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The Zero–Hopf bifurcations of the quadratic polynomial differential jerk systems in $\mathbb{R}^3$

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Abstract. We study the zero–Hopf bifurcations of all quadratic polynomial differential jerk systems in $\mathbb{R}^3$

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6xz + a_7y^2 + a_8yz + a_9z^2,
\end{aligned}$$

where the dot denotes derivative with respect to the independent variable t and the coefficients a_k, for $k = 0, 1, ..., 9$, are real.

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1. Introduction and statement of the main result

In mechanics, jerk is the rate of change of an object's acceleration over time. Thus a jerk equation is a differential equation of the form $\dddot{x} = f(x, \dot{x}, \ddot{x})$, where x ,$\dot{x}$, $\ddot{x}$ and $\dddot{x}$ represent the position. velocity, acceleration, and jerk, respectively. The jerk differential equation can be written as the jerk differential system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = f(x, y, z),$$

in $\mathbb{R}^3$.

In this paper we deal with the quadratic polynomial differential jerk systems in $\mathbb{R}^3$

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6xz + a_7y^2 + a_8yz + a_9z^2.
\end{aligned} \tag{1}$$

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A *zero-Hopf equilibrium* is an equilibrium point of a 3-dimensional autonomous differential system, whose linear part at it has a zero eigenvalue and a pair of purely imaginary eigenvalues.

We want to study the periodic orbits bifurcating from the zero-Hopf equilibrium points of the 3-dimensional autonomous differential jerk systems (1).

Without trying to be exhaustive in the references about differential jerk systems in dimension three the authors of the papers [3,7,15,22] studied the periodic orbits bifurcating from a zero-Hopf equilibrium of different families of cubic polynomial differential jerk systems in $\mathbb{R}^3$, and in the paper [16,17] of a particular family of quadratic polynomial differential jerk systems in $\mathbb{R}^3$. While in the papers [1,2,4–6,8–14,16–23,26] and in the books [24,25] the authors studied the chaotic dynamics of distinct kinds of differential jerk systems in $\mathbb{R}^3$.

The first objective of this paper is to determine the zero-Hopf equilibria of the quadratic polynomial differential jerk systems (1), and then study the periodic orbits that can bifurcate from these equilibria.

Our main results are the following ones.

Proposition 1. *The unique zero-Hopf equilibrium points of the quadratic polynomial differential jerk systems (1) are:*

(a) $(-\sqrt{a_0/a_4},\ 0,0)$ *if* $a_0a_4>0$, $a_1=2\sqrt{a_0a_4}$, $a_5=2a_4(a_2+\omega^2)/a_1$ *and* $a_6=2a_3a_4/a_1$*;*
(b) $(\sqrt{a_0/a_4},\ 0,0)$ *if* $a_0a_4>0$, $a_1=-2\sqrt{a_0a_4}$, $a_5=2a_4(a_2+\omega^2)/a_1$ *and* $a_6=2a_3a_4/a_1$*;*
(c) $(0,0,0)$ *if* $a_0=a_1=a_3=0$ *and* $a_2=-\omega^2$.

Theorem 2. *The following statements hold.*

(a) Assume that the quadratic polynomial differential jerk system (1) satisfies the assumptions of statement (a) and that

$$
\begin{aligned}
& a_1=2\sqrt{a_0a_4}, \quad a_5=2a_4(a_2+\omega^2)/a_1+\varepsilon a_{51}, \quad a_6=2a_3a_4/a_1+\varepsilon a_{61},\\
& a_0^{3/2}a_{61}^2/(2\sqrt{a_0}a_4\omega^2-a_3\sqrt{a_4}\omega^4)\neq 0,\\
& w_0=\frac{\omega(a_{11}-a_{31}\omega^2)}{a_6\omega^2-2a_4},\\
& r_0=\left|\frac{\sqrt{-2a_{31}^2a_4\omega^6+2a_{11}a_{31}a_6\omega^6+2a_{11}^2\omega^2(a_4-a_6\omega^2)-2a_{02}(-2a_4\omega+a_6\omega^3)^2}}{(2a_4-a_6\omega^2)\sqrt{a_4+\omega^2(-a_6+a_7+a_9\omega^2)}}\right|>0,
\end{aligned}
$$

and $\varepsilon\neq 0$ *is sufficiently small. Then the differential system (1) has the periodic solution*

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))=((-\sqrt{a_0/a_4}+\varepsilon(r_0\sin(\omega t)+w_0)/\omega,\varepsilon r_0\cos(\omega t),\varepsilon\omega r_0\sin(\omega t))+O(\varepsilon^2)$$

bifurcating from the zero-Hopf equilibrium $(-\sqrt{a_0/a_4},0,0)$.

(b) *Assume that the quadratic polynomial differential jerk system (1) satisfies the assumptions of statement (b), and that*

$$a_1 = -2\sqrt{a_0 a_4}, \quad a_5 = 2a_4(a_2+\omega^2)/a_1 + \varepsilon a_{51}, \quad a_6 = 3a_4/a_1 + \varepsilon a_{61},$$

$$w_0 = \frac{a_0 a_{61}\omega^3}{2\sqrt{a_0}a_4^{3/2} - a_3 a_4 \omega^2},$$

$$\frac{a_0^{3/2}a_{61}^2(-a_3^2\sqrt{a_4}\omega^4 + 2a_0\sqrt{a_4}(a_4 + a_7\omega^2 + a_9\omega^4) + \sqrt{a_0}a_3\omega^2(5a_4 + 3\omega^2(a_7 + a_9\omega^2)))}{\sqrt{a_4}(-2\sqrt{a_0 a_4}\omega + a_3\omega^3)^2(-a_3\sqrt{a_4}\omega^2 + \sqrt{a_0}(a_4 + a_7\omega^2 + a_9\omega^4))} \neq 0,$$

$$r_0 = \left| \frac{\sqrt{2}a_0^{5/4}a_{61}\omega^3\sqrt{a_3\sqrt{a_4}\omega^2 + \sqrt{a_0}(a_4 + a_7\omega^2 + a_9\omega^4)}}{\sqrt{a_4}(2\sqrt{a_0 a_4} - a_3\omega^2)\sqrt{a_3^2 a_4\omega^4 - a_0(a_4 + a_7\omega^2 + a_9\omega^4)^2}} \right| > 0,$$

and $\varepsilon \neq 0$ *is sufficiently small. Then the differential system (1) has the periodic solution*

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = ((\sqrt{a_0/a_4} + \varepsilon(r_0\sin(\omega t) + w_0)/\omega, \varepsilon r_0\cos(\omega t), \varepsilon\omega r_0\sin(\omega t)) + O(\varepsilon^2)$$

bifurcating from the zero-Hopf equilibrium $(\sqrt{a_0/a_4}, 0, 0)$.

(c) *Assume that the quadratic polynomial differential jerk system (1) satisfies the assumptions of statement (c) and that*

$$a_0 = \varepsilon^2 a_{02}, \quad a_1 = \varepsilon a_{11}, \quad a_2 = -\omega^2 + \varepsilon a_{21}, \quad a_3 = \varepsilon a_{31}, \quad w_0 = \frac{\omega(a_{11} - a_{31}\omega^2)}{a_6\omega^2 - 2a_4},$$

$$\frac{a_{31}^2 a_4\omega^4 - a_{11}a_{31}a_6\omega^4 + a_{02}(-2a_4 + a_6\omega^2)^2 + a_{11}^2(-a_4 + a_6\omega^2)}{+a_6\omega^8 - 2a_4\omega^6} \neq 0,$$

$$r_0 = \left| \frac{\sqrt{-2a_{31}^2 a_4\omega^6 + 2a_{11}a_{31}a_6\omega^6 + 2a_{11}^2\omega^2(a_4 - a_6\omega^2) - 2a_{02}(-2a_4\omega + a_6\omega^3)^2}}{(2a_4 - a_6\omega^2)\sqrt{a_4 + \omega^2(-a_6 + a_7 + a_9\omega^2)}} \right| > 0,$$

and $\varepsilon \neq 0$ *is sufficiently small. Then the differential system (1) has the periodic solution*

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = ((\varepsilon(r_0\sin(\omega t) + w_0)/\omega, \varepsilon r_0\cos(\omega t), \varepsilon\omega r_0\sin(\omega t)) + O(\varepsilon^2)$$

bifurcating from the zero-Hopf equilibrium $(0, 0, 0)$.

Proposition 1 and Theorem 2 are proved in section 2. Their proofs are based on the averaging theory for computing periodic orbits, see the appendix. For others applications of the averaging theory for studying limit cycles, see for instance [?, ?].

Now in Figure 1 we provide an example of the periodic orbit of the quadratic polynomial differential jerk system (1) whose existence was given in statement (a) of Theorem 2 for the values of the parameters $a_0 = 1/2$, $a_3 = a_4 = \omega = a_{61} = 1$, $a_2 = a_{51} = a_7 = a_8 = a_9 = 0$ and $\varepsilon = 1/100$. The image on the left in Figure 1 shows the analytic orbit $\gamma(t) = ((-\sqrt{a_0/a_4} + \varepsilon(r_0\sin(\omega t) + w_0)/\omega, \varepsilon r_0\cos(\omega t), \varepsilon\omega r_0\sin(\omega t))$ of statement (a) of Theorem 2. The image in the middle in Figure 1 shows the periodic orbit computed numerically starting with the intial condition $\gamma(0)$. Finally in the image on the right in Figure 1 we have combined the analytic and numerical periodic orbits, since the difference between them is of order $O(\varepsilon^2)$ almost undetectable to our eyes.

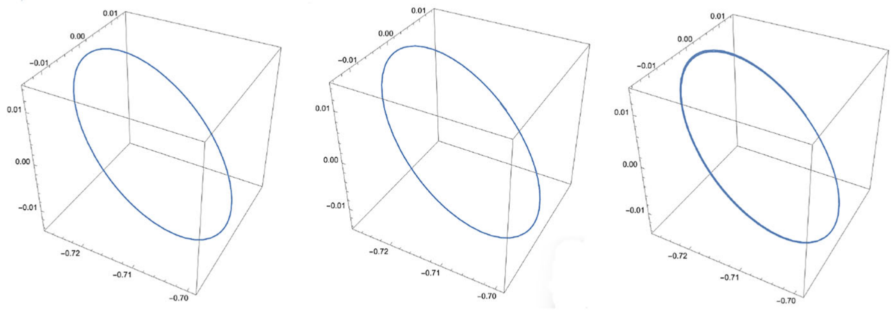

Figure 1. The periodic orbit of statement (a) of Theorem 2

2. Proofs of the results

Proof of Proposition 1. System (1) possesses the equilibrium points $(\dfrac{-a_1-\sqrt{a_1^2-4a_0a_4}}{2a_4},0,0)$ and $\dfrac{-a_1+\sqrt{a_1^2-4a_0a_4}}{2a_4},0,0)$ if $a_4\neq 0$.

The Jacobian matrix of system (1) at the equilibrium point $(\dfrac{-a_1-\sqrt{a_1^2-4a_0a_4}}{2a_4},0,0)$ is

$$\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -\sqrt{a_1^2-4a_0a_4} & a_2-\dfrac{(a_1+\sqrt{a_1^2-4a_0a_4})a_5}{2a_4} & a_3-\dfrac{(a_1+\sqrt{a_1^2-4a_0a_4})a_6}{2a_4} \end{pmatrix}.$$

The characteristic polynomial of this matrix is $p(\lambda)=-\lambda^3+(a_3-\dfrac{a_1a_6}{2a_4}-\dfrac{\sqrt{a_1^2-4a_0a_4}a_6}{2a_4})\lambda^2+(a_2-\dfrac{a_1a_5}{2a_4}-\dfrac{\sqrt{a_1^2-4a_0a_4}a_5}{2a_4})\lambda-\sqrt{a_1^2-4a_0a_4}$.

In order to have a zero-Hopf equilibrium we need that $a_1^2-4a_0a_4=0$. So we assume that $a_0a_4\geq 0$ and that $a_1=\pm 2\sqrt{a_0a_4}$. Then the characteristic polynomial becomes $p(\lambda)=-\lambda^3+(a_3-\dfrac{a_1a_6}{2a_4})\lambda^2+(a_2-\dfrac{a_1a_5}{2a_4})\lambda$. To look for possible zero-Hopf equilibria we impose that $p(\lambda)=-\lambda(\lambda^2+\omega^2)$ with $\omega>0$.

We obtain three families of zero-Hopf equilibrium points when either $a_0a_4>0$, $a_1=\pm\sqrt{a_0a_4}$, $a_5=\pm(a_4(a_2+\omega^2)/a_1$ and $a_6=\pm 2a_3a_4/a_1$ (here one of the two families has all positive signs and the other has all negative signs), or for third family $a_4\neq 0$, $a_0=a_1=a_3=0$ and $a_2=-\omega^2$.

Assume $a_4=0$ and $a_1\neq 0$. Then the unique equilibrium point of the quadratic polynomial differential jerk system (1) is $(-a_0/a_1,0,0)$. The Jacobian matrix of system (1) at the equilibrium point $(-a_0/a_1, 0, 0)$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 - \dfrac{a_0a_5}{a_1} & a_3 - \dfrac{a_0a_6}{a_1} \end{pmatrix}.$$

The characteristic polynomial of this matrix is $p(\lambda) = -\lambda^3 + (a_3 - \dfrac{a_0a_6}{a_1})\lambda^2 + (a_2 - \dfrac{a_0a_5}{a_1})\lambda + a_1$. Then the unique equilibrium cannot be zero-Hopf because when $a_4 = a_1 = 0$ there are no equilibrium points. This completes the proof of the proposition. □

Proof of statement (a) Theorem 2. In order to study the periodic orbits which can bifurcate from the zero-Hopf equilibrium $(-\sqrt{\frac{a_0}{a_4}}, 0, 0)$, first we translate the equilibrium $(-\sqrt{a_0/a_4}, 0, 0)$ to the origin of coordinates , and we will denote the new variables by x, y and z. So system (1) with the new parameters

$$a_1 = 2\sqrt{a_0a_4}, \quad a_5 = \frac{2a_4(a_2+\omega^2)}{a_1} + \varepsilon a_{51}, \quad a_6 = \frac{2a_3a_4}{a_1} + \varepsilon a_{61},$$

becomes the system

$$\begin{aligned} \dot{x} =& y, \\ \dot{y} =& z, \\ \dot{z} =& a_4x^2 + a_7y^2 + a_8yz + a_9z^2 - y\omega^2 + \frac{\sqrt{a_4x}(az + y(a_2+\omega^2)}{\sqrt{a_0}} \\ &+ \varepsilon\frac{(\sqrt{a_4}x - \sqrt{a_0})(a_{51}y + a_{61}z)}{\sqrt{a_4}}. \end{aligned} \tag{2}$$

Since we want to study the possible periodic orbits of the quadratic polynomial differential jerk systems in $\mathbb{R}^3$ in a neighborhood of the origin we rescale the variables as follows $x = \varepsilon X, y = \varepsilon Y, z = \varepsilon Z$, and we obtain the system

$$\begin{aligned} \dot{X} =& Y, \\ \dot{Y} =& Z, \\ \dot{Z} =& -Y\omega^2 + \varepsilon\frac{1}{\sqrt{a_0a_4}}(\sqrt{a_0}a_4^{3/2}X^2 - a_0a_{51}Y + a_2a_4XY + \sqrt{a_0a_4}a_7Y^2 \\ &- a_0a_{61}Z + a_3a_4XZ + \sqrt{a_0a_4}a_8YZ + \sqrt{a_0a_9}Z^2 + a_4XY\omega^2). \end{aligned} \tag{3}$$

Doing the change of variables $(X, Y, Z) \to (u, v, w)$ given by $X = \dfrac{u+w}{\omega}$, $Y = u$ and $Z = -\omega v$, we write the Jacobian matrix of system (3) at the origin of coordinates when $\varepsilon = 0$ into its real Jordan normal form, and the differential system (3) becomes

$$\dot{u} = -\omega v,$$

$$\dot{v} = \omega u + \frac{\varepsilon}{\sqrt{a_0 a_4}}(a_0\omega^2(a_{51}u - a_{61}\omega v) - \sqrt{a_0a_4}(a_4(v+w)^2 + a_7u^2\omega^2))$$
$$+v\omega^3(a_9v\omega - a_8u\omega)) + a_4(v+w)\omega(a_3v\omega - u(a_2+\omega^2))),$$
$$\dot{w} = \frac{\varepsilon}{\sqrt{a_0a_4}\omega^3}(a_0\omega^2(a_{61}v\omega - a_{51}u) + \sqrt{a_0a_4}(a_4(v+w)^2$$
$$+a_7u^2\omega^2 + v\omega^3(a_9v\omega - a_8u)) + a_4(v+w)\omega(u(a_2+\omega^2) - a_3v\omega)). \quad (4)$$

We pass from the coordinates (u, v, w) to cylindrical coordinates (r, θ, w) by taking $u = r\cos\theta$, $v = r\sin\theta$, $w = w$, and the differential system (4) in the new coordinates writes as

$$\dot{r} = \frac{\varepsilon}{\sqrt{a_0a_4}\omega^3}\sin\theta(a_0r\omega^2(a_{51}\cos\theta - a_{61}\omega\sin\theta) + a_4\omega(w + r\sin\theta)(a_3r\omega\sin\theta$$
$$- r(a_2+\omega^2)\cos\theta) - \sqrt{a_0a_4}(a_7r^2\omega^2\cos^2\theta + +a_4(w + r\sin^2\theta$$
$$+ r^2\omega^3\sin\theta(a_9\omega\sin\theta - a_8\cos\theta)),$$
$$\dot{\theta} = \frac{\varepsilon}{\sqrt{a_0a_4}\omega^3 r}\cos\theta(a_0r\omega^2(a_{51}\cos\theta - a_{61}\omega\sin\theta) + a_4\omega(w + r\sin\theta)$$
$$(a_3r\omega\sin\theta - r(a_2+\omega^2)\cos\theta) - r(a_2+\omega^2)\cos\theta - \sqrt{a_0a_4}(a_7r^2\omega^2\cos^2\theta$$
$$+ a_4(w + r\sin\theta)^2 + r^2\omega^3\sin\theta(a_9\omega\sin\theta - a_8\cos\theta),$$
$$\dot{w} = \frac{\varepsilon}{\sqrt{a_0a_4}\omega^3}(a_0r\omega^2(a_{61}\omega\sin\theta - a_{51}\cos\theta) + a_4\omega(w + r\sin\theta)(r(a_2+\omega^2)\cos\theta$$
$$- a_3r\omega\sin\theta) + \sqrt{a_0a_4}(a_7r^2\omega^2\cos^2\theta + a_4(w + r\sin\theta)^2$$
$$+ r^2\omega^3\sin\theta(a_9\omega\sin\theta - a_8\cos\theta)). \quad (5)$$

Now we take θ as the new independent variable and system (5) becomes

$$\dot{r} = \frac{\varepsilon}{\sqrt{a_0a_4}\omega^4}\sin\theta(a_0r\omega^2(a_{51}\cos\theta - a_{61}\omega\sin\theta) + a_4\omega(w + r\sin\theta)(a_3r\omega\sin\theta)$$
$$- r(a_2+\omega^2)\cos\theta) - \sqrt{a_0a_4}(a_7r^2\omega^2\cos^2\theta + a_4(w + r\sin\theta)^2$$
$$+ r^2\omega^3\sin\theta(a_9\omega\sin\theta - a_8\cos\theta)) + O(\varepsilon^2)$$
$$= \varepsilon F_{11}(r, \theta) + O(\varepsilon^2).$$
$$\dot{w} = \frac{\varepsilon}{\sqrt{a_0a_4}\omega^4}(a_0r\omega^2(a_{51}\cos\theta - a_{61}\omega\sin\theta) + a_4\omega(w + r\sin\theta)(r(a_2+\omega^2)\cos\theta$$
$$- a_3r\omega\sin\theta) + \sqrt{a_0a_4}(a_7r^2\omega^2\cos^2\theta + +a_4(w + r\sin\theta)^2$$
$$+ r^2\omega^3\sin\theta(a_9\omega\sin\theta - a_8\cos\theta+)) + O(\varepsilon^2)$$
$$= \varepsilon F_{12}(r, \theta) + O(\varepsilon^2). \quad (6)$$

Here the dot denotes the derivative with respect to θ.

The differential system (6) is periodic with respect to the independent variable θ with period 2π. So system (6) satisfies the assumptions of the averaging

Theorem 3 of the appendix. The corresponding averaged function is

$$f(r,w) = (f_1(r,w), f_2(r,w)) = \int_0^{2\pi} (F_{11}(r,\theta), F_{12}(r,\theta))d\theta.$$

Hence

$$f_1(r,w) = -\frac{r(2\sqrt{a_0}a_4^{3/2}w - a_3a_4w\omega^2 + a_0a_{61}\omega^3}{2\sqrt{a_0a_4}\omega^4},$$
$$f_2(r,w) = \frac{a_4(r^2+2w^2) - a_3\sqrt{a_4r^2\omega^2}/(\sqrt{a_0}+r^2\omega^2(a_7+a_9\omega^2))}{2\omega^4}.$$

The system $f_1(r,w) = f_2(r,w) = 0$ has at most a unique real solution (r_0, w_0) and the values of r_0 and w_0 are given in statement (a) of Theorem 2. For this real solution (r_0, w_0) we have that the condition (12) of the appendix, i.e when

$$\frac{-a_0^{3/2}a_{61}^2}{2\sqrt{a_0}a_4\omega^2 - a_3\sqrt{a_4}\omega^4} \neq 0.$$

Then, from Theorem 3 the differential system (6) for ε sufficiently small has a limit cycle $(r(\theta,\varepsilon),\, w(\theta,\varepsilon)) = (r_0, w_0) + O(\varepsilon)$.

The limit cycle $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ of system (6) in system (5) becomes the limit cycle $(r(t,\varepsilon),\theta(t,\varepsilon),w(t,\varepsilon)) = (r_0,\omega t, w_0)+O(\varepsilon)$ of the differential system (5).

The limit cycle $(r(t,\varepsilon),\theta(t,\varepsilon),w(t,\varepsilon))$ of system (5) in system (4) becomes the limit cycle $(u(t,\varepsilon),v(t,\varepsilon),w(t,\varepsilon)) = (r_0\cos(\omega t), r_0\sin(\omega t), w_0)+O(\varepsilon)$, when $\varepsilon \rightarrowtail 0$ of the differential system (4).

The limit cycle $(u(t,\varepsilon),v(t,\varepsilon),w(t,\varepsilon))$ of system (4) in system (3) becomes the limit cycle $(X(t,\varepsilon),Y(t,\varepsilon),Z(t,\varepsilon)) = ((r_0\cos(\omega t)+w_0)/\omega, r_0\cos(\omega t), -\omega r_0\sin(\omega t)) + O(\varepsilon)$ of the differential system (3).

The limit cycle $(X(t,\varepsilon),Y(t,\varepsilon),Z(t,\varepsilon))$ in system (3) becomes the limit cycle $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon)) = ((\varepsilon(r_0\sin(\omega t)+w_0)/\omega, \varepsilon r_0\cos(\omega t), \varepsilon\omega r_0\sin(\omega t)) + O(\varepsilon^2)$ of the differential system (2).

The limit cycle $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ in system (2) becomes the limit cycle $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon)) = ((-a_1/(2a_4)+\varepsilon(r_0\sin(\omega t)+w_0)/\omega, \varepsilon r_0\cos(\omega t), \varepsilon\omega r_0 \sin(\omega t)) + O(\varepsilon^2)$ of the differential system (1). This completes the proof of statement (a) of Theorem 2. □

Proof of statement (b) Theorem 2. Since the proof of statement (b) is essentially the same as the proof of statement (a) we omit it. □

Proof of statement (c) Theorem 2. Now we want to study the periodic orbits that bifurcate from the zero-Hopf equilibrium $(0,0,0)$ of the differential system (1) when $a_0 = a_1 = a_3 = 0$ and $a_2 = -\omega^2$. For this we perturb these parameters as follows:

$$a_0 = \varepsilon^2 a_{02}, \quad a_1 = \varepsilon a_{11}, \quad a_2 = -\omega^2 + \varepsilon a_{21}, \quad a_3 = \varepsilon a_{31}.$$

Since we want to study the possible periodic orbits of this quadratic polynomial differential jerk systems in a neighborhood of the origin we rescale the variables as follows $x = \varepsilon X, y = \varepsilon Y, z = \varepsilon Z$. In the new variables (X, Y, Z) the differential system writes as

$$\begin{aligned}
\dot{X} =& Y, \\
\dot{Y} =& Z, \\
\dot{Z} =& -\omega^2 Y + \varepsilon(a_{02} + a_{11}X + a_{21}Y + a_{13}Z + a_4X^2 + a_5XY + a_6XZ \\
&+ a_7Y^2 + a_8YZ + a_9Z^2).
\end{aligned}$$

By taking $X = (v + w)/\omega$, $Y = u$ and $Z = -\omega v$ we pass the linear part of this differential system at the origin to its real Jordan normal form, and we obtain the differential system

$$\begin{aligned}
\dot{u} =& -\omega v, \\
\dot{v} =& \omega u + \frac{\varepsilon}{\omega^3}(-a_4(v+w)^2 + \omega(-((a_{11} + a_5u)(v+w)) - (a_{02} + u(a_{21} + a_7u) \\
&- a_6v(v+w)\omega + (a_{31} + a_8u)v\omega^2 - a_9v^2\omega^3))i, \\
\dot{w} =& \frac{\varepsilon}{\omega^3}(a_4(v+w)^2 + \omega((a_{11} + a_5u)(v+w) + (a_{02} + u(a_{21} + a_7u) - a_6(v+w))\omega \\
&- (a_{31} + a_8u)v\omega^2 + a_9v^2\omega^3)).
\end{aligned}$$

Now we pass from the coordinates (u, v, w) to cylindrical coordinates (r, θ, w) by putting $u = r\cos\theta$, $v = r\sin\theta$, $w = w$, and we obtain the differential system

$$\begin{aligned}
\dot{r} =& \frac{\varepsilon}{\omega^3}\sin\theta(-a_4(w + r\sin\theta)^2 + \omega(r\omega^2(a_{31} + a_8r\cos\theta)\sin\theta - a_9r^2\omega^3\sin^2\theta \\
&- (a_{11} + a_5r\cos\theta)(w + r\sin\theta) - \omega(a_{02} + r\cos\theta(a_{21} + a_7r\cos\theta) \\
&- a_6r\sin\theta(w + r\sin\theta)))), \\
\dot{\theta} =& \omega + \frac{\varepsilon}{r\omega^3}\cos\theta(-a_4(w + r\sin\theta)^2 + \omega(r\omega^2(a_{31} + a_8r\cos\theta)\sin\theta - a_9r^2\omega^3\sin^2\theta \\
&- (a_{11} + a_5r\cos\theta)(w + r\sin\theta) - \omega(a_{02} + r\cos\theta(a_{21} + a_7r\cos\theta) \\
&- a_6r\sin\theta(w + r\sin\theta)))), \\
\dot{w} =& \frac{\varepsilon}{\omega^3}(a_4(w + r\sin\theta)^2 + \omega(-r\omega^2(a_{31} + a_8r\cos\theta)\sin\theta \\
&+ a_9r^2\omega^3\sin^2\theta + (a_{11} + a_5r\cos\theta)(w + r\sin\theta) + \omega(a_{02} + r\cos\theta(a_{21} + a_7r\cos\theta) \\
&- a_6r\sin\theta(w + r\sin\theta)))).
\end{aligned} \tag{7}$$

Now we take θ as the new independent variable and system (7) becomes

$$\begin{aligned}\dot{r} =&\frac{\varepsilon}{\omega^3}\sin\theta(-a_4(w+r\sin\theta)^2+\omega(r\omega^2 a_{31}+a_8 r\cos\theta)\sin\theta-a_9 r^2\omega^3\sin^2\theta\\ &-(a_{11}+a_5 r\cos\theta)(w+r\sin\theta)-\omega(a_{02}+r\cos\theta(a_{21}+a_7 r\cos\theta)-\\ &a_6 r\sin\theta(w+r\sin\theta))))+O(\varepsilon^2)\\ =&\varepsilon F_{11}(r,\theta)+O(\varepsilon^2)\\ \dot{w} =&\frac{\varepsilon}{\omega^3}(a_4(w+r\sin\theta)^2+\omega(-r\omega^2(a_{31}+a_8 r\cos\theta)\sin\theta\\ &+a_9 r^2\omega^3\sin^2\theta+(a_{11}+a_5 r\cos\theta)(w+r\sin\theta)\\ &+\omega(a_{02}+r\cos\theta(a_{21}+a_7 r\cos\theta)\\ &-a_6 r\sin\theta(w+r\sin\theta))))+O(\varepsilon^2)\\ =&\varepsilon F_{12}(r,\theta)+O(\varepsilon^2).\end{aligned} \tag{8}$$

Here the dot denotes the derivative with respect to θ. The differential system (8) is periodic in its independent variable θ of period 2π, so it satisfies all the assumptions of Theorem 3. Therefore, from Theorem 3 we obtain that its averaged function

$$f(r,w)=(f_1(r,w),f_2(r,w))=\int_0^{2\pi}(F_{11}(r,\theta),F_{12}(r,\theta))d\theta$$

is

$$\begin{aligned}f_1(r,w)&=\frac{r}{2\omega^4}(-2a_4 w+\omega(\omega(a_6 w+a_{31}\omega))-a_{11}),\\ f_2(r,w)&=\frac{1}{2\omega^4}(a_4(r^2+2w^2)+\omega(2a_{11}w+2a_{02}\omega+r^2\omega(-a_6+a_7+a_9\omega^2)).\end{aligned}$$

The system $f_1(r,w)=f_2(r,w)=0$ has at most a unique positive real solution (r_0,w_0), the values of r_0 and w_0 are given in statement (c) of Theorem 2. When this real solution (r_0,w_0) satisfies the condition (12) of the appendix, i.e when

$$\frac{a_{31}^2 a_4\omega^4-a_{11}a_{31}a_6\omega^4+a_{02}(-2a_4+a_6\omega^2)^2+a_{11}^2(-a_4+a_6\omega^2)}{+a_6\omega^8-2a_4\omega^6}\neq 0,$$

then the differential system (6) has a limit cycle $(r(\theta,\varepsilon),w(\theta,\varepsilon))=(r_0,w_0)+O(\varepsilon)$.

As in the proof of statement (a) of Theorem 2 the limit cycle $(r(\theta,\varepsilon),w(\theta,\varepsilon))$ of the differential system (8) becomes the limit cycle

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))=(\varepsilon(r_0\sin(\omega t)+w_0)/\omega,\varepsilon r_0\cos(\omega t),\varepsilon\omega r_0\sin(\omega t))+O(\varepsilon^2)$$

of the differential system (1). This completes the proof of statement (c) of Theorem 2. □

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Appendix: Averaging theory of first order

Now we shall present the basic results from the averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [32].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 G(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{9}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n, t \geq 0$. Moreover we assume that both $F(t, \mathbf{x})$ and $G(t, \mathbf{x}, \varepsilon)$ are T-periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0 \tag{10}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t,\mathbf{y})dt \tag{11}$$

Under certain conditions, equilibrium solutions of the averaged equation (10) provide T-periodic solutions of the differential equation (9).

Theorem 3. *Consider the two initial value problems (9) and (10). Suppose: (i) F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, G and its Jacobian $\partial G/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon \in (0,\varepsilon_0]$. (ii) F_1 and G are T-periodic in t (T independent of ε).*

Then the following statements hold. (a) If p is an equilibrium point of the averaged equation (10) and

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\bigg|_{\mathbf{y}=p} \neq 0 \tag{12}$$

then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of equation (9) such that $\varphi(0,\varepsilon)\to p$ as $\varepsilon\to 0$.
(b) The stability or instability of the periodic solution $\varphi(t,\varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system (10). In fact the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$.

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