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Linearity and classification of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes

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Abstract

The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes are subgroups of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$. A $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code is a Hadamard code which is the Gray map image of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. A recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ is known. In this paper, we generalize some known results for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes to $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. First, we show for which types the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^t are nonlinear. For these codes, we compute the kernel and its dimension, which allows us to give a partial classification of these codes. Moreover, for $3 \leq t \leq 11$, we give a complete classification by providing the exact amount of nonequivalent such codes. We also prove the existence of several families of infinite such nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes, which are not equivalent to any other constructed $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any previously constructed $\mathbb{Z}_{2^s}$-linear Hadamard code with $s \geq 2$, with the same length 2^t.

Keywords Hadamard code · Gray map · $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code · $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code · Kernel · Rank · Classification

Mathematics Subject Classification 94B25 · 94B60

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1 Introduction

Let $\mathbb{Z}_{2^s}$ be the ring of integers modulo 2^s with $s \geq 1$. The set of n-tuples over $\mathbb{Z}_{2^s}$ is denoted by $\mathbb{Z}_{2^s}^n$. In this paper, the elements of $\mathbb{Z}_{2^s}^n$ will also be called vectors. A code over $\mathbb{Z}_2$ of length n is a nonempty subset of $\mathbb{Z}_2^n$, and it is linear if it is a subspace of $\mathbb{Z}_2^n$. Similarly, a nonempty subset of $\mathbb{Z}_{2^s}^n$ is a $\mathbb{Z}_{2^s}$-additive code if it is a subgroup of the additive group of $\mathbb{Z}_{2^s}^n$. A $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a subgroup of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$. Note that a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a linear code over $\mathbb{Z}_2$ when $\alpha_2 = \alpha_3 = 0$, a $\mathbb{Z}_4$-additive or $\mathbb{Z}_8$-additive code when $\alpha_1 = \alpha_3 = 0$ or $\alpha_1 = \alpha_2 = 0$, respectively, and a $\mathbb{Z}_2\mathbb{Z}_4$-additive code when $\alpha_3 = 0$. The order of a vector $u \in \mathbb{Z}_{2^s}^n$, denoted by $o(u)$, is the smallest positive integer m such that $mu = (0, \dots, 0)$. Also, the order of a vector $\mathbf{u} \in \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$, denoted by $o(\mathbf{u})$, is the smallest positive integer m such that $m\mathbf{u} = (0, \dots, 0 \mid 0, \dots, 0 \mid 0, \dots, 0)$.

Two binary codes C_1 and C_2 are said to be equivalent if there is a vector $u \in \mathbb{Z}_2^n$ and a permutation of coordinates π such that $C_2 = \{u + \pi(\mathbf{c}) : u \in C_1\}$. The Hamming weight of a vector $u \in \mathbb{Z}_2^n$, denoted by $\mathrm{wt}_H(u)$, is the number of nonzero coordinates of u. The Hamming distance of two vectors $u, v \in \mathbb{Z}_2^n$, denoted by $d_H(u, v)$, is the number of coordinates in which they differ. Note that $d_H(u, v) = \mathrm{wt}_H(u - v)$. The minimum distance of a code C over $\mathbb{Z}_2$ is $d(C) = \min\{d_H(u, v) : u, v \in C, u \neq v\}$.

In [22], a Gray map from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is defined as $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$ and $\phi(3) = (1, 0)$. There exist different generalizations of this Gray map, which go from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$ [13, 15, 16, 23, 27]. In this paper, we focus on Carlet's Gray map [15], from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$, which is a particular case of the one given in [27, 35] satisfying $\sum \lambda_i \phi(2^i) = \phi(\sum \lambda_i 2^i)$. Specifically,

$$\phi_s(u) = (u_{s-1}, u_{s-1}, \dots, u_{s-1}) + (u_0, \dots, u_{s-2})Y_{s-1}, \tag{1}$$

where $u \in \mathbb{Z}_{2^s}$; $[u_0, u_1, \dots, u_{s-1}]_2$ is the binary expansion of u, that is, $u = \sum_{i=0}^{s-1} u_i 2^i$ with $u_i \in \{0, 1\}$; and Y_{s-1} is a matrix of size $(s-1) \times 2^{s-1}$ whose columns are all the vectors in $\mathbb{Z}_2^{s-1}$. Without loss of generality, we assume that the columns of Y_{s-1} are ordered in ascending order, by considering the elements of $\mathbb{Z}_2^{s-1}$ as the binary expansions of the elements of $\mathbb{Z}_{2^{s-1}}$. Note that ϕ_1 is the identity map. We define $\Phi_s : \mathbb{Z}_{2^s}^n \to \mathbb{Z}_2^{n2^{s-1}}$ as the component-wise extended map of ϕ_s. We can also define a Gray map Φ from $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ to $\mathbb{Z}_2^n$, where $n = \alpha_1 + 2\alpha_2 + 4\alpha_3$, as follows:

$$\Phi(u_1 \mid u_2 \mid u_3) = (u_1, \Phi_2(u_2), \Phi_3(u_3)),$$

for any $u_i \in \mathbb{Z}_{2^i}^{\alpha_i}$, where $1 \leq i \leq 3$.

Let $\mathcal{C} \subseteq \mathbb{Z}_{2^s}^n$ be a $\mathbb{Z}_{2^s}$-additive code of length n. We say that the Gray map image of $\mathcal{C}$, say $C = \Phi_s(\mathcal{C})$, is a $\mathbb{Z}_{2^s}$-linear code of length $n2^{s-1}$. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_{2^s}^n$, it is isomorphic to $\mathbb{Z}_{2^s}^{t_1} \times \mathbb{Z}_{2^{s-1}}^{t_2} \times \cdots \times \mathbb{Z}_2^{t_s}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi_s(\mathcal{C})$, is of type $(n; t_1, \dots, t_s)$. Similarly, if $\mathcal{C} \subseteq \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, we say that its Gray map image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length $\alpha_1 + 2\alpha_2 + 4\alpha_3$. Since $\mathcal{C}$ can be seen as a subgroup of $\mathbb{Z}_8^{\alpha_1+\alpha_2+\alpha_3}$, it is isomorphic to $\mathbb{Z}_8^{t_1} \times \mathbb{Z}_4^{t_2} \times \mathbb{Z}_2^{t_3}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$. Note that a $\mathbb{Z}_2\mathbb{Z}_4$-linear code $\mathcal{C}$ [11, 12] can be seen as a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of type $(\alpha_1, \alpha_2, 0; 0, t_2, t_3)$. In this case, we also say that the type of $\mathcal{C}$ is directly $(\alpha_1, \alpha_2; t_2, t_3)$. Unlike linear codes over finite fields, linear codes over rings do not have a basis, but there exists a generator matrix for these codes having minimum number of rows. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, then $|\mathcal{C}| = 8^{t_1}4^{t_2}2^{t_3}$ and there exists a generator matrix with $t_1 + t_2 + t_3$ rows.

Two structural properties of codes over $\mathbb{Z}_2$ are the rank and dimension of the kernel. The rank of a code C over $\mathbb{Z}_2$ is simply the dimension of the linear span, $\langle C\rangle$, of C. The kernel of a code C over $\mathbb{Z}_2$ is defined as $\mathrm{K}(C) = \{x \in \mathbb{Z}_2^n : x + C = C\}$ [2]. If the all-zero vector belongs to C, then $\mathrm{K}(C)$ is a linear subcode of C. Note also that if C is linear, then $K(C) = C = \langle C\rangle$. We denote the rank of C as $\mathrm{rank}(C)$ and the dimension of the kernel as $\ker(C)$. These parameters can be used to distinguish between nonequivalent codes, since equivalent ones have the same rank and dimension of the kernel.

A binary code of length n, $2n$ codewords and minimum distance $n/2$ is called a Hadamard code. Hadamard codes can be constructed from Hadamard matrices [1, 29]. Note that linear Hadamard codes are in fact first order Reed-Muller codes, or equivalently, the dual of extended Hamming codes [29]. The $\mathbb{Z}_{2^s}$-additive codes such that after the Gray map Φ_s give Hadamard codes are called $\mathbb{Z}_{2^s}$-additive Hadamard codes and the corresponding images are called $\mathbb{Z}_{2^s}$-linear Hadamard codes. Similarly, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes such that after the Gray map Φ give Hadamard codes are called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes and the corresponding images are called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes.

It is known that $\mathbb{Z}_4$-linear Hadamard codes (that is, $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 = 0$) and $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ can be classified by using either the rank or the dimension of the kernel [26, 31]. Moreover, in [28], it is shown that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$. Later, in [3, 5, 17, 19], a recursive construction for $\mathbb{Z}_{p^s}$-linear Hadamard codes, with p prime, is described, the linearity is established, and a partial classification by using the dimension of the kernel is obtained, giving the exact amount of nonequivalent such codes for some parameters. In [18], a complete classification of $\mathbb{Z}_8$-linear Hadamard codes by using the rank and dimension of the kernel is provided, giving the exact amount of nonequivalent such codes. For any $t \geq 2$, the full classification of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear Hadamard codes of length p^t, with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $p \geq 3$ prime, is given in [4, 6], by using just the dimension of the kernel.

The paper contributes to the study of codes over rings $\mathbb{Z}_{p^s}$, which were first studied by Blake [10] and Shankar [32] in 1975 and 1979, respectively. These codes have become more significant after the publication of [22]. It is also important to note that Hadamard codes are two weight codes, which have been widely studied in [33, 34]. On the other hand, the classification of nonlinear Hadamard codes is still an open problem. By giving an additive structure, as $\mathbb{Z}_{p^s}$-linear, $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear or $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes, to some of them, and showing whether they are equivalent or not among them, we are providing a partial classification for these codes.

From a more practical point of view, since Hadamard codes are optimal and have a high correction capability, they appear in different aspects related to the transmission of information, such as in digital communication with satellites [24], in CDMA phones to modulate the transmission of information and minimize interference with other transmissions [36] and, in general, in different OCDMA multiple access systems to allow access to multiple users asynchronously and simultaneously [25]. Other applications are found in cryptography [30] or in information hiding (steganography and watermarking) [37]. See [24] for more applications in other fields.

This paper is focused on $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, generalizing some results given for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ in [31] related to the linearity, kernel, its dimension, and the classification of such codes. These codes are also compared with the $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$ considered in [17]. This paper is organized as follows. In Sect. 2, we recall some properties of the generalized Gray map considered in this paper, the construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, and some known results, given in [8]. In Sects. 3 and 4, we establish for which types these codes are linear, and we give the kernel and its dimension whenever they are nonlinear. In Sect. 5, we prove the existence of several families of infinite such nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes, which are not equivalent to any other constructed $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any previously constructed $\mathbb{Z}_{2^s}$-linear Hadamard code with $s \geq 2$, with the same length 2^t. We also give a complete classification of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^t for $3 \leq t \leq 11$, by providing the exact amount of nonequivalent such codes. Finally, in Sect. 6, we give some conclusions and further research on this topic.

2 Preliminary results

In this section, we first recall some properties of the generalized Gray map ϕ_s. Then, we also recall the recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, and some known results given in [8].

Let $u', v' \in \mathbb{Z}_{2^s}$ and $[u'_0, u'_1, \dots, u'_{s-1}]_2$, $[v'_0, v'_1, \dots, v'_{s-1}]_2$ be the binary expansions of u' and v', respectively, i.e. $u' = \sum_{i=0}^{s-1} u'_i 2^i$ and $v' = \sum_{i=0}^{s-1} v'_i 2^i$. We define the operation "$\odot$" between elements u' and v' in $\mathbb{Z}_{2^s}$ as $u' \odot v' = \sum_{i=0}^{s-1} \xi_i 2^i$, where

$$\xi_i = \begin{cases} 1 \text{ if } & u'_i = v'_i = 1, \\ 0 \text{ otherwise.} \end{cases}$$

Note that the binary expansion of $u' \odot v'$ is $[\xi_0, \xi_1, \dots, \xi_{s-1}]_2$, where $\xi_i \in \{0, 1\}$. We denote in the same way, "$\odot$", the component-wise operation. For $\mathbf{u} = (u_1 \mid u_2 \mid u_3)$, $\mathbf{v} = (v_1 \mid v_2 \mid v_3) \in \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$, we denote $\mathbf{u} \odot \mathbf{v} = (u_1 \odot v_1 \mid u_2 \odot v_2 \mid u_3 \odot v_3)$. Note that $2(\mathbf{u} \odot \mathbf{v}) = (\mathbf{0} \mid 2(u_2 \odot v_2) \mid 2(u_3 \odot v_3))$.

From [5], we have the following results:

Corollary 1 *[5] Let $\lambda, \mu \in \mathbb{Z}_2$. Then, $\phi_s(\lambda\mu 2^{s-1}) = \lambda\phi_s(\mu 2^{s-1}) = \lambda\mu\phi_s(2^{s-1})$.*

Corollary 2 *[5] Let $u, v \in \mathbb{Z}_{2^s}$. Then, $\phi_s(u) + \phi_s(v) = \phi_s(u + v - 2(u \odot v))$.*

Corollary 3 *[5] Let $u, v \in \mathbb{Z}_{2^s}$. Then, $\phi_s(2^{s-1}u + v) = \phi_s(2^{s-1}u) + \phi_s(v)$.*

Corollary 4 *Let $u, v \in \mathbb{Z}_4$ and $[u_0, u_1]_2$, $[v_0, v_1]_2$ be the binary expansions of u and v, respectively. Then, $\phi_2(2(u \odot v)) = (\xi_0, \xi_0)$, where $\xi_0 = 1$ when $u_0 = v_0 = 1$ and $\xi_0 = 0$ otherwise.*

Corollary 5 *Let $u, v \in \mathbb{Z}_8$ and $[u_0, u_1, u_2]_2$, $[v_0, v_1, v_2]_2$ be the binary expansions of u and v, respectively. Then, $\phi_3(2(u \odot v)) = \phi_3(2\xi_0 + 4\xi_1) = \phi_3(2\xi_0) + (\xi_1, \xi_1, \xi_1, \xi_1)$, where $\xi_i = 1$ when $u_i = v_i = 1$ and 0 otherwise.*

Let $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{7}$ be the vectors having the elements $0, 1, 2, \dots, 7$ repeated in each coordinate, respectively. If A is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, that is, a subgroup of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ for some integers $\alpha_1, \alpha_2, \alpha_3 \geq 0$, then we denote by A_1 the submatrix of A with the first α_1 columns over $\mathbb{Z}_2$, A_2 the submatrix with the next α_2 columns over $\mathbb{Z}_4$, and A_3 the submatrix with the last α_3 columns over $\mathbb{Z}_8$. We have that $A = (A_1 \mid A_2 \mid A_3)$, where the number of columns of A_i is α_i for $i \in \{1, 2, 3\}$.

Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Now, we construct recursively matrices A^{t_1,t_2,t_3} having t_1 rows of order 8, t_2 rows of order 4, and t_3 rows of order 2 as follows. First, we consider the following matrix:

$$A^{1,0,1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right). \tag{2}$$

Then, we apply the following constructions. If we have a matrix $A^{\ell-1,0,1} = (A_1 \mid A_2 \mid A_3)$, with $\ell \geq 2$, we may construct the matrix

$$A^{\ell,0,1} = \left(\begin{array}{cc|ccccc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & M_2 & A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right), \tag{3}$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{\ell-1}\}$ and $M_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0,2,4,6\}^{\ell-1}\}$. We perform construction (3) until $\ell = t_1$. If we have a matrix $A^{t_1,\ell-1,1} = (A_1 \mid A_2 \mid A_3)$, with $t_1 \geq 1$ and $\ell \geq 1$, we may construct the matrix

$$A^{t_1,\ell,1} = \left(\begin{array}{cc|ccccc|cccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array}\right), \tag{4}$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1+\ell-1}\}$. We repeat construction (4) until $\ell = t_2$. Finally, if we have a matrix $A^{t_1,t_2,\ell-1} = (A_1 \mid A_2 \mid A_3)$, with $t_1 \geq 1$, $t_2 \geq 0$, and $\ell \geq 2$, we may construct the matrix

$$A^{t_1,t_2,\ell} = \left(\begin{array}{cc|cc|cc} A_1 & A_1 & A_2 & A_2 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{4} \end{array}\right). \tag{5}$$

We repeat construction (5) until $\ell = t_3$. Thus, in this way, we obtain A^{t_1,t_2,t_3}.

Summarizing, in order to achieve A^{t_1,t_2,t_3} from $A^{1,0,1}$, first we add $t_1 - 1$ rows of order 8 by applying construction (3) $t_1 - 1$ times, starting from $A^{1,0,1}$ up to obtain $A^{t_1,0,1}$; then we add t_2 rows of order 4 by applying construction (4) t_2 times, up to generate $A^{t_1,t_2,1}$; and, finally, we add $t_3 - 1$ rows of order 2 by applying construction (5) $t_3 - 1$ times to achieve A^{t_1,t_2,t_3}. Note that in the first row there is always the row $(\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})$.

Example 1 By using the constructions described in (3), (4), and (5), we obtain the following matrices $A^{2,0,1}$, $A^{1,1,1}$ and $A^{1,1,2}$, respectively, starting from $A^{1,0,1}$ given in (2):

$$A^{2,0,1} = \left(\begin{array}{c|c|c} 11\ 11 & 22\ 2222 & 4444\ 44444444 \\ 01\ 01 & 02\ 1111 & 0246\ 11111111 \\ 00\ 11 & 11\ 0123 & 1111\ 01234567 \end{array}\right), \tag{6}$$

$$A^{1,1,1} = \left(\begin{array}{c|c|c} 11\ 11 & 22\ 2222 & 4444 \\ 01\ 01 & 02\ 1111 & 1111 \\ 00\ 11 & 11\ 0123 & 0246 \end{array}\right), \tag{7}$$

$$A^{1,1,2} = \left(\begin{array}{c|c|c} 1111\ 1111 & 222222\ 222222 & 4444\ 4444 \\ 0101\ 0101 & 021111\ 021111 & 1111\ 1111 \\ 0011\ 0011 & 110123\ 110123 & 0246\ 0246 \\ 0000\ 1111 & 000000\ 222222 & 0000\ 4444 \end{array}\right).$$

The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by A^{t_1,t_2,t_3} is denoted by $\mathcal{H}^{t_1,t_2,t_3}$, and the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code $\Phi(\mathcal{H}^{t_1,t_2,t_3})$ by H^{t_1,t_2,t_3}.

Proposition 1 *[8] Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ generated by A^{t_1,t_2,t_3}. Then,*

$$\alpha_1 = 2^{t_1+t_2+t_3-1},$$
$$\alpha_1 + 2\alpha_2 = 4^{t_1+t_2}2^{t_3-1},$$
$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = 8^{t_1}4^{t_2}2^{t_3-1}.$$

Theorem 1 *[8] Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Then, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{t_1,t_2,t_3}$, generated by A^{t_1,t_2,t_3}, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of length 2^t, with $t + 1 = 3t_1 + 2t_2 + t_3$.*

Example 2 The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{1,0,1}$ generated by $A^{1,0,1}$, given in (2), is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type (2, 1, 1; 1, 0, 1). We can write $\mathcal{H}^{1,0,1} = \bigcup_{\alpha\in\{0,1\}}(\mathcal{A} + \alpha(1, 1 \mid 2 \mid 4))$, where $\mathcal{A} = \{\lambda(0, 1 \mid 1 \mid 1) : \lambda \in \mathbb{Z}_8\}$. Thus, $H^{1,0,1} = \Phi(\mathcal{H}^{1,0,1}) = \bigcup_{\alpha\in\{0,1\}}(\Phi(\mathcal{A}) + \alpha\mathbf{1})$, where $\Phi(\mathcal{A})$ consists of all the rows of the Hadamard matrix

$$H(2,4) = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1\\ 0&1&1&0&0&1&1&0\\ 0&0&0&0&1&1&1&1\\ 0&1&0&1&1&0&1&0\\ 0&0&1&1&1&1&0&0\\ 0&1&1&0&1&0&0&1 \end{pmatrix}.$$

Note that $\Phi(\mathcal{A})$ is linear and the minimum distance of $\Phi(\mathcal{A})$ is 4, so $H^{1,0,1}$ is a binary linear Hadamard code of length 8.

3 Linearity of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes

The linearity of $\mathbb{Z}_4$-linear Hadamard codes was studied in [26, 31]. In general, for $\mathbb{Z}_{p^s}$-linear Hadamard codes with $s \geq 2$ and p prime, the results on the linearity are given in [17] when $p = 2$ and in [5] when $p \geq 3$. In [31], it is shown that the $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of type $(\alpha_1, \alpha_2; 1, t_3)$, or equivalently, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of type $(\alpha_1, \alpha_2, 0; 0, 1, t_3)$, with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, are the only ones which are linear when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 = 0$. The next results show that the Hadamard codes $H^{1,0,t_3} = \Phi(\mathcal{H}^{1,0,t_3})$, where $t_3 \geq 1$, are the only $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$ which are linear.

Let $\mathcal{H}$ be a code over a ring. We denote by $(\mathcal{H}, \mathcal{H})$ the code where the codewords are all vectors (u, u) for $u \in \mathcal{H}$. Similarly, if $(\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, $(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3)$ is the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code having as codewords all the elements of the form $(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3)$ with $(u_1 \mid u_2 \mid u_3) \in (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$.

Proposition 2 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes $H^{1,0,t_3}$, with $t_3 \geq 1$, are linear.*

Proof We prove this theorem by induction on $t_3 \geq 1$. By Example 2, $H^{1,0,1}$ is linear. Assume that $H^{1,0,t_3} = \Phi(\mathcal{H}^{1,0,t_3})$, with $t_3 \geq 1$, is linear. Let $\mathcal{H} = \mathcal{H}^{1,0,t_3+1}$ and $H = \Phi(\mathcal{H})$. Now, we have to show that H is also linear. Let $\mathcal{H}^{1,0,t_3} = (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$. By construction (5),

$$\mathcal{H} = \bigcup_{\lambda\in\{0,1\}} ((\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1} \mid \mathbf{0}, \mathbf{2} \mid \mathbf{0}, \mathbf{4})).$$

By Corollaries 1 and 3,

$$\begin{aligned} H &= \bigcup_{\lambda\in\{0,1\}} (\Phi(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})) \\ &= A_0 \cup A_1, \end{aligned}$$

where $A_\lambda = \Phi(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$, $\lambda \in \{0, 1\}$. Any element in A_λ is of the form

$$\Phi(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}), \tag{8}$$

for $(u_1 \mid u_2 \mid u_3) \in (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$. Note that (8) is equal to $(u_1, u_1+\lambda\cdot\mathbf{1}, \Phi_2(u_2), \Phi_2(u_2)+\lambda\cdot\mathbf{1}, \Phi_3(u_3), \Phi_3(u_3)+\lambda\cdot\mathbf{1}) = (u_1, u_1+\lambda\cdot\mathbf{1}, u_2', u_2'+\lambda\cdot\mathbf{1}, u_3', u_3'+\lambda\cdot\mathbf{1})$, where $u_2' = \Phi_2(u_2)$ and $u_3' = \Phi_3(u_3)$. Thus,

$$\begin{aligned} A_0 &= \{(u_1, u_1, u_2', u_2', u_3', u_3') : (u_1, u_2', u_3') \in H^{1,0,t_3}\}, \\ A_1 &= \{(u_1, u_1 + \mathbf{1}, u_2', u_2' + \mathbf{1}, u_3', u_3' + \mathbf{1})) : (u_1, u_2', u_3') \in H^{1,0,t_3}\}. \end{aligned}$$

Since $H^{1,0,t_3}$ is linear, it is clear that if we take any two codewords from H, then their addition belongs to one of the blocks A_0 and A_1. Therefore, H is linear. □

Theorem 2 *The codes $H^{1,0,t_3}$ of type $(\alpha_1, \alpha_2, \alpha_3; 1, 0, t_3)$, with $t_3 \geq 1$, are the only $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, which are linear.*

Proof By Proposition 2, we have that the codes $H^{1,0,t_3}$, with $t_3 \geq 1$, are linear.

First, we show that $H^{2,0,1} = \Phi(\mathcal{H}^{2,0,1})$ is nonlinear. By Example 1, the code $\mathcal{H}^{2,0,1}$ is generated by the matrix $A^{2,0,1}$ given in (6). Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3)$ and $\mathbf{v} = (v_1 \mid v_2 \mid v_3)$ be the second and the third row vectors of $A^{2,0,1}$. Now, by Corollary 2,

$$\begin{aligned} &\Phi(\mathbf{u}) + \Phi(\mathbf{v}) \\ &= (u_1 + v_1, \Phi_2(u_2) + \Phi_2(v_2), \Phi_3(u_3) + \Phi_3(v_3)) \\ &= (u_1 + v_1, \Phi_2(u_2 + v_2 - 2(u_2 \odot v_2)), \Phi_3(u_3 + v_3 - 2(u_3 \odot v_3))) \\ &= \Phi(u_1 + v_1 \mid u_2 + v_2 - 2(u_2 \odot v_2) \mid u_3 + v_3 - 2(u_3 \odot v_3)) \\ &= \Phi((u_1 + v_1 \mid u_2 + v_2 \mid u_3 + v_3) - (0, 0, 0, 0 \mid 2(u_2 \odot v_2) \mid 2(u_3 \odot v_3))). \end{aligned}$$

Therefore, since $(u_1 + v_1 \mid u_2 + v_2 \mid u_3 + v_3) = \mathbf{u} + \mathbf{v} \in \mathcal{H}^{2,0,1}$, we just need to show that

$$\mathbf{w} = (0, 0, 0, 0 \mid 2(u_2 \odot v_2) \mid 2(u_3 \odot v_3)) \notin \mathcal{H}^{2,0,1}.$$

By Corollary 4, $\Phi_2(2(u_2 \odot v_2)) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$, where the vectors $\mathbf{0}$ and $\mathbf{1}$ are of length 2, so $\mathrm{wt}_H(\Phi_2(2(u_2 \odot v_2))) = 4$. By Corollary 5, $\Phi_3(2(u_3 \odot v_3)) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \phi_3(2), \mathbf{0}, \phi_3(2), \mathbf{0}, \phi_3(2), \mathbf{0}, \phi_3(2))$, where $\mathbf{0} = (0, 0, 0, 0)$, so $\mathrm{wt}_H(\Phi_3(2(u_3 \odot v_3))) = 8$. Therefore, $\mathrm{wt}_H(\Phi(\mathbf{w})) = \mathrm{wt}_H(\Phi_2(2(u_2 \odot v_2))) + \mathrm{wt}_H(\Phi_3(2(u_3 \odot v_3))) = 4 + 8 = 12 < 32$, and hence $\Phi(\mathbf{w}) \notin H^{2,0,1}$ since the minimum distance of $H^{2,0,1}$ is 32.

Second, we prove that if $H^{\ell,0,1}$, where $\ell \geq 2$, is nonlinear, then $H^{\ell+1,0,1}$ is also nonlinear. Assume that $H^{\ell+1,0,1}$ is linear. Then, by construction (3), for any $\mathbf{u} = (u_1 \mid u_2 \mid u_3)$,

$\mathbf{v} = (v_1 \mid v_2 \mid v_3) \in \mathcal{H}^{\ell,0,1}$, we have that

$$\bar{\mathbf{u}} = (u_1, u_1 \mid x_1, u_2, \overset{4}{\ldots}, u_2 \mid x_2, u_3, \overset{8}{\ldots}, u_3) \in \mathcal{H}^{\ell+1,0,1},$$
$$\bar{\mathbf{v}} = (v_1, v_1 \mid x_1', v_2, \overset{4}{\ldots}, v_2 \mid x_2', v_3, \overset{8}{\ldots}, v_3) \in \mathcal{H}^{\ell+1,0,1},$$

where $x_{i-1}, x_{i-1}' \in (2\mathbb{Z}_{2^i})^\ell$ for $i \in \{2, 3\}$. Since $H^{\ell+1,0,1}$ is linear, $\Phi(\bar{\mathbf{u}}) + \Phi(\bar{\mathbf{v}}) \in H^{\ell+1,0,1}$. Again, by construction (3), we have that $\Phi(\bar{\mathbf{u}}) + \Phi(\bar{\mathbf{v}}) = \Phi((a_1, a_1 \mid y_1, a_2, \overset{4}{\ldots}, a_2 \mid y_2, a_3, \overset{8}{\ldots}, a_3) + \lambda(\mathbf{0}, \mathbf{1} \mid \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3} \mid \mathbf{1}, \mathbf{0}, \mathbf{1}, \ldots, \mathbf{7})) \in H^{\ell+1,0,1}$, for some $\mathbf{a} = (a_1 \mid a_2 \mid a_3) \in \mathcal{H}^{\ell,0,1}$, $y_{i-1} \in (2\mathbb{Z}_{2^i})^\ell$ for $i \in \{2, 3\}$, and $\lambda \in \mathbb{Z}_8$. Considering the coordinates in positions 1, 4, and 9 of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$, we have that $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) = \Phi(\mathbf{a}) \in H^{\ell,0,1}$, and then $H^{\ell,0,1}$ is linear, which is a contradiction.

Third, we show that that $H^{1,1,1} = \Phi(\mathcal{H}^{1,1,1})$ is nonlinear. We consider the matrix $A^{1,1,1}$ given in (7) and, by using the same argument as before, we can see that $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) \notin H^{1,1,1}$, where $\mathbf{u}$ and $\mathbf{v}$ are the second and the third row vectors of $A^{1,1,1}$.

Finally, if $H^{t_1,\ell,1}$, with $\ell \geq 0$, (respectively, $H^{t_1,t_2,\ell}$, with $\ell \geq 1$) is nonlinear, then as above we can show that $H^{t_1,\ell+1,1}$ (respectively, $H^{t_1,t_2,\ell+1}$) is also nonlinear, and hence the result follows. □

4 Kernel of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes

The computation of the kernel and its dimension for $\mathbb{Z}_4$-linear Hadamard codes is given in [26, 31]. In general, for $\mathbb{Z}_{p^s}$-linear Hadamard codes with $s \geq 2$ and p prime, the results on the kernel are given in [17] when $p = 2$ and in [5] when $p \geq 3$. Regarding codes over mixed alphabets, it is proved that the kernel of a nonlinear $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear Hadamard code coincides with the Gray map image of the elements of order at most p of the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0, \alpha_2 \neq 0$ in [31] and [6], for $p = 2$ and $p \geq 3$ prime, respectively. Thus, the dimension of the kernel for these codes is equal to $t_1 + t_2$. In this section, we generalize some of these results and show that the dimension of the kernel for nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $\alpha_1 \neq 0, \alpha_2 \neq 0$, and $\alpha_3 \neq 0$ is equal to $t_1 + t_2 + t_3$.

When we include all the elements of $\mathbb{Z}_{2^i}$, where $1 \leq i \leq 3$, as coordinates of a vector, we place them in increasing order. For a set $S \subseteq \mathbb{Z}_{2^i}$ and $\lambda \in \mathbb{Z}_{2^i}$, where $i \in \{1, 2, 3\}$, we define $\lambda S = \{\lambda j : j \in S\}$ and $S + \lambda = \{j + \lambda : j \in S\}$. As before, when including all the elements in those sets as coordinates of a vector, we place them in increasing order. For example, $2\mathbb{Z}_8 = \{0, 4, 6, 8\}$, $(\mathbb{Z}_4, \mathbb{Z}_4) = (0, 1, 2, 3, 0, 1, 2, 3) \in \mathbb{Z}_4^8$ and $(\mathbb{Z}_2 \mid \mathbb{Z}_4 \mid 2\mathbb{Z}_8, 4\mathbb{Z}_8) = (0, 1 \mid 0, 1, 2, 3 \mid 0, 2, 4, 6, 0, 4) \in \mathbb{Z}_2^2 \times \mathbb{Z}_4^4 \times \mathbb{Z}_8^6$.

Theorem 3 *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H} = \mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ such that $\Phi(\mathcal{H})$ is nonlinear. Let $\mathcal{H}_2$ be the subcode of $\mathcal{H}$ which contains all the codewords of order at most two. Then, $K(\Phi(\mathcal{H})) = \Phi(\mathcal{H}_2)$.*

Proof By Corollary 3, for all $\mathbf{b} = (b_1 \mid b_2 \mid b_3) \in \mathcal{H}_2$ and $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{H}$, we have that $\Phi(\mathbf{b}) + \Phi(\mathbf{u}) = (b_1 + u_1, \Phi_2(b_2) + \Phi_2(u_2), \Phi_3(b_3) + \Phi_3(u_3)) = (b_1 + u_1, \Phi_2(b_2 + u_2), \Phi_3(b_3 + u_3)) = \Phi(b_1 + u_1 \mid b_2 + u_2 \mid b_3 + u_3) \in \Phi(\mathcal{H})$, so $\Phi(\mathcal{H}_2) \subseteq K(\Phi(\mathcal{H}))$.

Let $\Phi(\bar{\mathbf{u}}) \in K(\Phi(\mathcal{H}))$, where $\bar{\mathbf{u}} \neq \mathbf{0}$. We prove that $o(\bar{\mathbf{u}}) = 2$, and thus $K(\Phi(\mathcal{H})) \subseteq \Phi(\mathcal{H}_2)$. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_{t_1}\}$ (respectively, $\{\mathbf{c}_1, \ldots, \mathbf{c}_{t_2}\}$) be the collection of rows of order 8 (respectively, 4) of the generator matrix A^{t_1,t_2,t_3} of the code $\mathcal{H}$. Then, we can write $\bar{\mathbf{u}} = \mathbf{u} + \mathbf{u}_0$, where $\mathbf{u} = \sum_{i=1}^{t_1} \lambda_i \mathbf{b}_i + \sum_{j=1}^{t_2} \mu_j \mathbf{c}_j$, $\lambda_i \in \{0,1,2,3\} \subseteq \mathbb{Z}_8$, $\mu_j \in \{0,1\} \subseteq \mathbb{Z}_8$, and $o(\mathbf{u}_0) \leq 2$. By Corollary 3, $\Phi(\bar{\mathbf{u}}) = \Phi(\mathbf{u}) + \Phi(\mathbf{u}_0)$. Since $\Phi(\bar{\mathbf{u}}) \in K(\Phi(\mathcal{H}))$ and $\Phi(\mathcal{H}_2) \subseteq K(\Phi(\mathcal{H}))$, we have that $\Phi(\mathbf{u}) \in K(\Phi(\mathcal{H}))$. Assume that $o(\bar{\mathbf{u}}) > 2$, so $o(\mathbf{u}) > 2$. In order to obtain a contradiction, we just need to find an element $\tilde{\mathbf{u}} \in \mathcal{H}$ such that $\Phi(\mathbf{u}) + \Phi(\tilde{\mathbf{u}}) \notin \Phi(\mathcal{H})$.

Let $\mathbf{w}_2 = (w_1 \mid w_2 \mid w_3)$ be the second row of A^{t_1,t_2,t_3}. Note that the number of ones in w_3 is $m = 8^{t_1-1}4^{t_2}2^{t_3-1}$. Then, by Proposition 1, we have that $m = n/8$, where $n = \alpha_1 + 2\alpha_2 + 4\alpha_3$ is the length of $\Phi(\mathcal{H})$. Let I be the set of coordinate positions corresponding to these ones in w_3. Now, the proof is based on considering three different cases: when $o(\mathbf{u}) = 8$ and $\mathbf{u} \notin \{\mathbf{w}_2, 3\mathbf{w}_2\}$, when $o(\mathbf{u}) = 8$ and $\mathbf{u} \in \{\mathbf{w}_2, 3\mathbf{w}_2\}$, and when $o(\mathbf{u}) = 4$.

Assume that $o(\mathbf{u}) = 8$ and $\mathbf{u} \notin \{\mathbf{w}_2, 3\mathbf{w}_2\}$. Let $\mathbf{v} = 2\mathbf{w}_2 = (v_1 \mid v_2 \mid v_3)$. Note that v_3 contains the element 2 in the coordinates corresponding to I, and 0 or 4 in the other coordinates. By construction, the definition of $\mathbf{u}$, and the hypothesis of the considered case, in the coordinates corresponding to I, either u_3 is a permutation of the vector $(\mathbb{Z}_8, \overset{m/8}{\ldots}, \mathbb{Z}_8)$ (when $\lambda_i \in \{1,3\}$ for some $i > 1$) or it is a permutation of $(\mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{m/4}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ (when $\lambda_i \in \{0,2\}$ for all $i > 1$, and $\lambda_1 \in \{1,3\}$, with either $\lambda_i \neq 0$ for some $i > 1$ or $\mu_j \neq 0$ for some $j > 0$). Now, by Corollary 2,

$$\begin{aligned}
&\Phi(\mathbf{u}) + \Phi(\mathbf{v}) \\
&= (u_1 + v_1, \Phi_2(u_2) + \Phi_2(v_2), \Phi_3(u_3) + \Phi_3(v_3)) \\
&= (u_1 + v_1, \Phi_2(u_2 + v_2), \Phi_3(u_3 + v_3 - 2(u_3 \odot v_3))) \\
&= \Phi(u_1 + v_1 \mid u_2 + v_2 \mid u_3 + v_3 - 2(u_3 \odot v_3)) \\
&= \Phi((u_1 + v_1 \mid u_2 + v_2 \mid u_3 + v_3) - (\mathbf{0} \mid \mathbf{0} \mid 2(u_3 \odot v_3))) \\
&= \Phi(\mathbf{u} + \mathbf{v} - \mathbf{x}),
\end{aligned}$$

where $\mathbf{x} = (\mathbf{0} \mid \mathbf{0} \mid 2(u_3 \odot v_3))$. By Corollary 5, $\mathrm{wt}_H(\Phi(\mathbf{x})) = \mathrm{wt}_H(\Phi_3(2(u_3 \odot v_3))) = \frac{m}{8} \cdot 16 = 2m$ if u_3 is a permutation of $(\mathbb{Z}_8, \overset{m/8}{\ldots}, \mathbb{Z}_8)$, and $\mathrm{wt}_H(\Phi(\mathbf{x})) = \frac{m}{4} \cdot 8 = 2m$ otherwise. Since $\mathrm{wt}_H(\Phi(\mathbf{x})) = 2m = n/4$ and the minimum weight of $\Phi(\mathcal{H})$ is $n/2$, we have that $\Phi(\mathbf{x}) \notin \Phi(\mathcal{H})$. Therefore, $\Phi(\mathbf{u}) \notin K(\Phi(\mathcal{H}))$, which is a contradiction.

Assume that $o(\mathbf{u}) = 8$ and $\mathbf{u} \in \{\mathbf{w}_2, 3\mathbf{w}_2\}$. Let $\mathbf{w}_3 = (w_1' \mid w_2' \mid w_3')$ be the third row of A^{t_1,t_2,t_3}. First, consider that $t_1 > 1$. Let k be the number of ones in w_3' outside the coordinate positions of I. Then, $k = 4 \cdot 8^{t_1-2}4^{t_2}2^{t_3-1} = m/2$. Let $\mathbf{z} = 2\mathbf{w}_3 = (z_1 \mid z_2 \mid z_3)$. Note that in the coordinates corresponding to I, z_3 contains every element of $2\mathbb{Z}_8$ exactly $m/4$ times. Now, $\Phi(\mathbf{u}) + \Phi(\mathbf{z}) = \Phi(\mathbf{u} + \mathbf{z} - \mathbf{x}_1)$, where $\mathbf{x}_1 = (\mathbf{0} \mid \mathbf{0} \mid 2(u_3 \odot z_3))$. By Corollary 5,

$$\mathrm{wt}_H(\Phi(\mathbf{x}_1)) = \mathrm{wt}_H(\Phi_3(2(u_3 \odot z_3))) = \begin{cases} \frac{k}{4} \cdot 8 = m & \text{if } \mathbf{u} = \mathbf{w}_2, \\ \frac{k}{4} \cdot 8 + \frac{m}{4} \cdot 8 = 3m & \text{if } \mathbf{u} = 3\mathbf{w}_2. \end{cases}$$

Since $m = n/8$ and the minimum weight of $\Phi(\mathcal{H})$ is $n/2$, we have that $\Phi(\mathbf{x}_1) \notin \Phi(\mathcal{H})$. Therefore, $\Phi(\mathbf{u}) \notin K(\Phi(\mathcal{H}))$, which is a contradiction. Second, consider that $t_1 = 1$. Then, $m = \alpha_3$. Note that the number of ones in w_2 is $m_2 = 4^{t_2}2^{t_3-1}$. Then, by Proposition 1, we have that $m_2 = n/8$. Let I_2 be the set of coordinate positions to these ones in w_2. Let $\mathbf{y} = \mathbf{w}_2 + \mathbf{w}_3 = (y_1 \mid y_2 \mid y_3)$. Then, we have that $y_3 = (\mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{m/4}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$, and in the coordinates corresponding to I_2, y_2 is a permutations of the vector $(\mathbb{Z}_4, \overset{m_2/4}{\ldots}, \mathbb{Z}_4)$. Now, $\Phi(\mathbf{u}) + \Phi(\mathbf{y}) = \Phi(\mathbf{u} + \mathbf{y} - \mathbf{x}_2)$, where $\mathbf{x}_2 = (\mathbf{0} \mid 2(u_2 \odot y_2) \mid 2(u_3 \odot y_3))$. By Corollary 5,

$$\begin{aligned}
\mathrm{wt}_H(\Phi(\mathbf{x}_2)) &= \mathrm{wt}_H(\Phi_2(2(u_2 \odot y_2))) + \mathrm{wt}_H(\Phi_3(2(u_3 \odot y_3))) \\
&= \frac{m_2}{4} \cdot 2 \cdot 2 + \frac{m}{4} \cdot 8 \text{ if } \mathbf{u} \in \{\mathbf{w}_2, 3\mathbf{w}_2\}.
\end{aligned}$$

Since $m_2 = m = n/8$ and the minimum weight of $\Phi(\mathcal{H})$ is $n/2$, we have that $\Phi(\mathbf{x}_2) \notin \Phi(\mathcal{H})$. Therefore, $\Phi(\mathbf{u}) \notin K(\Phi(\mathcal{H}))$, which is a contradiction.

Assume that $o(\mathbf{u}) = 4$. Recall that $\mathbf{v} = 2\mathbf{w}_2$, $\mathbf{z} = 2\mathbf{w}_3$, and $\mathbf{y} = \mathbf{w}_2 + \mathbf{w}_3$. First, consider that $t_1 > 1$. By construction, in the coordinates corresponding to I, either u_3 contains every element of $2\mathbb{Z}_8$ exactly $m/4$ times or u_3 is equal to $\mathbf{2}$. In the first case, we consider $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) = \Phi(\mathbf{u} + \mathbf{v} - \mathbf{x}_3)$, where $\mathbf{x}_3 = (\mathbf{0} \mid \mathbf{0} \mid 2(u_3 \odot v_3))$. In the second case, we consider $\Phi(\mathbf{u}) + \Phi(\mathbf{z}) = \Phi(\mathbf{u} + \mathbf{z} - \mathbf{x}_4)$, where $\mathbf{x}_4 = (\mathbf{0} \mid \mathbf{0} \mid 2(u_3 \odot z_3)$. Then, in both cases, $\mathrm{wt}_H(\Phi(\mathbf{x}_3)) = \mathrm{wt}_H(\Phi(\mathbf{x}_4)) = \frac{m}{4} \cdot 8 = 2m = n/4$.

Second, consider that $t_1 = 1$. Note that $m = \alpha_3$, so u_3 is a permutation of $(2\mathbb{Z}_8, \overset{m/4}{\ldots}, 2\mathbb{Z}_8)$ or $u_3 = \mathbf{2}$. In the first case, we consider again $\Phi(\mathbf{u}) + \Phi(\mathbf{v}) = \Phi(\mathbf{u} + \mathbf{v} - \mathbf{x}_3)$, and in the second case, $\Phi(\mathbf{u}) + \Phi(\mathbf{y}) = \Phi(\mathbf{u} + \mathbf{y} - \mathbf{x}_5)$, where $\mathbf{x}_5 = (\mathbf{0} \mid \mathbf{0} \mid 2(u_3 \odot y_3)$. In both cases, $\mathrm{wt}_H(\Phi(\mathbf{x}_3)) = \mathrm{wt}_H(\Phi(\mathbf{x}_5)) = \frac{m}{4} \cdot 8 = 2m = n/4$. For all these cases, since the minimum weight of $\Phi(\mathcal{H})$ is $n/2$, we have that $\Phi(\mathbf{x}_3), \Phi(\mathbf{x}_4), \Phi(\mathbf{x}_5) \notin \Phi(\mathcal{H})$, which is a contradiction. This completes the proof. □

Corollary 6 *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H} = \mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ such that $\Phi(\mathcal{H})$ is nonlinear. Let $\mathbf{w}_k$ be the kth row of A^{t_1,t_2,t_3} and $Q = \{(o(\mathbf{w}_k)/2)\mathbf{w}_k\}_{k=1}^{t_1+t_2+t_3}$. Then, $\Phi(Q)$ is a basis of $K(\Phi(\mathcal{H}))$ and $\ker(\Phi(\mathcal{H})) = t_1 + t_2 + t_3$.*

5 Classification results

The classification of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of type $(\alpha_1, \alpha_2; t_2, t_3)$, (or equivalently, $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of type $(\alpha_1, \alpha_2, 0; 0, t_2, t_3)$) with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, of length 2^t, for any $t \geq 3$, using the rank or the dimension of the kernel is shown in [31]. For $\mathbb{Z}_4$-linear Hadamard codes of type $(n; t_2, t_3)$ (or equivalently, $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of type $(0, n, 0; 0, t_2, t_3)$), the classification is shown in [26, 31]. In [28], it is shown that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Some partial results on the classification of $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t, for any $t \geq 3$ and $s > 2$, are proved in [17, 19]. In this section, we present several families of infinite such nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3}, which are not equivalent to any other constructed $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any previously constructed $\mathbb{Z}_{2^s}$-linear Hadamard code [17], with $s \geq 2$, with the same length 2^t. Moreover, for $3 \leq t \leq 11$, we give a complete classification by providing the exact amount of nonequivalent such codes.

Next, we recall the construction of $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$, given in [17]. From [17, 19], we also recall for which types these codes are linear, what are their kernel dimensions when they are nonlinear, and which of them are equivalent to each other. Then, we recall the results on the rank of $\mathbb{Z}_8$-linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes given in [18] and [31], respectively. First, we recall the construction of $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$ and some results, given in [17–19]. Let $T_i = \{j \cdot 2^{i-1} : j \in \{0, 1, \ldots, 2^{s-i+1} - 1\}\}$ for all $i \in \{1, \ldots, s\}$. Note that $T_1 = \{0, \ldots, 2^s - 1\}$. Let $t_1, \ldots, t_s$ be non-negative integers with $t_1 \geq 1$. Consider a matrix $\bar{A}^{t_1,\ldots,t_s}$ whose columns are exactly all the vectors of the form $\mathbf{z}^T, \mathbf{z} \in \{1\} \times T_1^{t_1-1} \times T_2^{t_2} \times \cdots \times T_s^{t_s}$. See [17] for examples. Let $\bar{\mathcal{H}}^{t_1,\ldots,t_s}$ be the $\mathbb{Z}_{2^s}$-additive code of type $(n; t_1, \ldots, t_s)$ generated by $\bar{A}^{t_1,\ldots,t_s}$, where $t_1, \ldots, t_s$ are non-negative integers with $t_1 \geq 1$. Let $\bar{H}^{t_1,\ldots,t_s} = \Phi(\bar{\mathcal{H}}^{t_1,\ldots,t_s})$ be the corresponding $\mathbb{Z}_{2^s}$-linear Hadamard code. Let $\mathbf{w}_i$ be the ith row of $\bar{A}^{t_1,\ldots,t_s}$, $1 \leq i \leq t_1 + \cdots + t_s$. By construction, $\mathbf{w}_1 = \mathbf{1}$ and

$o(\mathbf{w}_i) \leq o(\mathbf{w}_j)$ if $i > j$. We define $\sigma \in \{1, \ldots, s\}$ as the integer such that $o(\mathbf{w}_2) = 2^{s+1-\sigma}$. For $\bar{\mathcal{H}}^{1,0,\ldots,0}$, we define $\sigma = s$. Note that

$$\sigma = \begin{cases} 1 & \text{if } t_1 \geq 2, \\ s & \text{if } t_1 = 1, t_2 = \cdots = t_s = 0, \\ \min\{i : t_i > 0, i \in \{2, \ldots, s\}\} & \text{otherwise.} \end{cases} \quad (9)$$

Theorem 4 *[17] Let $t_1, \ldots, t_s$ be non-negative integers with $s \geq 2$ and $t_1 \geq 1$. The $\mathbb{Z}_{2^s}$-linear code $\bar{H}^{t_1,\ldots,t_s}$ of type $(n; t_1, \ldots, t_s)$ is a binary Hadamard code of length 2^t, with $t = \left(\sum_{i=1}^{s}(s-i+1) \cdot t_i\right) - 1$ and $n = 2^{t-s+1}$.*

Theorem 5 *[17] The $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ of type $(n; t_1, \ldots, t_s)$, with $s \geq 3$, is linear if and only if $(t_1, \ldots, t_s) \in \{(1, 0, \ldots, 0, t_s), (1, 0, \ldots, 0, 1, t_s)\}$. Moreover, if $\bar{H}^{t_1,\ldots,t_s}$ is nonlinear, then $\ker(\bar{H}^{t_1,\ldots,t_s}) = \sigma + \sum_{i=1}^{s} t_i$.*

Theorem 6 *[18] Let $\bar{H}^{t_1,\ldots,t_s}$ be the $\mathbb{Z}_{2^s}$-linear Hadamard code of type $(n; t_1, \ldots, t_s)$. Then,* $\mathrm{rank}(\bar{H}^{t_1,\ldots,t_s+1}) = 1 + \mathrm{rank}(\bar{H}^{t_1,\ldots,t_s})$.

Theorem 7 *[18] Let $\bar{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_8$-linear Hadamard code of type $(n; t_1, t_2, t_3)$. Then,*

$$\mathrm{rank}(\bar{H}^{t_1,t_2,t_3}) = \frac{t_1^4}{24} - \frac{t_1^3}{12} + \frac{35t_1^2}{24} + \frac{7t_1}{12} + \frac{t_2}{2}(t_1^2 + t_1 + t_2 + 1) + t_3 + 1.$$

Theorem 8 *[19] Let $\bar{H}^{t_1,\ldots,t_s}$ be the $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$ and $t_s \geq 1$. Then, for all $\ell \in \{1, \ldots, t_s\}$, $\bar{H}^{t_1,\ldots,t_s}$ is permutation equivalent to the $\mathbb{Z}_{2^{s+\ell}}$-linear Hadamard code $\bar{H}^{1,\mathbf{0}^{\ell-1},t_1-1,t_2,\ldots,t_{s-1},t_s-\ell}$.*

Let $t'_1, \ldots, t'_{s'}$ be non-negative integers with $t'_1 \geq 2$ if $s' \geq 3$, or $t'_1 \geq 1$ if $s' = 2$. Let $C_H(t'_1, \ldots, t'_{s'}) = [H_1 = \bar{H}^{t'_1,\ldots,t'_{s'}}, H_2, \ldots, H_\rho]$ be the sequence of all $\mathbb{Z}_{2^s}$-linear Hadamard codes $\bar{H}^{t_1,\ldots,t_s}$ of length 2^t, where $t = (\sum_{i=1}^{s}(s-i+1)\cdot t_i)-1$, that are permutation equivalent to $\bar{H}^{t'_1,\ldots,t'_{s'}}$ by Theorem 8. We refer to $C_H(t'_1, \ldots, t'_s)$ as the equivalence chain of $\bar{H}^{t'_1,\ldots,t'_s}$. If we focus on nonlinear $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 3$, then we can assume that $t'_1 \geq 2$ [3].

Proposition 3 *[19] Let $t_1, \ldots, t_s$ be non-negative integers with $t_1 \geq 1$. Then, the $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ belongs to an unique equivalence chain $C_H(t'_1, \ldots, t'_{s'})$, where* $s' = s - \sigma + 1$.

Now, we recall the results about the rank and dimension of the kernel for nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, given in [31].

Theorem 9 *[31] Let $t \geq 3$ and $t_2 \in \{0, \ldots, \lfloor t/2 \rfloor\}$. Let H^{t_2,t_3} be the nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t and type $(\alpha_1, \alpha_2; t_2, t_3)$, where $\alpha_1 = 2^{t-t_2}$, $\alpha_2 = 2^{t-1} - 2^{t-t_2-1}$, and $t_3 = t + 1 - 2t_2$. Then,*

$$\mathrm{rank}(H^{t_2,t_3}) = t_3 + 2t_2 + \binom{t_2}{2} \quad \textit{and} \quad \ker(H^{t_2,t_3}) = t_2 + t_3.$$

The following example shows that the rank, the dimension of the kernel, or the pair of rank and dimension of the kernel cannot be used, in general, to completely classify all nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of a given length 2^t up to equivalence.

Example 3 By Theorems 1 and 2, the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^9 are the following: $H^{1,1,5}$, $H^{1,2,3}$, $H^{1,3,1}$, $H^{2,0,4}$, $H^{2,1,2}$ and $H^{3,0,1}$. Their kernels are of dimension 7, 6, 5, 6, 5 and 4, respectively, by Corollary 6. By using the computer algebra system Magma [14], we can check that $H^{1,2,3}$ and $H^{2,0,4}$ are nonequivalent. Therefore, the dimension of the kernel does not allow to classify these codes. By using Magma, we compute their ranks, which are 12, 15, 19, 15, 20 and 26, respectively. Therefore, the rank does not classify either. Note that $H^{1,2,3}$ and $H^{2,0,4}$ have the same pair of rank and dimension of the kernel, $(r, k) = (6, 15)$. Thus, the pair of rank and dimension of the kernel does not classify either.

Theorem 10 *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let H^{t_1,t_2,t_3} be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Then,* $\mathrm{rank}(H^{t_1,t_2,t_3+1}) = 1 + \mathrm{rank}(H^{t_1,t_2,t_3})$.

Proof Let $\mathcal{H}^{t_1,t_2,t_3} = (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code. Note that, by construction (5),

$$\mathcal{H}^{t_1,t_2,t_3+1} = \bigcup_{\lambda\in\{0,1\}} ((\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1} \mid \mathbf{0}, \mathbf{2} \mid \mathbf{0}, \mathbf{4})).$$

By Corollaries 1 and 3,

$$\begin{aligned} H^{t_1,t_2,t_3+1} &= \bigcup_{\lambda\in\{0,1\}} (\Phi(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})) \\ &= A_0 \cup A_1, \end{aligned}$$

where $A_\lambda = \Phi(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$, $\lambda \in \{0, 1\}$. Therefore, $\mathrm{rank}(\Phi(\mathcal{H}^{t_1,t_2,t_3+1})) = 1 + \mathrm{rank}(\Phi(\mathcal{H}^{t_1,t_2,t_3}))$. □

Now, we prove some technical lemmas.

Lemma 1 *Let t be a positive integer. Then,*

1. $\mathrm{rank}(H^{1,1,t-4}) = t + 3$, *where* $t \geq 5$.
2. $\mathrm{rank}(H^{1,2,t-6}) = \mathrm{rank}(H^{2,0,t-5}) = t + 6$, *where* $t \geq 7$.
3. $\mathrm{rank}(H^{1,3,t-8}) = t + 10$ *and* $\mathrm{rank}(H^{2,1,t-7}) = t + 11$, *where* $t \geq 9$.
4. $\mathrm{rank}(H^{2,2,t-9}) = \mathrm{rank}(H^{3,0,t-8}) = t + 17$, *where* $t \geq 10$.
5. $\mathrm{rank}(H^{1,4,t-10}) = t + 15$, *where* $t \geq 11$.
6. $\mathrm{rank}(\bar{H}^{2,0,0,t-7}) = t + 7$, *where* $t \geq 7$.
7. $\mathrm{rank}(\bar{H}^{1,1,2,t-10}) = t + 8$, *where* $t \geq 10$.
8. $\mathrm{rank}(\bar{H}^{1,0,5,t-13}) = t + 11$, *where* $t \geq 13$.
9. $\mathrm{rank}(\bar{H}^{1,2,0,t-9}) = t + 9$ *and* $\mathrm{rank}(\bar{H}^{2,0,1,t-9}) = t + 11$, *where* $t \geq 9$.
10. $\mathrm{rank}(\bar{H}^{1,1,3,t-12}) = t + 12$, *where* $t \geq 12$.
11. $\mathrm{rank}(\bar{H}^{1,2,1,t-11}) = t + 14$, *where* $t \geq 11$.
12. $\mathrm{rank}(\bar{H}^{1,0,6,t-15}) = t + 16$, *where* $t \geq 15$.

Proof By using Magma or as it is shown in Tables 2 and 3, we have that $\mathrm{rank}(H^{1,1,1}) = 8$, $\mathrm{rank}(H^{1,2,1}) = \mathrm{rank}(H^{2,0,2}) = 13$, $\mathrm{rank}(H^{1,3,1}) = \mathrm{rank}(H^{2,1,1}) = 19$, $\mathrm{rank}(H^{2,2,1}) = \mathrm{rank}(H^{3,0,2}) = 27$, $\mathrm{rank}(H^{1,4,1}) = 26$. For any H^{t_1,t_2,t_3}, we have that $t + 1 = 3t_1 + 2t_2 + t_3$ by Theorem 1. Thus, $H^{1,1,t_3} = H^{1,1,t-4}$, where $t \geq 5$. By Theorem 10, we have that $\mathrm{rank}(H^{1,1,t-4}) = t + 3$. Items 2-5 follow in a similar way.

Again, by using Magma [14], we also have that $\text{rank}(\bar{H}^{2,0,0,0}) = 14$, $\text{rank}(\bar{H}^{1,1,2,0}) = 18$, $\text{rank}(\bar{H}^{1,2,0,0}) = 18$, $\text{rank}(\bar{H}^{1,0,5,0}) = 24$, $\text{rank}(\bar{H}^{2,0,1,0}) = 20$, $\text{rank}(\bar{H}^{1,1,3,0}) = 24$, $\text{rank}(\bar{H}^{1,2,1,0}) = 25$, and $\text{rank}(\bar{H}^{1,0,6,0}) = 31$. The remaining Items 6-12 follow from the fact that $t + 1 = 4t_1 + 3t_2 + 2t_3 + t_4$ for any $\bar{H}^{t_1,t_2,t_3,t_4}$ and Theorem 6. □

Lemma 2 *Let t be a positive integer. Then, the following codes in each item are not equivalent to each other.*

1. $H^{1,2,t-6}$ *and* $H^{2,0,t-5}$, *where* $7 \leq t \leq 11$.
2. $H^{2,2,t-9}$ *and* $H^{3,0,t-8}$, *where* $10 \leq t \leq 11$.
3. $H^{1,1,t-4}$ *and* $\bar{H}^{2,0,t-5}$, *where* $5 \leq t \leq 11$.
4. $H^{5,t-9}$ *and* $H^{2,1,t-7}$, *where* $10 \leq t \leq 11$ *and* $H^{5,t-9}$ *is the* $\mathbb{Z}_2\mathbb{Z}_4$*-linear Hadamard code of length* 2^t *and type* $(\alpha_1, \alpha_2; 5, t-9)$ *with* $\alpha_1 = 2^{t-5}$, $\alpha_2 = 2^{t-1} - 2^{t-6}$.

Proof The inequivalence between these pairs of codes is established through computations in Magma [14]. Specifically, we use functions provided in the packages [20, 21] to construct the Hadamard codes, and then apply the function `IsHadamardEquivalent` to determine whether the codes are equivalent or not. □

Lemma 3 *Let k and t be two positive integers such that $t-k > 0$. Let $\bar{H}^{t_1,\dots,t_s}$ be a nonlinear $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t such that $\ker(\bar{H}^{t_1,\dots,t_s}) = t-k$. Then,*

1. *for $s \geq 2$ and σ as defined in (9),*

$$t_1 + \cdots + t_s = t - k - \sigma, \tag{10}$$
$$(s-1)t_1 + (s-2)t_2 + \cdots + t_{s-1} = \sigma + k + 1, \tag{11}$$

2. *for $s \geq 4$, $t_1 = 1$, σ as defined in (9) and $\lambda = (s-\sigma-1)t_{\sigma+1} + \cdots + t_{s-1}$,*

$$s - \sigma = \frac{k + 2 - \lambda}{t_\sigma + 1}. \tag{12}$$

Proof By Theorem 5, $\ker(\bar{H}^{t_1,\dots,t_s}) = \sigma + \sum_{i=1}^{s} t_i$, so we have that

$$t_1 + \cdots + t_s = t - k - \sigma. \tag{13}$$

By Theorem 4, we have that

$$st_1 + (s-1)t_2 + \cdots + t_s = t + 1. \tag{14}$$

From (13) and (14), we have that

$$(s-1)t_1 + (s-2)t_2 + \cdots + t_{s-1} = \sigma + k + 1. \tag{15}$$

For Item 2, since $s \geq 4$, $t_1 = 1$, and $\bar{H}^{t_1,\dots,t_s}$ is nonlinear, from the definition of σ, we have that $1 < \sigma < s$, $t_2 = \cdots = t_{\sigma-1} = 0$ and $t_\sigma \geq 1$. From (15), we have that $(s-1)+(s-\sigma)t_\sigma+\lambda = \sigma + k + 1$. Thus, we obtain (12). □

Lemma 4 *Let k and t be two positive integers such that $t-k > 0$. Let H^{t_1,t_2,t_3} be a nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of length 2^t such that $\ker(H^{t_1,t_2,t_3}) = t-k$. Then,*

$$t_1 + t_2 + t_3 = t - k, \tag{16}$$
$$2t_1 + t_2 = k + 1. \tag{17}$$

Proof By Corollary 6, $\ker(H^{t_1,t_2,t_3}) = t_1 + t_2 + t_3$, so we have (16). By Theorem 1,

$$3t_1 + 2t_2 + t_3 = t + 1. \tag{18}$$

From (16) and (18), we have that $2t_1 + t_2 = k + 1$. □

The following results show that there are several families of infinite $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^t, with $\alpha_1 \neq 0, \alpha_2 \neq 0$ and $\alpha_3 \neq 0$, which are not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear and $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$, of the same length 2^t.

Let k and t be two positive integers such that $t - k > 0$. We define the set

$$S_k = \{(t_1, t_2, t_3) : (t_1, t_2, t_3) \text{ satisfies (16) and (17)}\}.$$

By Corollary 6 and Lemma 4, we have that $\ker(H^{t_1,t_2,t_3}) = t-k$ if and only if $(t_1, t_2, t_3) \in S_k$. We also define the set

$$\bar{S}_{k,s} = \{(t_1, \ldots, t_s) : (t_1, \ldots, t_s) \text{ satisfies (10) and (11)}\}.$$

By Theorem 5 and Lemma 3, we have that $\ker(\bar{H}^{t_1,\ldots,t_s}) = t - k$ if and only if $(t_1, \ldots, t_s) \in \bar{S}_{k,s}$.

Theorem 11 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $H^{1,1,t-4}$ of length 2^t, with $t \geq 5$, is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_4$-linear or $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t.*

Moreover, $H^{1,1,t-4}$ is not equivalent to any $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t, with $s \geq 3$, except for the codes in the equivalence chain $C_H(2, 0, t-5)$.

Proof Let $H = H^{1,1,t-4}$. We have that $S_2 = \{(1, 1, t-4)\}$. Note that H is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, having the same length 2^t, $t \geq 5$, since the values of the kernel dimensions are different.

Let H^{t_2,t_3} be the nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t and type $(\alpha_1, \alpha_2; t_2, t_3)$, where $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. By Theorem 9, we have that $\text{rank}(H^{t_2,t_3}) = t + 1 + \binom{t_2}{2}$, and by Item 1 of Lemma 1, $\text{rank}(H) = t + 3$. Since $t + 1 + \binom{t_2}{2} \neq t + 3$ for any value of t_2, H is not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Since the family of $\mathbb{Z}_4$-linear Hadamard codes is included in the family of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ [28], we have that H is not equivalent to any $\mathbb{Z}_4$-linear Hadamard code of length 2^t.

Let $\bar{H}^{t_1,\ldots,t_s}$ be a nonlinear $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 3$, of length 2^t such that $\ker(\bar{H}^{t_1,\ldots,t_s}) = t - 2$. First, we show that, if $s \geq 4$, then $\bar{H}^{t_1,\ldots,t_s}$ belongs to an equivalence chain $C_H(t'_1, \ldots, t'_{s'})$, where $s' \in \{2, 3\}$. Note that, if $s \geq 4$ and $t_1 > 1$, then $\sigma = 1$. In this case, (11) has no solution. Assume that $s \geq 4$ and $t_1 = 1$. Thus, from (12),

$$s - \sigma = \frac{4 - \lambda}{t_\sigma + 1}.$$

Since $s - \sigma > 0$ and $t_\sigma + 1 \geq 2$, we have that $4 - \lambda \in \{2, 3, 4\}$, and hence, $s - \sigma \in \{1, 2\}$. Thus, from Proposition 3, $\bar{H}^{t_1,\ldots,t_s}$, with $s \geq 4$, belongs to an equivalence chain $C_H(t'_1, \ldots, t'_{s'})$, where $s' \in \{2, 3\}$. Therefore, we just need to show that H is not equivalent to any nonlinear $\mathbb{Z}_8$-linear Hadamard code $\bar{H}^{t_1,t_2,t_3}$ of length 2^t such that $\ker(\bar{H}^{t_1,t_2,t_3}) = t - 2$, except for $\bar{H}^{2,0,t-5}$.

If $t_1 = 1$, then $\sigma = 2$. In this case, we have that $(1, 3, t-8) \in \bar{S}_{2,3}$. Otherwise, if $t_1 > 1$, then $\sigma = 1$ and we have that $(2, 0, t-5) \in \bar{S}_{2,3}$. Thus, $\bar{S}_{2,3} = \{(2, 0, t-5), (1, 3, t-8)\}$.

By Theorem 7, $\text{rank}(\bar{H}^{2,0,t-5}) = t+3$ and $\text{rank}(\bar{H}^{1,3,t-8}) = t+4 \neq t+3$. Since $\text{rank}(H) = t+3$, H is not equivalent to any $\mathbb{Z}_8$-linear Hadamard code of length 2^t, except to $\bar{H}^{2,0,t-5}$ and the codes in its equivalence chain $C_H(2, 0, t-5)$. □

Theorem 12 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $H^{1,3,t-8}$ of length 2^t, with $t \geq 9$, is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear or $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t with $s \geq 2$.*

Proof Let $H = H^{1,3,t-8}$. We have that $S_4 = \{(1, 3, t-8), (2, 1, t-7)\}$. By Item 3 of Lemma 1, $\text{rank}(H) = t+10$ and $\text{rank}(H^{2,1,t-7}) = t+11$, so H is not equivalent to $H^{2,1,t-7}$. Moreover, H is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, having the same length 2^t, since the values of the kernel dimensions are different.

By the same argument as in the proof of Theorem 11, since $t+1+\binom{t_2}{2} \neq t+10$ for any value of t_2, we have that H is not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and neither to any $\mathbb{Z}_4$-linear Hadamard code of length 2^t.

Now, we show that any $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ with $s \geq 3$ and $\ker(\bar{H}^{t_1,\ldots,t_s}) = t-4$ is equivalent to a $\mathbb{Z}_8$-linear or $\mathbb{Z}_{16}$-linear Hadamard code. For $s \geq 4$ and $t_1 > 1$, we have that $\sigma = 1$, so (11) has solution only if $s = 4$ and hence $s - \sigma = 3$. For $s \geq 4$ and $t_1 = 1$, from (12), we have that $6 - \lambda \in \{2, 3, 4, 5, 6\}$, and hence $s - \sigma \in \{1, 2, 3\}$. Thus, from Proposition 3, $\bar{H}^{t_1,\ldots,t_s}$, with $s \geq 4$, belongs to an equivalence chain $C_H(t'_1, \ldots, t'_{s'})$, where $s' \in \{2, 3, 4\}$. Therefore, we just need to show that H is not equivalent to any nonlinear $\mathbb{Z}_8$-linear and $\mathbb{Z}_{16}$-linear Hadamard code of length 2^t. We have that

$$\bar{S}_{4,3} \cup \bar{S}_{4,4} = \{(1, 5, t-12), (2, 2, t-9), (3, 0, t-8), (1, 0, 5, t-13), \\ (1, 1, 2, t-10), (1, 2, 0, t-9), (2, 0, 0, t-7)\}.$$

By Theorem 7 and Items 6-9 of Lemma 1, $\text{rank}(\bar{H}^{t_1,\ldots,t_s})$ is $t+11$, $t+8$, $t+9$, $t+11$, $t+8$, $t+9$ and $t+7$, respectively. Since $\text{rank}(H) = t+10$, H is not equivalent to any $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ of length 2^t, with $s \geq 3$. □

Corollary 7 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $H^{2,1,t-7}$ of length 2^t, with $t \geq 8$, is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}.*

Theorem 13 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code $H^{1,4,t-10}$ of length 2^t, with $t \geq 11$, is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear or $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t with $s \geq 2$.*

Proof Let $H = H^{1,4,t-10}$. We have that $S_5 = \{(1, 4, t-10), (2, 2, t-9), (3, 0, t-8)\}$. Let $H_1 = H^{2,2,t-9}$ and $H_2 = H^{3,0,t-8}$. By Items 4 and 5 of Lemma 1, $\text{rank}(H) = t+15$ and $\text{rank}(H_1) = \text{rank}(H_2) = t+17$. Therefore, H is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, having the same length 2^t.

By the same argument as in the proof of Theorem 11, since $t+1+\binom{t_2}{2} \neq t+15$, we have that H is not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and neither to any $\mathbb{Z}_4$-linear Hadamard code of length 2^t.

Now, we show that any $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ with $s \geq 3$ and $\ker(\bar{H}^{t_1,\ldots,t_s}) = t-5$ is equivalent to a $\mathbb{Z}_8$-linear or $\mathbb{Z}_{16}$-linear Hadamard code. For $s \geq 4$ and $t_1 > 1$, we have that $\sigma = 1$, so (11) has a solution only if $s = 4$ and hence $s - \sigma = 3$. For $s \geq 4$ and $t_1 = 1$, from (12), we have that $7 - \lambda \in \{2, 3, 4, 5, 6, 7\}$, and hence $s - \sigma \in \{1, 2, 3\}$. Thus, from Proposition 3, $\bar{H}^{t_1,\ldots,t_s}$, with $s \geq 4$, belongs to an equivalence chain $C_H(t'_1, \ldots, t'_{s'})$, where $s' \in \{2, 3, 4\}$. Therefore, we just need to show that H is not equivalent to any nonlinear

$\mathbb{Z}_8$-linear and $\mathbb{Z}_{16}$-linear Hadamard code of length 2^t. We have that

$$\bar{S}_{5,3} \cup \bar{S}_{5,4} = \{(1,6,t-14),(2,3,t-11),(3,1,t-10),(1,0,6,t-15),\\ (1,1,3,t-12),(1,2,1,t-11),(2,0,1,t-9)\}.$$

By Theorem 7 and from Items 9 to 12 of Lemma 1, $\text{rank}(\bar{H}^{t_1,\ldots,t_s})$ is $t+16$, $t+12$, $t+14$, $t+16$, $t+12$, $t+14$ and $t+11$, respectively. Since $\text{rank}(H)=t+15$, H is not equivalent to any $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t, with $s \geq 3$. □

Theorem 14 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes $H^{1,2,t-6}$, with $t \geq 7$, and $H^{2,0,t-5}$, with $t \geq 6$, of length 2^t are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear or $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t with $s \geq 2$. Moreover, $H^{1,2,t-6}$ and $H^{2,0,t-5}$ are not equivalent for any $7 \leq t \leq 11$.*

Proof Let $H_1 = H^{1,2,t-6}$ and $H_2 = H^{2,0,t-5}$. We have that $S_3 = \{(1,2,t-6),(2,0,t-5)\}$. Then, H_1 and H_2 are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, having the same length 2^t, since the values of the kernel dimensions are different. By using Magma [14], we have that H_1 and H_2 are not equivalent for any $7 \leq t \leq 11$.

By Item 2 of Lemma 1, $\text{rank}(H_1) = \text{rank}(H_2) = t+6$. By the same argument as in the proof of Theorem 11, since $t+1+\binom{t_2}{2} \neq t+6$ for any value of t_2, we have that H_1 and H_2 are not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and neither to any $\mathbb{Z}_4$-linear Hadamard code of length 2^t.

Now, we show that, if $s \geq 3$, then any $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ with $\ker(\bar{H}^{t_1,\ldots,t_s}) = t-3$ is equivalent to a $\mathbb{Z}_8$-linear Hadamard code. We can assume that $s \geq 4$ and $t_1 = 1$, otherwise (11) has no solution. From (12), $5-\lambda \in \{2,3,4,5\}$, and hence, $s-\sigma \in \{1,2\}$. Thus, from Proposition 3, $\bar{H}^{t_1,\ldots,t_s}$, with $s \geq 4$, belongs to an equivalence chain $C_H(t'_1,\ldots,t'_{s'})$, where $s' \in \{2,3\}$. Therefore, we just need to show that H_1 and H_2 are not equivalent to any nonlinear $\mathbb{Z}_8$-linear Hadamard code $\bar{H}^{t_1,t_2,t_3}$ of length 2^t such that $\ker(\bar{H}^{t_1,t_2,t_3}) = t-3$. By using similar arguments as in the proof of Theorem 11, $\bar{S}_{3,3} = \{(2,1,t-7),(1,4,t-10)\}$. By Theorem 7, $\text{rank}(\bar{H}^{2,1,t-7}) = t+5$ and $\text{rank}(\bar{H}^{1,4,t-10}) = t+7$. This completes the proof, since $\text{rank}(H_1) = \text{rank}(H_2) = t+6$ by Item 2 of Lemma 1. □

Theorem 15 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes $H^{2,2,t-9}$, for $t \geq 10$, and $H^{3,0,t-8}$, for $t \geq 9$, of length 2^t are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear or $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t with $s \geq 2$. Moreover, $H^{2,2,t-9}$ and $H^{3,0,t-8}$ are not equivalent for any $10 \leq t \leq 11$.*

Proof Let $H_1 = H^{2,2,t-9}$ and $H_2 = H^{3,0,t-8}$. We have that $S_5 = \{(1,4,t-10),(2,2,t-9),(3,0,t-8)\}$. By Items 4 and 5 of Lemma 1, $\text{rank}(H^{1,4,t-10}) = t+15$ and $\text{rank}(H_1) = \text{rank}(H_2) = t+17$. Then, H_1 and H_2 are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, having the same length 2^t. By using Magma [14], H_1 and H_2 are not equivalent for any $7 \leq t \leq 11$.

As before, since $t+1+\binom{t_2}{2} \neq t+17$, we have that H_1 and H_2 are not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and neither to any $\mathbb{Z}_4$-linear Hadamard code of length 2^t.

Now, we show that any $\mathbb{Z}_{2^s}$-linear Hadamard code $\bar{H}^{t_1,\ldots,t_s}$ with $s \geq 3$ and $\ker(\bar{H}^{t_1,\ldots,t_s}) = t-5$ is equivalent to a $\mathbb{Z}_8$-linear or $\mathbb{Z}_{16}$-linear Hadamard code. For $s \geq 4$ and $t_1 > 1$, we have that $\sigma = 1$, so (11) has a solution only if $s = 4$ and hence $s - \sigma = 3$. For $s \geq 4$ and $t_1 = 1$, from (12), we have that $7-\lambda \in \{2,3,4,5,6,7\}$, and hence $s-\sigma \in \{1,2,3\}$. Thus, from Proposition 3, $\bar{H}^{t_1,\dots,t_s}$, with $s \geq 4$, belongs to an equivalence chain $C_H(t'_1,\dots,t'_{s'})$, where $s' \in \{2,3,4\}$. Therefore, we just need to show that H_1 and H_2 are not equivalent to any nonlinear $\mathbb{Z}_8$-linear and $\mathbb{Z}_{16}$-linear Hadamard code of length 2^t. We have that

$$\bar{S}_{5,3} \cup \bar{S}_{5,4} = \{(1,6,t-14),(2,3,t-11),(3,1,t-10),(1,0,6,t-15),\\ (1,1,3,t-12),(1,2,1,t-11),(2,0,1,t-9)\}.$$

By Theorem 7 and from Items 9 to 12 of Lemma 1, $\mathrm{rank}(\bar{H}^{t_1,\dots,t_s})$ is $t+16, t+12, t+14, t+16, t+12, t+14$ and $t+11$, respectively. Therefore, H_1 and H_2 are not equivalent to any $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^t, $s \geq 3$. □

Theorem 16 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes $H^{t/3,0,1}$ of length 2^t, with $t = 3m$ and $m \geq 2$, are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear, $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard code of length 2^t.*

Proof By Theorem 2, $H^{m,0,1}$ is nonlinear since $m \geq 2$. Then, by Corollary 6, $\ker(H^{m,0,1}) = m+1$. Let H^{t_1,t_2,t_3} be a nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of length 2^t such that $\ker(H^{t_1,t_2,t_3}) = m+1$. By Corollary 6, $\ker(H^{t_1,t_2,t_3}) = t_1+t_2+t_3$, so we have that

$$t_1 + t_2 + t_3 = m + 1. \tag{19}$$

By Theorem 1,

$$3t_1 + 2t_2 + t_3 = t + 1 = 3m + 1. \tag{20}$$

From (19) and (20),

$$t_2 + 2t_3 = 2. \tag{21}$$

Note that $\{(t_1,t_2,t_3) : (t_1,t_2,t_3) \text{ satisfies (19) and (21)}\} = \{(m,0,1)\}$. Therefore, $H^{m,0,1}$ is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3} of the same length 2^t, since the values of the kernel dimensions are different.

Let H^{t_2,t_3} be the nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t and type $(\alpha_1,\alpha_2;t_2,t_3)$, where $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, such that $\ker(H^{t_2,t_3}) = m+1$. Then, by Theorem 9,

$$t_2 + t_3 = m + 1, \tag{22}$$

and

$$2t_2 + t_3 = t + 1 = 3m + 1. \tag{23}$$

From (22) and (23),

$$t_2 + 2t_3 = 2. \tag{24}$$

From (22) and (24), we have that $t_3 = 1 - m < 0$, which is a contradiction, since $t_3 \geq 1$. Therefore, $H^{m,0,1}$ is not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Since the family of $\mathbb{Z}_4$-linear Hadamard codes is included in the family of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, we only need to show that $H^{m,0,1}$ is not equivalent to any $\mathbb{Z}_8$-linear Hadamard code of length 2^t.

Let $\bar{H}^{t_1,t_2,t_3}$ be a nonlinear $\mathbb{Z}_8$-linear Hadamard code of length 2^t such that $\ker(\bar{H}^{t_1,t_2,t_3}) = m+1$. By Theorem 5, $\ker(\bar{H}^{t_1,t_2,t_3}) = \sigma + \sum_{i=1}^{3} t_i$, so we have that

$$t_1 + t_2 + t_3 = m + 1 - \sigma. \tag{25}$$

By Theorem 4, we have that

$$3t_1 + 2t_2 + t_3 = t + 1 = 3m + 1. \tag{26}$$

From (25) and (26), we have that

$$t_2 + 2t_3 = 2 - 3\sigma. \tag{27}$$

Since $1 \leq \sigma \leq 3$, $2-3\sigma < 0$. Thus, (27) has no solution, since $t_2, t_3 \geq 0$. Therefore, $H^{m,0,1}$ is not equivalent to any $\mathbb{Z}_8$-linear Hadamard code. □

Theorem 17 *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes $H^{(t-2)/3,1,1}$ of length 2^t, with $t = 3m+2$ and $m \geq 2$, are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_4$-linear or $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t.*

Proof By Theorem 2, $H^{m,1,1}$ is nonlinear. Then, by Corollary 6, we have that $\ker(H^{m,1,1}) = m + 2$. Let H^{t_1,t_2,t_3} be a nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of length 2^t such that $\ker(H^{t_1,t_2,t_3}) = m + 2$. By Corollary 6, $\ker(H^{t_1,t_2,t_3}) = t_1 + t_2 + t_3$, so we have that

$$t_1 + t_2 + t_3 = m + 2. \tag{28}$$

By Theorem 1,

$$3t_1 + 2t_2 + t_3 = t + 1 = 3m + 3. \tag{29}$$

From (28) and (29),

$$t_2 + 2t_3 = 3. \tag{30}$$

Note that $\{(t_1, t_2, t_3) : (t_1, t_2, t_3)$ satisfies (28) and (30)$\} = \{(m, 1, 1)\}$. Therefore, $H^{m,1,1}$ is not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, having the same length 2^t, since the values of the kernel dimensions are different.

Let H^{t_2,t_3} be the nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t and type $(\alpha_1, \alpha_2; t_2, t_3)$, where $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, such that $\ker(H^{t_2,t_3}) = m+2$. Then, by Theorem 9,

$$t_2 + t_3 = m + 2, \tag{31}$$

and

$$2t_2 + t_3 = t + 1 = 3m + 3. \tag{32}$$

From (31) and (32),

$$t_2 + 2t_3 = 3. \tag{33}$$

From (31) and (33), we have that $t_3 = 1 - m < 0$, which is a contradiction, since $t_3 \geq 1$. Therefore, $H^{m,1,1}$ is not equivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Since the family of $\mathbb{Z}_4$-linear Hadamard codes is included in the family of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. This completes the proof. □

For $4 \leq t \leq 15$, Tables 2, 3 and 4, show all possible types corresponding to $\mathbb{Z}_4$-linear Hadamard, $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard (with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$), $\mathbb{Z}_8$-linear Hadamard $\bar{H}^{t_1,t_2,t_3}$ [17] and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$) Hadamard codes H^{t_1,t_2,t_3} of length 2^t. For each one of them, the values (r, k) are shown, where r is the rank and k is the dimension of the kernel. These values can be computed for $\mathbb{Z}_4$-linear Hadamard codes

Table 1 Number $\mathcal{A}_t$ of nonequivalent $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamrd code of length 2^t for $3 \leq t \leq 11$

t	3	4	5	6	7	8	9	10	11
$\mathcal{A}_t$	1	1	2	3	4	5	7	8	10

from [26, 31], for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes by Theorem 9, and for $\mathbb{Z}_8$-linear Hadamard codes by Theorems 5 and 7. The values of the rank for $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes are computed by using Magma [14], and the kernel dimensions are calculated by Theorem 6. Note that if two codes have different values (r, k), then they are not equivalent. As shown in Example 3 for $t = 9$, it is easy to see that taking only the values of the dimension of the kernel, or only the rank, given in these tables, it is not possible to classify completely the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes.

Note that, for $t \leq 4$, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^t are binary linear Hadamard codes. In the next theorem, we show that, for $5 \leq t \leq 11$, the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^t are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17] with $s \geq 2$, of the same length 2^t.

Theorem 18 *For $5 \leq t \leq 11$, the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^t are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any previously constructed $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$, of length 2^t.*

Proof It follows from Lemma 2, Theorems 11, 12, 13, 14 and 15, Tables 2 and 3, Tables 1 and 3 given in [17], and the fact that two codes are nonequivalent if they have different values of (r, k), where r is the rank and k is the dimension of the kernel. For example, consider $t = 11$. From Table 3, all nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^{11} are $H^{1,1,7}$, $H^{1,2,5}$, $H^{1,3,3}$, $H^{1,4,1}$, $H^{2,0,6}$, $H^{2,1,4}$, $H^{2,2,2}$, $H^{3,0,3}$ and $H^{3,1,1}$. All these codes, except $H^{2,2,2}$ and $H^{3,0,3}$, are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear and $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$, of length 2^{11}, since they have different values of (r, k). The codes $H^{2,2,2}$ and $H^{3,0,3}$ have the same (r, k), which is $(28, 6)$. By using Magma [14], we can check that they are not equivalent. Again, from Table 3, and Table 3 given in [17], we have that all the codes of length 2^{11} have values of (r, k) different from $(28, 6)$. This completes the proof. □

Let $\mathcal{A}_t$ be the number of nonequivalent $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^t. By Theorem 18, we can classify completely $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^t for $3 \leq t \leq 11$, by providing the number $\mathcal{A}_t$ of nonequivalent such codes, as shown in Table 1.

6 Conclusions and further research

In this paper, we study the linearity of the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, presented in [8]. We also determine the kernel and its dimension whenever they are nonlinear. The kernel does not give a complete classification, but it provides a lower bound on the number of such nonequivalent $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with the same length 2^t. We establish a complete classification for $3 \leq t \leq 11$ by using the dimension of the kernel and some Magma computations.

It is known that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ [28]. Unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, the family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes. Actually, we show that there are several families of infinite nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, which are not equivalent to any other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code H^{t_1,t_2,t_3}, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17] with $s \geq 2$, of the same length 2^t. Therefore, some nonlinear Hadamard codes, without any known structure, now can be seen as the Gray map image of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$.

Let $\mathcal{S}^{t_1-1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1-1, \alpha_2, \alpha_3; t_1-1, t_2, t_3)$ generated by a matrix obtained from A^{t_1,t_2,t_3} after deleting its first column and row. In [35, Theorem 12], it is proved that $\mathcal{S}^{t_1-1,t_2,t_3}$ is the additive dual of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive 1-perfect code. Moreover, it is shown that all such $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive 1-perfect codes are obtained from each other by monomial transformations. Thus, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes $\mathcal{H}^{t_1,t_2,t_3}$ are the only ones, up to monomial transformation, that can be obtained by extending the parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive 1-perfect code.

Let $\mathcal{H}$ be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$. Note that if $\alpha_3 = 0$, we obtain $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, which were studied in [26, 31]. Hence, we assume $\alpha_3 > 0$. If $\alpha_1 > 0$ and $\alpha_2 = 0$, then we could consider a generator matrix of $\mathcal{H}$ with $(\mathbf{1} \mid \mid \mathbf{4})$ in its first row and 0 in the first coordinate of the other rows, allowing us to construct $\mathcal{S}^{t_1-1,t_2,t_3}$ as before. Thus, this case is not possible. On the other hand, if $\alpha_1 = 0$ and $\alpha_2 > 0$, determining the existence of $\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes remains an open problem.

A Magma function to compute $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes and $\mathbb{Z}_{p^s}$-additive codes is included in a new Magma package to deal with $\mathbb{Z}_p\mathbb{Z}_{p^2}\dots\mathbb{Z}_{p^s}$-additive codes in general [21]. This new package also includes functions related to the generalized Gray map considered in this paper. Indeed, this package generalizes some of the functions for linear codes over $\mathbb{Z}_{p^s}$ [20]. The first version of this new package and a manual describing all functions will be released this year, and it will be available in a GitHub repository and in the CCSG web site (http://ccsg.uab.cat).

As further research, it would be interesting to prove that all nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [17] with $s \geq 2$, of the same length 2^t, as it is seen for $5 \leq t \leq 11$ by Theorem 18.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no Conflict of interest.

Table 2 Rank and kernel for all types of $\mathbb{Z}_4$-linear, $\mathbb{Z}_8$-linear, $\mathbb{Z}_2\mathbb{Z}_4$-linear, and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^t, $4 \leq t \leq 7$

	$t=4$		$t=5$		$t=6$		$t=7$	
	type	(r,k)	type	(r,k)	type	(r,k)	type	(r,k)
$\mathbb{Z}_4$	$(2^3;1,3)$	(5,5)	$(2^4;1,4)$	(6,6)	$(2^5;1,5)$	(7,7)	$(2^6;1,6)$	(8,8)
	$(2^3;2,1)$	(5,5)	$(2^4;2,2)$	(6,6)	$(2^5;2,3)$	(7,7)	$(2^6;2,4)$	(8,8)
			$(2^4;3,0)$	(7,4)	$(2^5;3,1)$	(8,5)	$(2^6;3,2)$	(9,6)
							$(2^6;4,0)$	(11,5)
$\mathbb{Z}_2\mathbb{Z}_4$	$(8,4;1,3)$	(5,5)	$(16,8;1,4)$	(6,6)	$(32,16;1,5)$	(7,7)	$(64,32;1,6)$	(8,8)
	$(4,6;2,1)$	(6,3)	$(8,12;2,2)$	(7,4)	$(16,24;2,3)$	(8,5)	$(32,48;2,4)$	(9,6)
					$(8,28;3,1)$	(10,4)	$(16,56;3,2)$	(11,5)
$\mathbb{Z}_8$	$(2^2;1,0,2)$	(5,5)	$(2^3;1,0,3)$	(6,6)	$(2^4;1,0,4)$	(7,7)	$(2^5;1,0,5)$	(8,8)
	$(2^2;1,1,0)$	(5,5)	$(2^3;1,1,1)$	(6,6)	$(2^4;1,1,2)$	(7,7)	$(2^5;1,1,3)$	(8,8)
			$(2^3;2,0,0)$	(8,3)	$(2^4;1,2,0)$	(8,5)	$(2^5;1,2,1)$	(9,6)
					$(2^4;2,0,1)$	(9,4)	$(2^5;2,0,2)$	(10,5)
							$(2^5;2,1,0)$	(12,4)
$\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$	$(4,2,2;1,0,2)$	(5,5)	$(8,4,4;1,0,3)$	(6,6)	$(16,8,8;1,0,4)$	(7,7)	$(32,16,16;1,0,5)$	(8,8)
			$(4,6,4;1,1,1)$	(8,3)	$(8,12,8;1,1,2)$	(9,4)	$(16,24,16;1,1,3)$	(10,5)
					$(4,6,12;2,0,1)$	(12,3)	$(8,28,16;1,2,1)$	(13,4)
							$(8,12,24;2,0,2)$	(13,4)

Table 3 Rank and kernel for all types of $\mathbb{Z}_4$-linear, $\mathbb{Z}_8$-linear, $\mathbb{Z}_2\mathbb{Z}_4$-linear, and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^t, $8 \leq t \leq 11$.

	$t=8$ type	(r,k)	$t=9$ type	(r,k)	$t=10$ type	(r,k)	$t=11$ type	(r,k)
$\mathbb{Z}_4$	$(2^7; 1, 7)$	(9,9)	$(2^8; 1, 8)$	(10,10)	$(2^9; 1, 9)$	(11,11)	$(2^{10}; 1, 10)$	(12,12)
	$(2^7; 2, 5)$	(9,9)	$(2^8; 2, 6)$	(10,10)	$(2^9; 2, 7)$	(11,11)	$(2^{10}; 2, 8)$	(12,12)
	$(2^7; 3, 3)$	(10,7)	$(2^8; 3, 4)$	(11,8)	$(2^9; 3, 5)$	(12,9)	$(2^{10}; 3, 6)$	(13,10)
	$(2^7; 4, 1)$	(12,6)	$(2^8; 4, 2)$	(13,7)	$(2^9; 4, 3)$	(14,8)	$(2^{10}; 4, 4)$	(15,9)
			$(2^8; 5, 0)$	(16,6)	$(2^9; 5, 1)$	(17,7)	$(2^{10}; 5, 2)$	(18,8)
							$(2^{10}; 6, 0)$	(22,7)
$\mathbb{Z}_2\mathbb{Z}_4$	(128, 64; 1, 7)	(9,9)	(256, 128; 1, 8)	(10,10)	(512, 256; 1, 9)	(11,11)	(1024, 512; 1, 10)	(12,12)
	(64, 96; 2, 5)	(10,7)	(128, 192; 2, 6)	(11,8)	(256, 384; 2, 7)	(12,9)	(512, 768; 2, 8)	(13,10)
	(32, 112; 3, 3)	(12,6)	(64, 224; 3, 4)	(13,7)	(128, 448; 3, 5)	(14,8)	(256, 896; 3, 6)	(15,9)
	(16, 120; 4, 1)	(15,5)	(32, 240; 4, 2)	(16,6)	(64, 480; 4, 3)	(17,7)	(128, 960; 4, 4)	(18,8)
					(32, 496; 5, 1)	(21,6)	(64, 992; 5, 2)	(22,7)
$\mathbb{Z}_8$	$(2^6; 1, 0, 6)$	(9,9)	$(2^7; 1, 0, 7)$	(10,10)	$(2^8; 1, 0, 8)$	(11,11)	$(2^9; 1, 0, 9)$	(12,12)
	$(2^6; 1, 1, 4)$	(9,9)	$(2^7; 1, 1, 5)$	(10,10)	$(2^8; 1, 1, 6)$	(11,11)	$(2^9; 1, 1, 7)$	(12,12)
	$(2^6; 1, 2, 2)$	(10,7)	$(2^7; 1, 2, 3)$	(11,8)	$(2^8; 1, 2, 4)$	(12,9)	$(2^9; 1, 2, 5)$	(13,10)
	$(2^6; 1, 3, 0)$	(12,6)	$(2^7; 1, 3, 1)$	(13,7)	$(2^8; 1, 3, 2)$	(14,8)	$(2^9; 1, 3, 3)$	(15,9)
	$(2^6; 2, 0, 3)$	(11,6)	$(2^7; 2, 0, 4)$	(12,7)	$(2^8; 1, 4, 0)$	(17,7)	$(2^9; 1, 4, 1)$	(18,8)
	$(2^6; 2, 1, 1)$	(13,5)	$(2^7; 2, 1, 2)$	(14,6)	$(2^8; 2, 0, 5)$	(13,8)	$(2^9; 2, 0, 6)$	(14,9)
	$(2^6; 3, 0, 0)$	(17,4)	$(2^7; 2, 2, 0)$	(17,5)	$(2^8; 2, 1, 3)$	(15,7)	$(2^9; 2, 1, 4)$	(16,8)
			$(2^7; 3, 0, 1)$	(18,5)	$(2^8; 2, 2, 1)$	(18,6)	$(2^9; 2, 2, 2)$	(19,7)
					$(2^8; 3, 0, 2)$	(19,6)	$(2^9; 2, 3, 0)$	(23,6)
					$(2^8; 3, 1, 0)$	(24,5)	$(2^9; 3, 0, 3)$	(20,7)
							$(2^9; 3, 1, 1)$	(25,6)
							$(2^9; 4, 0, 0)$	(32,5)

Table 3 continued

	$t=8$ type	(r,k)	$t=9$ type	(r,k)	$t=10$ type	(r,k)	$t=11$ type	(r,k)
$\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$	(64, 32, 32; 1, 0, 6)	(9,9)	(128, 64, 64; 1, 0, 7)	(10,10)	(256, 128, 128; 1, 0, 8)	(11,11)	(512, 256, 256; 1, 0, 9)	(12,12)
	(32, 48, 32; 1, 1, 4)	(11,6)	(64, 96, 64; 1, 1, 5)	(12,7)	(128, 192, 128; 1, 1, 6)	(13,8)	(256, 384, 256; 1, 1, 7)	(14,9)
	(16, 56, 32; 1, 2, 2)	(14,5)	(32, 112, 64; 1, 2, 3)	(15,6)	(64, 224, 128; 1, 2, 4)	(16,7)	(128, 448, 256; 1, 2, 5)	(17,8)
	(16, 24, 48; 2, 0, 3)	(14,5)	(16, 120, 64; 1, 3, 1)	(19,5)	(32, 240, 128; 1, 3, 2)	(20,6)	(64, 480, 256; 1, 3, 3)	(21,7)
	(8, 28, 48; 2, 1, 1)	(19,4)	(32, 48, 96; 2, 0, 4)	(15,6)	(64, 96, 192; 2, 0, 5)	(16,7)	(32, 496, 256; 1, 4, 1)	(26,6)
			(16, 56, 96; 2, 1, 2)	(20,5)	(32, 112, 192; 2, 1, 3)	(21,6)	(128, 192, 384; 2, 0, 6)	(17,8)
			(8, 28, 112; 3, 0, 1)	(26,4)	(16, 120, 192; 2, 2, 1)	(27,5)	(64, 224, 384; 2, 1, 4)	(22,7)
					(16, 56, 224; 3, 0, 2)	(27,5)	(32, 240, 384; 2, 2, 2)	(28,6)
							(32, 112, 448; 3, 0, 3)	(28,6)
							(16, 120, 448; 3, 1, 1)	(37,5)

Table 4 Rank and kernel for all types of $\mathbb{Z}_4$-linear, $\mathbb{Z}_8$-linear, $\mathbb{Z}_2\mathbb{Z}_4$-linear, and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of length 2^t, $12 \le t \le 15$

	$t = 12$ type	(r, k)	$t = 13$ type	(r, k)	$t = 14$ type	(r, k)	$t = 15$ type	(r, k)
$\mathbb{Z}_4$	$(2^{11}; 1, 11)$	(13,13)	$(2^{12}; 1, 12)$	(14,14)	$(2^{13}; 1, 13)$	(15,15)	$(2^{14}; 1, 14)$	(16,16)
	$(2^{11}; 2, 9)$	(13,13)	$(2^{12}; 2, 10)$	(14,14)	$(2^{13}; 2, 11)$	(15,15)	$(2^{14}; 2, 12)$	(16,16)
	$(2^{11}; 3, 7)$	(14,11)	$(2^{12}; 3, 8)$	(15,12)	$(2^{13}; 3, 9)$	(16,13)	$(2^{14}; 3, 10)$	(17,14)
	$(2^{11}; 4, 5)$	(16,10)	$(2^{12}; 4, 6)$	(17,11)	$(2^{13}; 4, 7)$	(18,12)	$(2^{14}; 4, 8)$	(19,13)
	$(2^{11}; 5, 3)$	(19,9)	$(2^{12}; 5, 4)$	(20,10)	$(2^{13}; 5, 5)$	(21,11)	$(2^{14}; 5, 6)$	(22,12)
	$(2^{11}; 6, 1)$	(23,8)	$(2^{12}; 6, 2)$	(24,9)	$(2^{13}; 6, 3)$	(25,10)	$(2^{14}; 6, 4)$	(26,11)
			$(2^{12}; 7, 0)$	(29,9)	$(2^{13}; 7, 1)$	(30,9)	$(2^{14}; 7, 2)$	(31,10)
							$(2^{14}; 8, 0)$	(37,9)
$\mathbb{Z}_2\mathbb{Z}_4$	$(2^{11}, 2^{10}; 1, 11)$	(13,13)	$(2^{12}, 2^{11}; 1, 12)$	(14,14)	$(2^{13}, 2^{12}; 1, 13)$	(15,15)	$(2^{14}, 2^{13}; 1, 14)$	(16,16)
	$(2^{10}, 1536; 2, 9)$	(14,11)	$(2^{11}, 3072; 2, 10)$	(15,12)	$(2^{12}, 6144; 2, 11)$	(16,13)	$(2^{13}, 12288; 2, 12)$	(17,14)
	$(2^{9}, 1792; 3, 7)$	(16,10)	$(2^{10}, 3584; 3, 8)$	(17,11)	$(2^{11}, 7168; 3, 9)$	(18,12)	$(2^{12}, 14336; 3, 10)$	(19,13)
	$(2^{8}, 1920; 4, 5)$	(19,9)	$(2^{9}, 3840; 4, 6)$	(20,10)	$(2^{10}, 7680; 4, 7)$	(21,11)	$(2^{11}, 15360; 4, 8)$	(22,12)
	$(2^{7}, 1984; 5, 3)$	(23,8)	$(2^{8}, 3968; 5, 4)$	(24,9)	$(2^{9}, 7936; 5, 5)$	(25,10)	$(2^{10}, 15872; 5, 6)$	(26,11)
	$(2^{6}, 2016; 6, 1)$	(28,7)	$(2^{7}, 4032; 6, 2)$	(29,8)	$(2^{8}, 8064; 6, 3)$	(30,9)	$(2^{9}, 16128; 6, 4)$	(31,10)
					$(2^{7}, 8128; 7, 1)$	(36,8)	$(2^{8}, 16256; 7, 2)$	(37,9)

Table 4 continued

	$t = 12$ type	(r, k)	$t = 13$ type	(r, k)	$t = 14$ type	(r, k)	$t = 15$ type	(r, k)
$\mathbb{Z}_8$	$(2^{10}; 1, 0, 10)$	(13,13)	$(2^{11}; 1, 0, 11)$	(14,14)	$(2^{12}; 1, 0, 12)$	(15,15)	$(2^{13}; 1, 0, 13)$	(16,16)
	$(2^{10}; 1, 1, 8)$	(13,13)	$(2^{11}; 1, 1, 9)$	(14,14)	$(2^{12}; 1, 1, 10)$	(15,15)	$(2^{13}; 1, 1, 11)$	(16,16)
	$(2^{10}; 1, 2, 6)$	(14,11)	$(2^{11}; 1, 2, 7)$	(15,12)	$(2^{12}; 1, 2, 8)$	(16,13)	$(2^{13}; 1, 2, 9)$	(17,14)
	$(2^{10}; 1, 3, 4)$	(16,10)	$(2^{11}; 1, 3, 5)$	(17,11)	$(2^{12}; 1, 3, 6)$	(18,12)	$(2^{13}; 1, 3, 7)$	(19,13)
	$(2^{10}; 1, 4, 2)$	(19,9)	$(2^{11}; 1, 4, 3)$	(20,10)	$(2^{12}; 1, 4, 4)$	(21,11)	$(2^{13}; 1, 4, 5)$	(22,12)
	$(2^{10}; 2, 0, 7)$	(15,10)	$(2^{11}; 2, 0, 8)$	(16,11)	$(2^{12}; 2, 0, 9)$	(17,12)	$(2^{13}; 2, 0, 10)$	(18,13)
	$(2^{10}; 2, 1, 5)$	(17,9)	$(2^{11}; 2, 1, 6)$	(18,10)	$(2^{12}; 2, 1, 7)$	(19,11)	$(2^{13}; 2, 1, 8)$	(20,12)
	$(2^{10}; 2, 2, 3)$	(20,8)	$(2^{11}; 2, 2, 4)$	(21,9)	$(2^{12}; 2, 2, 5)$	(22,10)	$(2^{13}; 2, 2, 6)$	(23,11)
	$(2^{10}; 2, 3, 1)$	(24,7)	$(2^{11}; 2, 3, 2)$	(25,8)	$(2^{12}; 2, 3, 3)$	(26,9)	$(2^{13}; 2, 3, 4)$	(27,10)
	$(2^{10}; 3, 0, 4)$	(21,8)	$(2^{11}; 3, 0, 5)$	(22,9)	$(2^{12}; 3, 0, 6)$	(23,10)	$(2^{13}; 3, 0, 7)$	(24,11)
	$(2^{10}; 3, 1, 2)$	(26,7)	$(2^{11}; 3, 1, 3)$	(27,8)	$(2^{12}; 3, 1, 4)$	(28,9)	$(2^{13}; 3, 1, 5)$	(29,10)
	$(2^{10}; 4, 0, 1)$	(33,6)	$(2^{11}; 4, 0, 2)$	(34,7)	$(2^{12}; 4, 0, 3)$	(35,8)	$(2^{13}; 4, 0, 4)$	(36,9)
	$(2^{10}; 1, 5, 0)$	(23,8)	$(2^{11}; 1, 5, 1)$	(24,9)	$(2^{12}; 1, 5, 2)$	(25,10)	$(2^{13}; 1, 5, 3)$	(26,11)
	$(2^{10}; 3, 2, 0)$	(32,6)	$(2^{11}; 3, 2, 1)$	(33,7)	$(2^{12}; 3, 2, 2)$	(34,8)	$(2^{13}; 3, 2, 3)$	(35,9)
			$(2^{11}; 2, 4, 0)$	(30,7)	$(2^{12}; 2, 4, 1)$	(31,8)	$(2^{13}; 2, 4, 2)$	(32,9)
			$(2^{11}; 4, 1, 0)$	(43,6)	$(2^{12}; 4, 1, 1)$	(44,7)	$(2^{13}; 4, 1, 2)$	(45,8)
					$(2^{12}; 1, 6, 0)$	(30,9)	$(2^{13}; 1, 6, 1)$	(31,10)
					$(2^{12}; 3, 3, 0)$	(41,7)	$(2^{13}; 3, 3, 1)$	(42,8)
					$(2^{12}; 5, 0, 0)$	(56,6)	$(2^{13}; 5, 0, 1)$	(57,7)
							$(2^{13}; 2, 5, 0)$	(38,8)
							$(2^{13}; 4, 2, 0)$	(55,7)

Table 4 continued

	$t=12$ type	(r,k)	$t=13$ type	(r,k)	$t=14$ type	(r,k)	$t=15$ type	(r,k)
$\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$	$(2^{10}, 2^9, 2^9; 1, 0, 10)$	(13,13)	$(2^{11}, 2^{10}, 2^{10}; 1, 0, 11)$	(14,14)	$(2^{12}, 2^{11}, 2^{11}; 1, 0, 12)$	(15,15)	$(2^{13}, 2^{12}, 2^{12}; 1, 0, 13)$	(16,16)
	$(2^9, 768, 2^9; 1, 1, 8)$	(15,10)	$(2^{10}, 1536, 2^{10}; 1, 1, 9)$	(16,11)	$(2^{11}, 3072, 2^{11}; 1, 1, 10)$	(17,12)	$(2^{12}, 6144, 2^{11}; 1, 1, 11)$	(18,13)
	$(2^8, 896, 2^9; 1, 2, 6)$	(18,9)	$(2^9, 1792, 2^{10}; 1, 2, 7)$	(19,10)	$(2^{10}, 3584, 2^{11}; 1, 2, 8)$	(20,11)	$(2^{11}, 7168, 2^{12}; 1, 2, 9)$	(21,12)
	$(2^7, 960, 2^9; 1, 3, 4)$	(22,8)	$(2^8, 1920, 2^{10}; 1, 3, 5)$	(23,9)	$(2^9, 3840, 2^{11}; 1, 3, 6)$	(24,10)	$(2^{10}, 7680, 2^{12}; 1, 3, 7)$	(25,11)
	$(2^6, 992, 2^9; 1, 4, 2)$	(27,7)	$(2^7, 1984, 2^{10}; 1, 4, 3)$	(28,8)	$(2^8, 3968, 2^{11}; 1, 4, 4)$	(29,9)	$(2^9, 7936, 2^{12}; 1, 4, 5)$	(30,10)
	$(2^8, 384, 768; 2, 0, 7)$	(18,9)	$(2^9, 768, 1536; 2, 0, 8)$	(19,10)	$(2^{10}, 1536, 3072; 2, 0, 9)$	(20,11)	$(2^{11}, 3072, 6144; 2, 0, 10)$	(21,12)
	$(2^7, 448, 768; 2, 1, 5)$	(23,8)	$(2^8, 896, 1536; 2, 1, 6)$	(24,9)	$(2^9, 1792, 3072; 2, 1, 7)$	(25,10)	$(2^{10}, 3584, 6144; 2, 1, 8)$	(26,11)
	$(2^6, 480, 768; 2, 2, 3)$	(29,7)	$(2^7, 960, 1536; 2, 2, 4)$	(30,8)	$(2^8, 1920, 3072; 2, 2, 5)$	(31,9)	$(2^9, 3840, 6144; 2, 2, 6)$	(32,10)
	$(2^5, 496, 768; 2, 3, 1)$	(36,6)	$(2^6, 992, 1536; 2, 3, 2)$	(37,7)	$(2^7, 1984, 3072; 2, 3, 3)$	(38,8)	$(2^8, 3968, 6144; 2, 3, 4)$	(39,9)
	$(2^6, 224, 896; 3, 0, 4)$	(29,7)	$(2^7, 448, 1792; 3, 0, 5)$	(30,8)	$(2^8, 896, 3584; 3, 0, 6)$	(31,9)	$(2^9, 1792, 7168; 3, 0, 7)$	(32,10)
	$(2^5, 240, 896; 3, 1, 2)$	(38,6)	$(2^6, 480, 1792; 3, 1, 3)$	(39,7)	$(2^7, 960, 3584, 3, 1, 4)$	(40,8)	$(2^8, 1920, 7168; 3, 1, 5)$	(41,9)
	$(2^4, 120, 960; 4, 0, 1)$	(49,5)	$(2^5, 240, 1920; 4, 0, 2)$	(50,6)	$(2^6, 480, 3840; 4, 0, 3)$	(51,7)	$(2^7, 960, 7680; 4, 0, 4)$	(52,8)
			$(2^6, 2016, 2^{10}; 1, 5, 1)$	(34,7)	$(2^7, 4032, 2^{11}; 1, 5, 2)$	(35,8)	$(2^8, 8064, 2^{12}; 1, 5, 3)$	(36,9)
			$(2^5, 496, 1792; 3, 2, 1)$	(49,6)	$(2^6, 992, 3584; 3, 2, 2)$	(50,7)	$(2^7, 1984, 7168; 3, 2, 3)$	(51,8)
					$(2^6, 2016, 3072; 2, 4, 1)$	(46,7)	$(2^7, 4032, 6144; 2, 4, 2)$	(47,8)
					$(2^5, 496, 3840; 4, 1, 1)$	(65,6)	$(2^6, 992, 7680; 4, 1, 2)$	(66,7)
							$(2^7, 8128, 4096; 1, 6, 1)$	(43,8)
							$(2^6, 2016, 7168; 3, 3, 1)$	(62,7)
							$(2^5, 496, 7936; 5, 0, 1)$	(85,6)

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