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Quadratic systems with two invariant real straight lines and an invariant ellipse

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Abstract. After the linear differential systems the easiest nonlinear differential systems are the quadratic polynomial differential systems or simply the quadratic systems. These systems have been studied intensively and there are more than one thousand papers published on them. Here we study all quadratic systems possessing two real invariant straight lines (taking into account their multiplicities) and one invariant ellipse. By analysing the relative positions of the finite and infinite equilibria and of their separatrices with respect to these invariant curves, we determine the four topologically distinct phase portraits of this class of quadratic systems in the Poincaré disc. These four phase portraits already have appeared before in the classification of the phase portraits of the quadratic systems having an invariant ellipse, and three of these phase portraits also appeared before in the classification of the phase portraits of the quadratic systems having the infinity in the Poincaré compactification filled with equilibria, but in these previous classifications it was unknown that such phase portraits can have exactly two real invariant straight lines (taking into account their multiplicities) and one invariant ellipse.

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1. Introduction

The study of quadratic systems has a history spanning more than one century. According to Coppel [10] Büchel [3] published the first paper on quadratic systems in 1904 that essentially presented a collection of examples. Later on, two foundational surveys were published, one by Coppel [10] in 1966 and another by Chicone and Tian [8] in 1982.

Quadratic systems have been intensively studied over the past few decades, yielding many valuable results (see [1,30] and the references therein).

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Although more than a thousand papers have been published on these systems, we are far from having them fully understood.

We consider the planar polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, the ring of real polynomials in the variables x and y. In this work we restrict our attention to quadratic systems, that is, systems for which $\max\{\deg P, \deg Q\} = 2$. This class of systems includes many classical models, such as certain Lotka-Volterra models in population dynamics, a class of Liénard-type systems in control theory, and many more. We focus on *non-degenerate* quadratic systems, in which the polynomials P and Q are coprime. This condition prevents that the quadratic systems can be reduced to linear differential systems, or to a constant differential systems.

Although defined by relatively simple algebraic expressions, quadratic systems occupy a central role in the qualitative theory of dynamical systems due to their rich dynamics. These systems may exhibit multiple invariant algebraic curves, limit cycles and a wide range of equilibrium types. The systematic classification and characterisation of such behaviors remains an active and challenging area of research.

Numerous authors have studied quadratic systems possessing invariant algebraic curves of degree 2, see [2,5,15,17–19,25,26,28,29]. Quadratic systems admitting invariant algebraic curves of degree 3 have been analysed in [4,7,13]. Moreover, quadratic systems possessing invariant straight lines of multiplicity four or five have been classified in [21–24], respectively.

In this paper we present a complete classification of the quadratic systems that admit two real invariant lines (taking into account their multiplicities) and one invariant ellipse. Building on earlier results on quadratic systems with invariant algebraic curves of degree 2 and 3 [6,13], our work contributes to the broader programme of classifying quadratic systems with invariant algebraic curves of total degree 4. We focus on this well-defined subclass, derive canonical normal forms, and analyse the associated equilibrium configurations. By combining algebraic criteria with geometric constructions and topological methods, we classify the four topologically distinct phase portraits of the quadratic systems with two real invariant lines (taking into account their multiplicities) and one invariant ellipse in the Poincaré disc.

Although these four phase portraits appeared in the classification of the quadratic systems with an invariant ellipse [15], and three of these four phase portraits appeared in the classification of the quadratic systems having the infinity in the Poincaré compactification filled with equilibria [27], in such classifications it was unknown that they can exhibit exactly two invariant straight lines (taking into account their multiplicities) and one invariant ellipse.

2. Preliminary results

We start by reviewing some fundamental concepts that will be used throughout this work. Let $X = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ denote the polynomial vector field associated to the differential system (1).

2.1. First integrals and integrating factors

Let U be an open and dense set in $\mathbb{R}^2$, and let $H : U \to \mathbb{R}$ be a continuously differentiable non-locally function. We say that H is a *first integral* of the vector field X (or of system (1)) if it remains constant along all the solutions $(x(t), y(t))$ of system (1) that are contained in U. That is, H is a first integral if and only if it satisfies the condition

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P(x,y) + \frac{\partial H}{\partial y}Q(x,y) = 0,$$

in all the points of U. Now consider a function $R : U \to \mathbb{R}$ of class C^1 that does not vanish on U. The function R is called an *integrating factor* of the vector field X on U if one of the following three equivalent conditions holds:

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \mathrm{div}(RP, RQ) = 0, \qquad XR = -R\,\mathrm{div}(P,Q),$$

where the divergence of the vector field (P, Q) is defined as

$$\mathrm{div}(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Given such an integrating factor R, one can construct a *first integral* H associated with it via the formula

$$H(x,y) = \int R(x,y)P(x,y)\,dy + h(x),$$

where the function h is determined by the condition

$$\frac{\partial H}{\partial x} = -RQ.$$

This construction ensures that the system can be rewritten as the Hamiltonian form:

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}.$$

2.2. Invariant algebraic curves

We say that system (1) possesses an *invariant algebraic curve* if there exists a nontrivial polynomial $f(x, y)$ such that

$$Xf = \frac{\partial f}{\partial x}P(x,y) + \frac{\partial f}{\partial y}Q(x,y) = K(x,y)\,f(x,y), \tag{2}$$

for some polynomial $K(x, y)$, called the *cofactor* associated with $f(x, y) = 0$. This condition is justified by the fact that, if an orbit of the system intersects the algebraic curve $f(x, y) = 0$, then Eq. (2) implies that the entire orbit is contained in the curve.

2.3. Extactic polynomial

Consider the vector field X associated with the polynomial differential system (1) of degree m. The *n-th extactic polynomial* of X, denoted by $E_n(X)$, is defined as the determinant

$$\det\begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l) \end{pmatrix},$$

where $\{v_1, v_2, \ldots, v_l\}$ is a basis of $\mathbb{R}_n[x, y]$, the $\mathbb{R}$-vector space of the polynomials in $\mathbb{R}[x, y]$ of degree at most n, and $l = \dfrac{(n+1)(n+2)}{2}$ is the dimension of this $\mathbb{R}$-vector space. Here we define derivatives $X^0(v_i) = v_i$ and $X^j(v_i) = X(X^{j-1}(v_i))$ for $j \geq 1$.

It is worth noting that this definition does not depend on the choice of basis in $\mathbb{R}_n[x, y]$.

Proposition 1 (Proposition 5.2 in [9]). *Let $f = 0$ be an invariant algebraic curve of degree n of the vector field X. Then the polynomial f divides the polynomial $E_n(X)$.*

Suppose that the invariant algebraic curve $f(x, y) = 0$, as introduced in Proposition 1, satisfies that f^k divides the extactic polynomial $E_n(X)$, but f^{k+1} does not divide. In this case the integer k is called the *multiplicity* of the curve $f = 0$.

From a geometric point of view, this multiplicity reflects the behaviour of the invariant algebraic curve $f = 0$ of the polynomial vector field X under perturbations of the vector field: within the space of polynomial vector fields of the same degree than the degree of X, the field X can be smoothly deformed into a family X_ε, in such a way that each perturbed field admits precisely k distinct invariant curves converging to $f = 0$ as $\varepsilon \to 0$. For a detailed discussion of this concept, see [9].

2.4. Poincaré compactification

To study the dynamics of the planar polynomial differential systems we extended their phase space to a compact space using the Poincaré compactification. This classical method extends the vector field from the plane $\mathbb{R}^2$ to the unit sphere $\mathbb{S}^2 = \left\{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\right\}$ by means of the central projection.

Let X be a polynomial vector field of degree d. Identify the tangent plane at the north pole $(0, 0, 1)$ of $\mathbb{S}^2$ with the plane $\mathbb{R}^2$, and project each point $(x, y, 1)$ on this plane along the straight line connecting it to the origin. This maps $\mathbb{R}^2$ onto $\mathbb{S}^2 \setminus \mathbb{S}^1$, where the equator $\mathbb{S}^1 := \{y \in \mathbb{S}^2 : y_3 = 0\}$ captures the points at infinity. The vector field X is then lifted to $\mathbb{S}^2 \setminus \mathbb{S}^1$, obtaining two symmetric copies of X denoted by X', one in the northern hemisphere and the other in the southern hemisphere.

To ensure an analytic extension to the entire sphere, we multiply the lifted vector field X' by y_3^d, obtaining a vector field defined in the whole sphere that we denote by $p(X)$. This process is called the *Poincaré compactification* . The extended vector field $p(X)$ contains both the finite behavior and the dynamics near infinity, which now correspond the neighborhood of the equator of $\mathbb{S}^2$.

The computations on the curved surface $\mathbb{S}^2$ are performed using local coordinate charts. Specifically we cover the sphere with six planar charts (U_i, ϕ_i) and (V_i, ψ_i) for $i = 1, 2, 3$, defined by:

$$U_i = \{y \in \mathbb{S}^2 : y_i > 0\}, \quad V_i = \{y \in \mathbb{S}^2 : y_i < 0\},$$
$$\phi_i(y) = \psi_i(y) = \left(\frac{y_j}{y_i}, \frac{y_k}{y_i}\right), \quad \text{with } j < k,\ j, k \neq i.$$

The local expressions of $p(X)$ in the three charts U_1, U_2, and U_3 are

$$(U_1): \quad \dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right),$$
$$(U_2): \quad \dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right),$$
$$(U_3): \quad \dot{u} = P(u, v), \quad \dot{v} = Q(u, v),$$

respectively. The charts V_i mirror the U_i expressions up to a factor $(-1)^{d-1}$, reflecting the symmetry of the compactified field.

Note that the lines $v = 0$ in the first two charts correspond to the equator $\mathbb{S}^1$, i.e. to the points at infinity of the original vector field X. An equilibrium of $p(X)$ located on $\mathbb{S}^1$ is called an *infinite equilibrium* of the planar vector field X. Due to the central symmetry of the construction, if $y \in \mathbb{S}^1$ is an equilibrium, then also $-y$ is an equilibrium, with their stability related by the parity of d: for odd d they share stability, while for even d they have opposite stability.

Because of this symmetry the flow of $p(X)$ on the whole sphere is completely determined by its restriction to the closed northern hemisphere $\{y \in \mathbb{S}^2 : y_3 \geq 0\}$. Projecting this closed hemisphere vertically onto the plane $y_3 = 0$ yields the *Poincaré disc* $\mathbb{D}$, on which the phase portrait of $\pi(p(X))$ is typically visualized.

For a more detailed discussion of this construction, see Chapter 5 of [11].

2.5. Topological equivalence via separatrix configuration

To compare dynamics on the Poincaré disc $\mathbb{D}$, we consider topological equivalence of compactified vector fields. Two such systems are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D} \to \mathbb{D}$, respecting the boundary $\mathbb{S}^1$ and mapping orbits of $\pi(p(X_1))$ onto those of $\pi(p(X_2))$, preserving or reversing the orientation of all the orbits.

A central tool in this classification is the *separatrix configuration*, which encodes the global structure of orbits. The *set of separatrices* is formed by all the orbits at infinity, all finite equilibria, limit cycles, the two orbits of the boundary of all the finite and infinite hyperbolic sectors. Every connected component of the Poincaré disc after removing the set of separatrices is a *canonical region*. The separatrix configuration is formed by the set of separatrices and one representative orbit from each canonical region. We denote the set of separatrices by Σ_X and the separatrix configuration by Σ'_X. We denote by S the number of separatrices in the phase portrait of $\pi(p(X))$ on the Poincaré disc $\mathbb{D}$, and by R the number of its canonical regions. If there are equilibria lying entirely on the circle at infinity we do not count them in the number of S.

As shown in the following theorem, which was proved by Markus [14], Neumann [16] and Peixoto [20], two vector fields on $\mathbb{D}$ are topologically equivalent if and only if there exists a homeomorphism $h : \Sigma'_{X_1} \to \Sigma'_{X_2}$ such that $h(\Sigma'_{X_1}) = \Sigma'_{X_2}$. This reduces the problem of classifying the phase portrait in the Poincaré disc to classify its separatrix configuration.

Theorem 2. *Two Poincaré compactified polynomial vector fields $\pi(p(X_1))$ and $\pi(p(X_2))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations Σ'_{X_1} and Σ'_{X_2} are topologically equivalent.*

2.6. Quadratic systems with an invariant ellipse and an invariant straight line

Lemma 3 (Theorem 2.3 in [13]). *After an affine change of coordinates and a time rescaling, any quadratic system with an invariant ellipse and an invariant straight line can be written in one of the following forms*

$$(E.1)\quad \dot{x} = -(x^2+y^2-1)-2b(ax+y+c)y, \qquad \dot{y} = a(x^2+y^2-1)+2b(y+ax+c)x,$$
$$(E.2)\quad \dot{x} = \frac{a}{2}(x^2+y^2-1)+(dx-by-bc)y, \qquad \dot{y} = (y+c)(dy+bx+d),$$

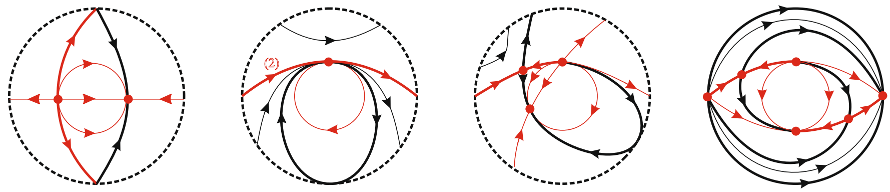
Here a, b, c, d are real parameters and $(c+1)d = 0$ in $(E.2)$.

3. Main results

The following results exhibit the normal forms and the corresponding phase portraits on the Poincaré disc for all planar quadratic systems with two invariant real straight lines (taking into account their multiplicities) and one invariant ellipse.

Theorem 4. *After an affine change of variables and a rescaling of time any quadratic system having two invariant real straight lines $f_j = f_j(x, y) = 0$ for $j = 2, 3$, and one invariant ellipse $f_1 = f_1(x, y) = 0$ with cofactors $K_j = K_j(x, y)$ for $j = 1, 2, 3$, can be written as one of the following systems, where $H = H(x, y)$ is a first integral and $R = R(x, y)$ is an integrating factor of the corresponding quadratic system.*

$$S(1)\quad \dot{x} = (ax+y+b)y-(x^2+y^2-1), \qquad \dot{y} = -(ax+y+b)x+a(x^2+y^2-1),$$
$$f_1 = x^2+y^2-1, \qquad K_1 = 2(ay-x),$$
$$f_2 = ax+y+b, \qquad K_2 = ay-x,$$
$$f_3 = Bx+(a-Ab)y+ab-A, \qquad K_3 = ay-x+A,$$
$$H = \frac{f_1}{f_2^2}, \qquad \text{where } A = \sqrt{a^2-b^2+1},\ B = b^2-1.$$



PP.1 S=6, R=3 System S(1,2)

PP.2 S=5, R=4 System S(1)

PP.3 S=9, R=4 System S(1)

PP.4 S=16, R=5 System S(2)

FIGURE 1. Phase portraits in the Poincaré disc of the systems in Theorem 4. Separatrices are represented by thick black or red lines, invariant curves are drawn in red, while trajectories within canonical regions are drawn with thin black lines

$$
\begin{aligned}
S(2)\ \ & \dot{x} = a(x^2+y^2-1)+2xy, \quad \dot{y} = 2(y^2-1),\\
& f_1 = x^2+y^2-1, \quad K_1 = 2(ax+2y),\\
& f_2 = y-1, \quad K_2 = 2(y+1),\\
& f_3 = y+1, \quad K_3 = 2(y-1), \quad R = f_1^{-1}(f_2f_3)^{-\frac{1}{2}}.
\end{aligned}
$$

Here $a, b \in \mathbb{R}$.

Theorem 5. *The phase portrait of each system from Theorem 4 in the Poincaré disc is topologically equivalent to one of the phase portraits shown in Fig. 1. Under each phase portrait in the Poincaré disc appears its name, the number of its separatrices S, the number of its canonical regions R, and the system having this phase portrait.*

4. Proof of Theorem 4

From Lemma 3 any quadratic system that admits an invariant real ellipse and an invariant straight line can be written in the form of $(E.1)$ or $(E.2)$.

System $(E.1)$**.** Suppose that system $(E.1)$ has an additional invariant straight line $g = 0$ with cofactor L, where

$$g = b_1x + b_2y + b_0, \quad L = l_1x + l_2y + l_0, \quad b_i, l_i \in \mathbb{R} \text{ for } i = 0, 1, 2.$$

We divide the analysis into the following two cases: $b_1 \neq 0$ and $b_1 = 0$.

Case 1. $b_1 \neq 0$**.** By rescaling the variable x we may assume that $b_1 = 1$. From the definition of invariant algebraic curve

$$\frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y} = Lg,$$

we obtain the following four solutions

$$E.1.\mathrm{s}_1 = \left\{ b = -\frac{l_2}{2a}, \quad b_0 = \frac{c}{a}, \quad b_2 = \frac{1}{a}, \quad l_0 = 0, \quad l_1 = -\frac{l_2}{a} \right\},$$

$$E.1.\mathrm{s}_2 = \left\{ b = 0, \quad b_2 = \frac{1}{a}, \quad l_0 = 0, \quad l_1 = 0, \quad l_2 = 0 \right\},$$

$$E.1.\mathrm{s}_3 = \left\{ b = -\frac{1}{2}, \quad b_0 = \frac{a\,c - A}{C}, \quad b_2 = \frac{a - c\,A}{C}, l_0 = A, \quad l_1 = -1, \quad l_2 = a \right\},$$
$$E.1.\mathrm{s}_4 = \left\{ b = -\frac{1}{2}, \quad b_0 = \frac{a\,c + A}{C}, \quad b_2 = \frac{a + c\,A}{C}, l_0 = -A, \quad l_1 = -1, \quad l_2 = a \right\}, \tag{3}$$

where $A = \sqrt{a^2 - c^2 + 1}$, $C = c^2 - 1$. The solution $E.1.\mathrm{s}_1$ corresponds to a system that admits a double invariant straight line. By imposing that g^2 is a factor of the extactic polynomial, we obtain the system

$$\begin{aligned}
&a^2(c^2-1)\big(1-c^2+a\big(a+ac^2-c^2l_2\big)\big)=0,\\
&2a^2c\big(2a^3c^2+l_2-c^2l_2+a^2l_2(1-2c^2)\big)=0,\\
&a(a^2+1)\big(6a^3c^2+l_2-c^2l_2+a^2l_2(1-6c^2)\big)=0,\\
&4a^2c(a^2+1)^2\,(a-l_2)=0,\\
&a(a^2+1)^3\,(a-l_2)=0,
\end{aligned}$$

which admit two solutions, denoted by $E.1.\mathrm{s}_1.k$ for $k = 1, 2$:

$$E.1.\mathrm{s}_1.1 = \left\{ c = -\sqrt{a^2+1},\ l_2 = a \right\}, \quad E.1.\mathrm{s}_1.2 = \left\{ c = \sqrt{a^2+1},\ l_2 = a \right\}.$$

However the solution $E.1.\mathrm{s}_1.1$ corresponds to the case $c = -\sqrt{a^2+1}$ of $E.1.\mathrm{s}_4$, while the solution $E.1.\mathrm{s}_1.2$ corresponds to the case $c = \sqrt{a^2+1}$ of $E.1.\mathrm{s}_3$.

The solution $E.1.\mathrm{s}_2$ corresponds to the system

$$\dot{x} = 1 - x^2 - y^2, \quad \dot{y} = a(x^2 + y^2 - 1).$$

This system can degenerate into a constant vector field.

The systems corresponding to the solutions $E.1.\mathrm{s}_3$ and $E.1.\mathrm{s}_4$ can be written into the form

$$\dot{x} = y(ax + y + c) - (x^2 + y^2 - 1), \quad \dot{y} = -x(ax + y + c) + a(x^2 + y^2 - 1),$$

which is precisely system $S(1)$.

Case 2. $b_1 = 0$**.** Since $b_2 \neq 0$, we may assume without loss of generality that $b_2 = 1$. Substituting g, L, and system $(E.1)$ into (2), we obtain the following four solutions

$$\begin{aligned}
E.1.\mathrm{s}_5 =& \{\, a = 0,\ b = 0,\ l_0 = 0,\ l_1 = 0,\ l_2 = 0\},\\
E.1.\mathrm{s}_6 =& \{\, a = 0,\ c = b_0,\ l_0 = 0,\ l_1 = 2b,\ l_2 = 0\},\\
E.1.\mathrm{s}_7 =& \left\{\, a = l_0,\ b = -\frac{1}{2},\ b_0 = -1,\ c = -1,\ l_1 = -1,\ l_2 = l_0 \right\},\\
E.1.\mathrm{s}_8 =& \left\{\, a = -l_0,\ b = -\frac{1}{2},\ b_0 = 1,\ c = 1,\ l_1 = -1,\ l_2 = -l_0 \right\}.
\end{aligned}$$

The solution $E.1.\mathrm{s}_5$ corresponds to the system

$$\dot{x} = 1 - x^2 - y^2, \quad \dot{y} = 0.$$

The solution $E.1.\mathrm{s}_6$ corresponds to a system that admits a double invariant straight line. By requiring that g^2 is a factor of the extactic polynomial, as in **Case 1**, we obtain the following system

$$1-2b_0^2+b_0^4=0,\quad 2b+b_0^2-(2b+1)\,b_0^2=0,\quad 2b+1=0.$$

However all solutions of these polynomials are special case of $E.1.\mathrm{s}_7$ or $E.1.\mathrm{s}_8$.

The solution $E.1.\mathrm{s}_7$ corresponds to the system

$$\dot{x}=-x^2+l_0xy-y+1,\quad \dot{y}=l_0y^2-xy+x-l_0.$$

This is the case $c=-1, a=l_0$ of system $S(1)$.

The solution $E.1.\mathrm{s}_8$ corresponds to the system

$$\dot{x}=-x^2-l_0xy+y+1,\quad \dot{y}=-l_0y^2-xy-x+l_0.$$

This is the case $c=1, a=-l_0$ of the system $S(1)$.

System $(E.2)$**.** Since $(c+1)d=0$ in $(E.2)$, and considering the invariant straight line $g=b_1x+b_2y+b_0=0$, we distinguish the following four cases: 1. $d\neq 0, b_1=1$; 2. $d\neq 0, b_1=0$; 3. $d=0, b_1=1$; 4. $d=0, b_1=0$.

If $d\neq 0$, then from $(E.2)$ we have $c+1=0$. Applying the transformation $(a,b,c)\to(ad,bd,-1)$ and the time rescaling $t\to\dfrac{t}{d}$ to system $(E.2)$, we obtain the system

$$\dot{x}=\frac{a}{2}(x^2+y^2-1)-y(by-x-b),\quad \dot{y}=(y-1)(y+bx+1),$$

denoted by $S(E.2.1)$.

If $d=0$, then by applying the transformation $(a,d)\to(ab,0)$ and the time rescaling $t\to\dfrac{t}{b}$, we obtain the system

$$\dot{x}=\frac{a}{2}(x^2+y^2-1)-y(y+c),\quad \dot{y}=x(y+c),$$

denoted by $S(E.2.2)$.

Case 1. $d\neq 0, b_1=1$. Substituting $g=x+b_2y+b_0$, the cofactor L, and system $S(E.2.1)$ into (2), we obtain the following three solutions

$$E.2.\mathrm{s}_1=\{\,a=-2\,b_2,\ b=-b_2,\ b_0=-b_2,\ l_0=0,\ l_1=-b_2,\ l_2=1\},$$

$$E.2.\mathrm{s}_2=\{\,a=-2(b_2+B),\ b=-b_2-B,\ b_0=-B,\ l_0=-1,\ l_1=-b_2-B,\ l_2=1\},$$

$$E.2.\mathrm{s}_3=\{\,a=-2(b_2-B),\ b=-b_2+B,\ b_0=B,\ l_0=-1,\ l_1=-b_2+B,\ l_2=1\},$$

where $B=\sqrt{b^2+1}$. Solutions $E.2.\mathrm{s}_i$ for $i=1,2,3$ provide $a=2b$, which means that these solutions correspond to the system $S(E.2.1.\mathrm{s}_1)$

$$\dot{x}=xy+b(x^2+y-1),\quad \dot{y}=(y-1)(bx+y+1).$$

Note that applying the transformation $(x,y,t,a,c)\to(-x,-y,-bt,-b^{-1},1)$ to $S(1)$ we obtain system $S(E.2.1.\mathrm{s}_1)$.

Case 2. $d\neq 0, b_1=0$. In this case we may assume $b_2=1$. Substituting $g=y+b_0$, the cofactor L, and system $S(E.2.1)$ into (2), we obtain the following two solutions

$$E.2.\mathrm{s}_4=\{\,b=l_1,\ b_0=-1,\ l_0=1,\ l_2=1\},$$

$$E.2.\mathrm{s}_5=\{\,b=0,\ b_0=1,\ l_0=-1,\ l_1=0,\ l_2=1\}.$$

The solution $E.2.\mathrm{s}_4$ corresponds to a system that admits a double invariant straight line. By requiring that g^2 is a factor of the extactic polynomial, as in **Case 1**, we find that no additional solution arises. Therefore there is only one system

$$\dot{x} = 2xy + a(x^2 + y^2 - 1), \quad \dot{y} = 2(y^2 - 1),$$

denoted by $S(2)$.

Case 3. $d = 0, b_1 = 1$. Substituting $g = x + b_2 y + b_0$, the cofactor L, and system $S(E.2.2)$ into (2), we obtain the following two solutions

$$E.2.\mathrm{s}_6 = \left\{a = 2,\ b_0 = -B,\ c = -\frac{b_2}{B},\ l_0 = \frac{1}{B},\ l_1 = 1,\ l_2 = 0\right\},$$
$$E.2.\mathrm{s}_7 = \left\{a = 2,\ b_0 = B,\ c = \frac{b_2}{B},\ l_0 = -\frac{1}{B},\ l_1 = 1,\ l_2 = 0\right\},$$

where $B = \sqrt{b^2 + 1}$. These solutions correspond to the system

$$\dot{x} = x^2 - cy - 1, \quad \dot{y} = x(y + c),$$

denoted by $S(E.2.2.\mathrm{s}_6)$. Note that applying the transformation $(x, y, a) \to (-x, y, 0)$ to $S(1)$ yields system $S(E.2.2.\mathrm{s}_6)$.

Case 4. $d = 0, b_1 = 0$. In this case we may assume $b_2 = 1$. Substituting $g = y + b_0$, the cofactor L, and system $S(E.2.2)$ into (2), and imposing that g^2 is a factor of the extactic polynomial, as in **Case 1**, we obtain the following solution

$$E.2.\mathrm{s}_8 = \{a = 2,\ b_0 = -1,\ c = -1,\ l_0 = 0,\ l_1 = 1,\ l_2 = 0\},$$

and the corresponding system $S(E.2.2.\mathrm{s}_8)$

$$\dot{x} = x^2 + y - 1, \quad \dot{y} = x(y - 1).$$

Applying the transformation $(x, y, t, a, c) \to (-x, -y, -t, 0, 1)$ to $S(1)$ yields system $S(E.2.2.\mathrm{s}_8)$. This completes the proof of Theorem 4.

5. Proof of Theorem 5

System $S(1)$**.** The system

$$\dot{x} = (ax + y + b)y - (x^2 + y^2 - 1), \quad \dot{y} = -(ax + y + b)x + a(x^2 + y^2 - 1),$$

admits the following invariant algebraic curves:

$$f_1 = x^2 + y^2 - 1, \quad f_2 = ax + y + b, \quad f_3 = Bx + (a - Ab)y + ab - A,$$

where $A = \sqrt{a^2 - b^2 + 1}$, $B = b^2 - 1$ and $a^2 - b^2 + 1 \geq 0$. We divide the parameter plane (a, b) into a series of zones as in Fig. 2. Note that if $(a, b) \in h_2 \cup h_3$, we have $f_3 = 0$ and then the system has only one invariant straight line. So we consider the system under condition $(a, b) \notin h_2 \cup h_3$.

To get the global phase portrait we divide the study of system $S(1)$ into the following five cases:

1. $(a, b) = (0, 0)$;
2. $(a, b) \in h_4 \setminus (0, 0)$;
3. $(a, b) \in \{(0, b) | b^2 < 1, b \neq 0\}$;
4. $(a, b) \in h_1 \setminus \{(0, 1), (0, -1)\}$;
5. $(a, b) \in Z_k$ for $k = 1, \cdots 4$,

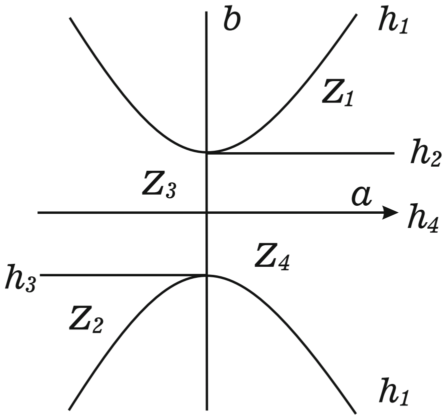


FIGURE 2. Zones of the parameter plane (a, b) for system $S(1)$, bounded by the straight lines $h_1 : a^2 - b^2 + 1 = 0$, $h_2 : b = 1$ and $a \geq 0$, $h_3 : b = -1$ and $a \leq 0$, and $h_4 : b = 0$

where

$$Z_1 : \{(a, b)|a^2 - b^2 + 1 > 0, b > 1, a > 0\},$$
$$Z_2 : \{(a, b)|a^2 - b^2 + 1 > 0, b < -1, a < 0\},$$
$$Z_3 : \{(a, b)|a^2 - b^2 + 1 > 0, b > 0\}\backslash(Z_1 \cup h_2),$$
$$Z_4 : \{(a, b)|a^2 - b^2 + 1 > 0, b < 0\}\backslash(Z_2 \cup h_3).$$

Case 1. $(a, b) = (0, 0)$. The system reduces to

$$\dot{x} = 1 - x^2, \quad \dot{y} = -xy.$$

There are two equilibria P_1 and P_2 in $\mathbb{R}^2$. We use the notation $P = (x, y) : \lambda_1, \lambda_2$ to denote an equilibrium point together with its corresponding eigenvalues. Thus,

$$P_1 = (-1, 0) : 1, 2, \quad P_2 = (1, 0) : -1, -2.$$

In the chart U_1 the Poincaré compactification is

$$\dot{u} = -uv^2, \quad \dot{v} = -v(v^2 - 1).$$

By eliminating the common factor v, we obtain the system on the line at infinity $v = 0$:

$$\dot{u}|_{v=0} = 0, \quad \dot{v}|_{v=0} = 1.$$

Then there are infinitely many equilibria on the line at infinity $v = 0$. By the normally hyperbolic theory at each infinite equilibrium starts one orbit going inside the Poincaré disc, see for more details [12].

In the chart U_2 the Poincaré compactification is

$$\dot{u} = v^2, \quad \dot{v} = uv.$$

After removing the common factor v the origin O_2 is an equilibrium point of the system: $O_2 : -1, 1$.

Case 2. $(a,b) \in h_4 \setminus (0,0)$. The system reduces to

$$\dot{x} = axy - x^2 + 1, \quad \dot{y} = ay^2 - xy - a,$$

where $a \neq 0$. There are two equilibria

$$P_1 = \left(\frac{1}{A_0}, -\frac{a}{A_0}\right) : -2A_0, -A_0, \quad P_2 = \left(-\frac{1}{A_0}, \frac{a}{A_0}\right) : 2A_0, A_0,$$

where $A_0 = \sqrt{a^2+1}$.

In the chart U_1 the Poincaré compactification is

$$\dot{u} = -v^2(u+a), \quad \dot{v} = -v(v^2 + au - 1).$$

By eliminating the common factor v, we obtain the system on the line at infinity $v = 0$:

$$\dot{u}|_{v=0} = 0, \quad \dot{v}|_{v=0} = 1 - au.$$

Then there are infinitely many equilibria on the line at infinity $v = 0$, and there exists an orbit that is tangent to $v = 0$ at $u = \dfrac{1}{a}$. By the normally hyperbolic theory at each infinite equilibrium with $1 - au > 0$ starts one orbit that enter inside the Poincaré disc, and at each infinite equilibrium with $1 - au < 0$ ends one orbit coming form the interior of the Poincaré disc.

In the chart U_2 the Poincaré compactification is

$$\dot{u} = v^2(au+1), \quad \dot{v} = v(av^2 + u - a).$$

After removing the common factor the origin is not an equilibrium point of the system. By the normally hyperbolic theory one orbit starts at the origin of U_2 if $a < 0$ that enters inside the Poincaré disc, and one orbit ends at the origin of U_2 if $a > 0$ coming from the interior of the Poincaré disc.

Case 3. $(a,b) \in \{(0,b)|b^2 < 1, b \neq 0\}$. The system reduces to

$$\dot{x} = by - x^2 + 1, \quad \dot{y} = -x(y+b),$$

where $b \neq 0$. There are three equilibria

$$P_1 = \left(0, -\frac{1}{b}\right) : -B_0, B_0, \quad P_2 = (-B_0, -b) : B_0, 2B_0,$$
$$P_3 = (B_0, -b) : -2B_0, -B_0,$$

where $B_0 = \sqrt{1-b^2}$.

In the chart U_1 the Poincaré compactification is

$$\dot{u} = -v(bu^2 + uv + b), \quad \dot{v} = -v(v^2 + buv - 1).$$

By eliminating the common factor v, we obtain the system on the line at infinity $v = 0$:

$$\dot{u}|_{v=0} = -b(u^2+1), \quad \dot{v}|_{v=0} = 1.$$

Then there are infinitely many equilibria on the line at infinity $v = 0$. By the normally hyperbolic theory at each infinite equilibrium starts one orbit going inside the Poincaré disc, see for more details [12].

In the chart U_2 the Poincaré compactification is

$$\dot{u} = v(bu^2 + v + b), \quad \dot{v} = uv(bv+1).$$

After removing the common factor the origin is not an equilibrium of the system. At the origin of the local chart U_2 there is a hyperbolic sector.

Case 4. $(a,b) \in h_1 \setminus \{(0,1),(0,-1)\}$. There is an equilibrium $P_1 = \left(-\frac{a}{b}, -\frac{1}{b}\right) : 0,0$ in $\mathbb{R}^2$, where $a^2 - b^2 + 1 = 0$. Since the linear part of the Jacobian matrix at P_1 is

$$\begin{pmatrix} \frac{a}{b} & -\frac{1}{b} \\ \frac{a^2}{b} & -\frac{a}{b} \end{pmatrix},$$

the point P_1 is a nilpotent equilibrium. It satisfies condition (4.iii.iii2) of Theorem 3.5 in [11]. Therefore, its phase portrait consists of one hyperbolic and one elliptic sector.

In the chart U_1 the Poincaré compactification is

$$\dot{u} = -v(bu^2 + uv + av + b), \quad \dot{v} = -v(v^2 + buv + au - 1).$$

By eliminating the common factor v, we obtain the system on the line at infinity $v = 0$:

$$\dot{u}|_{v=0} = -b(u^2 + 1), \quad \dot{v}|_{v=0} = 1 - au.$$

Then there are infinitely many equilibria on the line at infinity $v = 0$, and there exists an orbit that is tangent to $v = 0$ at $u = \frac{1}{a}$. By the normally hyperbolic theory at each infinite equilibrium with $1 - au > 0$ starts one orbit that enter inside the Poincaré disc, and at each infinite equilibrium with $1 - au < 0$ ends one orbit coming from the interior of the Poincaré disc.

In the chart U_2 the Poincaré compactification is

$$\dot{u} = v(bu^2 + auv + v + b), \quad \dot{v} = v(av^2 + cuv + u - a).$$

After removing the common factor the origin is not an equilibrium point of the system. By the normally hyperbolic theory one orbit starts at the origin of U_2 if $a < 0$ that enters inside the Poincaré disc, and one orbit ends at the origin of U_2 if $a > 0$ coming from the interior of the Poincaré disc.

Case 5. $(a,b) \in Z_k$ for $k = 1, \cdots 4$. There are three equilibria in $\mathbb{R}^2$:

$$P_1 = \left(-\frac{a}{b}, -\frac{1}{b}\right) : -A, A, \quad P_2 = \left(\frac{A - ab}{A_0^2}, -\frac{aA + b}{A_0^2}\right) : -2A, -A,$$
$$P_3 = \left(-\frac{A + ab}{A_0^2}, \frac{aA - b}{A_0^2}\right) : 2A, A.$$

The analysis of the infinite equilibria is the same as that of **Case 4**.

Since the curves $x^2 + y^2 - 1 = 0, ax + y + b = 0$ and $Bx + (a - Ab)y + ab - A = 0$ are invariant algebraic curves of system $S(1)$, we obtain the configurations $\mathrm{C}(S1.i)$ for the **Case** i, $i = 1, \cdots, 4$, and the configurations $\mathrm{C}(S1.5.Z_j)$ for **Case 5** where $(a,b) \in Z_j$, $j = 1, \cdots, 4$, as shown in Fig. 3.

Table 1 presents the types of finite and infinite equilibria, their corresponding configurations, and the phase portraits in the Poincaré disc shown in Fig. 1.

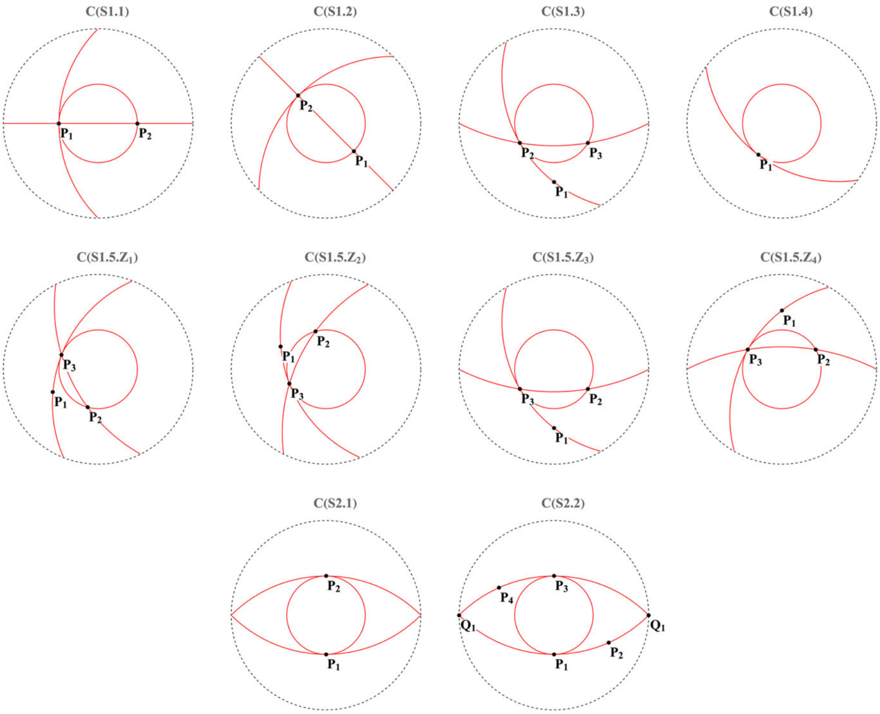


FIGURE 3. Configurations of the invariant algebraic curves for the systems in Theorem 4. Here the points P_i for $i = 1, 2, 3, 4$ are equilibria in the plane $\mathbb{R}^2$, and Q_1 is an equilibrium in the chart U_1

TABLE 1. The local phase portraits at the finite and infinite equilibria, as well as the corresponding configurations and the phase portraits in the Poincaré disc for system $S(1)$. The abbreviations "Ns", "Nu", "S" and "Ni" stand for "stable node", "unstable node", "saddle", and "nilpotent", respectively

Case	P_1	P_2	P_3	Config	PP
1	Nu	Ns	–	C(S1.1)	PP.1
2	Ns	Nu	–	C(S1.2)	PP.1
3	S	Nu	Ns	C(S1.3)	PP.3
4	Ni	–	–	C(S1.4)	PP.2
5, Z_1	S	Ns	Nu	C($S1.5.Z_1$)	PP.3
5, Z_2	S	Ns	Nu	C($S1.5.Z_2$)	PP.3
5, Z_3	S	Ns	Nu	C($S1.5.Z_3$)	PP.3
5, Z_4	S	Ns	Nu	C($S1.5.Z_4$)	PP.3

System $S(2)$**.** The system

$$\dot{x} = a(x^2 + y^2 - 1) + 2xy, \quad \dot{y} = 2(y^2 - 1),$$

admits the following invariant algebraic curves:

$$f_1 = x^2 + y^2 - 1, \quad f_2 = y - 1, \quad f_3 = y + 1.$$

We consider this system in the following two cases: $a = 0$ and $a \neq 0$.

Case 1. $a = 0$. The system reduces to

$$\dot{x} = 2xy, \quad \dot{y} = 2(y^2 - 1).$$

There are two finite equilibria:

$$P_1 = (0, -1) : -2, -4, \quad P_2 = (0, 1) : 2, 4.$$

In the chart U_1 the Poincaré compactification is

$$\dot{u} = -2v^2, \quad \dot{v} = -2uv.$$

By eliminating the common factor v, we find that there are infinitely many equilibria on the line at infinity $v = 0$. By the normally hyperbolic theory at each infinite equilibrium with $u < 0$ starts one orbit that enter inside the Poincaré disc, and at each infinite equilibrium with $u > 0$ ends one orbit coming form the interior of the Poincaré disc. At $u = 0$ there is a hyperbolic sector.

In the chart U_2 the Poincaré compactification is

$$\dot{u} = 2uv^2, \quad \dot{v} = 2v(v^2 - 1).$$

The origin is a non-isolated equilibrium point of this system. From the normally hyperbolic theory an orbit from the interior of the Poincaré disc ends at the origin of the chart U_2.

Case 2. $a \neq 0$. There are four finite equilibria:

$$P_1 = (0, -1) : -4, -2, \quad P_2 = \left(\frac{2}{a}, -1\right) : -4, 2,$$

$$P_3 = (0, 1) : 4, 2, \quad P_4 = \left(-\frac{2}{a}, 1\right) : 4, -2.$$

In the chart U_1 the Poincaré compactification is

$$\dot{u} = -au^3 + auv^2 - 2v^2 - au, \quad \dot{v} = av^3 - au^2v - 2uv - av.$$

The origin is an equilibrium point of this system, i.e. $O_1 = (0, 0) : -a, -a$.

In the chart U_2 the Poincaré compactification is

$$\dot{u} = 2uv^2 + au^2 - av^2 + a, \quad \dot{v} = 2v^3 - 2v.$$

The origin is not an equilibrium point.

Since the curves $x^2 + y^2 - 1 = 0, y - 1 = 0$ and $y + 1 = 0$ are invariant algebraic curves of system $S(2)$, we obtain the configurations $\mathrm{C}(S2.i)$ corresponding to **Case** i, $i = 1, 2$, according to the conditions of the above two cases, as shown in Fig. 3.

Table 2 presents the types of finite and infinite equilibria, their corresponding configurations, and the associated phase portraits in the Poincaré disc shown in Fig. 1.

Table 2. The local phase portraits at the finite and infinite equilibria, together with the corresponding configurations and phase portraits in the Poincaré disc for system $S(2)$

Zone	P_1	P_2	P_3	P_4	Q_1	Config	PP
$a = 0$	Ns	Nu	–	–	–	C($S2.1$)	PP.1
$a > 0$	Ns	S	Nu	S	Ns	C($S2.2$)	PP.4
$a < 0$	Ns	S	Nu	S	Nu	C($S2.2$)	PP.4

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Author contributions Huaxin Ou conducted the system classification and wrote the initial draft of the manuscript. Jaume Llibre supervised the research and contributed substantially to the revision and improvement of the manuscript. Both authors reviewed and approved the final version.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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