. Moreover, with this choice of the function f we have that the matrix A has Lipschitz coefficients.

Let v_1 be an eigenvector of A_1 with eigenvalue μ_1, i.e., $A_1v_1 = \mu_1u_1$, and let v_2 be an eigenvector of A_2 with eigenvalue μ_2, i.e., $A_2v_2 = \mu_2v_2$. Then, for $|x| < 1$ the vector u is an eigenvector of A,

$$A(x)v_1 = (Id + f(x)A_1)v_1 = (1 + f(x)\mu_1)v_1,$$

and for $|x| > 1$ the vector v is an eigenvector of A,

$$A(x)v_2 = (Id + f(x)A_2)v_2 = (1 + f(x)\mu_2)v_2.$$

From this we get that the matrix A diagonalizes with the same basis as A_1 if $|x| < 1$, and with the same basis as A_2 if $|x| > 1$, whence we obtain that the basis is not continuous.

In the following lemma we see that we can avoid the situation seen in the previous example by using a linear change of variables.

Lemma 3.7 *Let $U \subset \mathbb{R}^2$ be a set. Let $A \in C^{0,1}(U)$ be a uniformly elliptic and symmetric 2×2 matrix with ellipticity constant λ, and let $D = \begin{pmatrix} 1/K & 0 \\ 0 & 1 \end{pmatrix}$ with $K^2 \geq \lambda^2 + \lambda$. Then the matrix $\widetilde{A}(\cdot) = D^{-1}A(D\cdot)D^{-1}$ is of the form $\widetilde{A} = R^T BR \in C^{0,1}(D^{-1}(U))$, with $B \in C^{0,1}(D^{-1}(U))$ diagonal and $R \in C^{0,1}(D^{-1}(U))$ a rotation.*

Proof Denote the matrix $A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix}$, and let

$$\widetilde{A} = D^{-1}(A \circ D)D^{-1} = \begin{pmatrix} K^2\widetilde{a} & K\widetilde{b} \\ K\widetilde{b} & \widetilde{d} \end{pmatrix},$$

where we write $\widetilde{a}(x) = a(Dx)$ and the analogous expressions for the other elements of the matrix. We want to see that when $K^2 \geq \lambda^2 + \lambda$ we can write $\widetilde{A} = R^T BR$ where B is Lipschitz and diagonal, and R is a Lipschitz rotation matrix.

The eigenvalues $\lambda_\pm = \lambda_\pm(\cdot)$ of $\widetilde{A}$ are

$$\lambda_\pm = \frac{K^2\widetilde{a} + \widetilde{d} \pm \sqrt{\left(K^2\widetilde{a} - \widetilde{d}\right)^2 + 4K^2\widetilde{b}^2}}{2}, \tag{3.1}$$

and we want to see that they are Lipschitz if $K^2 \geq \lambda^2 + \lambda$. Note that

$$\lambda_+ + \lambda_- = K^2\widetilde{a} + \widetilde{d}. \tag{3.2}$$

For shortness, let $f = \left(K^2\widetilde{a} - \widetilde{d}\right)^2 + 4K^2\widetilde{b}^2$ be an auxiliar function. Note that since $a, b, d \in C^{0,1}(U) \cap L^\infty(U)$, and so $\widetilde{a}, \widetilde{b}, \widetilde{d} \in C^{0,1}(D^{-1}U) \cap L^\infty(D^{-1}U)$, we have $f \in C^{0,1}(D^{-1}U) \cap L^\infty(D^{-1}U)$.

For $x, y \in D^{-1}U$, i.e., $Dx, Dy \in U$,

$$\begin{aligned} 2|\lambda_\pm(x) - \lambda_\pm(y)| &\leq K^2|\widetilde{a}(x) - \widetilde{a}(y)| + |\widetilde{d}(x) - \widetilde{d}(y)| + |\sqrt{f(x)} - \sqrt{f(y)}| \\ &\leq C|x - y| + |\sqrt{f(x)} - \sqrt{f(y)}| = C|x - y| + \frac{|f(x) - f(y)|}{|\sqrt{f(x)} + \sqrt{f(y)}|}. \end{aligned} \tag{3.3}$$

Since $a \geq \lambda^{-1}$ and $d \leq \lambda$ by ellipticity (indeed $a, d \approx_\lambda 1$), and so $\widetilde{a} \geq \lambda^{-1}$ and $\widetilde{d} \leq \lambda$, in particular $K^2\widetilde{a} - \widetilde{d} \geq K^2\lambda^{-1} - \lambda$. Since $K^2 \geq \lambda^2 + \lambda$, we obtain that

$$K^2\widetilde{a} - \widetilde{d} \geq K^2\lambda^{-1} - \lambda \geq 1, \tag{3.4}$$

and with this we have

$$\sqrt{f} = \sqrt{\left(K^2\widetilde{a} - \widetilde{d}\right)^2 + 4K^2\widetilde{b}^2} \geq K^2\widetilde{a} - \widetilde{d} \geq 1. \tag{3.5}$$

Combining the estimates (3.3) and (3.5) we get

$$|\lambda_\pm(x) - \lambda_\pm(y)| \lesssim |x - y| + |f(x) - f(y)| \lesssim |x - y|,$$

i.e., $\lambda_\pm$ are Lipschitz.

It remains to see that the matrix diagonalizes in the form $R^T BR$ and that the eigenvectors are also Lipschitz. Let $u^\pm = (u_1^\pm, u_2^\pm)$ be the eigenvectors of the eigenvalues $\lambda_\pm$. Hence, $\left(\widetilde{A} - \lambda_\pm Id\right) u^\pm = 0$, i.e.,

$$\begin{cases} (K^2\widetilde{a} - \lambda_\pm)u_1^\pm + K\widetilde{b}u_2^\pm = 0, \\ K\widetilde{b}u_1^\pm + (\widetilde{d} - \lambda_\pm)u_2^\pm = 0. \end{cases} \tag{3.6}$$

Consider the vectors $v^+ = \left(\lambda_+ - \widetilde{d}, K\widetilde{b}\right)$ and $v^- = \left(K\widetilde{b}, \lambda_- - K^2\widetilde{a}\right)$, which are clearly Lipschitz by the preceding discussion. We claim that v^+ and v^- satisfy (3.6). Indeed, v^+ satisfy the second equality in (3.6) by the definition of v^+, and the first equality follows from the definition of the eigenvalues $\lambda_\pm$ and the equality $\lambda_+ - \widetilde{d} = K^2\widetilde{a} - \lambda_-$, see (3.2). The vector v^- satisfy (3.6) by the same reason.

By (3.2), we can write $v^+ = \left(K^2\widetilde{a} - \lambda_-, K\widetilde{b}\right)$ (and so v^- and v^+ are orthogonal), and hence $\|v^+\| = \|v^-\|$. Moreover,

$$\|v^\pm\|^2 = \left(K^2\widetilde{a} - \lambda_-\right)^2 + K^2\widetilde{b}^2 \geq \left(K^2\widetilde{a} - \lambda_-\right)^2 = \left(\frac{K^2\widetilde{a} - \widetilde{d} + \sqrt{f}}{2}\right)^2 \geq 1,$$

by (3.4) and (3.5). To conclude, set the unitary vectors $u^\pm := \frac{v^\pm}{\|v^\pm\|}$. They are orthonormal and hence we conclude that $\widetilde{A} = R^T BR$ with

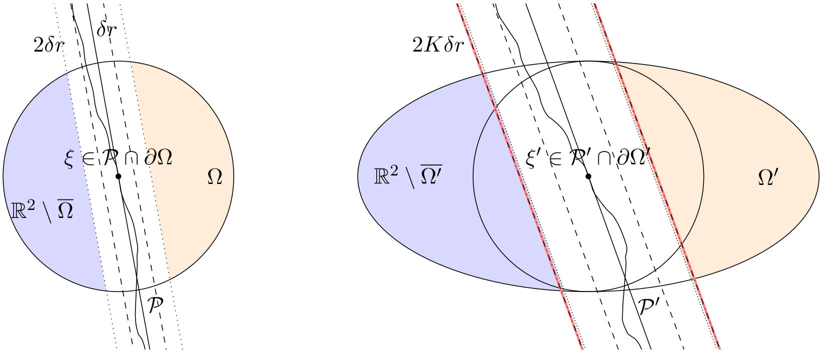
$$B = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \quad R = \begin{pmatrix} u_1^- & u_1^+ \\ u_2^- & u_2^+ \end{pmatrix} = \begin{pmatrix} u_2^+ & u_1^+ \\ -u_1^+ & u_2^+ \end{pmatrix},$$

and $u^\pm$ are Lipschitz since $v^\pm$ are Lipschitz and $\|v^\pm\| \geq 1$. □

We also need to control how Reifenberg flat sets change under the linear planar deformation in the previous lemma.

Lemma 3.8 *Let $K \geq 1$, $D = \begin{pmatrix} 1/K & 0 \\ 0 & 1 \end{pmatrix}$ and Ω be a (δ, r_0)-Reifenberg flat domain. If $\delta < \sqrt{15}/(16K)$ then $D^{-1}(\Omega)$ is a $\left(\frac{8\sqrt{15}}{15}K\delta, r_0\right)$-Reifenberg flat domain.*

Proof Let $0 < r \leq r_0$, $\xi \in \partial\Omega$ and $\mathcal{P} := \mathcal{P}(\xi, r) \ni \xi$. Denote $\Omega' = D^{-1}(\Omega)$, $\xi' = D^{-1}\xi$ and $\mathcal{P}' = D^{-1}(\mathcal{P})$. We want to check the conditions in Definition 2.7 with the point ξ', radius r and the hyperplane $\mathcal{P}'$. See Figure 1.



(A) Ω, $0 < r \le r_0$ and $\xi \in \mathcal{P} \cap \partial\Omega$.

(B) D^{-1} of fig. 1A and $\{x : \text{dist}(x, \mathcal{P}') = 2K\delta r\}$ in dash-dotted (red).

Fig. 1 Almost the worst situation: $\mathcal{P} \perp \{y = 0\}$

Claim 3.9 *For any* $x, y \in \mathbb{R}^2$, $K^{-1}\text{dist}(x, y) \le \text{dist}(Dx, Dy) \le \text{dist}(x, y)$.

Proof Indeed, since the minimum (resp. maximum) eigenvalue of D is K^{-1} (resp. 1), we have

$$\frac{\text{dist}(Dx, Dy)}{\text{dist}(x, y)} = \frac{\|Dx - Dy\|}{\|x - y\|} \in [K^{-1}, 1].$$

□

Claim 3.10 *One component of*

$$B_r(\xi') \cap \{x \in \mathbb{R}^2 : \text{dist}(x, \mathcal{P}') \ge 2K\delta r\} \tag{3.7}$$

is contained in Ω' *and the other is contained in* $\mathbb{R}^2 \setminus \Omega'$.

Proof By the previous claim we get

$$D^{-1}\left(\{x \in \mathbb{R}^2 : \text{dist}(x, \mathcal{P}) \ge 2\delta r\}\right) \supset \{x \in \mathbb{R}^2 : \text{dist}(x, \mathcal{P}') \ge 2K\delta r\}, \tag{3.8}$$

and from the definition of D^{-1} we have $B_r(\xi') \subset D^{-1}(B_r(\xi))$. By (3.8), and since $2K\delta < 2\sqrt{15}/16 < 1$, in particular $B_r(\xi') \cap D^{-1}(\{x : \text{dist}(x, \mathcal{P}) \ge 2\delta r\}) \ne \emptyset$, and the claim follows from the Reifenberg flat condition of Ω. □

First we check

$$\sup_{y \in \partial\Omega' \cap B_r(\xi')} \text{dist}(y, \mathcal{P}' \cap B_r(\xi')) \le K\delta r. \tag{3.9}$$

From the Reifenberg flatness of Ω, see Definition 2.7(1), we have

$$\sup_{x \in \partial\Omega \cap B_r(\xi)} \text{dist}(x, \mathcal{P} \cap B(\xi, r)) \le \delta r.$$

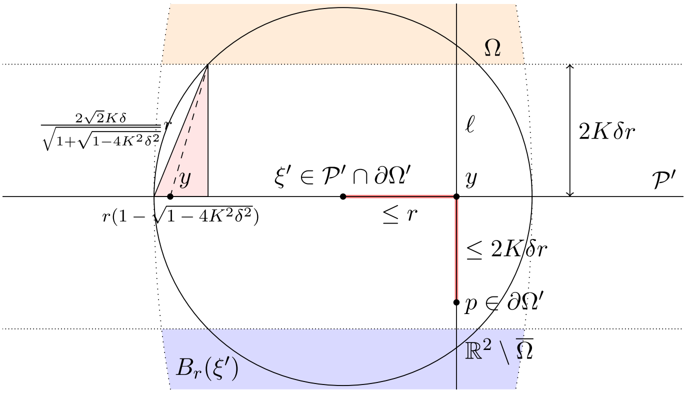


Fig. 2 Setting of (3.10)

Given $y \in \partial\Omega' \cap B_r(\xi')$, take $z \in \mathcal{P} \cap B_r(\xi)$ with $\operatorname{dist}(Dy, z) \leq \delta r$. Since $\mathcal{P}'$ is a line through ξ, the distance $\operatorname{dist}(y, \mathcal{P}')$ is attained at $B_r(\xi')$, that is,

$$\operatorname{dist}(y, \mathcal{P}' \cap B_r(\xi')) = \operatorname{dist}(y, \mathcal{P}') \leq \operatorname{dist}(y, D^{-1}z).$$

By Claim 3.9,

$$\operatorname{dist}(y, \mathcal{P}' \cap B_r(\xi')) \leq K \operatorname{dist}(Dy, z) \leq K\delta r,$$

and (3.9) follows.

Now we turn to prove

$$\sup_{y \in \mathcal{P}' \cap B_r(\xi')} \operatorname{dist}(y, \partial\Omega' \cap B_r(\xi')) \leq \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}} r. \tag{3.10}$$

By Claim 3.10 we have that for each line ℓ orthogonal to $\mathcal{P}'$ such that

$$\ell \cap B_r(\xi') \cap \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\} \neq \emptyset,$$

there is a point $p \in \ell \cap \partial\Omega' \cap \{x \in B_r(\xi') : \operatorname{dist}(x, \mathcal{P}') < 2K\delta r\}$, since $\partial\Omega'$ must separate each component in (3.7). Using this fact and $\xi' \in \mathcal{P}' \cap \partial\Omega'$ (the center of the ball), for every $y \in \mathcal{P}' \cap B_r(\xi')$ we obtain

$$\operatorname{dist}(y, \partial\Omega' \cap B_r(\xi')) \leq \min\left\{r, \operatorname{dist}\left(y, B_r(\xi') \cap \{x : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\}\right)\right\}.$$

Using basic trigonometric computations, see Fig. 2, we have that for every $y \in \mathcal{P}' \cap B_r(\xi')$,

$$\operatorname{dist}\left(y, B_r(\xi') \cap \{x : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\}\right) \leq \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}} r.$$

Since $K\delta < \sqrt{15}/16 < \sqrt{3}/4$, the previous value is less than r, and hence (3.10) is proved.

Notice that $K\delta r < 2K\delta r < \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r$. From this, (3.9) and (3.10) we get

$$\operatorname{dist}_{\mathcal{H}}(\partial\Omega' \cap B_r(\xi'), \mathcal{P}' \cap B_r(\xi')) \leq \frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r,$$

and Definition 2.7(1) is verified.

Since the last term is strictly larger than $2K\delta r$, and $2\frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}} < 1$ when $\delta < \frac{\sqrt{15}}{16K}$, we get

$$\begin{aligned} B_r(\xi') \cap \{x : \operatorname{dist}(x, \mathcal{P}') \geq 2K\delta r\} \supset B_r(\xi') \\ \cap \left\{x : \operatorname{dist}(x, \mathcal{P}') \geq 2\frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}r\right\} \neq \emptyset, \end{aligned}$$

and the second condition of Reifenberg flat, Definition 2.7(2), is achieved by Claim 3.10.

In conclusion, if $\delta < \sqrt{15}/(16K)$ then $\Omega' = D^{-1}(\Omega)$ is $\left(\frac{2\sqrt{2}K\delta}{\sqrt{1+\sqrt{1-4K^2\delta^2}}}, r_0\right)$-Reifenberg flat. In particular $\Omega' = D^{-1}(\Omega)$ is $\left(\frac{8\sqrt{15}}{15}K\delta, r_0\right)$-Reifenberg flat. □

3.2 Reduction 1: Lipschitz diagonalization of the symmetric part

Let us see how Lemma 3.3 implies Main Lemma 3.1. We will do this in two steps. First we show how to get rid of the assumption on the decomposition of A_0, using the linear transformation in Lemma 3.7.

Claim 3.11 *In Lemma 3.3, the hypothesis $A_0 = R^T BR$ is unnecessary.*

Proof For every square matrix X we define its symmetric part $X_0 := \frac{X+X^T}{2}$.

By Lemma 3.7 there exists a constant diagonal matrix $D = \begin{pmatrix} 1/K & 0 \\ 0 & 1 \end{pmatrix}$ with $K = K(\lambda) \geq 1$ such that $\widetilde{(A_0)}(\cdot) = D^{-1}A_0(D\cdot)D^{-1}$ can be written as $\widetilde{(A_0)} = R^T BR \in C^{0,1}(D^{-1}(U_{r_0}(\partial\Omega)))$ with $R \in C^{0,1}(D^{-1}(U_{r_0}(\partial\Omega)))$ a rotation, and $B \in C^{0,1}(D^{-1}(U_{r_0}(\partial\Omega)))$ diagonal.

Setting $\widetilde{A}(\cdot) := D^{-1}A(D\cdot)D^{-1}$, we have that the symmetric part of the matrix $\widetilde{A}$ is

$$\begin{aligned} \widetilde{A}_0 &:= \frac{D^{-1}A(D\cdot)D^{-1} + \left(D^{-1}A(D\cdot)D^{-1}\right)^T}{2} \\ &= \frac{D^{-1}A(D\cdot)D^{-1} + D^{-1}A^T(D\cdot)D^{-1}}{2} = \widetilde{(A_0)}, \end{aligned}$$

and hence $\widetilde{A}_0 = R^T BR$ as before.

Note that

$$U_{r_0}(\partial D^{-1}(\Omega)) := \{x : \mathrm{dist}(x, \partial D^{-1}\Omega) < r_0\} \subset D^{-1}(U_{r_0}(\partial\Omega)),$$

and so these matrices are Lipschitz in $U_{r_0}(\partial D^{-1}(\Omega))$.

Denoting $\widetilde{\omega} := \omega_{D^{-1}\Omega,\widetilde{A}}$ the elliptic measure in $D^{-1}\Omega$ with matrix $\widetilde{A}$, by Corollary 3.5 we have $\omega^x(\cdot) = \widetilde{\omega}^{D^{-1}x}(D^{-1}\cdot)$ for any $x \in \Omega$. By Lemma 3.8 we have that $D^{-1}\Omega$ is $(8\sqrt{15}K\delta/15, r_0)$-Reifenberg flat. Set $\widetilde{p} := D^{-1}p \in D^{-1}(\Omega)$. Since $\mathrm{dist}(p, \partial\Omega) > r_0$, $\mathrm{dist}(\widetilde{p}, \partial D^{-1}\Omega) > r_0$ and we are in position to apply Lemma 3.3 with this pole $\widetilde{p}$.

First, we need to compute the ellipticity constant, the Lipschitz seminorm, and the L^∞ norm of the matrix $\widetilde{A}$. Recall $\lambda \geq 1$ is the ellipticity constant of A. For $\xi, \eta \in \mathbb{R}^2$,

$$\langle \widetilde{A}\xi, \eta\rangle = \langle A(D\cdot)D^{-1}\xi, D^{-1}\eta\rangle \leq \lambda|D^{-1}\xi||D^{-1}\eta| \leq \lambda K^2|\xi||\eta|,$$
$$\langle \widetilde{A}\xi, \xi\rangle = \langle A(D\cdot)D^{-1}\xi, D^{-1}\xi\rangle \geq \lambda^{-1}|D^{-1}\xi|^2 \geq \lambda^{-1}|\xi|^2 \geq \left(\lambda K^2\right)^{-1}|\xi|^2,$$

i.e., the ellipticity constant of $\widetilde{A}$ is λK^2. If one seeks optimal constants, choosing $\widetilde{A}/K$ the ellipticity constant becomes λK. The Lipschitz seminorm of $\widetilde{A}$ in $D^{-1}(U_{r_0}(\partial\Omega))$ is at most $K^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))}$. Indeed, for any two points $x, y \in D^{-1}(U_{r_0}(\partial\Omega))$ with $x \neq y$,

$$\begin{aligned}\frac{|D^{-1}A(Dx)D^{-1} - D^{-1}A(Dy)D^{-1}|}{|x-y|} &\leq K^2\frac{|A(Dx) - A(Dy)|}{|x-y|}\\ &\leq K^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))}\frac{|Dx - Dy|}{|x-y|}\\ &\leq K^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))}.\end{aligned}$$

The L^∞ norm is $\|\widetilde{A}\|_{L^\infty(\mathbb{R}^2)} \leq K^2\|A\|_{L^\infty(\mathbb{R}^2)}$.

By Lemma 3.3 there exists

$$\delta_0 = \delta_0(K(\lambda)^2\lambda, K(\lambda)^2[A]_{C^{0,1}(U_{r_0}(\partial\Omega))} \cdot K(\lambda)^2\|A\|_{L^\infty(\mathbb{R}^2)}) > 0$$

such that if $0 < 8\sqrt{15}K(\lambda)\delta/15 \leq \delta_0$, i.e., $0 < \delta \leq 15\delta_0/(8\sqrt{15}K(\lambda))$, and taking M big enough such that Lemma 3.3 holds, then there is a set $\widetilde{F} \subset \partial D^{-1}(\Omega)$ such that $\widetilde{\omega}^{\widetilde{p}}(\widetilde{F}) \geq C^{-1}\tau$ and with a covering $\widetilde{F} \subset \bigcup_i B(\widetilde{z}_i, r_i)$ where

(1*) $\sum_i r_i \leq CM^\tau$,
(2*) $\sum_{\{i : r_i > \rho\}} r_i \leq CM^{-1}$,

with universal constant C.

Defining $F := D(\widetilde{F})$ we have $\omega^p(F) = \widetilde{\omega}^{\widetilde{p}}(\widetilde{F}) \geq C^{-1}\tau$, and $F \subset \bigcup_i D(B(\widetilde{z}_i, r_i))$. Finally, as $D(B(\widetilde{z}_i, r_i)) \subset B(D\widetilde{z}_i, r_i)$, then $\{B(D\widetilde{z}_i, r_i)\}_i$ is a covering of F satisfying the same properties. □

3.3 Reduction 2: The dependence on the Lipschitz seminorm

This is the second step to show that Lemma 3.3 implies Main Lemma 3.1. Note that the flatness constant of the Reifenberg flat domain on Lemma 3.3 (hence also on Claim 3.11) depends also on the Lipschitz seminorm κ of the matrix. Below, we see that in fact these results imply Main Lemma 3.1 by a rescaling argument. Here the flatness constant is determined solely by the ellipticity of the matrix.

Proof of Main Lemma 3.1 assuming Lemma 3.3 Fix $r \in (0, 1]$ be such that $r\kappa \|A\|_{L^\infty} \leq 1$, and let $\widetilde{A}(\cdot) = A(r\cdot)$, $\widetilde{\Omega} = \Omega/r$ and $\widetilde{\omega} := \omega_{\widetilde{\Omega},\widetilde{A}}$ be the elliptic measure with respect to the matrix $\widetilde{A}$ in $\widetilde{\Omega}$.

The matrix $\widetilde{A}$ is Lipschitz in $\{x/r : \operatorname{dist}(x, \partial\Omega) < r_0\} = \{x : \operatorname{dist}(x, \partial\widetilde{\Omega}) < r_0/r\}$, and $\widetilde{\Omega}$ is $(\delta, r_0/r)$-Reifenberg flat since Ω is (δ, r_0)-Reifenberg flat. By the uniqueness of the elliptic measure we have $\widetilde{\omega}^{z/r}(\cdot/r) = \omega^z(\cdot)$ for any $z \in \Omega$.

With this "zoom" the ellipticity constant of $\widetilde{A}$ becomes the same as the one of A, $\|\widetilde{A}\|_{L^\infty} = \|A\|_{L^\infty}$ and $[\widetilde{A}]_{C^{0,1}} = r[A]_{C^{0,1}}$, which implies $[\widetilde{A}]_{C^{0,1}} \cdot \|\widetilde{A}\|_{L^\infty} = r\kappa\|A\|_{L^\infty} \leq 1$.

This allows us to invoke Claim 3.11 for the elliptic measure $\widetilde{\omega}^{p/r}$ since $\operatorname{dist}(p/r, \Omega/r) > r_0/r$, as $\operatorname{dist}(p, \Omega) > r_0$. Note that now we don't have the dependence on the Lipschitz seminorm and L^∞ norm of the matrix since we are in the case $[\widetilde{A}]_{C^{0,1}}\|\widetilde{A}\|_{L^\infty} \leq 1$. Hence, there exists $\delta_0 = \delta_0(\lambda) > 0$ such that if $0 < \delta \leq \delta_0$, then for M big enough (to satisfy Claim 3.11) and setting ρ such that $0 < \rho/r < 1/M$, we can find a set $F' \subset \partial\Omega/r$ such that $\widetilde{\omega}^{p/r}(F') \geq C^{-1}\tau$ and with a covering $F' \subset \bigcup_i B(z_i', r_i)$ such that

(1*) $\sum_i r_i \leq CM^\tau$,
(2*) $\sum_{\{i\,:\,r_i>\rho/r\}} r_i \leq CM^{-1}$.

Set $F = rF'$. Then $\omega^p(F) = \widetilde{\omega}^{p/r}(F') \geq C^{-1}\tau$, and $F = rF' \subset \bigcup_i B(z_i, rr_i)$, which implies

(1) $\sum_i rr_i \leq rCM^\tau \leq CM^\tau$,
(2) $\sum_{\{i\,:\,rr_i>\rho\}} rr_i = \sum_{\{i\,:\,r_i>\rho/r\}} rr_i \leq rCM^{-1} \leq CM^{-1}$,

as claimed. □

4 Elliptic measures in CDC domains

In this section we collect the key properties of elliptic measures in CDC domains for the proof of Main Lemma 3.1. The first one, frequently called Bourgain's lemma (see [7, Lemma 1] for the harmonic case), is the following lemma.

Lemma 4.1 *([21, Lemma 11.21]) Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded CDC domain with constant c_0 and radius s_0, and let A be a real uniformly elliptic (not necessarily symmetric) matrix. Then there exists a constant $\tau \in (0, 1)$, depending only on n, c_0 and the ellipticity constant of the matrix A, such that for $E \subset \partial\Omega$, $x_0 \in \partial\Omega$ and $0 < r \leq s_0$, we have the following:*

(1) if $B(x_0, 2r) \cap E = \emptyset$, then $\omega^p_{\Omega,A}(E) \leq 1 - \tau < 1$, and
(2) if $B(x_0, 2r) \cap \partial\Omega \subset E$, then $\omega^p_{\Omega,A}(E) \geq \tau > 0$,

for any point $p \in B(x_0, r) \cap \Omega$.

The proof of Main Lemma 3.1 is based on a modification of the domain, without losing the initial information. In the following lemma we obtain the first step in that modification. This is the analogue of [44, Lemma 1.1].

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a bounded CDC domain with constants c_0 and radius s_0, let $x \in \partial\Omega$ and $0 < r \leq s_0$, and let A be a real uniformly elliptic (not necessarily symmetric) matrix. Then for any $k > 2$, there exists a constant C depending only on n, c_0 and the ellipticity constant of the matrix A such that*

$$\omega^p_{\Omega\setminus\overline{B(x,r)},A}\left(\overline{B(x,r)}\right) \leq C\omega^p_{\Omega,A}(B(x,kr)), \text{ for all } p \in \Omega\setminus\overline{B(x,r)}.$$

Proof The proof follows the same argument as [44, Lemma 1.1].

Since $k > 2$, by Lemma 4.1(2) there exists $C > 1$, depending on n, c_0 and the ellipticity constant of the matrix, such that $\omega^p_{\Omega,A}(B(x,kr)) \geq \omega^p_{\Omega,A}(\overline{B(x,2r)}) \geq C^{-1}$ for any $p \in \overline{B(x,r)} \cap \Omega$. The lemma follows by the maximum principle in $\Omega\setminus\overline{B(x,r)}$ by standard techniques. □

Later on we will need to have some control on the Radon-Nikodym derivative of the elliptic measure of the modified domain with respect to its surface measure, see Lemma 4.5 below. First we compute the CDC constants of an annulus, which will be used later to control this density in a modified domain.

Lemma 4.3 *Let $k > 3$. The annulus $\mathcal{A}_k = B(0, k^2 r) \setminus \overline{B(0,r)} \subset \mathbb{R}^{n+1}$, $n \geq 1$, satisfies the CDC with constant $c_0 = c_0(k)$ and radius $s_0 := \left(\frac{k+1}{2}\right)^2 r$, and moreover*

(1) $\overline{B(0,kr)} \subset B(x_0, s_0)$ and
(2) $\overline{B(x_0, 2s_0)} \subset B(0, k^2 r)$,

for any $x_0 \in \partial B(0,r)$, i.e., the inner circle.

Proof In the following computations we find the radius s_0 to have the CDC on the annulus $\mathcal{A}_k$ with the conditions (1) and (2). From the first condition we get $s_0 > (k+1)r$, and from the second $2s_0 + r < k^2 r$, i.e., $2s_0 < (k^2 - 1)r$. In order to have existence in s_0 we need $2(k+1)r < (k^2 - 1)r$, whence we need $k > 3$. Let s_0 be the middle point in $\left((k+1)r, \frac{k^2-1}{2}r\right)$,

$$s_0 := \frac{(k+1)r + \frac{k^2-1}{2}r}{2} = \frac{k^2 + 2k + 1}{4}r =: C(k) \cdot r.$$

Now we want to see that for $k > 3$, the annulus $\mathcal{A}_k$ satisfies the capacity density condition with $s_0 = \frac{k^2+2k+1}{4}r = C(k) \cdot r$. By definition of $C(k)$, given a point

$x_0 \in \partial \mathcal{A}_k$, the ball $B(x_0, C(k) \cdot r)$ does not intersect the other component of $\partial \mathcal{A}_k$, by condition (2).

We want to see that there exists $c_0 = c_0(k)$ such that

$$\operatorname{Cap}\left(\overline{B(x_0, s)} \cap \mathcal{A}_k^c, B(x_0, 2s)\right) \geq c_0(k) \cdot s^{n-1},$$

for all $x_0 \in \partial \mathcal{A}_k$ and $0 < s \leq s_0$.

Case 1. Suppose $x_0 \in \partial B(0, r) \subset \partial \mathcal{A}_k$. Let $0 < s \leq s_0 = C(k) \cdot r$. By the choice of $C(k)$ we have $B(x_0, s) \cap \mathcal{A}_k^c = B(x_0, s) \cap \overline{B(0, r)}$.

Set $\xi = x_0 - x_0 \frac{s}{2s_0}$. So $|\xi - x_0| = \left| x_0 \frac{s}{2s_0} \right| = \frac{rs}{2s_0} = \frac{s}{2C(k)}$, and note that $|\xi - x_0| \leq \frac{r}{2}$. In particular $B\left(\xi, \frac{s}{2C(k)}\right) \subset B(x_0, s) \cap \mathcal{A}_k^c$. Also, $B(x_0, 2s) \subset B(\xi, 2s + |x_0 - \xi|) = B\left(\xi, 2s + \frac{s}{2C(k)}\right)$. From these two inclusions, the monotonicity of the capacity and [21, (2.13)], we have

$$\operatorname{Cap}\left(\overline{B(x_0, s)} \cap \mathcal{A}_k^c, B(x_0, 2s)\right) \geq \operatorname{Cap}\left(\overline{B\left(\xi, \frac{s}{2C(k)}\right)}, B\left(\xi, 2s + \frac{s}{2C(k)}\right)\right) \approx_k s^{n-1}.$$

Case 2. Suppose $x_0 \in \partial B(0, k^2 r) \subset \partial \mathcal{A}_k$. Let $0 < s \leq s_0 = C(k) \cdot r$. Define $\xi = x_0 + \frac{x_0}{|x_0|} \frac{s}{2}$. Hence $B\left(\xi, \frac{s}{2}\right) \subset B(x_0, s) \cap \mathcal{A}_k^c$ and $B(x_0, 2s) \subset B\left(\xi, \frac{5}{2} s\right)$. Arguing as before we get

$$\operatorname{Cap}\left(\overline{B(x_0, s)} \cap \mathcal{A}_k^c, B(x_0, 2s)\right) \gtrsim s^{n-1},$$

as claimed. □

Now we study the density of the elliptic measure in an annulus. For a Hölder matrix A, here we use that L_A-harmonic functions are Hölder continuous up to the boundary, see Theorem 2.12.

Lemma 4.4 *Let $k > 3$, $0 < r \leq 1$ and $\mathcal{A}_k = B(x, k^2 r) \setminus \overline{B(x, r)} \subset \mathbb{R}^{n+1}$, $n \geq 1$. Let A be a real uniformly elliptic (not necessarily symmetric) matrix. Suppose also that $A \in C^\alpha\left(B(x, 2k^2 r)\right)$ with $0 < \alpha < 1$. Then the elliptic measure in the annulus $\mathcal{A}_k$ (arising from the matrix A) satisfies*

$$\omega^z_{\mathcal{A}_k, A}(Y) \lesssim \frac{\sigma(Y)}{r^n}, \text{ for any } z \in \partial B(x, kr) \text{ and any } Y \subset \partial B(x, r),$$

with constant depending only on k, $[A]_{C^\alpha}$ and the ellipticity of A.

Proof Suppose without loss of generality that the annulus is centered at the origin and denote $B_t := B(0, t)$. We can also assume that $r = 1$. Indeed, denote $\omega := \omega_{\mathcal{A}_k, A}$ and $\widetilde{\omega} := \omega_{\widetilde{\mathcal{A}}_k, \widetilde{A}}$ the elliptic measure associated to the matrix $\widetilde{A}(\cdot) := A(r \cdot)$, where the rescaled annulus is $\widetilde{\mathcal{A}}_k = B_{k^2}\backslash\overline{B_1}$. After rescaling and by the uniqueness of the elliptic measure,

$$\omega^z(Y) = \widetilde{\omega}^{z'}(Y') \text{ where } z' = z/r \text{ and } Y' = Y/r,$$

see Corollary 3.5. The matrix $\widetilde{A}$ has the same ellipticity constant as A, and the Hölder seminorm is improved because $[\widetilde{A}]_{C^\alpha} = [A]_{C^\alpha} \cdot r^\alpha$ whenever $0 < r < 1$. If the lemma were true with $r = 1$ then writing $p = z/r$ we would get

$$\omega^z(Y) = \widetilde{\omega}^p(Y') \le C_k \sigma(Y') = C_k \frac{\sigma(Y)}{r^n},$$

as claimed.

Let $p \in \partial B_k$ and let g_p^T the Green function of the annulus $\mathcal{A}_k$ with pole at p. Then, using (2.8),

$$\omega^p(Y) = -\int_Y \langle A^T(\xi)\nabla g_p^T(\xi), \nu(\xi)\rangle \, d\sigma(\xi) \lesssim \int_Y |\nabla g_p^T(\xi)| \, d\sigma(\xi). \tag{4.1}$$

We would be done if we can bound $|\nabla g_p^T| \le C_k$. To obtain this we apply Theorem 2.12. In the next paragraphs we check its hypothesis.

The function g_p^T is L_{A^T}-harmonic in $B_2\backslash B_1$ since we are in the case $k > 3$. Moreover $g_p^T \equiv 0$ in ∂B_1. We need to verify that $g_p^T \in W^{1,2}(B_2\backslash B_1)$. Recall that the Green function is constructed in (2.5) as $g_p^T = -\mathcal{E}_p^T + h$ where h is a $L_{\widetilde{A}^T}$-harmonic function with $h = \mathcal{E}_p^T$ on $\partial\mathcal{A}_k$, and $\mathcal{E}_p^T$ is the fundamental solution with pole at p. Hence,

$$\begin{aligned} \|g_p^T\|_{L^2(B_2\backslash B_1)} &\le \|g_p^T\|_{L^2(B_{5/2}\backslash B_1)} \le \max_{y\in\partial B_{5/2}} g_p^T(y) \le \max_{y\in\partial B_{5/2}} |\mathcal{E}_p^T(y)| + \max_{y\in\partial B_{5/2}} |h(y)| \\ &\le \max_{y\in\partial B_{5/2}} |\mathcal{E}_p^T(y)| + \max_{y\in\partial B_{k^2}\cup\partial B_1} |\mathcal{E}_p^T(y)|. \end{aligned} \tag{4.2}$$

In the planar case, we have $|\mathcal{E}_p^T(y)| \lesssim 1 + |\log|y-p||$ by Corollary 2.17, and in particular, from (4.2) we obtain

$$\|g_p^T\|_{L^2(B_2\backslash B_1)} \lesssim \max_{y\in\partial B_{5/2}} \left[1 + |\log|y-p||\right] + \max_{y\in\partial B_{k^2}\cup\partial B_1} \left[1 + |\log|y-p||\right] \le C_k.$$

In higher dimensions, $n \ge 2$, the fundamental solution is bounded by $|\mathcal{E}_p^T(y)| \lesssim |p-y|^{1-n}$ (see [20, Theorem 3.1 (3.55)]). From this bound and (4.2) we get

$$\|g_p^T\|_{L^2(B_2\backslash B_1)} \lesssim \max_{y\in\partial B_{5/2}} |y-p|^{n-1} + \max_{y\in\partial B_{k^2}\cup\partial B_1} |y-p|^{n-1} \le C_k.$$

In fact, we have obtained $\|g_p^T\|_{L^2(B_{5/2}\setminus B_1)} \lesssim C_k$, and hence by Caccioppoli's inequality in the annulus $B_2\setminus B_1$ we also obtain the upper bound for the gradient,

$$\|\nabla g_p^T\|_{L^2(B_2\setminus B_1)} \lesssim \|g_p^T\|_{L^2(B_{5/2}\setminus B_1)} \lesssim C_k,$$

implying $g_p^T \in W^{1,2}(B(0,2) \setminus B(0,1))$ depending only on k and the ellipticity constant.

Consider $\Omega = B_2\setminus\overline{B}_1$, $T = \partial B_1$ and $\Omega' = B_{3/2} \setminus \overline{B}_1$ in Theorem 2.12. Note that $\Omega' \subset \Omega$, $T \subset \partial\Omega'$ and $\operatorname{dist}\big((B_{3/2}\setminus\overline{B}_1) \cup \partial B_1, \partial B_2\big) = 1/2$. Then we get $g_p^T \in C^{1,\alpha}\big((B_{3/2}\setminus\overline{B}_1) \cup \partial B_1\big)$ with

$$\max_{y\in\partial B_1} |\nabla g_p^T(y)| \le \|g_p^T\|_{1;\Omega'} \overset{\text{Thm 2.12}}{\lesssim} \max_{y\in B_2\setminus\overline{B}_1} g_p^T(y) \le \max_{y\in\partial B_{5/2}} g_p^T(y) \le C_k,$$

and the lemma follows. □

Lemma 4.2 relates the elliptic measure on the initial domain with the elliptic measure on the domain minus a fixed ball. Next in Lemma 4.5, which is the analogue of [44, Lemma 1.2], we study the elliptic measure on this last setting. Combining Lemmas 4.2 and 4.5 we will obtain density properties of the elliptic measure on a modified domain.

Lemma 4.5 *Set $k > 3$ and $0 < r \le 1$. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 1$, be a bounded Wiener regular domain, $x \in \partial\Omega$ and let A be a real uniformly elliptic (not necessarily symmetric) matrix. Suppose also that $A \in C^\alpha(\{y \in \mathbb{R}^{n+1} : \operatorname{dist}(y, \partial\Omega) < 2k^2 r\})$ with $0 < \alpha < 1$.*

Set $\widetilde{\Omega} = \Omega\setminus\overline{B}$ where $B = B(x, r)$. Then $\omega^p_{\widetilde{\Omega},A}\big|_{\partial B}$ is absolutely continuous with respect to σ for any $p \in \Omega\setminus k\overline{B}$, and for $z \in \partial B$,

$$\frac{d\omega^p_{\widetilde{\Omega},A}}{d\sigma}(z) \le \frac{C}{r^n}\omega^p_{\Omega\setminus k\overline{B},A}\left(k\overline{B}\right), \text{ for any } p \in \Omega \setminus k\overline{B},$$

with constant C depending only on n, k, $[A]_{C^\alpha}$ and the ellipticity of A.

Following the scheme of the proof of [44, Lemma 1.2], to obtain Lemma 4.5 we study the elliptic measure of the annulus $\mathcal{A}_k := B(x, k^2 r)\setminus\overline{B(x, r)}$ when $k > 3$ (in order to apply Lemma 4.3) and $0 < r \le 1$ (to have a control on the Hölder seminorm of the matrix $A \in C^\alpha$). However, some technicalities are needed due to the variability of the coefficients of the matrix.

Proof of Lemma 4.5 During the proof we write $\omega_\cdot$ instead of $\omega_{\cdot,A}$.

To obtain the result it suffices to prove $\omega^p_{\widetilde{\Omega}}(Y) \lesssim \frac{\sigma(Y)}{r^n}\omega^p_{\Omega\setminus k\overline{B}}(k\overline{B})$ for all $p \in \Omega\setminus k\overline{B}$ and every $Y \subset \partial B$, and in fact, it is enough to assume that Y is open. Indeed, fixed $p \in \Omega\setminus k\overline{B}$, for $\varepsilon > 0$ let $U \supset Y$ be an open set (relative to ∂B) such that $\sigma(U) \le \sigma(Y)+\varepsilon$. If the lemma were true for open sets, then $\omega^p_{\widetilde{\Omega}}(Y) \le \omega^p_{\widetilde{\Omega}}(U) \lesssim \frac{\sigma(U)}{r^n}\omega^p_{\Omega\setminus k\overline{B}}(k\overline{B}) = \frac{\sigma(Y)+\varepsilon}{r^n}\omega^p_{\Omega\setminus k\overline{B}}(k\overline{B})$ and the general case would follow taking $\varepsilon \to 0$.

Let us assume Y is an open set relative to ∂B, and fix $p \in \Omega \setminus k\overline{B}$. Again, for $\varepsilon > 0$ let $V \supset k\overline{B}$ be an open set such that $\omega^p_{\Omega\setminus k\overline{B}}(V) \leq \omega^p_{\Omega\setminus k\overline{B}}(k\overline{B}) + \varepsilon$, $\psi \in C_c(V)$ such that $\mathbf{1}_{k\overline{B}} \leq \psi \leq \mathbf{1}_V$, and let $v_{\Omega\setminus k\overline{B}}$ denote the L_A-harmonic extension of ψ in $\Omega \setminus k\overline{B}$.

Let $N := \max_{x\in\partial kB} \omega^x_{\widetilde{\Omega}}(Y)$ and let $x_0 \in \partial kB$ such that $\omega^{x_0}_{\widetilde{\Omega}}(Y) = N$. Define the annulus $\mathcal{A}_k := k^2B\setminus\overline{B}$. By Lemma 4.6 below we obtain

$$N - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) = \omega^{x_0}_{\widetilde{\Omega}}(Y) - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) = \int_{\Omega\cap\partial k^2B} \omega^{\xi}_{\widetilde{\Omega}}(Y)\, d\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\xi). \tag{4.3}$$

By the maximum principle in $\widetilde{\Omega}\setminus k\overline{B}$ we have that $\omega^{\xi}_{\widetilde{\Omega}}(Y) \leq \omega^{x_0}_{\widetilde{\Omega}}(Y) = N$ for $\xi \in \partial k^2B$, and hence

$$\int_{\Omega\cap\partial k^2B} \omega^{\xi}_{\widetilde{\Omega}}(Y)\, d\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\xi) \leq N\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\Omega\cap\partial k^2B). \tag{4.4}$$

All in all, from (4.3) and (4.4) then

$$N - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \leq N\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\Omega\cap\partial k^2B). \tag{4.5}$$

Also, by the maximum principle and the fact that the annulus $\mathcal{A}_k = k^2B\setminus\overline{B}$ when $k > 3$ satisfies the CDC with the precise conditions in Lemma 4.3, we have that the right-hand side of (4.5) is controlled by

$$\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\Omega\cap\partial k^2B) \leq \omega^{x_0}_{\mathcal{A}_k}(\partial k^2B) \leq 1 - \tau < 1, \tag{4.6}$$

for $\tau \in (0, 1)$, depending also on k. Indeed, this last step follows by applying Lemma 4.1(1) to any $y \in \partial B$ with the choice of s_0 in Lemma 4.3, because in that case $x_0 \in \partial kB \subset B(y, s_0)$ and $B(y, 2s_0) \cap \partial k^2B = \emptyset$.

From (4.5) and (4.6) we obtain $N - \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \leq N\omega^{p_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(\partial k^2B) \leq (1-\tau)N$, equivalently $\omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \geq \tau N = \tau\omega^{x_0}_{\widetilde{\Omega}}(Y)$. From this and the maximum principle,

$$\tau\omega^{x_0}_{\widetilde{\Omega}}(Y) \leq \omega^{x_0}_{\widetilde{\Omega}\cap\mathcal{A}_k}(Y) \leq \omega^{x_0}_{\mathcal{A}_k}(Y).$$

We have reduced to the elliptic measure in the annulus $\mathcal{A}_k$. By Lemma 4.4 we have $\omega^{x_0}_{\mathcal{A}_k}(Y) \lesssim \sigma(Y)/r^n$, and so $\omega^{x_0}_{\widetilde{\Omega}}(Y) \lesssim \omega^{x_0}_{\mathcal{A}_k}(Y) \lesssim \sigma(Y)/r^n$. Since $x_0 \in \partial kB$ was chosen to achieve the maximum of $\omega^{\cdot}_{\widetilde{\Omega}}(Y)$ in ∂kB, we obtain that for any $x \in \widetilde{\Omega}\cap\partial kB$,

$$\omega^x_{\widetilde{\Omega}}(Y) \leq \omega^{x_0}_{\widetilde{\Omega}}(Y) \lesssim \frac{\sigma(Y)}{r^n} = \frac{\sigma(Y)}{r^n} \cdot v_{\Omega\setminus k\overline{B}}(x),$$

where the last equality is just because $v_{\Omega\setminus k\overline{B}}(\xi) = \psi = 1$ for $\xi \in \widetilde{\Omega}\cap\partial kB$. Moreover, $\omega^x_{\widetilde{\Omega}}(Y) = 0$ if $x \in \partial\widetilde{\Omega}\setminus kB$. The same inequality follows for $x \in \widetilde{\Omega}\setminus k\overline{B}$ by the maximum principle in $\widetilde{\Omega} \setminus k\overline{B}$. Evaluating at the fixed pole $p \in \Omega \setminus k\overline{B}$, by the choice of V we have

$$\omega^p_{\widetilde{\Omega}}(Y) \lesssim \frac{\sigma(Y)}{r^n} \cdot v_{\Omega\setminus k\overline{B}}(p) \leq \frac{\sigma(Y)}{r^n} \cdot \omega^p_{\Omega\setminus k\overline{B}}(V) \leq \frac{\sigma(Y)}{r^n} \cdot (\omega^p_{\Omega\setminus k\overline{B}}(k\overline{B}) + \varepsilon),$$

and the lemma follows by taking $\varepsilon \to 0$. □

For the sake of completeness here we provide a proof of (4.3).

Lemma 4.6 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a Wiener regular domain and A be a real uniformly elliptic (not necessarily symmetric) matrix. Let $\widetilde{\Omega} \subset \Omega$ be a Wiener regular domain. For any Borel set $E \subset \partial\Omega \cap \partial\widetilde{\Omega}$ and $p \in \widetilde{\Omega}$, there holds*

$$\omega^p_{\Omega,A}(E) - \omega^p_{\widetilde{\Omega},A}(E) = \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi_{\Omega,A}(E)\, d\omega^p_{\widetilde{\Omega},A}(\xi).$$

Proof During the proof we write $\omega = \omega_{\Omega,A}$ and $\widetilde{\omega} = \omega_{\widetilde{\Omega},A}$.

Fixed $p \in \widetilde{\Omega}$, for $m \geq 1$, by the inner (just for Borel sets) and outer regularity of Radon measures, let $K_m \subset E$ be a compact set and $U_m \supset E$ be an open set such that

$$\begin{aligned} &\omega^p(U_m) - 1/m \leq \omega^p(E) \leq \omega^p(K_m) + 1/m, \text{ and} \\ &\widetilde{\omega}^p(U_m) - 1/m \leq \widetilde{\omega}^p(E) \leq \widetilde{\omega}^p(K_m) + 1/m. \end{aligned} \tag{4.7}$$

Moreover, we take $K_m \subset K_{m+1}$ and $U_m \supset U_{m+1}$ by redefining the sequences suitably. Finally, take $K := \bigcup_{m\geq 1} K_m$ and $U := \bigcap_{m\geq 1} U_m$.

Let $\varphi = \varphi_m \in C_c(U_m)$ be such that $\mathbf{1}_{K_m} \leq \varphi \leq \mathbf{1}_{U_m}$, and let $u = u_m$ and $\widetilde{u} = \widetilde{u}_m$ denote the L_A-harmonic extension of φ in Ω and $\widetilde{\Omega}$ respectively. By the monotonicity of the integral,

$$\omega^\xi(K_m) \leq u(\xi) \leq \omega^\xi(U_m), \text{ for all } \xi \in \Omega, \text{ and} \tag{4.8a}$$

$$\widetilde{\omega}^\xi(K_m) \leq \widetilde{u}(\xi) \leq \widetilde{\omega}^\xi(U_m), \text{ for all } \xi \in \widetilde{\Omega}. \tag{4.8b}$$

Using (4.7), we get

$$|\omega^p(E) - \widetilde{\omega}^p(E) - (u(p) - \widetilde{u}(p))| \leq 2/m. \tag{4.9}$$

On the other hand, note that $u(\xi) - \widetilde{u}(\xi) = (u(\xi) - \varphi(\xi)) \cdot \mathbf{1}_{\partial\widetilde{\Omega}\setminus\partial\Omega}(\xi)$ if $\xi \in \partial\widetilde{\Omega}$, and in particular, writing $u - \widetilde{u}$ as the L_A-harmonic extension in $\widetilde{\Omega}$ of its boundary values in $\partial\widetilde{\Omega}$ and evaluating at the point $p \in \Omega$ we get

$$u(p) - \widetilde{u}(p) = \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} (u(\xi) - \varphi(\xi))\, d\widetilde{\omega}^p(\xi).$$

From this, (4.8a) and $0 \leq \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \varphi(\xi)\, d\widetilde{\omega}^p(\xi) \leq \widetilde{\omega}^p(U_m \setminus K_m) \leq 2/m$, we therefore obtain

$$\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi(K_m)\, d\widetilde{\omega}^p(\xi) - \frac{2}{m} \leq u(p) - \widetilde{u}(p) \leq \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^\xi(U_m)\, d\widetilde{\omega}^p(\xi),$$

which together with (4.9) gives

$$\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^{\xi}(K_m)\, d\widetilde{\omega}^p(\xi) - \frac{4}{m} \le \omega^p(E) - \widetilde{\omega}^p(E) \le \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^{\xi}(U_m)\, d\widetilde{\omega}^p(\xi) + \frac{2}{m}.$$

Note that the set $K = \bigcup_{m\ge 1} K_m \subset E$ satisfies $\omega^p(E) = \omega^p(K)$, and since K is measurable, then $\omega^p(E\setminus K) = 0$. Since elliptic measures are mutually absolutely continuous for any two different poles, this implies that $\omega^{\xi}(E\setminus K) = 0$ for any $\xi \in \Omega$. Again, since K is measurable we obtain that $\omega^{\xi}(E) = \omega^{\xi}(K)$ for any $\xi \in \Omega$. By the same argument, using now that E is Borel and so measurable for every ω^{ξ} with $\xi \in \Omega$, the set $U = \bigcap_{m\ge 1} U_m$ satisfies $\omega^{\xi}(E) = \omega^{\xi}(U)$ for any $\xi \in \Omega$.

Taking $m \to \infty$, by the monotone convergence theorem, the equation above becomes

$$\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \lim_{m\to\infty} \omega^{\xi}(K_m)\, d\widetilde{\omega}^p(\xi) \le \omega^p(E) - \widetilde{\omega}^p(E) \le \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \lim_{m\to\infty} \omega^{\xi}(U_m)\, d\widetilde{\omega}^p(\xi),$$

and the lemma follows since $\lim_{m\to\infty} \omega^{\xi}(K_m) = \omega^{\xi}(K) = \omega^{\xi}(E) = \omega^{\xi}(U) = \lim_{m\to\infty} \omega^{\xi}(U_m)$. □

5 Proof of Lemma 3.3, the weak version of the Main Lemma

According to the previous reductions in Section 3, to obtain the Main Lemma 3.1 it suffices to prove Lemma 3.3, which we intend to do in this section modulo the proof of (1.2) which is deferred to Section 7.

In this section we work with bounded (δ, r_0)-Reifenberg flat domains $\Omega \subset \mathbb{R}^2$. Recall that for $\delta > 0$ small enough we have that Ω is an NTA domain (see [28, Section 3]), and hence it satisfies the capacity density condition. See Remark 2.5.

We now turn to the proof of Lemma 3.3. Let $M > 0$ be big enough and $0 < \rho < 1/M$. Denote $\omega := \omega^p_{\Omega,A}$.

For $x \in \partial\Omega$ define the 'high density value' as

$$h(x) := \sup\{r \ge \rho : \omega(B(x,r)) \ge Mr\}, \tag{5.1}$$

and $h(x) = \rho$ if the supremum runs over an empty set. Note that $\rho \le h(x) \le 1/M$ for every $x \in \partial\Omega$, because ω is a probability measure.

Definition 5.1 (Good balls) For $x \in \partial\Omega$, we say that the ball $B(x,r)$ is good, $B(x,r) \in$ Good, if $r > h(x)$, i.e., for any $s \ge r$ we have $\omega(B(x,s)) < Ms$.

For $x \in \mathbb{R}^2$ define

$$d(x) := \inf_{B\in \mathsf{Good}} \left[|x - c(B)| + r(B)\right],$$

where $c(B)$ is the center of the ball and $r(B)$ its radius. For any $B \in$ Good we have that $r(B) \ge h(c(B)) \ge \rho$, which implies $d(x) \ge \rho$.

Note that for every ball $B(\xi, r) \in \mathsf{Good}$ we have $r \geq h(\xi)$. Therefore

$$d(x) = \inf_{\xi \in \partial\Omega} [|x - \xi| + h(\xi)].$$

This function is 1-Lipschitz as it is the infimum of 1-Lipschitz functions.

Remark 5.2 For $x \in \partial\Omega$ it follows $\rho \leq d(x) \leq h(x)$, and hence $h(x) = \rho$ implies $d(x) = h(x)$. Moreover, if $d(x) = \rho$ then $x \in \partial\Omega$.

Let $0 < \varepsilon < 1$ be small enough (to be fixed in (5.8) below) depending on the ellipticity constant λ and the product $\kappa \|A\|_{L^\infty(\mathbb{R}^2)}$. Let $\mathcal{I} = \mathcal{I}_{\varepsilon^{-2}}$ be the family of maximal dyadic cubes $Q \in \mathcal{D}(\mathbb{R}^2)$ such that $Q \cap \partial\Omega \neq \emptyset$ and $\ell(Q) \leq \varepsilon^2 d(x)$ for all $x \in Q$.

Lemma 5.3 *Let $\mathcal{I}$ be the family defined above. Then:*

(1) Every $Q \in \mathcal{I}$ satisfy $\frac{\varepsilon^2 d(x)}{3} < \ell(Q) \leq 2\varepsilon^2 d(x)$ for all $x \in \frac{\varepsilon^{-2}}{4} Q$.
(2) If $Q_1, Q_2 \in \mathcal{I}$ and $\frac{\varepsilon^{-2}}{4} Q_1 \cap \frac{\varepsilon^{-2}}{4} Q_2 \neq \emptyset$, then $\frac{\ell(Q_1)}{6} < \ell(Q_2) < 6\ell(Q_1)$.
(3) $\{\frac{\varepsilon^{-2}}{4} Q\}_{Q \in \mathcal{I}}$ has finite superposition, with superposition number $N = N_\varepsilon$ depending on ε only.

Proof Let $x \in \frac{\varepsilon^{-2}}{4} Q$. We start by proving $\ell(Q) \leq 2\varepsilon^2 d(x)$ in (1). Take any $y \in Q$. By the election of y and since d is 1-Lipschitz,

$$d(x) = d(y) + d(x) - d(y) \geq \varepsilon^{-2}\ell(Q) - |x - y| \geq \varepsilon^{-2}\ell(Q) - \operatorname{diam} \frac{\varepsilon^{-2}}{4} Q \geq \frac{\ell(Q)}{2\varepsilon^2}.$$

For the other inequality in (1), let $\widehat{Q}$ be the dyadic father of Q, i.e., the unique $\widehat{Q} \in \mathcal{D}(\mathbb{R}^2)$ such that $Q \subset \widehat{Q}$ and $\ell(\widehat{Q}) = 2\ell(Q)$. Since Q is maximal, there exists $y \in \widehat{Q}$ such that $2\ell(Q) = \ell(\widehat{Q}) > \varepsilon^2 d(y)$, and hence

$$d(x) = d(x) - d(y) + d(y) < |x - y| + \frac{2\ell(Q)}{\varepsilon^2} \leq \operatorname{diam} \frac{\varepsilon^{-2}}{4} Q + \frac{2\ell(Q)}{\varepsilon^2} \leq \frac{3\ell(Q)}{\varepsilon^2},$$

and with this we conclude the proof of (1).

Let $Q_1, Q_2 \in \mathcal{I}$ such that $\frac{\varepsilon^{-2}}{4} Q_1 \cap \frac{\varepsilon^{-2}}{4} Q_2 \neq \emptyset$. Take $x \in \frac{\varepsilon^{-2}}{4} Q_1 \cap \frac{\varepsilon^{-2}}{4} Q_2$, and then (2) follows from (1) by

$$\ell(Q_1) \leq 2\varepsilon^2 d(x) < 6\ell(Q_2).$$

Given $Q \in \mathcal{I}$, there is only a finite number $N = N_\varepsilon$ of cubes $P \in \mathcal{D}(\mathbb{R}^2)$ such that $\ell(Q)/6 < \ell(P) < 6\ell(Q)$ and $\frac{\varepsilon^{-2}}{4} Q \cap \frac{\varepsilon^{-2}}{4} P \neq \emptyset$, which gives (3). □

Lemma 5.4 *There exists $\gamma \geq 1$ such that for every $Q \in \mathcal{I}$ there exists a ball $G_Q \in \mathsf{Good}$ with $r(G_Q) \approx \varepsilon^{-2}\ell(Q)$, satisfying the inclusions $\varepsilon^{-2} Q \subset \gamma G_Q$ and $G_Q \subset \gamma\varepsilon^{-2} Q$.*

Proof Given $Q \in \mathcal{I}$, fix any $x \in Q \cap \partial\Omega$, and take $\xi \in \partial\Omega$ such that

$$d(x) \le |x-\xi| + h(\xi) \le 1.1 d(x). \tag{5.2}$$

Define the ball $G_Q := B(\xi, 2(|x-\xi| + h(\xi)))$, and hence $G_Q \in \mathsf{Good}$, since $r(G_Q) \ge 2h(\xi) \ge h(\xi) + \rho > h(\xi)$.

We claim that $r(G_Q) \approx \varepsilon^{-2}\ell(Q)$. Indeed, from (5.2) and (1) in Lemma 5.3,

$$\frac{1}{2} r(G_Q) = |x-\xi| + h(\xi) \overset{(5.2)}{\approx} d(x) \overset{(1)}{\approx} \varepsilon^{-2}\ell(Q).$$

Also, using the previous comparability, the distance between x and ξ is controlled above by

$$|x-\xi| \le |x-\xi| + h(\xi) = \frac{1}{2} r(G_Q) \approx \varepsilon^{-2}\ell(Q),$$

which implies that there exists a universal constant $\gamma \ge 1$ such that $\varepsilon^{-2}Q \subset \gamma G_Q$ and $G_Q \subset \gamma\varepsilon^{-2}Q$. $\square$

For each cube $Q \in \mathcal{I}$, fix a point $z_Q \in Q \cap \partial\Omega$ and define $B_Q := B(z_Q, r_Q)$ with $r_Q := \varepsilon d(z_Q)$. Next, we modify the domain as in [44], but using the family $\{B_Q\}_{Q\in\mathcal{I}}$. To do so, define

$$\widetilde{\Omega} := \Omega \setminus \bigcup_{Q\in\mathcal{I}} B_Q,$$

and denote $\widetilde{\omega} := \omega^p_{\widetilde{\Omega},A}$ its elliptic measure with pole p. From (1) in Lemma 5.3 and $d(\cdot) \ge \rho$ on $\partial\Omega$ we have that $\mathcal{I}$ is finite. In particular, the family $\{B_Q\}_{Q\in\mathcal{I}}$ is finite, and $\partial\widetilde{\Omega}$ is smooth except at finitely many points.

Recall $\mathcal{P}$ is the approximating hyperplane in Definition 2.7. Since the function $d(\cdot)$ is 1-Lipschitz, and so $\varepsilon d(\cdot)$ is ε-Lipschitz, the same proof as in [3, Lemma 2.2] applies to obtain the following lemma.

Lemma 5.5 *Let $r_0 \in (0,\infty]$ and let $\varepsilon > 0$ be small enough. Then:*

(1) (Analogue of [3, Lemma 2.2]) There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\Omega \subset \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat with $0 < \delta < \delta_0$, then the modified domain $\widetilde{\Omega}$ as above is $(c\varepsilon^{1/2}, r_0/2)$-Reifenberg flat.

(2) (See [3, Lemma 2.3(c)]) For every $Q \in \mathcal{I}$, there exists a Lipschitz function $f_Q : \mathcal{P}(z_Q, 30r(B_Q)) \cap 10B_Q \to \mathcal{P}(z_Q, 30r(B_Q))^{\perp}$ with Lipschitz constant at most $c\varepsilon^{1/2}$.

For any $Q \in \mathcal{I}$, by the maximum principle and Lemma 4.2 (with $k = 10$) respectively,

$$\widetilde{\omega}\left(\overline{B_Q}\right) \le \omega_{\Omega\setminus\overline{B_Q}}\left(\overline{B_Q}\right) \lesssim \omega(10B_Q). \tag{5.3}$$

Moreover, for all $z \in \partial\widetilde{\Omega} \cap \partial B_Q$, by Lemma 4.5 (with $k = \sqrt{10}$), the maximum principle and Lemma 4.2 (with $k = \sqrt{10}$) respectively,

$$\frac{d\widetilde{\omega}}{d\sigma}(z) \lesssim \frac{\omega_{\widetilde{\Omega}\setminus\overline{\sqrt{10}B_Q}}\left(\overline{\sqrt{10}B_Q}\right)}{r(B_Q)} \leq \frac{\omega_{\Omega\setminus\overline{\sqrt{10}B_Q}}\left(\overline{\sqrt{10}B_Q}\right)}{r(B_Q)} \lesssim \frac{\omega\left(10B_Q\right)}{r(B_Q)}. \tag{5.4}$$

By the existence of a good ball $G_Q \in \mathsf{Good}$ with $r(G_Q) \approx \varepsilon^{-2}\ell(Q)$ and $\varepsilon^{-2}Q \subset \gamma G_Q$ (see Lemma 5.4), the ball $10B_Q$ has bounded density with respect to the initial elliptic measure:

$$\begin{aligned}\omega\left(10B_Q\right) = \omega\left(B(z_Q, 10\varepsilon d(z_Q))\right) &\overset{\text{L.5.3(1)}}{\leq} \omega(3\cdot 3\cdot 10\varepsilon^{-1}Q) \overset{\varepsilon\ll 1}{\leq} \omega(\varepsilon^{-2}Q) \leq \omega(\gamma G_Q)\\ &\overset{\gamma G_Q\in\mathsf{Good}}{\leq} \gamma r(G_Q)M \approx \gamma\varepsilon^{-2}\ell(Q)M \overset{\text{L.5.3(1)}}{\approx} \gamma d(z_Q)M = \gamma\varepsilon^{-1}r(B_Q)M.\end{aligned}$$

In particular, combined with (5.4) this implies

$$\frac{d\widetilde{\omega}}{d\sigma}(z) \lesssim M \text{ for all } z \in \partial\widetilde{\Omega}, \tag{5.5}$$

where the involved constant depends on γ and ε.

Let us smooth out the domain $\widetilde{\Omega}$. Recall that $\widetilde{\Omega}$ is smooth except at finitely many points $\{\xi_j\}_{j\in J}$, with $\#J < \infty$ depending on M and ρ. Let $0 < s < \min_{Q\in\mathcal{I}} r(B_Q)/1000$ be a small enough parameter to be fixed later. For each point ξ_j, $j \in J$, let $B_1, B_2 \in \{B_Q : Q \in \mathcal{I}\}$ be the two intersecting balls such that $\xi_j \in \partial B_1 \cap \partial B_2$, and let c_1, c_2 be their centers and r_1, r_2 be their radii respectively. Take the unique point $q_j \in \widetilde{\Omega} \cap \partial B(c_1, r_1 + s) \cap \partial B(c_2, r_2 + s)$, and denote $B_j := B(q_j, s)$. The ball B_j is tangent to B_1 and B_2, and define T_j to be the bounded open region enclosed between B_1, B_2 and B_j. We define the new smooth domain

$$\widetilde{\widetilde{\Omega}} = \widetilde{\widetilde{\Omega}}_s := \widetilde{\Omega}\setminus\bigcup_{j\in J}\overline{T_j},$$

taking small enough s such that

(1) for each $Q \in \mathcal{I}$, $\sigma(\partial B_Q \cap \partial\widetilde{\widetilde{\Omega}}) \geq 0.9\sigma(\partial B_Q \cap \partial\widetilde{\Omega})$, and
(2) $\widetilde{\widetilde{\Omega}}$ is a Lipschitz domain with the same Lipschitz character as $\widetilde{\Omega}$.

Note that $\sigma(\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}) \leq \#J \cdot 2\pi s$, and so we can take this value to be as small as needed.

Denote $\widetilde{\widetilde{\omega}} := \omega^p_{\widetilde{\widetilde{\Omega}},A}$. By the maximum principle, $\widetilde{\widetilde{\omega}}(E) \leq \widetilde{\omega}(E)$ for any $E \subset \partial\widetilde{\Omega} \cap \partial\widetilde{\widetilde{\Omega}}$. Consequently,

$$\widetilde{\widetilde{\omega}}(\overline{B_Q}) \leq \widetilde{\omega}(\overline{B_Q}) \overset{(5.3)}{\lesssim} \omega(10B_Q) \text{ for any } Q \in \mathcal{I}, \tag{5.6}$$

and

$$\frac{d\widetilde{\widetilde{\omega}}}{d\sigma}(z) \leq \frac{d\widetilde{\omega}}{d\sigma}(z) \overset{(5.5)}{\lesssim} M \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}. \tag{5.7}$$

Let $K(\cdot) := \frac{d\widetilde{\widetilde{\omega}}}{d\sigma}(\cdot)$ be the Radon-Nikodym derivative. By (2.8) we have

$$K(\xi) = -\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle \text{ for } \xi \in \partial\widetilde{\widetilde{\Omega}},$$

where g^T is the Green function in $\widetilde{\widetilde{\Omega}}$ with respect to the matrix A^T. By (1.2) (proved in Section 7), if $\varepsilon > 0$ is small enough depending on the ellipticity constant λ and the product $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$, then there is a constant $\text{const}_{\Omega,A} > 0$ such that

$$\begin{aligned} -\infty < -\text{const}_{\Omega,A} &\leq \int_{\partial\widetilde{\widetilde{\Omega}}} \log|S(\xi)\nabla g^T(\xi)|\, d\widetilde{\widetilde{\omega}}(\xi) \\ &= \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log|S(\xi)\nabla g^T(\xi)|\, d\sigma(\xi), \end{aligned} \tag{5.8}$$

where $A_0 = \frac{A+A^T}{2}$ and $S = A_0^{1/2}$, i.e., $S^T S = A_0$.

For every $\xi \in \partial\widetilde{\widetilde{\Omega}}$ we can write

$$\begin{aligned} \langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle &= \langle \nabla g^T(\xi), A(\xi)\nu(\xi)\rangle \\ &= \langle \nabla g^T(\xi), c_1(\xi)\nu(\xi)\rangle + \langle \nabla g^T(\xi), c_2(\xi)t(\xi)\rangle, \end{aligned}$$

where ν (resp. t) is the outward normal (resp. tangential) vector of $\partial\widetilde{\widetilde{\Omega}}$, and $c_1(\xi)$ (resp. $c_2(\xi)$) is the projection of $A(\xi)\nu(\xi)$ into $\nu(\xi)$ (resp. $t(\xi)$). In particular $c_1(\xi) = \langle A(\xi)\nu(\xi), \nu(\xi)\rangle \approx 1$, and since $\partial\widetilde{\widetilde{\Omega}}$ is smooth, we have that $\langle \nabla g^T(\xi), c_2(\xi)t(\xi)\rangle = c_2(\xi)\partial_t g^T(\xi) = 0$ in $\partial\widetilde{\widetilde{\Omega}}$ as $g^T|_{\partial\Omega} = 0$. Hence $-\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle \approx |\nabla g^T(\xi)|$, and since $|\nabla g^T(\xi)|^2 \approx \langle A_0\nabla g^T(\xi), \nabla g^T(x)\rangle = |S(\xi)\nabla g^T(\xi)|^2$, we obtain

$$|S(\xi)\nabla g^T(\xi)| \leq -C\langle A^T(\xi)\nabla g^T(\xi), \nu(\xi)\rangle = CK(\xi). \tag{5.9}$$

From (5.8) and (5.9),

$$\begin{aligned} -\infty < -\text{const}_{\Omega,A} &\leq \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log|S(\xi)\nabla g^T(\xi)|\, d\sigma(\xi) \leq \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log CK(\xi)\, d\sigma(\xi) \\ &= \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log C\, d\sigma(\xi) + \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log K(\xi)\, d\sigma(\xi) \\ &= \log C + \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log K(\xi)\, d\sigma(\xi), \end{aligned}$$

whence we obtain

$$-\infty < -C_{\Omega,A} \leq \int_{\partial\widetilde{\widetilde{\Omega}}} K(\xi)\log K(\xi)\, d\sigma(\xi). \tag{5.10}$$

In view of (5.10) and the fact that $K(\cdot) \lesssim M$ on $\partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$ by (5.7), if M is big enough (provided s is small enough depending on M and ρ), then the set of points

$\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$ with density $K(\xi) < M^{-\tau}$ indeed has elliptic measure at most $1 - \tau/4$, uniformly in ρ and M. More precisely, we will obtain

$$\widetilde{\widetilde{\omega}}\left(\left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}\right) = \int_{\left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}} K(\xi)\, d\sigma(\xi) \geq \frac{\tau}{4} \tag{5.11}$$

whenever M is big enough depending on τ and the constant $\mathrm{const}_{\Omega,A}$, taking small enough s depending on M and ρ. Note that if we had $K(\cdot) \lesssim M$ in the whole boundary $\partial\widetilde{\widetilde{\Omega}}$, using Tchebyshoff's inequality as in [44, p. 170] we would directly get (5.11) from (5.10). However, we only have the bound of K on $\partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$ and we will need to estimate the elliptic measure (and the $K \log^+ K\, d\sigma$ measure) of the "bad" set $\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}$.

Proof of (5.11) Let $\log^+(\cdot) := \max\{0, \log(\cdot)\}$ and $\log^-(\cdot) := -\min\{0, \log(\cdot)\}$, and recall that $\widetilde{\widetilde{\Omega}} = \widetilde{\widetilde{\Omega}}_s$. We first establish that

$$\int_{\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}} K \log^+ K\, d\sigma \to 0 \text{ as } s \to 0, \tag{5.12}$$

which in particular implies

$$\widetilde{\widetilde{\omega}}(\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}) = \int_{\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}} K\, d\sigma \leq e\sigma(\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}) + \int_{\partial\widetilde{\widetilde{\Omega}} \setminus \partial\widetilde{\Omega}} K \log^+ K\, d\sigma \to 0 \text{ as } s \to 0. \tag{5.13}$$

As previously noted, for sufficiently small $s > 0$, the Lipschitz character of $\widetilde{\widetilde{\Omega}}$ is controlled by the one of $\widetilde{\Omega}$, and therefore we have that it is NTA with uniform constants. Thus, the involved constants in the subsequent computations will be independent of s.

We claim that there exists $\alpha > 0$ such that

$$\widetilde{\widetilde{\omega}}(B(x, r)) \lesssim \left(\frac{r}{r_0}\right)^{\alpha} \text{ for all } x \in \partial\Omega \text{ and } 0 < r \leq r_0/1000, \tag{5.14}$$

where r_0 is the scale where Reifenberg flatness of Ω is granted. To prove this, note that there exists $\beta \in (0, 1)$ such that $\widetilde{\widetilde{\omega}}(B \setminus B/2) \geq \beta\widetilde{\widetilde{\omega}}(B/2)$ for any ball B centered at $\partial\widetilde{\widetilde{\Omega}}$ with radius $r_B \leq r_0/100$, as a consequence of the doubling property of elliptic measure in NTA domains (see [25, (1.3.7)]) and the fact that $B \setminus B/2 \neq \emptyset$ by Reifenberg flatness. It follows that $\widetilde{\widetilde{\omega}}(B/2) \leq \widetilde{\widetilde{\omega}}(B)/(1 + \beta)$. Iterating this inequality for a fixed $x \in \partial\Omega$ and $B_0 = B(x, r_0/200)$, we have $\widetilde{\widetilde{\omega}}(2^{-k} B_0) \leq \widetilde{\widetilde{\omega}}(B_0)/(1+\beta)^k \leq 1/(1+\beta)^k$ for all $k \geq 0$ integer. This readily implies the existence of $\alpha = \alpha(\beta) > 0$ such that (5.14) holds.

Let us fix $j \in J$ momentarily. For B_j, let $q_j = c(B_j)$ denote its center. By the change of pole formula in NTA domains (see [25, Corollary 1.3.8]), we have

$$K(z) \approx \frac{d\widetilde{\widetilde{\omega}}^{q_j}}{d\sigma}(z)\widetilde{\widetilde{\omega}}(\overline{B_j}) \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j. \tag{5.15}$$

Let $\widetilde{\widetilde{g}}^T_{q_j}$ be the Green function of $\widetilde{\widetilde{\Omega}}$ with pole at q_j, and $u(\cdot) = \widetilde{\widetilde{g}}^T_{q_j}(s\cdot+q_j)$, which is the Green function of $\{(z-q_j)/s : z \in \widetilde{\widetilde{\Omega}}\}$ with pole at 0 for the matrix $A(s\cdot+q_j)^T$. Arguing as in Lemma 4.4 (using Theorem 2.12 with $T = \{(z-q_j)/s : z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j\}$) we obtain $s|\nabla\widetilde{\widetilde{g}}^T_{q_j}(z)| = |\nabla u((z-q_j)/s))| \lesssim \max_{\partial B_{1/2}(0)} u \approx 1$ for all $z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j$, where we used [16, Lemma 5.4] in the last step[2]. Consequently,

$$\frac{d\widetilde{\widetilde{\omega}}^{q_j}}{d\sigma}(z) \lesssim \frac{1}{s} \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial B_j.$$

Using both this and (5.14) in (5.15), since $j \in J$ was arbitrary, we get

$$K(z) \lesssim \frac{s^{\alpha-1}}{r_0^\alpha} \text{ for all } z \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}.$$

This bound suffices to establish (5.12), as we can now estimate

$$\int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K \log^+ K\, d\sigma \lesssim \frac{1}{r_0^\alpha}\int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} s^{\alpha-1}\log^+(s^{\alpha-1})\, d\sigma \lesssim \frac{\#J s^\alpha \log^+(s^{\alpha-1})}{r_0^\alpha},$$

which goes to zero as $s \to 0$.

Taking $s > 0$ small enough in (5.13), to prove (5.11) it suffices to see

$$\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}: K(\xi)\geq M^{-\tau}\right\}} K\, d\sigma \geq \frac{\tau}{2}. \tag{5.16}$$

We now prove (5.16). First we bound

$$\begin{aligned}\int_{\partial\widetilde{\widetilde{\Omega}}} K \log^+ K\, d\sigma &= \int_{\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}} K \log^+ K\, d\sigma + \int_{\partial\widetilde{\widetilde{\Omega}}\setminus\partial\widetilde{\Omega}} K \log^+ K\, d\sigma \\ &\overset{(5.12)}{\leq} \int_{\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}} K \log^+ K\, d\sigma + 1, \end{aligned} \tag{5.17}$$

where in the last step we took $s > 0$ small enough in (5.12). Writing $K(\cdot) \leq e^C M$ on $\partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega}$, see (5.7),

$$\begin{aligned}\int_{\partial\widetilde{\widetilde{\Omega}}\cap\partial\widetilde{\Omega}} K \log^+ K\, d\sigma &\leq (\log M + C)\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}: K(\xi)\geq 1\right\}} K\, d\sigma \\ &\leq \log M \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}: K(\xi)\geq 1\right\}} K\, d\sigma + C. \end{aligned} \tag{5.18}$$

[2] In fact, it would suffice to show that $|\nabla\widetilde{\widetilde{g}}^T_{q_j}(z)| \lesssim \frac{|\log s|}{s} \leq s^{-\alpha/2-1}$ for small enough $s = s(\alpha, \operatorname{diam}\widetilde{\widetilde{\Omega}})$, which follows from the pointwise estimate in Corollary 2.17.

By (5.17) and (5.18) we can control the term with $K \log^+ K$ as

$$\int_{\partial\widetilde{\widetilde{\Omega}}} K \log^+ K \, d\sigma \leq \log M \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq 1\right\}} K \, d\sigma + C + 1.$$

From this and (5.10),

$$\begin{aligned}\int_{\partial\widetilde{\widetilde{\Omega}}} K \log^- K \, d\sigma &= \int_{\partial\widetilde{\widetilde{\Omega}}} K \log^+ K \, d\sigma - \int_{\partial\widetilde{\widetilde{\Omega}}} K \log K \, d\sigma \\ &\leq \log M \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq 1\right\}} K \, d\sigma + C_{\Omega,A} \\ &\leq \log M \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K \, d\sigma + C_{\Omega,A}.\end{aligned}$$

Using this in the last inequality of the following computations, we obtain

$$\begin{aligned}&\tau \log M - \tau \log M \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K \, d\sigma = \log M^\tau \int_{\partial\widetilde{\widetilde{\Omega}}} K \, d\sigma - \log M^\tau \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K \, d\sigma \\ &\quad= \log M^\tau \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)< M^{-\tau}\right\}} K \, d\sigma \leq \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)< M^{-\tau}\right\}} K \log^- K \, d\sigma \\ &\quad\leq \int_{\partial\widetilde{\widetilde{\Omega}}} K \log^- K \, d\sigma \leq \log M \int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K \, d\sigma + C_{\Omega,A},\end{aligned}$$

which gives

$$\int_{\left\{\xi\in\partial\widetilde{\widetilde{\Omega}}:K(\xi)\geq M^{-\tau}\right\}} K \, d\sigma \geq \frac{\tau \log M - C_{\Omega,A}}{(1+\tau)\log M} = \frac{\tau}{1+\tau} - \frac{C_{\Omega,A}}{(1+\tau)\log M} \geq \frac{\tau}{2},$$

as claimed in (5.16), by letting $M > 0$ be big enough depending on τ and $C_{\Omega,A}$. □

We are now in position to find the final set F with the claimed properties. Let $\{B_{Q_n}\}_n \subset \{B_Q\}_{Q\in\mathcal{I}}$ be the subfamily satisfying either (HD) or (TS):

(HD) $h(z_{Q_n}) > \rho$, where $z_{Q_n} = c(B_{Q_n})$ is the center of the ball B_{Q_n}.
(TS) $h(z_{Q_n}) = \rho$ and $\partial B_{Q_n} \cap \left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\} \neq \emptyset$.

Here HD stands for 'high density' and TS for 'touching set'.

Notation We write $B_Q \in$ HD if B_Q satisfies (HD), and $B_Q \in$ TS if B_Q satisfies (TS).

With this choice,

$$\frac{\tau}{4} \overset{(5.11)}{\leq} \widetilde{\widetilde{\omega}}\left(\left\{\xi \in \partial\widetilde{\widetilde{\Omega}} \cap \partial\widetilde{\Omega} : K(\xi) \geq M^{-\tau}\right\}\right) \leq \widetilde{\widetilde{\omega}}\left(\overline{\bigcup_n B_{Q_n}}\right) \leq \sum_n \widetilde{\widetilde{\omega}}\left(\overline{B_{Q_n}}\right),$$

and by (5.6),

$$\frac{\tau}{4} \le \sum_n \widetilde{\widetilde{\omega}}\left(\overline{B_{Q_n}}\right) \lesssim \sum_n \omega(10B_{Q_n}).$$

If ε is small enough, then $10B_Q = B\left(z_Q, 10\varepsilon d(z_Q)\right) \subset \frac{\varepsilon^{-2}}{4}Q$ for each $Q \in \mathcal{I}$. Moreover, since $\{\frac{\varepsilon^{-2}}{4}Q\}_{Q\in\mathcal{I}}$ has finite overlapping by Lemma 5.33, $\{10B_Q\}_{Q\in\mathcal{I}}$ has also finite overlapping with constant depending on ε only. From this we obtain,

$$\tau \lesssim \sum_n \omega(10B_{Q_n}) \lesssim \omega\left(\bigcup_n 10B_{Q_n}\right). \tag{5.19}$$

At this point we have found a subset of $\partial\Omega$ (covered by balls) with elliptic measure bounded uniformly from below. Moreover, the radii $\varepsilon d(z_{Q_n})$ of these balls B_{Q_n} are smaller than the 'high density value' $h(z_{Q_n})$, which will allow us to have a control on the sum of the radii.

First we need to define the set $F \subset \partial\Omega$ and its covering. For each $B_{Q_n} \in \mathsf{HD}$, we have $10B_{Q_n} = B\left(z_{Q_n}, 10\varepsilon d(z_{Q_n})\right)$, and since $d(\cdot) \le h(\cdot)$ on $\partial\Omega$, $10B_{Q_n} \subset B\left(z_{Q_n}, 10\varepsilon h(z_{Q_n})\right) \subset B\left(z_{Q_n}, 10h(z_{Q_n})\right)$. Since the family $\left\{B\left(z_{Q_n}, 10h(z_{Q_n})\right)\right\}_{B_{Q_n}\in\mathsf{HD}}$ is finite, by means of the $3R$-covering theorem consider a disjoint subfamily

$$\{B_m\}_m \subset \left\{B\left(z_{Q_n}, 10h(z_{Q_n})\right)\right\}_{B_{Q_n}\in\mathsf{HD}}$$

such that

$$\bigcup_{B_{Q_n}\in\mathsf{HD}} B\left(z_{Q_n}, 10h(z_{Q_n})\right) \subset \bigcup_m 3B_m.$$

Let us define

$$F := \left(\bigcup_m 3B_m \cup \bigcup_{B_{Q_n}\in\mathsf{TS}} 10B_{Q_n}\right) \cap \partial\Omega.$$

Note that $\bigcup_n 10B_{Q_n} \subset \bigcup_m 3B_m \cup \bigcup_{B_{Q_n}\in\mathsf{TS}} 10B_{Q_n}$ implies $\omega(F) \gtrsim \tau$ by (5.19). Next we show that the covering

$$\{3B_m\}_m \cup \left\{10B_{Q_n}\right\}_{B_{Q_n}\in\mathsf{TS}}$$

satisfies the properties (1*) and (2*) in Lemma 3.3.

We can control the radius of the balls with high density

$$\sum_m r(3B_m) = 30\sum_m h\left(c(B_m)\right) \overset{(5.1)}{\le} \frac{30}{M}\sum_m \omega\left(B_m\right) = \frac{30}{M}\omega\left(\bigcup_m B_m\right) \lesssim \frac{1}{M},$$

by the definition of $h(\cdot)$ in (5.1), and the fact that the balls $\{B_m\}_m$ are pairwise disjoint. Also, $r(3B_m) = 30h(c(B_m)) > 30\rho$. We have shown the second property of the covering.

Recall that the balls in TS intersect the set $\left\{\xi \in \partial\widetilde{\Omega} \cap \partial\widetilde{\widetilde{\Omega}} : K(\xi) \geq M^{-\tau}\right\}$. For each $B_{Q_n} \in \mathsf{TS}$ consider a point $x_{Q_n} \in \partial B_{Q_n} \cap \left\{\xi \in \partial\widetilde{\Omega} \cap \partial\widetilde{\widetilde{\Omega}} : K(\xi) \geq M^{-\tau}\right\}$. Then, by (5.7) and (5.4) we have

$$M^{-\tau} \leq \frac{d\widetilde{\widetilde{\omega}}}{d\sigma}(x_{Q_n}) \overset{(5.7)}{\leq} \frac{d\widetilde{\omega}}{d\sigma}(x_{Q_n}) \overset{(5.4)}{\lesssim} \frac{\omega\left(10B_{Q_n}\right)}{r(B_{Q_n})},$$

which implies

$$\sum_{B_{Q_n} \in \mathsf{TS}} r\left(10B_{Q_n}\right) \lesssim M^{\tau} \sum_{B_{Q_n} \in \mathsf{TS}} \omega\left(10B_{Q_n}\right) \lesssim M^{\tau} \omega\left(\bigcup_{Q_n \in \mathsf{TS}} 10B_{Q_n}\right) \leq M^{\tau},$$

obtaining the first property of the covering. Note that for $B_{Q_n} \in \mathsf{TS}$ we have $h(z_{Q_n}) = \rho$, and in particular $d(z_{Q_n}) = \rho$, which implies $r(10B_{Q_n}) = 10\varepsilon d(z_Q) = 10\varepsilon\rho$. This concludes the proof of Lemma 3.3 modulo the proof of (1.2) □

6 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. First we make the reduction to the case of bounded domains (Claim 6.4) and then we prove the theorem using Main Lemma 3.1.

6.1 Reduction to bounded domains

First, we state some lemmas.

Let $\Omega \subset \mathbb{R}^{n+1}$ with $n \geq 1$, and let B be a ball centered at $\partial\Omega$. By the maximum principle [18, p. 46] we have

$$\omega^{z}_{\Omega\cap B, A}(E) \leq \omega^{z}_{\Omega, A}(E) \text{ for any } E \subset \partial\Omega \cap B \text{ and } z \in \Omega \cap B. \tag{6.1}$$

The converse inequality may fail. However, the following weaker relation holds.

Lemma 6.1 *Let $\Omega \subset \mathbb{R}^2$ be a (possibly unbounded) Wiener regular domain, A be a real uniformly elliptic matrix and B be a ball centered at $\partial\Omega$ with $\mathrm{Cap}(B \cap \partial\Omega, 4B) \neq 0$. Let $E \subset \partial\Omega \cap B$ be a Borel set and $z_E \in \partial 1.5B$ such that $\omega^{z_E}_{\Omega, A}(E) = \max_{z \in \partial 1.5B} \omega^{z}_{\Omega, A}(E)$. Then*

$$\omega^{z_E}_{\Omega, A}(E) \lesssim \frac{\mathrm{Cap}(2B, 4B)}{\mathrm{Cap}(B \cap \partial\Omega, 4B)} \omega^{z_E}_{\Omega\cap 4B, A}(E),$$

where the constant involved depends on the ellipticity constant of the matrix A and the dimension. The same also holds for bounded Wiener regular domains $\Omega \subset \mathbb{R}^{n+1}$ when $n \geq 2$.

Proof During this proof we write $\omega_{\cdot}$ instead of $\omega_{\cdot, A}$.

By Lemma 4.6 we have

$$\omega_{\Omega}^{z_E}(E) - \omega_{\Omega\cap 4B}^{z_E}(E) = \int_{\partial 4B\cap\Omega} \omega_{\Omega}^{\xi}(E)\, d\omega_{\Omega\cap 4B}^{z_E}(\xi). \tag{6.2}$$

By the maximum principle[3] in $\Omega\backslash 1.5\overline{B}$, $\omega_{\Omega}^{\xi}(E) \leq \omega_{\Omega}^{z_E}(E)$ for $\xi \in 4\partial B \cap \Omega$. From this and (6.2), we get

$$\omega_{\Omega}^{z_E}(E) \leq \omega_{\Omega\cap 4B}^{z_E}(4\partial B \cap \Omega) \cdot \omega_{\Omega}^{z_E}(E) + \omega_{\Omega\cap 4B}^{z_E}(E). \tag{6.3}$$

It remains to bound $\omega_{\Omega\cap 4B}^{z_E}(4\partial B \cap \Omega)$. By the maximum principle we have

$$\omega_{\Omega\cap 4B}^{z_E}(4\partial B \cap \Omega) \leq \omega_{4B\backslash(B\cap\partial\Omega)}^{z_E}(4\partial B).$$

By [21, Lemma 6.21], since $z_E \in 1.5\partial B$, we have

$$1 - \omega_{4B\backslash(B\cap\partial\Omega)}^{z_E}(4\partial B) \geq \widetilde{c} \cdot \mathrm{Cap}(B \cap \partial\Omega, 4B)/\,\mathrm{Cap}(2B, 4B) =: c \in (0, 1). \tag{6.4}$$

In particular

$$\omega_{\Omega\cap 4B}^{z_E}(4\partial B \cap \Omega) \leq \omega_{4B\backslash(B\cap\partial\Omega)}^{z_E}(4\partial B) \leq 1 - c. \tag{6.5}$$

From (6.3) and (6.5) we get

$$\omega_{\Omega}^{z_E}(E) \leq \omega_{\Omega\cap 4B}^{z_E}(4\partial B \cap \Omega) \cdot \omega_{\Omega}^{z_E}(E) + \omega_{\Omega\cap 4B}^{z_E}(E) \leq (1-c) \cdot \omega_{\Omega}^{z_E}(E) + \omega_{\Omega\cap 4B}^{z_E}(E),$$

obtaining

$$c \cdot \omega_{\Omega}^{z_E}(E) \leq \omega_{\Omega\cap 4B}^{z_E}(E),$$

as claimed, with c as in step (6.4). □

We remark that $\mathrm{Cap}(B \cap \partial\Omega, 4B) = 0$ would imply $\omega_D(B \cap \partial\Omega) = 0$ for any domain D with $B \cap \partial\Omega \subset \partial D$, see [21, Theorems 10.1 and 11.14]. For (δ, r_0)-Reifenberg flat domains, $\mathrm{Cap}(B \cap \partial\Omega, 4B)/\,\mathrm{Cap}(2B, 4B) \approx 1$ whenever $r_B \leq r_0$. In fact it is only needed the exterior Corkscrew condition, see Remark 2.5.

Lemma 6.2 *Let $\Omega \subset \mathbb{R}^2$ be a (possibly unbounded) Wiener regular domain and A be a real uniformly elliptic matrix. Let $\{B_i\}_i$ be a pairwise disjoint collection of balls centered at $\partial\Omega$ with $\omega_{\Omega,A}\left(\partial\Omega \setminus \bigcup_i B_i\right) = 0$, and $F_i \subset \partial(\Omega \cap 4B_i)$ with $\omega_{\Omega\cap 4B_i,A}(F_i) = 1$. Then $F := \partial\Omega \cap \bigcup_i F_i$ satisfies $\omega_{\Omega,A}(F) = 1$. The same also holds for bounded Wiener regular domains $\Omega \subset \mathbb{R}^{n+1}$ when $n \geq 2$.*

[3] In the planar case the maximum principle on $\Omega\backslash 1.5\overline{B}$ holds even if Ω is not bounded, since $\omega_{\Omega}^{\cdot}(E) \in [0, 1]$.

Proof In this proof we denote $\omega := \omega_{\Omega,A}$ and $\omega_{\Omega\cap 4B_i} := \omega_{\Omega\cap 4B_i,A}$.

Let $p \in \Omega\setminus\{\infty\}$. Abusing notation, we write $F^c = \partial\Omega\setminus F$. Since $\omega^p(F^c\setminus\bigcup_i B_i) \le \omega^p(\partial\Omega\setminus\bigcup_i B_i) = 0$ and the balls are pairwise disjoint, we can conclude

$$\omega^p(F^c) = \omega^p\left(\bigcup_i F^c \cap B_i\right) = \sum_i \omega^p\left(F^c \cap B_i\right). \tag{6.6}$$

We claim that each term in the right-hand side is zero. Indeed, for each i fix a pole $p_i \in \Omega\cap 4B_i$, by the Borel regularity of $\omega^{p_i}_{\Omega\cap 4B_i}$ let $E_i \supset F^c \cap B_i$ be a Borel set with $\omega^{p_i}_{\Omega\cap 4B_i}(E_i) = \omega^{p_i}_{\Omega\cap 4B_i}(F^c \cap B_i)$, and finally let $z_i \in 1.5\partial B_i$ such that $\omega^{z_i}(E_i) = \max_{z\in 1.5\partial B_i}\omega^z(E_i)$. With this choice of z_i, by Lemma 6.1 we have

$$\omega^{z_i}(E_i) \lesssim \omega^{z_i}_{\Omega\cap 4B_i}(E_i). \tag{6.7}$$

Since the balls are pairwise disjoint we have that $F^c \cap B_i \subset (F_i^c \cap B_i)\cap\partial\Omega \subset F_i^c$, so

$$\omega^{p_i}_{\Omega\cap 4B_i}(E_i) = \omega^{p_i}_{\Omega\cap 4B_i}(F^c \cap B_i) = 0.$$

Hence by the Harnack inequality (denoting its use with H), (6.7) and $\omega^{p_i}_{\Omega\cap 4B_i}(E_i) = 0$, for every index i there holds

$$\omega^p(F^c \cap B_i) \le \omega^p(E_i) \overset{(H)}{\lesssim} \omega^{z_i}(E_i) \overset{(6.7)}{\lesssim} \omega^{z_i}_{\Omega\cap 4B_i}(E_i) \overset{(H)}{\lesssim} \omega^{p_i}_{\Omega\cap 4B_i}(E_i) = 0. \tag{6.8}$$

Notice that for each i the constants involved in the use of Harnack inequality and Lemma 6.1 in (6.7) and (6.8) depend on i, but the right-hand side in (6.7) is zero.

By (6.8) we have that the sum in the right-hand side of (6.6) is zero as claimed. Therefore the set $F := \partial\Omega\cap\bigcup_i F_i$ satisfies $\omega^p(F) = 1$. □

Lemma 6.3 *Let $r_0 \in (0,\infty]$ and let $\varepsilon > 0$ be small enough. There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\Omega \in \mathbb{R}^2$ is (δ, r_0)-Reifenberg flat with $\delta \in (0, \delta_0)$ and B is a ball centered at $\partial\Omega$ with radius $r_B \le r_0/100$, then there exists a bounded (ε, r)-Reifenberg flat domain $D \subset \{z : \operatorname{dist}(z, \partial\Omega) < r_0/2\}$ (for some $r \in (0, r_0/2)$) with $\Omega\cap 4B = D\cap 4B$.*

Note that we are not interested in the precise dependence of r with respect to r_0, because we seek for a qualitative result in Theorem 1.1. It is quite likely that with some care the previous result could be made quantitative.

Proof The proof uses the construction in [3, Definition 2.1 and Lemma 2.2]. Set $0 < \varepsilon < 1/100$ and $E := \partial\Omega\cap 5\overline{B}$. Let $\mathcal{W}_{\varepsilon^{-2}}(E^c)$ be the set of maximal dyadic cubes $Q \subset E^c$ such that $\operatorname{diam}(\varepsilon^{-2}Q) \le r_0$ and $\varepsilon^{-2}Q \cap E = \emptyset$.

Denote $\mathcal{I}$ the family of cubes $Q \in \mathcal{W}_{\varepsilon^{-2}}(E^c)$ such that $Q\cap\partial\Omega \ne \emptyset$. For each cube $Q \in \mathcal{I}$ fix a point $z_Q \in Q\cap\partial\Omega$, and set $r_Q := \varepsilon\min\{r_0, \operatorname{dist}(z_Q, E)\}$ and

$B_Q := B(z_Q, r_Q)$. Consider the enlarged domain

$$\Omega_\varepsilon^+ := \Omega \cup \bigcup_{Q \in \mathcal{I}} B_Q \supset \Omega.$$

By [3, Lemma 2.2], this new domain is $(c\varepsilon^{1/2}, r_0/2)$-Reifenberg flat, where the constant c depends only on the dimension, provided the initial domain is (δ, r_0)-Reifenberg flat with $\delta \in (0, \delta_0)$ and δ_0 is small enough depending on ε.

Consider the domain $D_0 := \Omega_\varepsilon^+ \cap 10B$. Clearly $D_0 \subset \{z : \operatorname{dist}(z, \partial\Omega) < r_0/2\}$ as $10r_B \le r_0/10$. Let us smooth the corners of D_0 out, where the $(c\varepsilon^{1/2}, s)$-Reifenberg flat condition fails for all $s > 0$. Note that this may only happen in a finite number of points $\{\xi_j\}_{j\in J} \in \partial D_0 \cap \partial 10B$ because Ω_ε^+ is contructed as a countable union of balls. Fix a small parameter τ. For each ξ_j of these points in $\partial D_0 \cap \partial 10B$, let $B_j \in \{B_Q : Q \in \mathcal{I}\}$ such that $\xi_j \in \partial 10B \cap \partial B_j$, and let c_j and r_j denote its centers and radii respectively. Consider now the unique point $p_j \in D_0 \cap (\partial B(c_B, 10r_B - \tau) \cap \partial B(c_j, r_j - \tau))$. In particular, the ball $B(p_j, \tau)$ is tangent to $\partial 10B$ and ∂B_j. Let $T_j = 10B \cap B_j \cap B(p_j, \tau)^c$ the bounded open region enclosed between the previous balls. Taking τ to be small enough, the final domain

$$D := D_0 \backslash \bigcup_{j\in J} T_j \subset D_0$$

satisfies $D \cap 4B = \Omega_\varepsilon^+ \cap 4B = \Omega \cap 4B$ and is $(c\varepsilon^{1/2}, r)$-Reifenberg flat for some $r > 0$ depending on τ. □

Claim 6.4 *If Theorem 1.1 holds for bounded (δ_0, r_0)-Reifenberg flat domains, then there exists $\delta_1 = \delta_1(\delta_0) > 0$ such that Theorem 1.1 holds for unbounded (δ_1, r_0)-Reifenberg flat domains.*

Proof First we want to remark that if Theorem 1.1 holds for (δ, r_0)-Reifenberg flat domains for a fixed $r_0 > 0$, then by means of a dilation it holds for (δ, r)-Reifenberg flat domains for any $r > 0$.

Let $\varepsilon > 0$ given by Lemma 6.3 and let $\varepsilon' := \min\{\varepsilon, \delta_0/2\}$ be small enough. Let $\delta_1 = \delta_1(\varepsilon')$ given by Lemma 6.3. Let $\Omega \subset \mathbb{R}^2$ be an unbounded (δ_1, r_0)-Reifenberg flat domain. Let $\{B_i\}_i \subset \{B(\xi, r)\}_{\xi\in\partial\Omega, 0<r<r_0/100}$ be a disjoint family with $\omega_{\Omega,A}(\partial\Omega \setminus \bigcup_i B_i) = 0$, by Vitali's covering theorem. For each ball B_i let D_i be the bounded (ε, r_i)-Reifenberg flat domain from Lemma 6.3, for some $r_i \in (0, r_0/2)$. As $\varepsilon' < \delta_0$, in particular each D_i is a bounded (δ_0, r_i)-Reifenberg flat domain.

As we are assuming that Theorem 1.1 holds for bounded (δ_0, r)-Reifenberg flat domains for any $r > 0$, for each i take $F_i \subset \partial D_i$ with $\omega_{D_i,A}(F_i) = 1$ and σ-finite one-dimensional Hausdorff measure. From $\Omega \cap 4B_i = D_i \cap 4B_i$, the maximum principle and $\omega_{D_i}(F_i) = 1$ we get

$$\begin{aligned}\omega_{\Omega\cap 4B_i,A}((\partial\Omega\backslash F_i) \cap 4B_i) &= \omega_{D_i\cap 4B_i,A}((\partial D_i\backslash F_i) \cap 4B_i)\\ &\le \omega_{D_i,A}((\partial D_i\backslash F_i) \cap 4B_i) = 0.\end{aligned}$$

In particular $\omega_{\Omega\cap 4B_i,A}((F_i \cap 4B_i) \cup \partial 4B_i) = 1$. As F_i has σ-finite length, so does $\widetilde{F}_i := (F_i \cap 4B_i) \cup \partial 4B_i$. By Lemma 6.2, the set $F = \partial\Omega \cap \bigcup_i \widetilde{F}_i$ satisfies $\omega_\Omega(F) = 1$, and clearly has σ-finite length. □

6.2 Proof for bounded domains

Theorem 1.1 follows from Main Lemma 3.1 as it is done in [44, Proof of Theorem 1], with some small modifications. For the sake of completeness we give the detailed proof.

Proof of Theorem 1.1 By Claim 6.4 we can assume without loss of generality that Ω is bounded. We denote $\omega := \omega_{\Omega,A}$.

Let $\phi : [0,\infty) \to [0,\infty)$ be any increasing function with $\lim_{t\to 0}\phi(t)/t = 0$, and consider the ϕ-Hausdorff content

$$h_\phi(E) = \inf\left\{\sum_i \phi(r_i) : E \subset \bigcup_i B(z_i, r_i)\right\}.$$

Now we claim that there exists $F_\phi \subset \partial\Omega$ with $\omega(F_\phi) = 1$ and $h_\phi(F_\phi) = 0$. Indeed, suppose that the pole $z \in \Omega$ is such that $\text{dist}(z, \partial\Omega) > r_0$. Set $\tau = 1/2$ and fix $0 < r \leq 1$ to be small enough as in Main Lemma 3.1. Let $\varepsilon > 0$ and fix $M > \varepsilon^{-1}$ satisfying the hypothesis in Main Lemma 3.1. Take ρ small enough such that $0 < \rho/r < 1/M$ and $\phi(\gamma) < \varepsilon M^{-1/2}\gamma$ for all $\gamma \leq \rho$. Then, by Lemma 3.1 with these parameters, we obtain a set $F_\varepsilon \subset \partial\Omega$ such that $\omega^z(F_\varepsilon) \geq C^{-1}$ and with a covering $F_\varepsilon \subset \bigcup_i B(z_i, r_i)$ with $\sum_i r_i \leq CM^{1/2}$ and $\sum_{\{i\,:\,r_i>\rho\}} r_i \leq CM^{-1}$. This covering satisfies

$$\begin{aligned} h_\phi(F_\varepsilon) &\leq \sum_{r_i\leq\rho}\phi(r_i) + \sum_{r_i>\rho}\phi(r_i) \leq \varepsilon M^{-1/2}\sum_{r_i\leq\rho} r_i + \sum_{r_i>\rho} r_i \\ &\leq \varepsilon M^{-1/2} C M^{1/2} + CM^{-1} \leq C\varepsilon. \end{aligned}$$

Define now $F_\infty \subset \partial\Omega$ as

$$F_\infty := \limsup_{j\to\infty} F_{1/j^2} = \bigcap_{j\geq 1}\bigcup_{k\geq j} F_{1/k^2}.$$

With this choice we have

$$\omega^z(F_\infty) = \lim_{j\to\infty}\omega^z\left(\bigcup_{k\geq j} F_{1/k^2}\right) \geq \limsup_{j\to\infty}\omega^z(F_{1/j^2}) \geq C^{-1},$$

and as $F_\infty \subseteq \bigcup_{k\geq j} F_{1/k^2}$ for any $j \geq 1$, then

$$h_\phi(F_\infty) \leq h_\phi\left(\bigcup_{k\geq j} F_{1/k^2}\right) \leq \sum_{k\geq j} h_\phi(F_{1/k^2}) \leq C\sum_{k\geq j}\frac{1}{k^2},$$

which gives $h_\phi(F_\infty) = 0$ letting $j \to \infty$.

Let $\{z_k\}_{k=1}^\infty$ be a countable dense subset of Ω. Fix z_k and set $d_k = \operatorname{dist}(z_k, \partial\Omega)$. As Ω is (δ, r_0)-Reifenberg flat and the matrix A is Lipschitz in $\{x : \operatorname{dist}(x, \partial\Omega) < r_0\}$, then Ω is (δ, r_k)-Reifenberg flat and A is Lipschitz in $\{x : \operatorname{dist}(x, \partial\Omega) < r_k\}$ with $r_k = \min\{r_0, d_k/2\}$. By the choice of r_k we are in the situation $\operatorname{dist}(z_k, \partial\Omega) > r_k$. By the same argument done in the previous paragraphs we get a set $F_k \subset \partial\Omega$ (relative to z_k) such that

$$\omega^{z_k}(F_k) \geq C^{-1} \text{ and } h_\phi(F_k) = 0.$$

Define $F_\phi := \bigcup_{k=1}^\infty F_k$. The condition $h_\phi(F_k) = 0$ for every $k \geq 1$ means that for every $\epsilon > 0$ there exists a covering $F_k \subset \bigcup_i B(z_i^{k,\epsilon}, r_i^{k,\epsilon})$ such that $\sum_i \phi(r_i^{k,\epsilon}) \leq \epsilon/2^k$. Hence, $F_\phi = \bigcup_{k=1}^\infty F_k \subset \bigcup_{k=1}^\infty \bigcup_i B(z_i^{k,\epsilon}, r_i^{k,\epsilon})$ which gives $h_\phi(F_\phi) = 0$ because

$$h_\phi(F_\phi) \leq \sum_{k=1}^\infty \sum_i \phi(r_i^{k,\epsilon}) \leq \sum_{k=1}^\infty \frac{\epsilon}{2^k} = \epsilon.$$

Moreover $\omega^{z_k}(F_\phi) \geq \omega^{z_k}(F_k) \geq C^{-1}$ for any z_k. As $\{z_k\}_{k=1}^\infty \subset \Omega$ is dense and $\omega^z(F_\phi)$ is L_A-harmonic with respect to z (in particular continuous), then $\omega^p(F_\phi) \geq C^{-1}$ for any $p \in \Omega$. By [21, Lemma 11.16] we conclude $\omega^p(F_\phi) = 1$ for any $p \in \Omega$.

Finally let

$$F = \left\{\xi \in \partial\Omega : \limsup_{r\to 0} \frac{\omega^p(B(\xi, r))}{r} > 0\right\}.$$

We claim that this set has σ-finite length and $\omega(F) = 1$. We start by proving $\omega(F) = 1$, and later we will show that it has σ-finite length.

Suppose to get a contradiction that $\omega^p(F) \neq 1$, that is,

$$\omega^p(F^c) = \omega^p\left(\left\{\xi \in \partial\Omega : \lim_{r\to 0} \frac{\omega^p(B(\xi, r))}{r} = 0\right\}\right) > 0.$$

Egorov's theorem ensures that for every $s > 0$ there exists a measurable set $V = V_s \subset F^c$ such that $\omega^p(V) < s$ and $\omega^p(B(\xi, r))/r \to 0$ as $r \to 0$ uniformly on $F^c \setminus V$. Since $0 < \omega^p(F^c) = \omega^p(V) + \omega^p(F^c \setminus V) < s + \omega^p(F^c \setminus V)$, if we take $s > 0$ small enough, say $s = \omega^p(F^c)/2$, then $\omega^p(F^c \setminus V) > 0$. The set $Y := F^c \setminus V$ has non zero elliptic measure and the limit

$$\lim_{r\to 0}\left\|\frac{\omega^p(B(\cdot,r))}{r}\right\|_{L^\infty(Y)}=0.$$

Note that the function $\phi(r):=\sup_{\xi\in Y}\omega^p(B(\xi,r))$ satisfies the conditions of the rate function in the beginning of this proof, and by definition it satisfies $\omega^p(B(\xi,r))\leq\phi(r)$ for all $\xi\in Y$ and all $r>0$. All in all,

(1) $\omega^p(B(\xi,r))\leq\phi(r)$ for all $\xi\in Y$ and all $r>0$,
(2) ϕ is increasing, and
(3) $\phi(r)/r\to 0$ as $r\to 0$.

For this particular function ϕ, let F_ϕ be the set constructed in the beginning of this proof. That is, a set F_ϕ with $h_\phi(F_\phi)=0$ and $\omega^p(F_\phi)=1$ for every $p\in\Omega$. Hence $\omega^p(Y\cap F_\phi)=\omega^p(Y)>0$, and moreover $h_\phi(Y\cap F_\phi)\leq h_\phi(F_\phi)=0$. Consequently, we can cover $Y\cap F_\phi$ with balls $B(\xi_i,r_i)$ centered at $Y\cap F_\phi$ such that $\sum_i\phi(r_i)<\omega^p(Y\cap F_\phi)/2$. With this we get

$$\omega^p(Y\cap F_\phi)\leq\sum_i\omega^p(B(\xi_i,r_i))\overset{(1)}{\leq}\sum_i\phi(r_i)<\frac{\omega^p(Y\cap F_\phi)}{2}.$$

This is a contradiction, and hence $\omega^p(F)=1$.

It remains to prove that F has σ-finite one-dimensional Hausdorff measure. The set F can be written as

$$F=\bigcup_{j\geq 1}F^{1/j}\text{ with }F^{1/j}:=\left\{\xi\in\partial\Omega:\limsup_{r\to 0}\frac{\omega^p(B(\xi,r))}{r}>1/j\right\}.$$

Therefore, it suffices to see that every $F^{1/j}$ has finite one-dimensional Hausdorff measure. Each point $\xi\in F^{1/j}$ has arbitrarily small neighborhoods $B(\xi,r)$ such that $\omega^p(B(\xi,r))\geq r/j$. Given $\varepsilon>0$ small, consider the family of these balls centered at $F^{1/j}$ with radius at most ε, i.e.,

$$\mathcal{B}_\varepsilon:=\{B(\xi,r):\xi\in F^{1/j},r<\varepsilon,\text{ and }\omega^p(B(\xi,r))\geq r/j\}.$$

By the Besicovitch covering theorem there is a countable subfamily $\{B(\xi_i,r_i)\}_i\subset\mathcal{B}_\varepsilon$ such that no point belongs to more than a fixed finite number C (it depends on the dimension only) of these balls. Hence,

$$\sum_i r_i\leq j\sum_i\omega^p(B(\xi_i,r_i))\leq Cj,$$

and letting $\varepsilon\to 0$ we obtain $\mathcal{H}^1(F^{1/j})\leq Cj$, as claimed. □

Hence Theorem 1.1 is proved under the assumption that (1.2) holds.

7 $L \log L(d\sigma)$ type estimate for small densities: Proof of (1.2)

The purpose of this section is to prove (1.2), under the hypothesis of Lemma 3.3. More specifically, we prove the following result.

Lemma 7.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded (δ, r_0)-Reifenberg flat domain, $p \in \Omega$ with* $\operatorname{dist}(p, \partial\Omega) > r_0$*, and A be a real uniformly elliptic (not necessarily symmetric) matrix with ellipticity constant λ. Suppose also that A is κ-Lipschitz in $U_{r_0}(\partial\Omega) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial\Omega) < r_0\}$ and that its symmetric part $A_0 = \frac{A+A^T}{2}$ is of the form $A_0 = R^T B R$, with $R \in C^{0,1}(U_{r_0}(\partial\Omega))$ a rotation and $B \in C^{0,1}(U_{r_0}(\partial\Omega))$ diagonal.*

Then there exists $\delta_0 = \delta_0(\lambda, \kappa\|A\|_{L^\infty(\mathbb{R}^2)}) > 0$ and $C = C(\lambda, \kappa, r_0, \operatorname{diam}\partial\Omega) \in (0, \infty)$ such that if $\delta \leq \delta_0$, then for any (δ, r_0)-Reifenberg flat domain $\widetilde{\Omega} \subset \Omega$ with smooth boundary $\partial\widetilde{\Omega}$ and small enough $\operatorname{dist}_{\mathcal{H}}(\partial\Omega, \partial\widetilde{\Omega})$, we have

$$\left| \int_{\partial\widetilde{\Omega}} \log |S\nabla g_p^T(\xi)|^2 \, d\widetilde{\omega}^p(\xi) \right| \leq C < +\infty, \tag{7.1}$$

where $\widetilde{\omega}^p = \omega^p_{\widetilde{\Omega}, A}$ is the elliptic measure in $\widetilde{\Omega}$ with respect to the matrix A, g_p^T is the Green function in $\widetilde{\Omega}$ with respect to the matrix A^T, and S is the square root matrix of the symmetric part $A_0 = (A + A^T)/2$, i.e., $S^T S = A_0$.

Remark 7.2 As argued from (5.8) to (5.10), this implies that the Radon-Nykodym derivative $\frac{d\widetilde{\omega}^p}{d\sigma}$ satisfies the following $L \log L(d\sigma)$ type estimate

$$-\infty < C'(\lambda, \kappa, \operatorname{diam}\partial\Omega) \leq \int_{\partial\widetilde{\Omega}} \frac{d\widetilde{\omega}^p}{d\sigma}(\xi) \log \frac{d\widetilde{\omega}^p}{d\sigma}(\xi) \, d\sigma(\xi).$$

By symmetry, estimate (7.1) is equivalent to the existence of a constant $C = C(\lambda, \kappa, \operatorname{diam}\partial\Omega)$ depending only on the ellipticity constant, the Lipschitz seminorm of the matrix A, and the diameter $\operatorname{diam}\partial\Omega$ (but not on $\widetilde{\Omega}$) such that

$$\left| \int_{\partial\widetilde{\Omega}} \log |S\nabla g(\xi)|^2 \, d\widetilde{\omega}^p_T(\xi) \right| \leq C < \infty, \tag{7.2}$$

where $S = A_0^{1/2}$, i.e., $S^T S = A_0$, $g = g_p$ is the Green function in $\widetilde{\Omega}$ with respect to the matrix A with pole p, and $\widetilde{\omega}_T = \omega^p_{\widetilde{\Omega}, A^T}$ denotes the elliptic measure in $\widetilde{\Omega}$ with respect to the matrix A^T and a pole $p \in \widetilde{\Omega}$ such that $\operatorname{dist}(p, \partial\Omega) > r_0$. The existence of the matrix S is granted by the fact that the symmetric matrix A_0 is uniformly elliptic with the same ellipticity constant as A, and hence positive definite. In fact, $S = R^T\sqrt{B}R$, where $\sqrt{B} = (\delta_{i=j}\sqrt{b_{ij}})_{1\leq i,j\leq 2}$.

Throughout all this section, when dealing with terms within solid integrals we will use the following notation:

- We write $h(x) = \mathcal{O}(f(x))$ to denote "$|h(x)| \leq Cf(x)$ almost everywhere with respect to the Lebesgue measure".

- Given a Lipschitz function h, we write $|\nabla h(x)| \leq C$ instead of "$|\nabla h(x)| \leq C$ almost everywhere with respect to the Lebesgue measure". Recall that Lipschitz functions are differentiable almost everywhere by Rademacher's theorem, see [34, Theorem 7.3].

7.1 Directional derivatives and the dual space

We introduce some extra notation regarding the directional derivatives.

Notation Definition of directional derivatives.

- The (vertical) vector $e_i := (0, \dots, 0, \overset{i}{\smile}{1}, 0, \dots, 0)^T$ has 1 in position i and 0's otherwise. In $\mathbb{R}^2$ there are only two such vectors, namely $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$.
- $R_i := e_i^T \cdot R$ corresponds to the i-th row of R.
- The ∂^i-directional derivative $\partial^i := \partial_{R_i^T}$ (superscript) is defined as

$$\partial^i f := \partial_{R_i^T} f = \langle \nabla f, R_i^T \rangle = R_i \cdot \nabla f.$$

  Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. The derivative $\partial_i = \partial_{e_i}$ (subscript) is the usual one in the direction e_i.
- The R-directional gradient ∇_R is defined as

$$\nabla_R f := R \cdot \nabla f = \left(\partial^i f\right)_{i=1}^{n+1}.$$

Remark 7.3 The directional derivative ∂^i preserves the usual properties in sums, products, and logarithms, i.e., $\partial^i (f + g) = \partial^i f + \partial^i g$, $\partial^i (fg) = g\partial^i f + f\partial^i g$, and $\partial^i \log f = \frac{\partial^i f}{f}$.

Note that

$$\left|\partial^{ij} f\right| = \left|\partial^i (R_j \cdot \nabla f)\right| = \left|\partial^i R_j \cdot \nabla f + R_j \partial^i \nabla f\right| \lesssim |\nabla f| + \left|\nabla^2 f\right|, \qquad (7.3)$$

where we denote $|\nabla^2 f|^2 := \sum_{i,j=1}^{n+1} (\partial_{i,j} f)^2$.

Since the directions R_i change depending on the point, the usual integration by parts formula does not apply. Instead we have the following formula.

Claim 7.4 *(Integration by parts formula) Let $U \subset \mathbb{R}^{n+1}$ be an open set. For $f, h \in W^{1,2}(U)$, if $fh \in W_0^{1,2}(U)$ then*

$$\int_U \partial^i f(x) h(x)\, dx = -\int_U f(x)h(x) \operatorname{div} R_i^T(x)\, dx - \int_U f(x)\partial^i h(x)\, dx. \qquad (7.4)$$

Proof The product rule gives $\partial^i f(x)h(x) = \partial^i(f(x)h(x)) - f(x)\partial^i h(x)$. By definition of the directional derivative ∂^i we can write

$$\begin{aligned}\partial^i(f(x)h(x)) &= R_i(x)\cdot\nabla(f(x)h(x)) = \sum_{j=1}^{n+1}\partial_j(f(x)h(x))R_{i,j}(x)\\ &= \sum_{j=1}^{n+1}\partial_j\left(f(x)h(x)R_{i,j}(x)\right) - f(x)h(x)\sum_{j=1}^{n+1}\partial_j R_{i,j}(x)\\ &= \operatorname{div}(fhR_i^T)(x) - f(x)h(x)\operatorname{div}R_i^T(x),\end{aligned}$$

which imply

$$\begin{aligned}\int\partial^i f(x)h(x)\,dx = \int\operatorname{div}(fhR_i^T)(x)\,dx - \int f(x)h(x)\operatorname{div}R_i^T(x)\,dx\\ -\int f(x)\partial^i h(x)\,dx.\end{aligned}$$

Using that $C_c^\infty(U)$ is dense in $W_0^{1,2}(U)$ and the divergence theorem, the first element in the right-hand side is $\int\operatorname{div}(fhR_i^T)(x)\,dx = 0$. □

Again, since the directions R depend on the point, the directional derivatives do not commute. Instead, the following formula relating $\partial^{\alpha,\beta}$ and $\partial^{\beta,\alpha}$ is available, where $\partial^{\alpha,\beta} := \partial^\alpha\partial^\beta$. We refer to this as the 'almost'-commutative property.

Claim 7.5 *Let $U\subset\mathbb{R}^{n+1}$ be an open set and $f\in W^{2,2}(U)$. For $\alpha,\beta\in\{1,\ldots,n+1\}$,*

$$\partial^{\alpha,\beta}f - \partial^{\beta,\alpha}f = (\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot\nabla f.$$

Proof By definition we have $\partial^\beta f = R_\beta\cdot\nabla f = \sum_{i=1}^{n+1}\partial_i f R_{\beta,i}$. Applying ∂^α gives

$$\partial^{\alpha,\beta}f = R_\alpha\cdot\nabla\left(\sum_{i=1}^{n+1}\partial_i f R_{\beta,i}\right) = \sum_{j=1}^{n+1}\partial_j\left(\sum_{i=1}^{n+1}\partial_i f R_{\beta,i}\right)R_{\alpha,j}.$$

Expanding the derivative of this last expression and arranging we obtain

$$\partial^{\alpha,\beta}f = \sum_{i,j=1}^{n+1}\partial_{j,i}f R_{\beta,i}R_{\alpha,j} + \sum_{i=1}^{n+1}\partial_i f\sum_{j=1}^{n+1}\partial_j R_{\beta,i}R_{\alpha,j}.$$

The sum inside the second term in the right-hand side is precisely $\sum_{j=1}^{n+1}\partial_j R_{\beta,i}R_{\alpha,j} = R_\alpha\cdot\nabla R_{\beta,i} = \partial^\alpha R_{\beta,i}$. Hence

$$\partial^{\alpha,\beta}f = \sum_{i,j=1}^{n+1}\partial_{j,i}f R_{\beta,i}R_{\alpha,j} + \sum_{i=1}^{n+1}\partial_i f\partial^\alpha R_{\beta,i} = \sum_{i,j=1}^{n+1}\partial_{j,i}f R_{\beta,i}R_{\alpha,j} + \partial^\alpha R_\beta\cdot\nabla f.$$

By symmetry we obtain $\partial^{\beta,\alpha} f = \sum_{i,j=1}^{n+1} \partial_{j,i} f R_{\alpha,i} R_{\beta,j} + \partial^{\beta} R_{\alpha} \cdot \nabla f$, and subtracting $\partial^{\alpha,\beta} f - \partial^{\beta,\alpha} f$ we get Claim 7.5. □

In particular, setting $\alpha = 1$ and $\beta = 2$ in the planar case we have

$$\partial^{1,2} f - \partial^{2,1} f = (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla f,$$

and applying it to the Green function g, we get

$$\partial^{1,2} g - \partial^{2,1} g = (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla g = \mathcal{O}(|\nabla g|). \tag{7.5}$$

7.1.1 Definition of the dual space

Let $U \subset \widetilde{\Omega}$ and $u \in L^1_{\mathrm{loc}}(U)$. Define the linear functional

$$T_u(\psi) := \int_U u\psi,$$

whenever it makes sense. In particular, $T_u \in L^\infty_c(U)'$. To simplify the notation, we will also write $u(\psi)$ to denote $T_u(\psi)$.

The ∂_i-derivative functional is defined as

$$T_{\partial_i u}(\psi) := -\int_U u \partial_i \psi,$$

whenever it makes sense. Note that $T_{\partial_i u} \in W^{1,\infty}_c(U)'$ for $u \in L^1_{\mathrm{loc}}(U)$. Moreover, we also have $T_{\partial_i u} \in W^{1,2}_c(U)'$ whenever $u \in L^2_{\mathrm{loc}}(U)$. As before, we will also write $Du(\psi)$ to denote $T_{Du}(\psi)$ for any differential operator D. In general, the functional

$$T_{fD(u)}(\psi) := T_{D(u)}(f\psi),$$

is in $W^{1,\infty}_c(U)'$ as long as either $f \in W^{1,2}_{\mathrm{loc}}(U)$ and $u \in L^2_{\mathrm{loc}}(U)$, or $f \in W^{1,\infty}_{\mathrm{loc}}(U)$ and $u \in L^1_{\mathrm{loc}}(U)$.

We claim that whenever $u \in L^2_{\mathrm{loc}}(U)$ and $f \in W^{1,2}_{\mathrm{loc}}(U)$, in the dual space $W^{1,\infty}_c(U)'$, the ∂_i-derivative functional satisfies the product rule

$$T_{\partial_i(fu)} = T_{\partial_i f u} + T_{f \partial_i u}. \tag{7.6}$$

Indeed, for any $\psi \in W^{1,\infty}_c(U)$, the previous equality reads as

$$-\int_U fu\partial_i \psi = \int_U \partial_i f u \psi - \int_U u \partial_i (f\psi),$$

which holds by the Leibniz's rule a.e. for $W^{1,2}$-functions.

Using the definitions of T_{∂_i} and the directional derivative $\partial^i(\cdot) = R_i\nabla(\cdot)$, see the beginning of Section 7.1, the directional ∂^i-derivative functional $T_{\partial^i u} : W_c^{1,\infty}(U) \to \mathbb{R}$, for $u \in L^1_{\mathrm{loc}}(U)$, is defined as

$$T_{\partial^i u} := T_{R_i \nabla u} = \sum_j T_{R_{i,j}\partial_j u}. \tag{7.7}$$

Equivalently, for $\psi \in W_c^{1,\infty}(U)$,

$$\begin{aligned} T_{\partial^i u}(\psi) &= \sum_j T_{\partial_j u}(R_{i,j}\psi) = -\sum_j \int_U u\partial_j(R_{i,j}\psi) \\ &= -\sum_j \int_U u\psi\partial_j R_{i,j} - \sum_j \int_U uR_{i,j}\partial_j\psi \\ &= -\int_U u\psi \operatorname{div} R_i^T - \int_U u\partial^i\psi. \end{aligned} \tag{7.8}$$

This agrees with the integration by parts formula in (7.4) when $u \in W^{1,2}(U)$.

Next we claim that, in the dual space $W_c^{1,\infty}(U)'$, for $u \in L^2_{\mathrm{loc}}(U)$ and $f \in W^{1,2}_{\mathrm{loc}}(U)$, the directional ∂^i-derivative functional satisfies the product rule

$$T_{\partial^i(fu)} = T_{\partial^i f u} + T_{f\partial^i u}. \tag{7.9}$$

Indeed, $R_{i,j}\psi \in W_c^{1,\infty}(U)$ when $\psi \in W_c^{1,\infty}(U)$, and hence, for $f \in W^{1,2}_{\mathrm{loc}}(U)$,

$$\begin{aligned} T_{\partial^i(fu)}(\psi) &\overset{(7.7))}{=} \sum_j T_{\partial_j(fu)}(R_{i,j}\psi) \overset{(7.6))}{=} \sum_j T_{\partial_j f u}(R_{i,j}\psi) + T_{f\partial_j u}(R_{i,j}\psi) \\ &\overset{(7.7))}{=} T_{\partial^i f u}(\psi) + T_{f\partial^i u}(\psi), \end{aligned}$$

as claimed.

Another important property is the following 'almost'-commutative property of the directional derivatives (compare to Claim 7.5). For $u \in W^{1,1}_{\mathrm{loc}}(U)$, in the dual space $W_c^{1,\infty}(U)'$, we claim that

$$T_{\partial^1(\partial^2 u)} - T_{\partial^2(\partial^1 u)} = T_{(\partial^1 R_2 - \partial^2 R_1)\cdot\nabla u}. \tag{7.10}$$

Indeed, for $\psi \in W_c^{1,\infty}(U)$ and $\alpha, \beta \in \{1, 2\}$,

$$T_{\partial^\alpha(\partial^\beta u)}(\psi) \overset{(7.8)}{=} -\int_U \partial^\beta u\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta u\partial^\alpha\psi.$$

Given $\{u_k\}_{k\geq 1} \subset C_c^\infty(U)$ with $\lim_{k\to\infty} \|u_k - u\|_{W^{1,1}(\operatorname{supp}\psi)} = 0$, this equals

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &-\int_U \partial^\beta u_k \psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta u_k \partial^\alpha \psi \\ &- \int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta (u-u_k)\partial^\alpha \psi. \end{aligned} \tag{7.11}$$

By the integration by parts in (7.4) applied to $\partial^\beta u_k$, we have that the first row in the right-hand side is precisely $\int_U \partial^\alpha(\partial^\beta u_k)\psi$. Applying now the 'almost'-commutative property (Claim 7.5) here we obtain

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &\int_U \partial^\beta(\partial^\alpha u_k)\psi + \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla u_k)\psi \\ &- \int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta (u-u_k)\partial^\alpha \psi. \end{aligned}$$

Again, by the integration by parts in (7.4), we can replace the first term in the right-hand side to obtain

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &-\int_U \partial^\alpha u_k \psi \operatorname{div} R_\beta^T - \int_U \partial^\alpha u_k \partial^\beta \psi \\ &+ \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla u_k)\psi \\ &- \int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta (u-u_k)\partial^\alpha \psi. \end{aligned}$$

Now, adding and subtracting u in the second row we get

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) = &-\int_U \partial^\alpha u_k \psi \operatorname{div} R_\beta^T - \int_U \partial^\alpha u_k \partial^\beta \psi \\ &+ T_{(\partial^\alpha R_\beta - \partial^\beta R_\alpha)\nabla u}(\psi) + \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla (u_k - u))\psi, \\ &- \int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta (u-u_k)\partial^\alpha \psi. \end{aligned}$$

By symmetry in (7.11) and subtracting we obtain

$$\begin{aligned} T_{\partial^\alpha(\partial^\beta u)}(\psi) - T_{\partial^\beta(\partial^\alpha u)}(\psi) = &\, T_{(\partial^\alpha R_\beta - \partial^\beta R_\alpha)\nabla u}(\psi) + \int_U ((\partial^\alpha R_\beta - \partial^\beta R_\alpha)\cdot \nabla (u_k - u))\psi, \\ &- \int_U \partial^\beta (u-u_k)\psi \operatorname{div} R_\alpha^T - \int_U \partial^\beta (u-u_k)\partial^\alpha \psi \\ &+ \int_U \partial^\alpha (u-u_k)\psi \operatorname{div} R_\beta^T + \int_U \partial^\alpha (u-u_k)\partial^\beta \psi. \end{aligned}$$

By the convergence $\lim_{k\to\infty} \|u_k - u\|_{W^{1,1}(\operatorname{supp}\psi)} = 0$ we get (7.10).

7.1.2 Properties of directional derivatives and the dual space

The following claim allows us to move from the initial matrix A to its symmetric part A_0, which we will relate to the directional derivatives in Claim 7.7.

Claim 7.6 *Let $U \subset \widetilde{\Omega}$ be an open set and $u \in W^{1,2}_{\mathrm{loc}}(U)$. Every $\psi \in W^{1,\infty}_c(U)$ satisfies*

$$(\operatorname{div} A\nabla u)(\psi) = (\operatorname{div} A_0\nabla u)(\psi) + \left(\sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j} - a_{j,i}}{2}\right) \partial_j u\right)(\psi).$$

Proof Let $u \in C^\infty(U)$. By definition of the divergence and differentiating we have

$$\operatorname{div} A\nabla u = \sum_{i,j=1}^{n+1} \partial_i a_{i,j} \partial_j u + \sum_{i,j=1}^{n+1} a_{i,j} \partial_{i,j} u = \sum_{i,j=1}^{n+1} \partial_i a_{i,j} \partial_j u + \sum_{i,j=1}^{n+1} \frac{a_{i,j} + a_{j,i}}{2} \partial_{i,j} u.$$

Note that $a^0_{i,j} := \frac{a_{i,j}+a_{j,i}}{2}$ are precisely the coefficients of the symmetric matrix A_0. The product derivative rule gives

$$a^0_{i,j} \partial_{i,j} u = \partial_i \left(a^0_{i,j} \partial_j u\right) - \partial_i a^0_{i,j} \partial_j u,$$

and applying this relation to each pair of indexes $i, j \in \{1, \ldots, n+1\}$ we get

$$\sum_{i,j=1}^{n+1} a^0_{i,j} \partial_{i,j} u = \operatorname{div} A_0 \nabla u - \sum_{i,j=1}^{n+1} \partial_i a^0_{i,j} \partial_j u.$$

Summing up,

$$\begin{aligned} \operatorname{div} A\nabla u &= \operatorname{div} A_0\nabla u + \sum_{i,j=1}^{n+1} \partial_i a_{i,j} \partial_j u - \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j} + a_{j,i}}{2}\right) \partial_j u \\ &= \operatorname{div} A_0\nabla u + \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j} - a_{j,i}}{2}\right) \partial_j u, \end{aligned}$$

as claimed, for functions in $C^\infty(U)$.

Let us check this in the dual sense for a function $u \in W^{1,2}_{\mathrm{loc}}(U)$. Let $\psi \in W^{1,\infty}_c(U)$, and hence $u \in W^{1,2}(\operatorname{supp}\psi)$. Take $\{u_k\}_{k\geq 1} \subset C^\infty_c(U)$ such that $\lim_{k\to\infty} \|u_k - u\|_{W^{1,2}(\operatorname{supp}\psi)} = 0$. As we have the claim for each function u_k, in particular

$$\begin{aligned} \int_U A\nabla u \nabla\psi &= \int_U A\nabla(u - u_k)\nabla\psi + \int_U A\nabla u_k \nabla\psi \\ &= \int_U A\nabla(u - u_k)\nabla\psi + \int_U A_0\nabla u_k \nabla\psi \end{aligned}$$

$$
\begin{aligned}
&- \int_U \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j}-a_{j,i}}{2}\right) \partial_j u_k \cdot \psi \\
&= \int_U A_0 \nabla u \nabla \psi - \int_U \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j}-a_{j,i}}{2}\right) \partial_j u \cdot \psi \\
&+ \int_U A \nabla (u-u_k) \nabla \psi + \int_U A_0 \nabla (u_k - u) \nabla \psi \\
&- \int_U \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j}-a_{j,i}}{2}\right) \partial_j (u_k - u) \psi .
\end{aligned}
$$

Now, since the matrix is Lipschitz and $\lim_{k\to\infty} \|u_k - u\|_{W^{1,2}(\operatorname{supp}\psi)} = 0$, we conclude

$$
\int_U A \nabla u \nabla \psi = \int_U A_0 \nabla u \nabla \psi - \int_U \sum_{i,j=1}^{n+1} \partial_i \left(\frac{a_{i,j}-a_{j,i}}{2}\right) \partial_j u \cdot \psi ,
$$

as claimed. □

We treat one of the main terms in (7.1) by means of a perturbation argument. To do that, we note that we can write $\operatorname{div}(A_0 \nabla u)$ as in the diagonal case using the R-directional derivatives, plus an error term. More precisely, we have the following claim.

Claim 7.7 *Let $A_0 = R^T B R$ satisfy the conditions in Lemma 7.1, let $U \subset \widetilde{\Omega}$ be an open set and $u \in W^{1,2}_{\text{loc}}(U)$. Every $\psi \in W^{1,\infty}_c(U)$ satisfies*

$$
(\operatorname{div}(A_0 \nabla u))(\psi) = \left(\sum_{i=1}^{n+1} \partial^i (b_i \partial^i u)\right)(\psi) + \left(\sum_{i=1}^{n+1} b_i \partial^i u \cdot \operatorname{div} R_i^T\right)(\psi).
$$

Proof Assume first $u \in C^\infty(U)$. Let us write $R\nabla u = (R_j \cdot \nabla u)_{j=1}^{n+1} = (\partial^j u)_{j=1}^{n+1}$, and so $BR\nabla u = (b_j \partial^j u)_{j=1}^{n+1}$. Hence,

$$
R^T B R \nabla u = R^T (b_j \partial^j u)_{j=1}^{n+1} = \left(\sum_{j=1}^{n+1} R^T_{i,j} b_j \partial^j u\right)_{i=1}^{n+1} .
$$

Therefore,

$$
\operatorname{div}(A_0 \nabla u) = \sum_{i=1}^{n+1} \partial_i \left(\sum_{j=1}^{n+1} R^T_{i,j} b_j \partial^j u\right) = \sum_{j=1}^{n+1} b_j \partial^j u \sum_{i=1}^{n+1} \partial_i (R_{j,i}) + \sum_{j=1}^{n+1}\sum_{i=1}^{n+1} R_{j,i} \partial_i (b_j \partial^j u).
$$

Note that the sum inside the first term in the right-hand side is precisely $\sum_{i=1}^{n+1} \partial_i (R_{j,i}) = \operatorname{div} R_j^T$, and the sum inside the second term in the right-hand side is

$\sum_{i=1}^{n+1} R_{j,i}\partial_i(b_j\partial^j u) = R_j \cdot \nabla(b_j\partial^j u) = \partial^j(b_j\partial^j u)$. Thus, the claim follows for $u \in C^\infty(U)$.

From the definition of the directional ∂^i-derivative in (7.7) and (7.8), the claim follows by a density argument as in Claim 7.6. □

7.2 Sketch of the proof

Throughout this section the pole $p \in \Omega$ so that $\operatorname{dist}(p, \partial\Omega) > r_0$ is fixed unless it is otherwise stated (see Lemma 7.10 below). In any case, it will be far from $\partial\Omega$ and so from $\partial\widetilde{\Omega}$. Recall also that $S := A_0^{1/2}$, i.e., $S^T S = A_0$.

Fix $R := \min\{1, r_0/2\}/2$, and so we have $0 < R < \min\{1, r_0/2\}$. By [30, Lemma 3.35], see Lemma 2.13 and Remark 2.14 above, if $\widetilde{\Omega}$ is Reifenberg flat enough, depending on the ellipticity constant λ and the value $\kappa\|A\|_{L^\infty(\mathbb{R}^{n+1})}$, then there exists a constant $c = c(\lambda, \kappa\|A\|_{L^\infty(\mathbb{R}^{n+1})}) \geq 1$ such that

$$c^{-1}|\nabla g(y)| \leq \frac{g(y)}{\operatorname{dist}\big(y, \partial\widetilde{\Omega}\big)} \leq c|\nabla g(y)| \text{ for every } y \in B(\xi, R/c) \text{ with } \xi \in \partial\widetilde{\Omega}.$$

Here we need to work with (δ, r_0)-Reifenberg flat domains with flatness parameter δ smaller than some constant depending on the ellipticity constant λ and the value $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$.

Let us now fix the support function φ with $\varphi(p) = 0$:

Remark 7.8 Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ with

- $\varphi = 1$ in $U_{\frac{R}{2000c}}(\partial\Omega) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial\Omega) \leq \frac{R}{2000c}\}$,
- $\varphi = 0$ in $U_{\frac{R}{1500c}}(\partial\Omega)^c$, and
- $|\nabla\varphi| \lesssim 1$.

So $\operatorname{supp}\varphi \subseteq U_{\frac{R}{1500c}}(\partial\Omega)$. Note that $\partial\widetilde{\Omega} \subset \Omega \cap \operatorname{supp}\varphi$ if we assume that $\operatorname{dist}_{\mathcal{H}}(\partial\widetilde{\Omega}, \partial\Omega)$ is small enough. With this choice of φ we have the comparability

$$|\nabla g(y)| \approx \frac{g(y)}{\operatorname{dist}(y, \partial\widetilde{\Omega})} \text{ for } y \in \widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega) \supset \widetilde{\Omega} \cap \operatorname{supp}\varphi. \tag{7.12}$$

We claim

$$\log|S\nabla g|^2 \in W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega)). \tag{7.13}$$

Indeed, this follows as $|\nabla g| \approx g/\operatorname{dist}(\cdot, \partial\widetilde{\Omega}) > 0$ in $\widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega)$ and $g \in W^{2,2}(\widetilde{\Omega} \cap U_{\frac{R}{1500c}}(\partial\Omega))$, see (7.20).

Notation From now on the variables and the region of integration will not be written unless they are not clear from the context.

Estimate (7.2) will follow from the following partial results.

For $a \geq 10$, let $\log_{(a)} x := \max\{\log x, -a\}$. Note that in the weak sense $(\log_{(a)}(x))' = \mathbf{1}_{\{x \geq e^{-a}\}}(x) \log x$.

Lemma 7.9 *(Step 1) For $a \geq 10$ big enough,*

$$\int_{\partial\widetilde{\Omega}} \log|S\nabla g|^2 \, d\widetilde{\omega}_T^p = \int_{\partial\widetilde{\Omega}} \log_{(a)} |S\nabla g|^2 \, d\widetilde{\omega}_T^p + \sigma(\partial\widetilde{\Omega})\mathcal{O}(ae^{-a/2}).$$

Since $\sigma(\partial\widetilde{\Omega}) < \infty$, the second term in the right-hand side tends to zero as $a \to \infty$.

We prove the preceding lemma in Section 7.5.1.

Lemma 7.10 *(Step 2) For any $a \geq 10$ so that $e^{-a} < \min_{z \in \operatorname{supp} \nabla\varphi} |S\nabla g(z)|^2$, and for a.e. $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$,*

$$\int_{\partial\widetilde{\Omega}} \log_{(a)} |S\nabla g|^2 \, d\widetilde{\omega}_T^p = -\int_{\widetilde{\Omega}} \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle + \mathcal{O}\left(1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g\right).$$

We prove the preceding lemma in Section 7.5.2. Note that the left-hand side integral is supported on the boundary, while the one on the right-hand side is a solid integral.

Next, fix $\widetilde{\Omega}_\varphi \subset \widetilde{\Omega}$ a smooth domain satisfying

$$\widetilde{\Omega} \cap \operatorname{supp}\varphi \subset \widetilde{\Omega}_\varphi \subset U_{\frac{R}{1000c}}(\partial\Omega) \cap \widetilde{\Omega}.$$

With this choice we have the comparability $|\nabla g| \approx g/\operatorname{dist}(\cdot, \partial\widetilde{\Omega})$ in $\widetilde{\Omega}_\varphi$, see (7.12). We prove the following two lemmas in Section 7.5.1.

Lemma 7.11 *(Step 3) For $a \geq 10$ big enough,*

$$-\int_{\widetilde{\Omega}} \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle = -\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle$$
$$+ \mathcal{O}\left(\mathcal{H}^2(\Omega)e^{-a/2} + \int_{\widetilde{\Omega}\cap\operatorname{supp}\varphi\cap\{|S\nabla g|^2 \leq e^{-a}\}} |\nabla^2 g|\right).$$

Note that the last term in the right-hand side also tends to zero as $a \to \infty$ because $g \in W^{2,2}(\widetilde{\Omega}_\varphi)$ and $\bigcap_{a\geq 10}\{|S\nabla g|^2 \leq e^{-a}\} \cap \widetilde{\Omega}_\varphi = \emptyset$.

For $\varepsilon > 0$ as small as desired, we consider a given function $\psi_\varepsilon \in C^\infty(\mathbb{R}^2)$ satisfying

(1) $0 \leq \psi_\varepsilon \leq 1$ everywhere,
(2) $\psi_\varepsilon = 0$ in $U_\varepsilon(\partial\widetilde{\Omega})$,
(3) $\psi_\varepsilon = 1$ in $U_{3\varepsilon}(\partial\widetilde{\Omega})^c$, and
(4) $|\nabla\psi_\varepsilon| \lesssim \frac{1}{\varepsilon}$.

Lemma 7.12 *(Step 4) For ψ_ε as above we have*

$$\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle = \lim_{\varepsilon\to 0} \left\{ \int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\psi_\varepsilon \varphi g)\rangle \right.$$
$$\left. + \mathcal{O}\left(\frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g\right)\right\}.$$

In the weak sense, we can write the first term in the right-hand side in Lemma 7.12 as $\operatorname{div}(A^T \nabla \log |S\nabla g|^2)$ acting on $\psi_\varepsilon \varphi g \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)$, which can be understood as test functions. When studying the harmonic measure in the plane, i.e., $A = Id$, this term is 0 by the harmonicity of $\log |\nabla g|$ far from the critical points, a property that does not hold in general in higher dimensions. This was a key point to establish (1.2) for the Laplacian in [24, Lemma 3.1].

In the following remark we discuss this argument in the constant coefficient case.

Remark 7.13 (Constant matrix and L_A-harmonic functions) Suppose now that the matrix A is constant. Given any function u, by Claim 7.6,

$$\operatorname{div}(A^T \nabla u) = \operatorname{div}(A_0 \nabla u),$$

i.e., we can reduce the study to the symmetric part. Moreover, by means of a linear change of variables, see Lemma 3.4, we have that $\operatorname{div}(A^T \nabla u) = 0$ if and only if $\widetilde{u} = u \circ D$ satisfies $\operatorname{div}\left(\widetilde{A_0} \nabla \widetilde{u}\right) = 0$ where $\widetilde{A_0} = D^{-1} A_0 (D^{-1})^T$, for any constant matrix D with $\det D \neq 0$. Since A is constant now, if we choose $D = S^T$ where $S = A_0^{1/2}$, i.e., $S^T S = A_0$, then $\widetilde{A_0} = (S^T)^{-1} A_0 S^{-1} = (S^T)^{-1} S^T S S^{-1} = Id$, and so $\Delta \widetilde{u} = 0$. It is known that if $\Delta \widetilde{u} = 0$, then $\Delta \log |\nabla \widetilde{u}|^2 = 0$ (only) in the plane, whenever $\nabla \widetilde{u} \neq 0$. If we undo the previous change of variables then we obtain

$$\operatorname{div}\left(A_0 \nabla \left(\log |S\nabla u|^2\right)\right) = 0,$$

off the critical points.

Back to the general case, when the matrix A is non-constant, instead of moving from our matrix to the Laplacian by means of a change of variables as it is done in the constant case (meaning that we would need a non-constant change of variables), we will work with its symmetric part. Using the particular form $A_0 = R^T B R$, the rotation matrices will play the role of the directional derivatives, so we will be left with the diagonal matrix B, and this allows us to apply the previous strategy.

Now we study the first term appearing in the right-hand side in Lemma 7.12,

$$-\int_{\widetilde{\Omega}} \langle A^T \nabla \log |S\nabla g|^2, \nabla(\psi_\varepsilon \varphi g)\rangle,$$

which is the most delicate, and the key point in the different behavior between the planar case and higher dimensions. Recall that, in the dual space, this term is

$$\left(\operatorname{div}(A^T \nabla \log |S\nabla g|^2)\right)(\psi_\varepsilon \varphi g) \text{ where } \psi_\varepsilon \varphi g \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi).$$

Notation From now on, a boxed term $\boxed{X}$ (representing the functional T_X) can be understood as a pointwise function, while a functional written as T_X needs to be understood strictly in the dual sense.

In the following lemma we decompose this last term.

Lemma 7.14 *(Step 5) In the dual space $W_c^{1,\infty}(\widetilde{\Omega}_\varphi)'$, we have*

$$\operatorname{div}(A^T \nabla \log |S\nabla g|^2) = \boxed{M1} + \boxed{M2} + T_{M3} + T_E,$$

where

$$\boxed{M1} := -\sum_{i=1}^{2} \frac{4b_i}{|S\nabla g|^4} \langle \partial^i \nabla_R g, B\nabla_R g\rangle^2,$$

$$\boxed{M2} := \sum_{i=1}^{2} \frac{2b_i}{|S\nabla g|^2} \langle \partial^i \nabla_R g, B\partial^i \nabla_R g\rangle,$$

$$T_{M3} := \frac{2}{|S\nabla g|^2} \left\langle \sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g \right), B\nabla_R g \right\rangle,$$

and T_E is an error term (involving derivatives of the matrix A) of the form

$$T_E = \sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i \left(\mathcal{O}(|\nabla g|^2) \right) + \mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|} \right). \tag{7.14}$$

We prove the preceding lemma in Section 7.5.3.

Note that, using (7.3), we have that the two main terms satisfy

$$\left| \boxed{M1} \right| + \left| \boxed{M2} \right| \lesssim 1 + \left(\frac{|\nabla^2 g|}{|\nabla g|} \right)^2, \tag{7.15}$$

which is bad for our purposes. A similar behavior occurs with the term T_{M3}. Instead, we need to exploit their cancellation, as illustrated in the constant matrix case:

Remark 7.15 (Constant matrix and main terms) If the matrix A were constant, and hence R and B were constant as well, then we would have $T_E = 0$. That is,

$$\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2 \right) = \boxed{M1} + \boxed{M2} + T_{M3}.$$

This suggests that these 3 terms are the main terms, and the others must be bounded error terms. Moreover, we have the following points:

- The key point in the plane is that $\sum_{i=1}^{2} \partial^i (b_i \partial^i g) = 0$ would imply $b_1 \partial^{1,1} g = -b_2 \partial^{2,2} g$ (off the critical points). One can show that, as a consequence, the main terms $\boxed{M1}$ and $\boxed{M2}$ would cancel each other in the sense $\boxed{M1} + \boxed{M2} = 0$ off the critical points of g.
- Since the derivatives would commute, i.e., $\partial^{1,2} = \partial^{2,1}$, we would have

$$\sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g \right) = \nabla_R \left(\sum_{i=1}^{2} \partial^i \left(b_i \partial^i g \right) \right) = \nabla_R(0) = 0, \tag{7.16}$$

where we used that the Green function g satisfies $\sum_{i=1}^{2} \partial^i (b_i \partial^i g) = 0$. This is why we call T_{M3} the "zero" term. In contrast to the previous point, here it is not needed that we are in the plane. However, we use that the Green function is L_A-harmonic, that is, this term is zero even in higher dimensions.

All in all, in the constant matrix case we would have

$$\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) = \boxed{M1} + \boxed{M2} + T_{M3} = 0$$

off the critical points of g.

The strategy explained in the previous remark is, morally, what we will do to prove the following two lemmas.

Lemma 7.16 *(Step 6) Pointwise* $\left|\boxed{M1} + \boxed{M2}\right| \lesssim 1 + |\nabla^2 g|/|\nabla g|$ *in* $\widetilde{\Omega}_\varphi$.

We prove the preceding lemma in Section 7.5.4.

Lemma 7.17 *(Step 7) In the dual space* $W_c^{1,\infty}(\widetilde{\Omega}_\varphi)'$*, the functional* T_{M3} *is of the form*

$$T_{M3} = \sum_{j=1}^{2} \frac{2b_j \partial^j g}{|S\nabla g|^2} \left(\mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla^2 g|) + \partial^1 \left(\mathcal{O}(|\nabla g|)\right) + \partial^2 \left(\mathcal{O}(|\nabla g|)\right)\right).$$

Recall that this equality must be understood at a functional level, see (7.36) below.

We prove the preceding lemma in Section 7.5.5.

In the following lemma, we bound the error terms that have appeared in the previous computations.

Lemma 7.18 *(Step 8) For small enough* $\varepsilon > 0$,

$$|T_{M3}(\psi_\varepsilon \varphi g)| + |T_E(\psi_\varepsilon \varphi g)| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g.$$

We prove the preceding lemma in Section 7.5.6.

Lemma 7.19 *(Step 9)* $\int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g \leq C < \infty$*, and for small enough* $\varepsilon > 0$, $\int_{U_\varepsilon(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g \lesssim \varepsilon$.

We prove the preceding lemma at the end of Section 7.4.

Using the previous lemmas we obtain (7.2).

Proof of (7.2) Let $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$ so that Lemma 7.10 holds for all $a > 10$ integer. By Lemma 7.9 to 7.11 we have

$$\left|\int_{\partial\widetilde{\Omega}} \log |S\nabla g|^2 \, d\widetilde{\omega}_T^p\right| \leq \left|-\int \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle\right| + \mathcal{O}\left(1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g\right).$$

By Lemmas 7.12, 7.14, 7.16 and 7.18, we get

$$\left| -\int \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle \right| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g + \mathcal{O}\left(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g \right).$$

In particular,

$$\left| \int_{\partial\widetilde{\Omega}} \log |S\nabla g|^2 \, d\widetilde{\omega}_T^p \right| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g + \mathcal{O}\left(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g \right).$$

Finally, (7.2) follows from this and Lemma 7.19. Note that by Harnack inequality this is true for every $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$. □

7.3 Second derivatives of the Green function

In this subsection, we collect a fundamental property of the Green function in our situation, and some useful equalities and bounds that we will use later on involving the second order derivatives of the Green function.

The following claim points out the main difference between the planar and the higher dimensional case. This is a key point in the proof of Lemmas 7.16 and 7.17.

Claim 7.20 *The Green function g satisfies*

$$\sum_{i=1}^{2} \partial^i (b_i \partial^i g) = -\sum_{i=1}^{2} b_i \partial^i g \operatorname{div} R_i^T - \sum_{i,j=1}^{2} \partial_i \left(\frac{a_{ij} - a_{ji}}{2} \right) \partial_j g = \mathcal{O}(|\nabla g|).$$

a.e. in $U_{r_0}(\partial\Omega) \cap \widetilde{\Omega}$. In particular,

$$b_1 \partial^{1,1} g = -b_2 \partial^{2,2} g + \mathcal{O}(|\nabla g|). \tag{7.17}$$

Proof By Claims 7.7 and 7.6,

$$\begin{aligned} \operatorname{div} A\nabla g &= \sum_{i=1}^{2} \partial^i (b_i \partial^i g) + \sum_{i=1}^{2} b_i \partial^i g \operatorname{div} R_i^T + \sum_{i,j=1}^{2} \partial_i \left(\frac{a_{ij} - a_{ji}}{2} \right) \partial_j g \\ &= \sum_{i=1}^{2} \partial^i (b_i \partial^i g) + \mathcal{O}(|\nabla g|), \end{aligned}$$

where we used that the matrices A, B and R are Lipschitz. The claim follows from the fact that the Green function satisfies $\operatorname{div} A\nabla g = 0$ a.e. in $U_{r_0}(\partial\Omega) \cap \widetilde{\Omega}$, see Remark 2.11.

As a consequence of $\partial^i (b_i \partial^i g) = \partial^i b_i \partial^i g + b_i \partial^{i,i} g$ and $\partial^i b_i \partial^i g \in \mathcal{O}(|\nabla g|)$, we have (7.17). □

A close look to the proof reveals that if the matrix A were constant, then we would have $b_1\partial^{1,1}g = -b_2\partial^{2,2}g$ in the planar case, as it was explained in Remark 7.15. Instead, we get the analog with some error terms in (7.17).

In the following claim we collect some basic equalities and estimates. These equalities will be useful in the decomposition procedure in Lemma 7.14.

Claim 7.21 *Using the notation $\nabla_R f = R\cdot\nabla f$, see Section 7.1, we have*

$$\partial^i|S\nabla g|^2 = 2\langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle. \tag{7.18}$$

Consequently,

$$\partial^i\left(\frac{1}{|S\nabla g|^2}\right) = \frac{-\partial^i|S\nabla g|^2}{|S\nabla g|^4} \overset{((7.18))}{=} -\frac{2\langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle}{|S\nabla g|^4}, \tag{7.19}$$

and

$$|\nabla\log|S\nabla g|^2| = \frac{|\nabla|S\nabla g|^2|}{|S\nabla g|^2} \lesssim 1 + \frac{|\nabla^2 g|}{|\nabla g|} \text{ in } \widetilde{\Omega}_\varphi. \tag{7.20}$$

Proof Since $S^T S = A_0$ and $A_0 = R^T B R$, we can write

$$|S\nabla g|^2 = \langle S\nabla g, S\nabla g\rangle = \langle\nabla g, A_0\nabla g\rangle = \langle R\nabla g, BR\nabla g\rangle = \langle\nabla_R g, B\nabla_R g\rangle.$$

Recall that R and B are Lipschitz regular, thus we can apply the usual derivative rules a.e. to obtain

$$\partial^i|S\nabla g|^2 = \langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle + \langle\nabla_R g, B\partial^i\nabla_R g\rangle,$$

and (7.18) follows from the symmetry of B.

The inequality in (7.20) follows from

$$\begin{aligned}\Big|\nabla|S\nabla g|^2\Big| &\overset{(7.18))}{\approx} \sum_{i=1}^{2}\Big|2\langle\partial^i\nabla_R g, B\nabla_R g\rangle + \langle\nabla_R g, \partial^i B\nabla_R g\rangle\Big| \\ &\le \sum_{i=1}^{2} 2|\langle\partial^i\nabla_R g, B\nabla_R g\rangle| + \sum_{i=1}^{2}|\langle\nabla_R g, \partial^i B\nabla_R g\rangle| \\ &\lesssim |\nabla g|^2 + \sum_{i=1}^{2}|\partial^i\nabla_R g||\nabla g| \overset{(7.3))}{\lesssim} |\nabla g|^2 + |\nabla g||\nabla^2 g|,\end{aligned}$$

and this concludes the proof of the claim. □

Remark 7.22 $1/|S\nabla g|^2 \in W^{1,2}_{\text{loc}}(\widetilde{\Omega}_\varphi)$ because of (7.19), $|\nabla g| \approx g/\text{dist}(\cdot, \partial\widetilde{\Omega}) > 0$ in $\widetilde{\Omega}_\varphi$ and $g \in W^{2,2}(\widetilde{\Omega}_\varphi)$. By the same reason, $\log|S\nabla g|^2 \in W^{1,2}_{\text{loc}}(\widetilde{\Omega}_\varphi)$.

7.4 Whitney cubes and proof of Step 9

A classical way to compute an integral over a given set is to discretize it. We will do so using Whitney cubes, that is, we divide the domain into regions (cubes) which have diameter comparable to their distance to the boundary, so that Harnack's and Caccioppoli's inequalities can be used locally.

Let us start by defining the Whitney covering of $\widetilde{\Omega}$, and then we will move to study the properties of those cubes touching $\operatorname{supp}\varphi$.

Definition 7.23 There exists a collection $W_{\widetilde{\Omega}}$ of dyadic cubes satisfying the following properties:

(W1) $\sqrt{2}\ell(Q) \le \operatorname{dist}(Q, \partial\widetilde{\Omega}) \le 4\sqrt{2}\ell(Q)$, equivalently, $\operatorname{diam} Q \le \operatorname{dist}(Q, \partial\widetilde{\Omega}) \le 4 \operatorname{diam} Q$.
(W2) $1.5Q \cap \partial\widetilde{\Omega} = \emptyset$, since $\operatorname{dist}(1.5Q, \partial\widetilde{\Omega}) \ge \operatorname{dist}(Q, \partial\widetilde{\Omega}) - \sqrt{2} \cdot 0.25\ell(Q) \ge 0.75\sqrt{2}\ell(Q) > 0$,

which we call *Whitney cubes*, see [17, Appendix J]. We define $W_\varphi := \{Q \in W_{\widetilde{\Omega}} : Q \cap \operatorname{supp}\varphi \ne \emptyset\}$.

Lemma 7.24 *Every $Q \in W_\varphi$ satisfies also:*

(W3) $\operatorname{dist}(x, \partial\widetilde{\Omega}) \le 7\frac{R}{2000c} \le R/c$ *for* $x \in 1.5Q$.
(W4) $\ell(Q) \lesssim 1$.
(W5) If $Q \cap \operatorname{supp}\nabla\varphi \ne \emptyset$, *then* $\ell(Q) \gtrsim 1$.
(W6) For $x \in 1.5Q$ *we have*

$$g(x) \approx g(A(\xi, 10\ell(Q))) \lesssim \widetilde{\omega}_T(B(\xi, 10\ell(Q))) \le \widetilde{\omega}_T(50Q) \le 1,$$

where $\xi \in \partial\widetilde{\Omega}$ *is such that* $\operatorname{dist}(\xi, Q) = \operatorname{dist}(\partial\widetilde{\Omega}, Q)$, *and* $A(\xi, 10\ell(Q))$ *is the Corkscrew point at* ξ *with radius* $10\ell(Q)$.

Proof Note that for $x \in 1.5Q$, Harnack's inequality gives $g(x) \approx g(A(\xi, 10\ell(Q)))$. Using the relation between the elliptic measure and Green's function in NTA domains (see [25, Lemma 1.3.3]), we get

$$g(A(\xi, 10\ell(Q))) \lesssim \widetilde{\omega}_T(B(\xi, 10\ell(Q))).$$

Here we need $p \notin B(\xi, 3 \cdot 10\ell(Q))$, which is granted as long as $\partial\widetilde{\Omega}$ and $\partial\Omega$ are close enough.

The remaining estimates in the lemma follow from the definition by standard arguments. The details are left to the reader. □

In order to bound the error terms arising in the proofs of Section 7.2, we will use (without mention) the following remark.

Remark 7.25 If $f \in \mathcal{O}(|\nabla g|^k)$ then $\left|\int \frac{f}{|\nabla g|^k}\varphi g\right| \le C < \infty$.

Proof Since $g \lesssim 1$ in $\operatorname{supp}\varphi$ (see Lemma 7.24(W6)), we obtain

$$\left|\int \frac{f}{|\nabla g|^k}\varphi g\right| \le \int \frac{|f|}{|\nabla g|^k}|\varphi g| \lesssim \int |\varphi g| \lesssim \mathcal{H}^2(\Omega) \le (\operatorname{diam}\partial\Omega)^2 < \infty,$$

where we used that Ω is bounded. □

Next we continue by controlling the Green function over Whitney cubes:

Lemma 7.26 *We have:*

(1) $\sum_{Q\in W_\varphi} \ell(Q)\cdot\widetilde{\omega}_T(50Q) \lesssim 1$.
(2) *For* $\varepsilon > 0$, $\sum_{\{Q\in W_{\widetilde{\Omega}}: Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset\}} \ell(Q)\cdot\widetilde{\omega}_T(50Q) \lesssim \varepsilon$.
(3) *For* $\varepsilon > 0$, $\int_{U_\varepsilon(\partial\widetilde{\Omega})} g \lesssim \varepsilon^2$.
(4) *For each* $Q \in W_\varphi$, $\left(\int_{1.5Q} g^2\right)^{1/2} \lesssim \ell(Q)\cdot\widetilde{\omega}_T(50Q)$.
(5) *For each* $Q \in W_\varphi$, $\int_Q |\nabla g| \lesssim \ell(Q)\cdot\widetilde{\omega}_T(50Q)$.
(6) *For each* $Q \in W_\varphi$, $\int_Q |\nabla^2 g| \lesssim \widetilde{\omega}_T(50Q)$.

Proof Item (1): As $\ell(Q) \lesssim 1$ for every $Q \in W_\varphi$ (see Lemma 7.24(W4)), there is k_0 such that every $Q \in W_\varphi$ satisfies $\ell(Q) = 2^{-k}$ with $k \ge k_0$. Note also that, for each scale $k \ge k_0$, the family $\{50Q\}_{\{Q\in W_\varphi\,:\,\ell(Q)=2^{-k}\}}$ has finite overlapping. Therefore,

$$\sum_{Q\in W_\varphi} \ell(Q)\cdot\widetilde{\omega}_T(50Q) = \sum_{k\ge k_0}\sum_{\substack{Q\in W_\varphi\\ \ell(Q)=2^{-k}}} \ell(Q)\cdot\widetilde{\omega}_T(50Q) = \sum_{k\ge k_0} 2^{-k}\sum_{\substack{Q\in W_\varphi\\ \ell(Q)=2^{-k}}} \widetilde{\omega}_T(50Q)$$
$$\lesssim \sum_{k\ge k_0} 2^{-k}\widetilde{\omega}_T\Bigg(\bigcup_{\substack{Q\in W_\varphi\\ \ell(Q)=2^{-k}}} 50Q\Bigg) \le \sum_{k\ge k_0} 2^{-k} < \infty.$$

Item (2): Recall that the cubes $Q \in W_{\widetilde{\Omega}}$ with $Q \cap U_\varepsilon(\partial\widetilde{\Omega}) \neq \emptyset$ satisfy $\ell(Q) \lesssim \varepsilon$, and $\ell(Q) \approx \varepsilon$ if $Q\cap(\partial U_\varepsilon(\partial\widetilde{\Omega})\setminus\partial\widetilde{\Omega}) \neq \emptyset$, see (W1). Hence, as in the proof of item (1), taking $k_0(\varepsilon) \approx -\log_2\varepsilon$ we have

$$\sum_{\substack{Q\in W_{\widetilde{\Omega}}\\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q)\widetilde{\omega}_T(50Q) \le \sum_{k\ge k_0(\varepsilon)} 2^{-k}\sum_{\substack{Q\in W_{\widetilde{\Omega}}\\ \ell(Q)=2^{-k}}} \widetilde{\omega}_T(50Q) \lesssim \sum_{k\ge k_0(\varepsilon)} 2^{-k} \approx 2^{\log_2\varepsilon} = \varepsilon.$$

Item (3): Using that $g \lesssim \widetilde{\omega}_T(50Q)$ in Q, see (W6), that the cubes $Q \in W_{\widetilde{\Omega}}$ with $Q \cap U_\varepsilon(\partial\widetilde{\Omega}) \neq \emptyset$ satisfy $\ell(Q) \lesssim \varepsilon$, see (W1), and item (2), we have

$$\int_{U_\varepsilon(\partial\widetilde{\Omega})} g \le \sum_{\substack{Q\in W_{\widetilde{\Omega}}\\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \int_Q g \overset{(W6),(W1)}{\lesssim} \varepsilon \sum_{\substack{Q\in W_{\widetilde{\Omega}}\\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q)\widetilde{\omega}_T(50Q) \overset{(2)}{\lesssim} \varepsilon^2.$$

Item (4) follows since $g(x) \lesssim \widetilde{\omega}_T(50Q)$ for $x \in 1.5Q$ (see Lemma 7.24 (W6)). Item (5) follows from Cauchy-Schwarz and Caccioppoli's inequalities and item (4).

Item (6): Since the Green function g of $\widetilde{\Omega}$ is solution of $\operatorname{div}(A\nabla\cdot) = 0$ in $1.1Q$, the function $\widehat{g}(\cdot) = g(\ell(Q)\cdot)$ is solution of $\operatorname{div}(\widehat{A}\nabla\cdot)$ in $1.1\widehat{Q}$, where $\widehat{Q} := \{x/\ell(Q) : x \in Q\}$ has side-length one, and $\widehat{A}(\cdot) = A(\ell(Q)\cdot)$. Moreover, since $\partial_i\partial_j\widehat{g}(\cdot) = \ell(Q)^2\partial_i\partial_j g(\ell(Q)\cdot)$, we get

$$\int_Q |\nabla^2 g(x)|^2\,dx = \ell(Q)^2 \int_{\widehat{Q}} |\nabla^2 g(\ell(Q)x)|^2\,dx = \frac{1}{\ell(Q)^2}\int_{\widehat{Q}} |\nabla^2\widehat{g}(x)|^2\,dx.$$

The matrix $\widehat{A}$ has the same ellipticity constant as A, but as $\ell(Q) \lesssim 1$ for every $Q \in W_\varphi$, see Lemma 7.24 (W4), the Lipschitz norm cannot grow too much:

$$\begin{aligned}\|\widehat{A}\|_{C^{0,1}(1.1\widehat{Q})} &:= \|\widehat{A}\|_{L^\infty(1.1\widehat{Q})} + [\widehat{A}]_{C^{0,1}(1.1\widehat{Q})}\\ &= \|A\|_{L^\infty(1.1Q)} + \ell(Q)[A]_{C^{0,1}(1.1Q)} \le C\|A\|_{C^{0,1}(1.1Q)}.\end{aligned}$$

Moreover, $\operatorname{dist}\left(\widehat{Q}, (1.1\widehat{Q})^c\right) \approx 1$. Therefore we can apply (2.2) in Theorem 2.9 to obtain

$$\int_Q |\nabla^2 g|^2\,dx = \frac{1}{\ell(Q)^2}\int_{\widehat{Q}} |\nabla^2\widehat{g}|^2\,dx \overset{(2.2))}{\lesssim} \frac{1}{\ell(Q)^2}\left[\left(\int_{1.1\widehat{Q}} |\nabla\widehat{g}|^2\right)^{1/2} + \left(\int_{1.1\widehat{Q}} \widehat{g}^2\right)^{1/2}\right]^2.$$

Applying Caccioppoli's inequality and item (4), this implies

$$\int_Q |\nabla^2 g|^2\,dx \lesssim \frac{1}{\ell(Q)^2}\int_{1.5\widehat{Q}} \widehat{g}^2 = \frac{1}{\ell(Q)^4}\int_{1.5Q} g^2 \overset{(4)}{\lesssim} \frac{\widetilde{\omega}_T(50Q)^2}{\ell(Q)^2}.$$

Using the Cauchy-Schwarz inequality and this, we get

$$\int_Q |\nabla^2 g| \le \ell(Q)\left(\int_Q |\nabla^2 g|^2\right)^{1/2} \lesssim \widetilde{\omega}_T(50Q).$$

□

Using this lemma we obtain Lemma 7.19.

Proof of Lemma 7.19 Recall that, by (7.12) we have

$$|\nabla g(x)| \gtrsim \frac{g(x)}{\operatorname{dist}\left(x, \partial\widetilde{\Omega}\right)} \text{ for } x \in \operatorname{supp}\varphi,$$

provided the domain $\widetilde{\Omega}$ is $(\delta, r_0/2)$-Reifenberg flat with δ small enough depending only on λ and $\kappa\|A\|_{L^\infty(\mathbb{R}^2)}$. With this it suffices to show

$$\int_{\operatorname{supp}\varphi} |\nabla^2 g(x)|\operatorname{dist}\left(x, \partial\widetilde{\Omega}\right)\,dx \le C.$$

Summing in W_φ, by (6) and (1) in Lemma 7.26 we obtain

$$\int_{\operatorname{supp}\varphi} |\nabla^2 g(x)| \operatorname{dist}(x, \partial\Omega)\, dx \lesssim \sum_{Q\in W_\varphi} \ell(Q) \int_Q |\nabla^2 g| \overset{(6)}{\lesssim} \sum_{Q\in W_\varphi} \ell(Q)\cdot \widetilde{\omega}_T(50Q) \overset{(1)}{\lesssim} 1,$$

as claimed.

Let us see the localized version. As above, using (7.12), and (6) and (2) in Lemma 7.26,

$$\int_{U_\varepsilon(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g \overset{(7.12))}{\lesssim} \int_{U_\varepsilon(\partial\widetilde{\Omega})} |\nabla^2 g| \operatorname{dist}(x, \partial\widetilde{\Omega}) \le \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q) \int_Q |\nabla^2 g|$$
$$\overset{(6)}{\lesssim} \sum_{\substack{Q\in W_{\widetilde{\Omega}} \\ Q\cap U_\varepsilon(\partial\widetilde{\Omega})\neq\emptyset}} \ell(Q)\widetilde{\omega}_T(50Q) \overset{(2)}{\lesssim} \varepsilon.$$

□

7.5 Proof of Steps 1 to 8

In this subsection we prove remaining Steps 1 to 8 in Section 7.2.

7.5.1 Proof of Lemmas 7.9, 7.11 and 7.12

Proof of Lemma 7.9 We have

$$\left|\int_{\partial\widetilde{\Omega}} \log|S(\xi)\nabla g(\xi)|^2\, d\widetilde{\omega}_T^p(\xi) - \int_{\partial\widetilde{\Omega}} \log_{(a)} |S(\xi)\nabla g(\xi)|^2\, d\widetilde{\omega}_T^p(\xi)\right|$$
$$= \left|\int_{\{\xi\in\partial\widetilde{\Omega}: |S(\xi)\nabla g(\xi)|^2 \le e^{-a}\}} (\log|S(\xi)\nabla g(\xi)|^2 + a)\, d\widetilde{\omega}_T^p(\xi)\right|$$

Since the domain is smooth, we can use (2.8) to write and bound the right-hand side term as

$$\left|\int_{\{\xi\in\partial\widetilde{\Omega}: |S(\xi)\nabla g(\xi)|^2 \le e^{-a}\}} -\langle A(\xi)\nabla g(\xi), \nu\rangle (\log|S(\xi)\nabla g(\xi)|^2 + a)\, d\sigma(\xi)\right|$$
$$\lesssim \int_{\{\xi\in\partial\widetilde{\Omega}: |S(\xi)\nabla g(\xi)|^2 \le e^{-a}\}} |S(\xi)\nabla g(\xi)|(|\log|S(\xi)\nabla g(\xi)||)\, d\sigma(\xi)$$
$$\le \sigma(\partial\widetilde{\Omega}) \left(e^{-a/2}|\log e^{-a/2}|\right).$$

□

Proof of Lemma 7.11 We have

$$
\begin{aligned}
&-\int \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle + \int \langle A^T \nabla \log |S\nabla g|^2, \nabla(\varphi g)\rangle \\
&\quad = \int \langle A^T \nabla \min\{0, a + \log |S\nabla g|^2\}, \nabla(\varphi g)\rangle.
\end{aligned}
$$

On the other hand, by (7.12) we have that if $a \geq 10$ is big enough then $\varphi = 1$ in $\widetilde{\Omega}_\varphi \cap \{|S\nabla g|^2 \leq e^{-a}\}$, and hence

$$
\int \langle A^T \nabla \min\{0, a + \log |S\nabla g|^2\}, \nabla(\varphi g)\rangle = \int \langle A^T \nabla \log |S\nabla g|^2, \nabla g\rangle \mathbf{1}_{\{|S\nabla g|^2 \leq e^{-a}\}}.
$$

By (7.20), this is controlled in absolute value by

$$
\int_{\{|S\nabla g|^2 \leq e^{-a}\}} \left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right) |\nabla g| \leq \int_{\{|S\nabla g|^2 \leq e^{-a}\}} |\nabla g| + \int_{\{|S\nabla g|^2 \leq e^{-a}\}} |\nabla^2 g|.
$$

The lemma follows because the first term in the right-hand side is bounded by a constant times $\mathcal{H}^2(\Omega)e^{-a/2}$. □

Proof of Lemma 7.12 For $\varepsilon > 0$ small we have $(1 - \psi_\varepsilon)\varphi = (1 - \psi_\varepsilon)$. Hence

$$
\begin{aligned}
&-\int \langle A^T \nabla \log |S\nabla g|^2, \nabla((1 - \psi_\varepsilon)\varphi g)\rangle = -\int \langle A^T \nabla \log |S\nabla g|^2, \nabla((1 - \psi_\varepsilon)g)\rangle \\
&\quad = \int \langle A^T \nabla \log |S\nabla g|^2, \nabla \psi_\varepsilon\rangle g - \int \langle A^T \nabla \log |S\nabla g|^2, \nabla g\rangle (1 - \psi_\varepsilon).
\end{aligned}
$$

Using (7.20), the second term in the right-hand side is controlled in absolute value by

$$
\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right) |\nabla g| = \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} |\nabla g| + \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} |\nabla^2 g| \xrightarrow{\varepsilon \to 0} 0,
$$

and the first term is controlled in absolute value by

$$
\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right) g |\nabla \psi_\varepsilon| = \frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} g + \frac{1}{\varepsilon} \int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} \frac{|\nabla^2 g|}{|\nabla g|} g.
$$

The lemma follows by applying (3) in Lemma 7.26 in the last line. □

7.5.2 Proof of Lemma 7.10

Recall that the Green function g in $\widetilde{\Omega}$ for the operator $L_A u = \operatorname{div} A\nabla u$ satisfies, for every $\phi \in C^0(\overline{\widetilde{\Omega}}) \cap W^{1,2}(\widetilde{\Omega})$,

$$
\int_{\partial\widetilde{\Omega}} \phi(\xi)\, d\widetilde{\omega}_T^x(\xi) - \phi(x) = -\int_{\widetilde{\Omega}} \langle A^T(y)\nabla\phi(y), \nabla g_x(y)\rangle\, dy, \text{ for a.e. } x \in \widetilde{\Omega}, \tag{7.21}
$$

see (2.4).

We claim that

$$\varphi \log_{(a)} |S\nabla g|^2 \in C^0(\overline{\widetilde{\Omega}}) \cap W^{1,2}(\widetilde{\Omega}).$$

Indeed, $\varphi \log_{(a)} |S\nabla g|^2 \in C^0(\overline{\widetilde{\Omega}})$ because the function φ avoids the pole and ∇g is continuous up to the boundary, see Theorem 2.12. Let us see that $\varphi \log_{(a)} |S\nabla g|^2 \in W^{1,2}(\widetilde{\Omega})$. Using Jensen's inequality as the function $x \mapsto |\log x|^2$ is concave for $x > e$, the L^2-norm is controlled (depending on a) by

$$\begin{aligned}
&\int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} \left|\log_{(a)} |S\nabla g|^2\right|^2 \\
&\quad \leq \int_{\widetilde{\Omega}\cap\{|S\nabla g|^2 \leq e\}} \left|\log_{(a)} |S\nabla g|^2\right|^2 + \int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi\cap\{|S\nabla g|^2 > e\}} \left|\log |S\nabla g|^2\right|^2 \\
&\quad \lesssim \mathcal{H}^2(\widetilde{\Omega}) \left(a^2 + \left|\log\left(\fint_{\widetilde{\Omega}\cap \operatorname{supp}\varphi\cap\{|S\nabla g|^2 > e\}} |S\nabla g|^2\right)\right|^2\right) < \infty,
\end{aligned} \tag{7.22}$$

which is bounded since $g \in W^{1,2}(\widetilde{\Omega} \cap \operatorname{supp}\varphi)$. The L^2 norm of $\nabla(\varphi \log_{(a)} |S\nabla g|^2)$ is

$$\int_{\widetilde{\Omega}} |\nabla(\varphi \log_{(a)} |S\nabla g|^2)|^2 \leq \int_{\widetilde{\Omega}} |\nabla\varphi|^2 \left|\log_{(a)} |S\nabla g|^2\right|^2 + \int_{\widetilde{\Omega}} \varphi^2 |\nabla \log_{(a)} |S\nabla g|^2|^2.$$

The first term is finite by (7.22) because $\varphi \in C_c^\infty(\mathbb{R}^2)$. From the definition of $\log_{(a)} x := \max\{\log x, -a\}$ and by (7.20), we get

$$|\nabla \log_{(a)} |S\nabla g|^2| \lesssim \mathbf{1}_{\{x \geq e^{-a}\}}(|S\nabla g|^2)\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right),$$

and hence the second term is controlled by

$$\begin{aligned}
\int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} |\nabla \log_{(a)} |S\nabla g|^2|^2 &\lesssim \int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi\cap\{|S\nabla g|^2 \geq e^{-a}\}} 1 + \frac{|\nabla^2 g|^2}{|\nabla g|^2} \\
&\lesssim \mathcal{H}^2(\widetilde{\Omega}) + e^a \int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} |\nabla^2 g|^2 < \infty,
\end{aligned}$$

which is bounded as $g \in W^{2,2}(\widetilde{\Omega} \cap \operatorname{supp}\varphi)$.

By the choice of φ with $\varphi(p) = 0$ and $\varphi|_{\partial\widetilde{\Omega}} = 1$, and plugging $\phi = \varphi \log_{(a)} |S\nabla g|^2 \in C^0(\overline{\widetilde{\Omega}}) \cap W^{1,2}(\widetilde{\Omega})$ in (7.21), for a.e. $p \in \Omega$ with $\operatorname{dist}(p, \partial\Omega) > r_0$

we have

$$\begin{aligned}\int_{\partial\widetilde{\Omega}} \log_{(a)} |S\nabla g|^2 \, d\widetilde{\omega}_T^p &= -\int_{\widetilde{\Omega}\cap \operatorname{supp}\varphi} \langle A^T \nabla \left(\varphi \log_{(a)} |S\nabla g|^2\right), \nabla g\rangle \\ &= -\int \langle A^T \nabla\varphi, \nabla g\rangle \cdot \log_{(a)} |S\nabla g|^2 \\ &\quad - \int \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \varphi\nabla g\rangle \\ &= \boxed{1} + \boxed{2} + \boxed{3},\end{aligned} \tag{7.23}$$

with

$$\begin{aligned}\boxed{1} &:= -\int \langle A^T \nabla\varphi, \nabla g\rangle \cdot \log_{(a)} |S\nabla g|^2, \\ \boxed{2} &:= \int \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla\varphi\rangle \cdot g, \text{ and} \\ \boxed{3} &:= -\int \langle A^T \nabla \log_{(a)} |S\nabla g|^2, \nabla(\varphi g)\rangle.\end{aligned}$$

We claim that $\left|\boxed{1}\right| \lesssim 1$ and $\left|\boxed{2}\right| \lesssim 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g$, and hence Lemma 7.10 follows.

We start by controlling the term $\boxed{1}$. Recall that we have fixed φ so that $|\nabla\varphi| \lesssim 1$, and assumed that a is big enough so that $e^{-a} < \min_{z\in \operatorname{supp}\nabla\varphi} |S\nabla g(z)|^2$, that is, $\log_{(a)} |S\nabla g(z)|^2 = \log |S\nabla g(z)|^2$ for $z \in \operatorname{supp}\nabla\varphi$. Thus,

$$\left|\boxed{1}\right| \lesssim \int_{\operatorname{supp}\nabla\varphi} |\nabla g| \, |\log |S\nabla g|| \approx \int_{\operatorname{supp}\nabla\varphi} |S\nabla g| \, |\log |S\nabla g|| \,.$$

Since $\operatorname{supp}\nabla\varphi$ is far from $\partial\widetilde{\Omega}$, in particular $\operatorname{dist}(x, \partial\widetilde{\Omega}) \approx 1$ for $x \in \operatorname{supp}\nabla\varphi$. Consequently, from this and (7.12) we have $|\nabla g| \approx g/\operatorname{dist}(\cdot, \partial\widetilde{\Omega}) \approx g$ in the support of $\nabla\varphi$, and hence

$$|S\nabla g(x)| \approx |\nabla g(x)| \approx g(x) \overset{(W6)}{\lesssim} 1 \text{ for } x \in \operatorname{supp}\nabla\varphi.$$

Note that for $0 < t < t_0$ we have $t\,|\log t| \le \max\{1, t_0\,|\log t_0|\}$. Therefore,

$$\boxed{1} \lesssim \int_{\operatorname{supp}\nabla\varphi} |S\nabla g| \, |\log |S\nabla g|| \lesssim \mathcal{H}^2(\operatorname{supp}\nabla\varphi) \le C < \infty.$$

Now we study the term $\boxed{2}$. Since φ is fixed, we have $|\nabla\varphi| \lesssim 1$ and so

$$\left|\boxed{2}\right| \lesssim \int_{\operatorname{supp}\nabla\varphi} g \cdot |\nabla \log_{(a)} |S\nabla g|^2| \le \int_{\operatorname{supp}\varphi} g \cdot |\nabla \log_{(a)} |S\nabla g|^2|.$$

Plugging (7.20) in here we get

$$\left|\boxed{2}\right| \lesssim \int_{\operatorname{supp}\varphi} g + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g \overset{(W6)}{\lesssim} 1 + \int_{\operatorname{supp}\varphi} \frac{|\nabla^2 g|}{|\nabla g|} g.$$

□

7.5.3 Proof of Lemma 7.14

In this subsection we study, via a perturbation argument, the functional term

$$\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)'.$$

Note that the action of this functional on φg gives rise to a modified version of the term $\boxed{3}$ in (7.23).

First, we move from the matrix A to its symmetric part A_0 by Claim 7.6, and we write its divergence in terms of the directional derivatives using Claim 7.7 (see Remark 7.22):

$$\begin{aligned}
\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) &= \operatorname{div}\left(A_0 \nabla \log |S\nabla g|^2\right) + \sum_{i,j=1}^{2} \partial_i \left(\frac{a_{ij}^T - a_{ji}^T}{2}\right) \partial_j \log |S\nabla g|^2 \\
&= \sum_{i=1}^{2} \partial^i (b_i \partial^i (\log |S\nabla g|^2)) + \sum_{i=1}^{2} b_i \partial^i (\log |S\nabla g|^2) \operatorname{div} R_i^T \\
&\quad + \sum_{i,j=1}^{2} \partial_i \left(\frac{a_{ji} - a_{ij}}{2}\right) \partial_j (\log |S\nabla g|^2) =: T_{3.A} + \boxed{3.B} + \boxed{3.C}.
\end{aligned}$$

Note that the terms $\boxed{3.B}$ and $\boxed{3.C}$ contain derivatives of the matrix A. In particular, if the matrix were constant then these two terms would be zero, which suggests that the terms $\boxed{3.B}$ and $\boxed{3.C}$ must be bounded error terms.

Next we deal with $T_{3.A}$. As $1/|S\nabla g|^2 \in W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}_\varphi)$ (see Remark 7.22), by (7.9) we have

$$\begin{aligned}
T_{3.A} &= \sum_{i=1}^{2} \partial^i \left(b_i \frac{\partial^i |S\nabla g|^2}{|S\nabla g|^2}\right) \\
&= \sum_{i=1}^{2} \partial^i \left(\frac{1}{|S\nabla g|^2}\right) b_i \partial^i |S\nabla g|^2 + \sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i (b_i \partial^i |S\nabla g|^2).
\end{aligned} \tag{7.24}$$

We infer that each element in the sum in the first term in the right-hand side in (7.24) is

$$\partial^i \left(\frac{1}{|S\nabla g|^2}\right) b_i \partial^i |S\nabla g|^2 \overset{(7.19)}{=} \frac{-b_i \left(\partial^i |S\nabla g|^2\right)^2}{|S\nabla g|^4}. \tag{7.25}$$

Therefore, using identities (7.18) and (7.25) and expanding, the first term in the right-hand side of (7.24) can be written as

$$\sum_{i=1}^{2} \partial^i \left(\frac{1}{|S\nabla g|^2} \right) b_i \partial^i |S\nabla g|^2 = \boxed{M1} + \boxed{E1} + \boxed{E2},$$

where

$$\boxed{M1} := -\sum_{i=1}^{2} \frac{4b_i}{|S\nabla g|^4} \langle \partial^i \nabla_R g, B\nabla_R g \rangle^2,$$

$$\boxed{E1} := -\sum_{i=1}^{2} \frac{4b_i}{|S\nabla g|^4} \langle \partial^i \nabla_R g, B\nabla_R g \rangle \langle \nabla_R g, \partial^i B\nabla_R g \rangle, \text{ and}$$

$$\boxed{E2} := -\sum_{i=1}^{2} \frac{b_i}{|S\nabla g|^4} \langle \nabla_R g, \partial^i B\nabla_R g \rangle^2.$$

Note that $\boxed{E1}$ and $\boxed{E2}$ can be considered "error terms" from a perturbation point of view, since in case B were constant we would get $\boxed{E1} = \boxed{E2} = 0$, while $\boxed{M1}$ can be considered a "main term".

Next we perform a similar decomposition for $\partial^i (b_i \partial^i |S\nabla g|^2)$, in the second term in the right-hand side in (7.24). In this case, we need to pay attention to the first term in the right-hand side of (7.18):

$$\partial^i \left(b_i \langle \partial^i \nabla_R g, B\nabla_R g \rangle \right) = \left\langle \partial^i \left(b_i \partial^i \nabla_R g \right), B\nabla_R g \right\rangle + b_i \langle \partial^i \nabla_R g, \partial^i B\nabla_R g \rangle$$
$$+ b_i \langle \partial^i \nabla_R g, B\partial^i \nabla_R g \rangle. \tag{7.26}$$

Then using (7.18) and (7.26) we get that the second term in the right-hand side of (7.24) can be written as

$$\sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i (b_i \partial^i |S\nabla g|^2) = \boxed{M2} + T_{M3} + \boxed{E3} + T_{E4},$$

where

$$\boxed{M2} := \sum_{i=1}^{2} \frac{2b_i}{|S\nabla g|^2} \langle \partial^i \nabla_R g, B\partial^i \nabla_R g \rangle,$$

$$T_{M3} := \frac{2}{|S\nabla g|^2} \left\langle \sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g \right), B\nabla_R g \right\rangle,$$

$$\boxed{E3} := \sum_{i=1}^{2} \frac{2b_i}{|S\nabla g|^2} \langle \partial^i \nabla_R g, \partial^i B\nabla_R g \rangle, \text{ and}$$

$$T_{E4} := \sum_{i=1}^{2} \frac{1}{|S\nabla g|^2} \partial^i \left(b_i \langle \nabla_R g, \partial^i B \nabla_R g \rangle \right).$$

All together we have

$$\begin{aligned}
\operatorname{div}\left(A^T \nabla \log |S\nabla g|^2\right) &= T_{3.A} + \boxed{3.B} + \boxed{3.C} \\
&= \boxed{M1} + \boxed{M2} + T_{M3} && \text{(Main terms)} \\
&\quad + \boxed{3.B} + \boxed{3.C} + \boxed{E1} + \boxed{E2} + \boxed{E3} + T_{E4}, && \text{(Error terms)}
\end{aligned}$$

and the first part of the lemma follows taking

$$T_E := \boxed{3.B} + \boxed{3.C} + \boxed{E1} + \boxed{E2} + \boxed{E3} + T_{E4}.$$

Let us see now the second part of the lemma. Indeed, $\boxed{E2}$ is $\mathcal{O}(1)$ because $|\langle \nabla_R g, \partial^i B \nabla_R g \rangle| \lesssim |\nabla g|^2$. By (7.20) we get that $\boxed{3.B}$ and $\boxed{3.C}$ are in $\mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right)$, and (7.3) implies that $\boxed{E1}$ and $\boxed{E3}$ are of the form $\mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right)$. All in all,

$$T_E = T_{E4} + \mathcal{O}\left(1 + \frac{|\nabla^2 g|}{|\nabla g|}\right),$$

as claimed. □

7.5.4 Proof of Lemma 7.16

In this subsection we prove that the two main terms cancel out in the sense

$$\left| \boxed{M1} + \boxed{M2} \right| \lesssim 1 + \frac{|\nabla^2 g|}{|\nabla g|}.$$

This cancellation is related to what happens in the constant case, see Remark 7.15.

We will see that they both have a common term in opposite sign, and exploiting this cancellation we will obtain a sum of error terms. The key identity is (7.17), which only applies in the plane.

The key idea to prove the lemma is that using (7.5), which relates $\partial^{1,2} g$ and $\partial^{2,1} g$, and (7.17), which relates $b_1 \partial^{1,1} g$ and $b_2 \partial^{2,2} g$, we will be able to write $\partial^{1,1}$, $\partial^{2,2}$, $\partial^{1,2}$ and $\partial^{2,1}$ in terms of $\partial^{1,1}$ and $\partial^{2,1}$ only.

We start by studying the term $\boxed{M1} := -\sum_i \frac{4b_i}{|S\nabla g|^4}\langle \partial^i \nabla_R g, B\nabla_R g\rangle^2$ in $\widetilde{\Omega}_\varphi$. Expanding the numerator,

$$\begin{aligned}\sum_{i=1}^{2} b_i\langle \partial^i \nabla_R g, B\nabla_R g\rangle^2 &= b_1\langle \partial^1 \nabla_R g, B\nabla_R g\rangle^2 + b_2\langle \partial^2 \nabla_R g, B\nabla_R g\rangle^2\\ &= b_1(b_1\partial^1 g\partial^{1,1}g + b_2\partial^2 g\partial^{1,2}g)^2\\ &\quad + b_2(b_1\partial^1 g\partial^{2,1}g + b_2\partial^2 g\partial^{2,2}g)^2.\end{aligned}$$

By (7.5) and (7.17), which read as $\partial^{1,2}g = \partial^{2,1}g + \mathcal{O}(|\nabla g|)$ and $b_2\partial^{2,2}g = -b_1\partial^{1,1}g + \mathcal{O}(|\nabla g|)$ respectively, we have

$$\begin{aligned}\sum_{i=1}^{2} b_i\langle \partial^i \nabla_R g, B\nabla_R g\rangle^2 &= b_1(b_1\partial^1 g\partial^{1,1}g + b_2\partial^2 g\partial^{2,1}g + \mathcal{O}(|\nabla g|^2))^2\\ &\quad + b_2(b_1\partial^1 g\partial^{2,1}g - b_1\partial^2 g\partial^{1,1}g + \mathcal{O}(|\nabla g|^2))^2.\end{aligned}$$

Expanding, and using $b_1(\partial^1 g)^2 + b_2(\partial^2 g)^2 = \langle \nabla_R g, B\nabla_R g\rangle = |S\nabla g|^2$ and the cancellation of cross terms, we get

$$\begin{aligned}\sum_{i=1}^{2} b_i\langle \partial^i \nabla_R g, B\nabla_R g\rangle^2 &= |S\nabla g|^2\left(b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2\right)\\ &\quad + (\partial^{1,1}g + \partial^{2,1}g)\mathcal{O}(|\nabla g|^3) + \mathcal{O}(|\nabla g|^4).\end{aligned}$$

In conclusion, $\boxed{M1}$ is of the form

$$\boxed{M1} = -4\frac{b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2}{|S\nabla g|^2} + \frac{\mathcal{O}(\partial^{1,1}g + \partial^{2,1}g)}{|S\nabla g|} + \mathcal{O}(1). \tag{7.27}$$

With the same strategy we study the term $\boxed{M2} := \sum_i \frac{2b_i}{|S\nabla g|^2}\langle \partial^i \nabla_R g, B\partial^i \nabla_R g\rangle$ in $\widetilde{\Omega}_\varphi$. Using (7.5) and (7.17) as before and expanding, the numerator can be written as

$$\begin{aligned}\sum_{i=1}^{2} b_i\langle \partial^i \nabla_R g, B\partial^i \nabla_R g\rangle &= b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{1,2}g)^2 + b_1b_2(\partial^{2,1}g)^2 + b_2^2(\partial^{2,2}g)^2\\ &= b_1^2(\partial^{1,1}g)^2 + b_1b_2\left(\partial^{2,1}g + \mathcal{O}(|\nabla g|)\right)^2 + b_1b_2(\partial^{2,1}g)^2 + \left(-b_1\partial^{1,1}g + \mathcal{O}(|\nabla g|)\right)^2\\ &= 2\left(b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2\right) + \left(\partial^{1,1}g + \partial^{2,1}g\right)\mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla g|^2).\end{aligned}$$

Hence, $\boxed{M2}$ is of the form

$$\boxed{M2} = 4\frac{b_1^2(\partial^{1,1}g)^2 + b_1b_2(\partial^{2,1}g)^2}{|S\nabla g|^2} + \frac{\mathcal{O}(\partial^{1,1}g + \partial^{2,1}g)}{|S\nabla g|} + \mathcal{O}(1). \tag{7.28}$$

Note that both $\boxed{M1}$ in (7.27) and $\boxed{M2}$ in (7.28) have the term

$$4\frac{b_1^2(\partial^{1,1}g)^2 + b_1 b_2(\partial^{2,1}g)^2}{|S\nabla g|^2}$$

in common with opposite sign, which allows us to remove the square in (7.15). That is, adding up both terms,

$$\boxed{M1} + \boxed{M2} = \frac{\mathcal{O}(\partial^{1,1}g + \partial^{2,1}g)}{|S\nabla g|} + \mathcal{O}(1),$$

whence we obtain

$$\left|\boxed{M1} + \boxed{M2}\right| \overset{(7.3))}{\lesssim} 1 + \frac{|\nabla^2 g|}{|\nabla g|},$$

as claimed. □

7.5.5 Proof of Lemma 7.17

We study, for $\psi \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)$, the term

$$T_{M3}(\psi) := \left(\frac{2}{|S\nabla g|^2}\left\langle \sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g\right), B\nabla_R g\right\rangle\right)(\psi).$$

In this subsection we don't use the fact that we are in the plane. Instead, the key ingredient is the L_A-harmonicity of the Green function far from the pole, and so the computations in this subsection could be done also in higher dimensions.

The term T_{M3} must be read as

$$\begin{aligned} T_{M3} &= \frac{2}{|S\nabla g|^2}\left\langle \sum_{i=1}^{2} \begin{pmatrix} \partial^i(b_i\partial^{i,1}g) \\ \partial^i(b_i\partial^{i,2}g)\end{pmatrix}, \begin{pmatrix} b_1\partial^1 g \\ b_2\partial^2 g\end{pmatrix}\right\rangle \\ &= \frac{2}{|S\nabla g|^2}\sum_{i=1}^{2}\left\{\partial^i(b_i\partial^{i,1}g)b_1\partial^1 g + \partial^i(b_i\partial^{i,2}g)b_2\partial^2 g\right\} \\ &= \frac{2}{|S\nabla g|^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\partial^i(b_i\partial^{i,j}g)b_j\partial^j g =: \sum_{i=1}^{2}\sum_{j=1}^{2} T_{M3}^{(i,j)}. \end{aligned} \tag{7.29}$$

Here, for each $i, j \in \{1, 2\}$,

$$T_{M3}^{(i,j)}(\psi) = T_{\partial^i(b_i\partial^{i,j}g)}\left(\frac{2}{|S\nabla g|^2}b_j\partial^j g\psi\right). \tag{7.30}$$

We want to study the functionals $T_{M3}^{(i,j)}$ for $i, j \in \{1, 2\}$. First note that

$$\frac{2}{|S\nabla g|^2} b_j \partial^j g \psi \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi),$$

while $b_i \partial^{i,j} g \in L^2(\widetilde{\Omega}_\varphi)$. Hence, (7.30) makes sense and it suffices to study the functionals $T_{\partial^i(b_i \partial^{i,j} g)} \in W_c^{1,\infty}(\widetilde{\Omega})'$. In fact, fixed $j \in \{1, 2\}$, we will exploit the cancellation of the functional $\sum_{i=1}^2 T_{\partial^i(b_i \partial^{i,j} g)}$. For simplicity we will write $\partial^i(b_i \partial^{i,j} g)$ instead of $T_{\partial^i(b_i \partial^{i,j} g)}$.

Compared to the strategy seen in Remark 7.15, now the matrix is not constant and hence the directional derivatives do not commute. For this reason, some error terms will appear in the procedure of extracting the gradient ∇_R outside from

$$\sum_{i=1}^{2} \partial^i \left(b_i \partial^i \nabla_R g \right) = \begin{pmatrix} \partial^1(b_1 \partial^{1,1} g) + \partial^2(b_2 \partial^{2,1} g) \\ \partial^1(b_1 \partial^{1,2} g) + \partial^2(b_2 \partial^{2,2} g) \end{pmatrix}, \tag{7.31}$$

as it is done in the constant matrix case in (7.16). The idea is the same as in the proof of Lemma 7.16, see Section 7.5.4. That is, using the 'almost'-commutative property (7.5) (relating $\partial^{1,2} g$ and $\partial^{2,1} g$) of the directional derivatives, and its functional version in (7.10), we manage to extract the gradient ∇_R outside.

Let us fix $j \in \{1, 2\}$. If $i = j$ (clearly $\partial^{i,j} = \partial^{j,i}$ in this case), then

$$\partial^i(b_i \partial^{i,j} g) = \partial^j \left(\partial^i(b_i \partial^i g) \right) - \partial^i(\partial^j b_i \partial^i g), \tag{7.32}$$

and when $i \neq j$, using (7.5) and (7.10) we get

$$\begin{aligned} \partial^i(b_i \partial^{i,j} g) &\overset{(7.5))}{=} \partial^i \left(b_i \partial^{j,i} g \right) + \partial^i \left(b_i (\partial^i R_j - \partial^j R_i) \nabla g \right) \\ &= \partial^i \left(\partial^j (b_i \partial^i g) \right) - \partial^i \left(\partial^j b_i \partial^i g \right) + \partial^i \left(b_i (\partial^i R_j - \partial^j R_i) \nabla g \right) \\ &\overset{(7.10))}{=} \partial^j \left(\partial^i (b_i \partial^i g) \right) - (\partial^j R_i - \partial^i R_j) \nabla (b_i \partial^i g) - \partial^i \left(\partial^j b_i \partial^i g \right) \\ &\quad + \partial^i \left(b_i (\partial^i R_j - \partial^j R_i) \nabla g \right). \end{aligned} \tag{7.33}$$

Plugging (7.32) and (7.33) in (7.31), we get, for $j = 1$,

$$\begin{aligned} \sum_{i=1}^{2} \partial^i(b_i \partial^{i,1} g) = \partial^1 \left(\sum_{i=1}^{2} \partial^i \left(b_i \partial^i g \right) \right) - \sum_{i=1}^{2} \partial^i(\partial^1 b_i \partial^i g) - (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla(b_2 \partial^2 g) \\ + \partial^2 \left(b_2 (\partial^2 R_1 - \partial^1 R_2) \cdot \nabla g \right), \end{aligned} \tag{7.34}$$

and for $j = 2$,

$$\sum_{i=1}^{2} \partial^i (b_i \partial^{i,2} g) = \partial^2 \left(\sum_{i=1}^{2} \partial^i \left(b_i \partial^i g \right) \right) - \sum_{i=1}^{2} \partial^i (\partial^2 b_i \partial^i g) - (\partial^2 R_1 - \partial^1 R_2) \cdot \nabla(b_1 \partial^1 g) + \partial^1 \left(b_1 (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla g \right). \tag{7.35}$$

Note that in both cases there is the term $\sum_{i=1}^{2} \partial^i \left(b_i \partial^i g \right)$, which is studied in Claim 7.20. By (7.29) and using Claim 7.20 in both (7.34) and (7.35), in the dual space we get

$$\begin{aligned} T_{M3} &= \frac{2}{|S\nabla g|^2} \sum_{j=1}^{2} b_j \partial^j g \left(\sum_{i=1}^{2} \partial^i (b_i \partial^{i,j} g) \right) \\ &= \sum_{j=1}^{2} \frac{2 b_j \partial^j g}{|S\nabla g|^2} \left\{ \partial^j \left(- \sum_{i=1}^{2} b_i \partial^i g \operatorname{div} R_i^T - \sum_{i,k=1}^{2} \partial_i \left(\frac{a_{ik} - a_{ki}}{2} \right) \partial_k g \right) \right. \\ &\qquad \left. - \sum_{i=1}^{2} \partial^i (\partial^j b_i \partial^i g) \right\} \\ &\quad + \frac{2 b_1 \partial^1 g}{|S\nabla g|^2} \left\{ -(\partial^1 R_2 - \partial^2 R_1) \cdot \nabla(b_2 \partial^2 g) + \partial^2 \left(b_2 (\partial^2 R_1 - \partial^1 R_2) \cdot \nabla g \right) \right\} \\ &\quad + \frac{2 b_2 \partial^2 g}{|S\nabla g|^2} \left\{ -(\partial^2 R_1 - \partial^1 R_2) \cdot \nabla(b_1 \partial^1 g) + \partial^1 \left(b_1 (\partial^1 R_2 - \partial^2 R_1) \cdot \nabla g \right) \right\}. \end{aligned} \tag{7.36}$$

As a consequence of this, the right-hand equality in Claim 7.20 and $|\nabla \partial^i g| = \mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla^2 g|)$, see (7.3), we conclude that the functional T_{M3} is of the form

$$T_{M3} = \sum_{j=1}^{2} \frac{2 b_j \partial^j g}{|S\nabla g|^2} \left(\mathcal{O}(|\nabla g|) + \mathcal{O}(|\nabla^2 g|) + \partial^1 \left(\mathcal{O}(|\nabla g|) \right) + \partial^2 \left(\mathcal{O}(|\nabla g|) \right) \right),$$

as claimed. □

7.5.6 Proof of Lemma 7.18

First, we need the following claim, which we will prove later.

Claim 7.27 *Let $\psi \in W_c^{1,\infty}(\widetilde{\Omega}_\varphi)$. Then, for $i, j \in \{1, 2\}$, both*

$$T_{\frac{1}{|S\nabla g|^2} \partial^i (\mathcal{O}(|\nabla g|^2))}(\psi) \text{ and } T_{\frac{b_j \partial^j g}{|S\nabla g|^2} \partial^i (\mathcal{O}(|\nabla g|))}(\psi)$$

are of the form

$$\int \frac{\mathcal{O}(|\nabla^2 g|)}{|\nabla g|} \psi + \int \mathcal{O}(1) \psi + \int \mathcal{O}(1) \partial^i \psi.$$

Granted this, by Lemma 7.17 and (7.14) we have

$$|T_{M3}(\psi_\varepsilon\varphi g)| + |T_E(\psi_\varepsilon\varphi g)| \lesssim \int \frac{|\nabla^2 g|}{|\nabla g|}\psi_\varepsilon\varphi g + \int \psi_\varepsilon\varphi g + \int |\partial^i(\psi_\varepsilon\varphi g)|.$$

The second term in the right-hand side is bounded by a constant times $\mathcal{H}^2(\Omega) < \infty$ as Ω is bounded and $g \lesssim 1$ in $\operatorname{supp}\varphi$, see (W6). Hence, it suffices to prove

$$\int \left|\partial^i(\psi_\varepsilon\varphi g)\right| \lesssim 1, \text{ for } i \in \{1, 2\}.$$

By the product derivative rule and the notation $\partial^i f = R_i \cdot \nabla f$, we can use

$$\int \left|\partial^i\,(\psi_\varepsilon\varphi g)\right| \lesssim \int g|\nabla\varphi| + \int_{\operatorname{supp}\varphi} |\nabla g| + \frac{1}{\varepsilon}\int_{\operatorname{supp}\nabla\psi_\varepsilon} g.$$

Since $|\nabla\varphi| \lesssim 1$ and $g(x) \lesssim 1$ in $\operatorname{supp}\varphi$ (see (W6)), we get $\int g|\nabla\varphi| \lesssim 1$. For the second integral on the right-hand side we sum over Whitney cubes in W_φ and apply items (5) and (1) in Lemma 7.26

$$\int_{\operatorname{supp}\varphi} |\nabla g| \le \sum_{Q\in W_\varphi} \int_Q |\nabla g| \overset{(5)}{\lesssim} \sum_{Q\in W_\varphi} \ell(Q)\cdot\widetilde{\omega}_T(50Q) \overset{(1)}{\lesssim} 1.$$

On the other hand, using $\operatorname{supp}\nabla\psi_\varepsilon \subset U_{3\varepsilon}(\partial\widetilde{\Omega})$ and Lemma 7.26(3) respectively, we have

$$\frac{1}{\varepsilon}\int_{\operatorname{supp}\nabla\psi_\varepsilon} g \le \frac{1}{\varepsilon}\int_{U_{3\varepsilon}(\partial\widetilde{\Omega})} g \lesssim \varepsilon \le 1,$$

and Lemma 7.18 follows. □

We now turn to the proof of Claim 7.27.

Proof of Claim 7.27 First we study $T_{\frac{1}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|^2))}(\psi)$. We have

$$\begin{aligned} T_{\frac{1}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|^2))}(\psi) &= T_{\partial^i(\mathcal{O}(|\nabla g|^2))}\left(\frac{1}{|S\nabla g|^2}\psi\right) \\ &\overset{(7.8)}{=} -\int \mathcal{O}(|\nabla g|^2)\frac{1}{|S\nabla g|^2}\psi \operatorname{div} R_i^T - \int \mathcal{O}(|\nabla g|^2)\partial^i\left(\frac{1}{|S\nabla g|^2}\psi\right). \end{aligned}$$

The first term in the right-hand side is of the form $\int \mathcal{O}(1)\psi$. Let us study the second term in the right-hand side. By (7.19),

$$\partial^i\left(\frac{1}{|S\nabla g|^2}\psi\right) = -\frac{2\langle\partial^i\nabla_R g, B\nabla_R g\rangle}{|S\nabla g|^4}\psi - \frac{\langle\nabla_R g, \partial^i B\nabla_R g\rangle}{|S\nabla g|^4}\psi + \frac{\partial^i\psi}{|S\nabla g|^2},$$

and hence we get

$$-\int \mathcal{O}(|\nabla g|^2)\partial^i\left(\frac{1}{|S\nabla g|^2}\psi\right) = \int \frac{\mathcal{O}(\partial^i \nabla_R g)}{|S\nabla g|}\psi + \int \mathcal{O}(1)\psi + \int \mathcal{O}(1)\partial^i\psi$$
$$\overset{(7.3)}{=} \int \frac{\mathcal{O}(|\nabla^2 g|)}{|\nabla g|}\psi + \int \mathcal{O}(1)\psi + \int \mathcal{O}(1)\partial^i\psi.$$

Next we study $T_{\frac{b_j\partial^j g}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|))}(\psi)$, similarly as we did with the previous term. Now we have

$$T_{\frac{b_j\partial^j g}{|S\nabla g|^2}\partial^i(\mathcal{O}(|\nabla g|))}(\psi) = T_{\partial^i(\mathcal{O}(|\nabla g|))}\left(\frac{b_j\partial^j g}{|S\nabla g|^2}\psi\right)$$
$$\overset{(7.8)}{=} -\int \frac{\mathcal{O}(|\nabla g|)b_j\partial^j g}{|S\nabla g|^2}\psi \operatorname{div} R_i - \int \mathcal{O}(|\nabla g|)\partial^i\left(\frac{b_j\partial^j g}{|S\nabla g|^2}\psi\right).$$

As before, the first term in the right-hand side is of the form $\int \mathcal{O}(1)\psi$ and the second term is

$$-\int \mathcal{O}(|\nabla g|)\partial^i\left(\frac{b_j\partial^j g}{|S\nabla g|^2}\psi\right) = \int \frac{\mathcal{O}(1)\psi}{|S\nabla g|}\partial^i\left(b_j\partial^j g\right)$$
$$+\int \mathcal{O}(|\nabla g|^2)\partial^i\left(\frac{1}{|S\nabla g|^2}\right)\psi + \int \mathcal{O}(1)\partial^i\psi.$$

The claim follows by using (7.3) and (7.19) in the previous line. □

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