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Solvability of the Poisson-Dirichlet problem with interior data in $L^{p'}$ -Carleson spaces and its applications to the L^p -regularity problem

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Abstract

We prove that the $L^{p'}$ -solvability of the homogeneous Dirichlet problem for an elliptic operator $L = -\operatorname{div} A \nabla$ with real and merely bounded coefficients is equivalent to the L^p -solvability of the Poisson Dirichlet problem $Lw = H - \operatorname{div} F$, which is defined in terms of an $L^{p'}$ estimate on the non-tangential maximal function, assuming that $\operatorname{dist}(\cdot, \partial Q)H$ and F lie in certain $L^{p'}$ -Carleson-type spaces, and that the domain $Q \subset \mathbb{R}^{n+1}$, $n \geq 2$, satisfies the corkscrew condition and has n -Ahlfors regular boundary. In turn, we use this result to show that, in a bounded domain with uniformly n -rectifiable boundary that satisfies the corkscrew condition, $L^{p'}$ -solvability of the homogeneous Dirichlet problem for an operator $L = -\operatorname{div} A \nabla$ satisfying the Dahlberg-Kenig-Pipher condition (of arbitrarily large constant) implies solvability of the L^p -regularity problem for the adjoint operator $L^* = -\operatorname{div} A^T \nabla$, where $1/p + 1/p' = 1$ and A^T is the transpose matrix of A . This result for Dahlberg-Kenig-Pipher operators is new even if Q is the unit ball, despite the fact that the $L^{p'}$ -solvability of the Dirichlet problem for these operators in Lipschitz domains has been known since 2001.

Further novel applications include i) new local estimates for the Green's function and its gradient in rough domains, ii) a local $W^{1,p}$ -type theorem for the L^p -solvability of the ‘‘Poisson-Regularity problem’’, itself equivalent to the $L^{p'}$ -solvability of the homogeneous Dirichlet problem, in terms of certain gradient estimates for local landscape functions, and iii) new L^p estimates for the eigenfunctions (and their gradients) of symmetric operators L on bounded rough domains.

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Introduction

The L^p -solvability of homogeneous boundary value problems for second-order linear elliptic partial differential equations with non-smooth coefficients on rough domains has been an area of intense and fruitful research in the last few decades, culminating in shocking equivalences between certain geometric properties of domains in the Euclidean space and L^p -solvability of the homogeneous Dirichlet problem for the Laplacian. However, the literature regarding the L^p -solvability of corresponding *inhomogeneous problems* is far more scant. Here, by L^p -solvability of a boundary value problem, we mean solvability of the problem together with a natural, scale-invariant bound on the L^p -norm of an appropriate non-tangential maximal function. For instance, a natural question to consider is to find the quantitative conditions which should be put on H and F so that the solution w to the problem

$$\begin{cases} \operatorname{div}(\operatorname{eq.poisson}) \left(\begin{array}{l} \operatorname{div} A \nabla w = H - \operatorname{div} F, \\ \text{in } \Omega, \\ u = g, \end{array} \right) \text{ on } \partial \Omega, \end{cases}$$

satisfies that $\square(w) \in L^p(\partial Q)$, where $\square$ is the non-tangential maximal function, defined in Section 1.1. Assuming that Q has reasonable geometry, if $H \equiv 0$, $F \equiv 0$, and $A \equiv I$, then it is known that $g \in L^p(\partial Q)$ implies that $\square(w) \in L^p(\partial Q)$, but the problem for non-trivial H and F is not well understood, although energy estimates for solutions of the Poisson problem are well-known if the data lie in Besov spaces (see also Section 1.2.4 for some related results in the literature).

In this paper, we prove that the L^p -solvability of the homogeneous Dirichlet problem (denoted (D_p^L))

$$\begin{cases} \operatorname{div}(\operatorname{eq.dirichlet}) \left(\begin{array}{l} \operatorname{div} A \nabla u = 0, \\ \text{in } \Omega, \\ u = g, \end{array} \right) \text{ on } \partial \Omega, \end{cases}$$

is equivalent to the L^p -solvability of the *Poisson problem* ([eq.poisson](#)), whenever H and F lie in certain Carleson spaces (see Theorems 1 and 1). Furthermore, we will see a non-trivial application of this result to the regularity problem for elliptic PDEs with rough coefficients, which was our initial motivation to study the Poisson problem and shows that the question of the L^p -solvability of the Poisson problem is far from being an isolated curiosity.

We use the L^p -solvability of the Poisson problem to study the Dirichlet regularity problem for Dahlberg-Kenig-Pipher operators, which are operators $L = -\operatorname{div} A \nabla$ where $\operatorname{dist}(\cdot, \partial Q)^2 |\nabla A|^2$ satisfies a Carleson measure condition. More precisely, we show that if $Q \subset \mathbb{R}^{n+1}$, $n \geq 2$ is bounded and uniformly n -rectifiable, and if A is a real, not necessarily symmetric, DKP matrix in Q (see Definition 1), then L^p -solvability of the Dirichlet problem ([eq.dirichlet](#)) for the adjoint operator $L^* = -\operatorname{div} A^T \nabla$ implies L^p -solvability of the Dirichlet *regularity problem* for the operator L (see Theorem 1). In particular, if Q is a bounded domain satisfying the corkscrew condition which has n -Ahlfors regular boundary and interior big pieces of chord-arc domains, then for any DKP operator associated with Q , there exists $p > 1$ such that the regularity problem for L is L^p -solvable (Corollary 1). Note that Corollary 1 is new even when Q is the unit ball, and our result complements the study of the Dirichlet problem for DKP operators, initiated in 2001.

Briefly, we mention several further novel applications of the L^p -solvability of the Poisson problem:

- We present new local L^p estimates on the n.t. maximal function of the Green's function and its gradient (see ([eq.trueint1](#)) and ([eq.trueint2](#))), which are equivalent to

the solvability of $(D_p L)$ (Theorem 1(g),(h)).

- We establish an equivalence between the solvability of $(D_p L)$ and a certain L^p estimate on the non-tangential maximal function of the gradient of (local) *landscape functions* of the operator $-\operatorname{div} A^T \nabla$ (Theorem 1(f) and Remark 1). This result may be interpreted as a local T1 theorem for the L^p -solvability of a problem we call the *Poisson-regularity problem*, itself also equivalent to the solvability of $(D_p L)$ (Theorem 1(d),(e)).
- We furnish new estimates (Corollary 1) for eigenfunctions and their gradients on bounded rough domains, in terms of the landscape function (also known as the *torsion function*), loosely related to the Hassell-Tao inequality.

Definitions and main results

Let us now state the main results more precisely, and in order to do so, we will need to record some definitions. We always consider domains $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$ satisfying the corkscrew condition and with n -Ahlfors regular boundary (see Section 2.1.2 for definitions of these classical geometric concepts), while $A = (A_{ij})$ is always a real, not necessarily symmetric $(n+1) \times (n+1)$ matrix of merely bounded measurable coefficients in Ω verifying the strong ellipticity conditions

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n+1} A_{ij}(\xi) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^{n+1}, \quad \lambda, \Lambda > 0$$

for some $\lambda \in (0, 1)$. We denote $L = -\operatorname{div} A \nabla$, and $L^* = -\operatorname{div} A^T \nabla$ where A^T is the transpose of A . Moreover, we denote $\delta(x) = \delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$.

For $\alpha > 0$, $U \subseteq \mathbb{R}^{n+1}$, and $\xi \in \partial U$, we define the *cone with vertex ξ and aperture $\alpha > 0$* by

$$\gamma_\alpha^U(\xi) := \{x \in U : |x - \xi| < (1 + \alpha)\delta_U(x)\},$$

and if $U = \Omega$, we write $\gamma_\alpha = \gamma_\alpha^\Omega$. Define the *non-tangential maximal function* of $u \in L_{\text{loc}}^q(\Omega)$ by

$$\square_d u(\xi) := \sup_{x \in \gamma_d(\xi)} |u(x)|, \quad \text{for } \xi \in \partial\Omega.$$

Following [1], we introduce the *modified non-tangential maximal function*

$\widetilde{M}_\alpha^r(u)$ for a given aperture $\alpha > 0$, a parameter $\hat{c} \in (0, 1/2]$, and $r \geq 1$: for any $u \in L_{\text{loc}}^q(\Omega)$, we write

$$\widetilde{M}_\alpha^r(u) = \sup_{x \in \gamma_\alpha(x)} \left(\frac{1}{|B(x, \delta(x))|} \int_{B(x, \delta(x))} |u(y)|^r dy \right)^{1/r} \quad \text{for } x \in \partial\Omega.$$

The L^p norms of non-tangential maximal functions are equivalent under changes of α or the averaging parameter $\hat{c}$ (see Lemma 1). For ease of notation, we will often write $\widetilde{M}_\alpha^r(u)$ for $\widetilde{M}_{\alpha, \hat{c}}^r(u)$ if we do not wish to specify $\hat{c}$. When we do not need to specify neither α nor $\hat{c}$, we will write $\square = \square_\alpha$, $\gamma = \gamma_\alpha$ and $\widetilde{M}_r = \widetilde{M}_r$.

Definition 1 (The L^p Dirichlet and regularity problems). Let $p \in (1, \infty)$ and p' its Hölder conjugate. We say that the (homogeneous) *Dirichlet problem for the operator L with L^p data in Ω is solvable* (write $(D_p L)$ is solvable in Ω), if there exists $C \geq 1$ so that for each $g \in C_c(\partial\Omega)$, the solution u to the continuous Dirichlet problem ([eq.dirichlet]) with boundary data g satisfies the estimate

$$\square(u) \leq C \|g\|_{L^p(\partial\Omega, \sigma)}$$

where σ is the restriction of the n -dimensional Hausdorff measure to $\partial\Omega$. We call C in ([eq.dirichlet]) the $(D_p L)$ constant.

Let Ω be a bounded domain, and $p \in (1, \infty)$. We say that the (homogeneous) *Dirichlet regularity problem* (or just *regularity problem*) for the operator L with L^p data is solvable in Ω (write $(R_p L)$ is solvable in Ω), if there exists $C \geq 1$ so that for each $f \in \operatorname{Lip}(\partial\Omega)$, the solution u to the continuous Dirichlet problem for L ([eq.dirichlet]) with boundary data f satisfies the estimate

$$\square(u) \leq C \|f\|_{L^p(\partial\Omega, \sigma)}$$

Here, $\operatorname{Lip}(\partial\Omega)$ is the *Hajlasz-Sobolev space* on $\partial\Omega$; we defer its definition to Section 2.1.4. We call the constant C in ([eq.regest]) the $(R_p L)$ constant.

From now on we take $\sigma = \mathbb{P}|_{\partial\Omega}$ to be the underlying measure on $\partial\Omega$. The space $\operatorname{Lip}(\partial\Omega)$ was identified in as the correct space to consider the L^p regularity problem in rough domains beyond the Lipschitz setting; indeed, if Ω is Lipschitz, then $(R_p L)$ has usually been defined with the estimate $\square(u) \leq C \|f\|_{L^p(\partial\Omega, \sigma)}$ taking the place of ([eq.regest]), where ∇_τ denotes the tangential derivative. In two-sided chord arc domains, one has that $\square(u) \approx \square_\tau(u)$ for more information on the geometric conditions that guarantee this bound, see and [1].

L^p -solvability of the Poisson-Dirichlet problem

Let $q \geq 1$ and $\hat{c} \in (0, 1/2]$. Define the *q -Carleson functional* of a function $H : \Omega \rightarrow \mathbb{R}$ by $\mathcal{C}_q(H)$ by

$C_{\{2,p\}}$ and H such that $\delta_\Omega \in C_{\{2,p\}}$; see Theorem 1.

Theorem 1 is new even when $L = -\Delta$ or Q is a ball. This is the first time that an estimate such as (eq:ntmaxn) or (eq:ntmaxtrue) has appeared explicitly in the literature for the solution to a Poisson-Dirichlet problem on a domain $Q \subseteq \mathbb{R}^{n+1}$. On the other hand, certain related estimates may be found in the literature, and Sobolev and Besov space estimates for solutions to the Poisson problem with right-hand side in Sobolev-Besov spaces are well-studied and have been considered many times; we will look into the connections of Theorem 1 with these results a little deeper in Section 1.2.4.

The proof of Theorem 1 is presented in Section 3.1. The argument contains several novel components; let us give a very brief sketch of the ideas for the case that $g = 0$, $H = 0$. First, we may use Green's representation formula to see that we must find appropriate bounds for $\sup_x \int_{\partial Q} \nabla_y G(x,y) F(y) dm(y)$, for any $\zeta \in \partial Q$. To achieve the desired bounds, we split Q into four regions tailored to the pole x and a non-tangential cone $\gamma(\zeta)$. Namely, the first region is a small Whitney ball around x , the second region is a small Carleson tent over ζ , the third region is the rest of a large cone $\gamma(\zeta)$, and the last region corresponds to a far-away, tangential (with respect to ζ) portion of Q .

Each of the regions described above is treated separately, using harmonic analysis techniques: for the first, we use the properties of the solution operators L_Q^{-1} and $L_Q^{-1} \text{div}$ (eq:opl); for the second, we use a dyadic Carleson embedding result and the weak- RH_p property of the Poisson kernel; for the third, we relate the bound to an area integral estimate and use (eq:tentandcar); and finally, we break up the fourth region into annular regions on which we apply the same ideas as for the second region, but since these annular regions are tangential and far-away from ζ and x , we use the boundary Hölder inequality and a global weak- RH_p inequality for the Poisson kernel to furnish summable decay. The bounds corresponding to the second and fourth regions are the most difficult, since it is here where a delicate Carleson embedding is used; indeed, this is one of the main technical innovations of our paper.

It is natural to wonder how sharp Theorem 1 is. This question is solved by our second main result, which gives several new characterizations of the solvability of (D_p^L) , a new estimate for the gradient of the Green's function, and a certain local $T1$ theorem, under very mild geometric assumptions of the domain. First, we need a few more definitions.

Definition 1 (Landscape functions). Given a Whitney cube I in a Whitney decomposition of Q (see Section 2.1.6 for definitions), we say that the unique weak solution $u_I \in Y_0^{1,2}(Q)$ to the equation $Lu_I = \chi_I$ is the *local landscape function* subordinate to I . If Q is bounded, then the unique weak solution $u \in Y_0^{1,2}(Q)$ to the equation $Lu = \chi_Q$ is the *landscape function* of L .

Up to a rescaling, the landscape function is known in the literature of shape optimization as the *torsion function* of L (see, for instance,). The landscape function (introduced with this name by Filoche and Mayboroda in) has recently been a subject of heavy interest when L is the Schrödinger operator $L = -\text{div} A \nabla + V$, $V \geq 0$, due to applications to mathematical physics and the phenomenon of localization of eigenfunctions. For related literature, see .

Definition 1 ((PD_p^L) and (PR_p^L)). For any $p \in (1, \infty)$, we say that (PD_p^L) is solvable in Q if there exists $C > 0$ so that for each $H, F \in L_c^\infty(Q)$, the unique weak solution $w \in Y_0^{1,2}(Q)$ (see Section 2.2 for the definition of this space) to the equation $Lw = H - \text{div} F$ satisfies the estimate

$$\| \widetilde{\mathcal{M}}_{\{2,p\}}(w) \|_{L^p(\partial Q)} \leq C \left(\| H \|_{L^\infty(Q)} + \| F \|_{L^p(\partial Q)} \right).$$

We say that (PD_p^L) is solvable in Q for $H = 0$ if the estimate (eq:pd1) holds for $H = 0$ and arbitrary $F \in L_c^\infty(Q)$.

We say that (PR_p^L) is solvable in Q if there exists $C > 0$ so that for each $H, F \in L_c^\infty(Q)$, the unique weak solution $v \in Y_0^{1,2}(Q)$ to the equation $Lv = H - \text{div} F$ satisfies the estimate

$$\| \widetilde{\mathcal{M}}_{\{2,p\}}(v) \|_{L^p(\partial Q)} \leq C \left(\| H \|_{L^\infty(Q)} + \| F \|_{L^p(\partial Q)} \right).$$

We say that (PR_p^L) is solvable for *local landscape functions* in Q if there exist $C > 0$, $C \geq 2$ so that for each Whitney cube I in a Whitney decomposition of Q , the local landscape function u_I satisfies the estimate

$$\| \widetilde{\mathcal{M}}_{\{2,p\}}(u_I) \|_{L^p(\partial I)} \leq C \left(\| \chi_Q \|_{L^\infty(Q)} + \| \chi_I \|_{L^p(\partial I)} \right).$$

where Q is a boundary cube of I (see (eq:truncated1) for the definition of the truncated non-tangential maximal function $\widetilde{\mathcal{M}}_{\{2,p\}}$).

Remark 1. If (PR_p^L) is solvable in Q , then an easy variant of allows for solvability with general data in the space $C_{\{2,p\}}$. More precisely, for any $\mathbf{H} \in C_{\{2,p\}}$, there exists a weak solution $v \in W_{\text{loc}}^{1,2}(Q)$ to the equation $Lv = H$ satisfying (eq:estreg1) and the non-tangential limit

$$\lim_{\substack{\gamma \rightarrow 0 \\ \gamma \in \Gamma}} \gamma \int_{\partial Q} \nabla_y G(x,y) F(y) dm(y) = 0, \quad \text{for } x \in \partial Q.$$

Similarly, if (R_p^L) is solvable in Q , then by essentially the same argument of , we can extend the solvability to general data in the space $W^{1,p}(\partial Q)$: for any $f \in W^{1,p}(\partial Q)$, there exists a weak solution $u \in W_{\text{loc}}^{1,2}(Q)$ to the equation $Lu = 0$ satisfying (eq:regest1) and that $u \rightarrow f$ non-tangentially σ -a.e. on ∂Q .

Theorem 1 (Characterizations of (D_p^L)). Let $Q \subseteq \mathbb{R}^{n+1}$, $n \geq 2$ be a domain satisfying the corkscrew condition and with n -Ahlfors regular boundary; such that either Q is bounded, or ∂Q is unbounded. Let $p \in (1, \infty)$, p' its Hölder conjugate, and $L = -\text{div} A \nabla$. The

following are equivalent.

- $(D_p L)$ is solvable in Ω .
- $(PD_p L)$ is solvable in Ω .
- $(PD_p L)$ is solvable in Ω for $H \equiv 0$.
- $(PR_p L^*)$ is solvable in Ω .
- $(PR_p L^*)$ is solvable in Ω for $F \equiv 0$.
- $(PR_p L^*)$ is solvable for local landscape functions in Ω .
- There exist $C > 0$, $C' \geq 2$ so that for some $\delta \in (1, \frac{1}{n+1})$ and for each $x \in \Omega$, we have the estimate
$$\frac{\| \nabla \tilde{G}_L(x, \cdot) \|_{L^p(B(x, C'\delta))}}{\| \tilde{G}_L(x, \cdot) \|_{L^p(B(x, C'\delta))}} \leq C \delta^{n-p},$$
where $\tilde{G}_L$ is the Green's function for the operator L (see Definition [\[def.green\]](#)).
- There exist $C > 0$, $C' \geq 2$ so that for some $\delta \in (1, \frac{1}{n+1})$ and for each $x \in \Omega$, we have the estimate
$$\frac{\| \nabla \tilde{G}_L(x, \cdot) \|_{L^p(B(x, C'\delta))}}{\| \tilde{G}_L(x, \cdot) \|_{L^p(B(x, C'\delta))}} \leq C \delta^{n-p}.$$

Theorem [1](#) is proved in Section [3.2](#). Note that (a) $\square$ (b) is directly implied by Theorem [1](#), while (b) $\square$ (c), (d) $\square$ (e) and (e) $\square$ (f) follow by definition. The implications (c) $\square$ (d) and (e) $\square$ (c) follow from an application of the $\square$ - $\square$ duality, while the direction (c) $\square$ (a) is proved by combining this duality with an argument in [\[1\]](#). The statements (f) $\square$ (a), (g) $\Leftrightarrow$ (a) and (h) $\Leftrightarrow$ (a) are obtained via similar arguments to the proof of (c) $\square$ (a).

Let us discuss Theorem [1](#). It shows that one loses no information when switching from the study of the homogeneous Dirichlet problem with L^p data, to the study of the Poisson problem with inhomogeneous data $\text{div } F$ with F in the Carleson space $\mathcal{B}C_{\text{loc}}^{2,p}$. Perhaps surprisingly, both of these problems are also equivalent to the ‘‘Poisson regularity’’ problem $(PR_p L^*)$, even though the implication $(D_p L) \square (R_p L^*)$ remains an open problem for arbitrary elliptic operators L on Lipschitz domains. We now emphasize two observations, important enough to merit their own remarks.

Remark 1. The statements (g) and (h) give quantitative characterizations of $(D_p L)$ in terms of new local boundary estimates on the Green's function and its gradient. We have not been able to find either estimate [\(eq.truente1\)](#) or [\(eq.truente2\)](#) written explicitly in the literature even in significantly simpler geometric settings, or for $L = -\Delta$. On the other hand, both estimates are natural since they are closely related to the RH_p property of the Poisson kernel. Moreover, observe that a global version of the estimate [\(eq.truente1\)](#) follows formally from applying that $(PR_p L^*)$ is solvable in Ω , since $L^* G_L(x, \cdot) = \delta_x$ and a straightforward computation shows that, if $\mu \in \mathcal{M}^+(\Omega)$ and μ is a measure on $\overline{\Omega}$, then $\| \nabla G_L(x, \cdot) \|_{L^p(\Omega)} \approx \delta(x)^{-n/p}$. This argument can be made rigorous by approximating δ_x by averages, along the lines of [\[1\]](#).

There has recently been strong interest in studying how estimates for the Green's function are related to the geometry of domains, and in this connection, let us point out that, when coupled with the literature on free boundary results for $(D_p L^{-\Delta})$, Theorem [1](#) implies that under its background assumptions, the estimates [\(eq.truente1\)](#), [\(eq.truente2\)](#) for $L = -\Delta$ characterize the IBPCAD condition (or uniform rectifiability plus the weak local John condition; see [\[1\]](#)).

Remark 1. Here, we give context to the statement (f) and why its equivalence to the statements (a) and (e) is interesting in its own right. As remarked before, if (e) holds, then (f) holds immediately, since

$$\| \nabla \tilde{G}_L(x, \cdot) \|_{L^p(\Omega)} \approx \delta(x)^{-n/p}, \quad \text{for each Whitney box } I \in \mathcal{W}_\delta.$$

So (f) $\square$ (e) tells us that in order to ascertain that $(PR_p L^*)$ is solvable for arbitrary $\mathcal{B}C_{\text{loc}}^{2,p}$ and $F \equiv 0$, it is enough to obtain the estimate [\(eq.estreg2\)](#) (which itself is a local version of the global estimate [\(eq.estreg1\)](#)) for the countable collection $H = \{I\}_I$, where I runs over all Whitney boxes in Ω . It is easy to interpret this result as a local $T1$ theorem, as follows: if $T = \nabla (L_Q^*)^{-1}$ (see [\(eq.op1\)](#)), then the statement (e) is equivalent to asking that the map $S: \mathcal{B}C_{\text{loc}}^{2,p} \rightarrow \mathcal{B}C_{\text{loc}}^{2,p}$ is bounded (by Remark [1](#)). But (f) $\square$ (e) gives that this map is bounded if the local estimate [\(eq.estreg2\)](#) holds for $\{T\}_I \in \mathcal{H}$, which is exactly a local $T1$ theorem.

The implications (f) $\square$ (a) and (f) $\square$ (d) join a family of results in the literature about the landscape function which say, loosely speaking, that several distinct properties of the operator L can be a priori controlled by properties of the landscape function; see for instance [\[1\]](#), or [\[2\]](#). Finally, we mention that the equivalence (f) $\Leftrightarrow$ (a) gives the first robust connection between the theory of the landscape/torsion function (primarily developed for the study of questions in mathematical physics and shape optimization) and the theory of solvability of the Dirichlet problem with singular data.

We now give an immediate application of Theorems [1](#) and [2](#) toward estimates for eigenfunctions and their gradients of elliptic operators $L = -\text{div } A \nabla$ on rough domains.

Corollary 1 (Boundary estimates for eigenfunctions). *Retain the setting and assumptions of Theorem [1](#), and moreover, assume that Ω is bounded, that A is symmetric, and that $(D_p L)$ is solvable in Ω . Let $\psi \in W_0^{1,2}(\Omega)$ satisfy $L\psi = E\psi$ in Ω for some eigenvalue $E > 0$, and let u be the landscape function of L on Ω . Then there exists $C > 0$, depending only on n, λ, p , the corkscrew condition and n -Ahlfors regularity constants, and the $(D_p L)$ constant, such that*

$$\frac{\| \nabla \psi \|_{L^p(\Omega)}}{\| \psi \|_{L^p(\Omega)}} \leq C \frac{\| \nabla u \|_{L^p(\Omega)}}{\| u \|_{L^p(\Omega)}}.$$

$$\begin{aligned} & \|\text{label}{eq.efn2} \|\text{Vert}\widetilde{\text{mathcal{N}}}\{2(\nabla\psi)\}\text{Vert}_{L^p\{p\}} \\ & (\partial\Omega)\|\leq C E\text{Vert}\text{mathscr{C}}_{2_*}(\psi)\text{Vert}_{L^p\{p\}}(\partial\Omega)\|\leq \\ & C E^2\text{Vert}\psi\text{Vert}_{L^p\{\infty\}}(\Omega)\|\text{Vert}\text{mathscr{C}}_{2_*}(u)\text{Vert}_{L^p\{p\}} \\ & (\partial\Omega)\}. \end{aligned}$$

In particular, if Ω is a bounded Lipschitz domain and $L = -\Delta$, then both estimates (eq.efn1) and (eq.efn2) hold with $p = p' = 2$.

Proof. We prove (eq.efn2). Since $(D_p L)$ is solvable in Ω , then by Theorem 1 we have that $(P_p L)$ is solvable in Ω , and so we see that

$$\begin{aligned} & \|\text{Vert}\widetilde{\text{mathcal{N}}}\{2(\nabla\psi)\}\text{Vert}_{L^p\{p\}} \\ & (\partial\Omega)\|\leq E\text{Vert}\text{mathscr{C}}_{2_*}(\psi)\text{Vert}_{L^p\{p\}}(\partial\Omega)\|\leq \\ & E\text{Vert}\text{mathscr{C}}_{2_*}(\psi)\text{Vert}_{L^p\{p\}}(\partial\Omega)\}. \end{aligned}$$

Since (see) $|\psi(x)| \leq E|\psi|_{L^\infty(\partial\Omega)}\rho(x)$ for each $x \in \Omega$, the desired estimate (eq.efn2) follows. The estimate (eq.efn1) follows by an analogous argument, using Theorem 1.

The estimates (eq.efn1) and (eq.efn2) are new even for the eigenfunctions of the Laplacian; although estimates related to (eq.efn2) controlling the normal derivative of Dirichlet eigenfunctions have been studied heavily, such as the Hassell-Tao inequality , when the domain is smooth. On the other hand, the usual techniques in the literature rely on Rellich identities, and thus do not extend to the optimally rough geometric setting that we consider here. Indeed, since we merely assume that $(D_p L)$ is solvable (in addition to the background assumptions that the domain has Corkscrew points and its boundary is n -Ahlfors regular), the estimates (eq.efn1) and (eq.efn2) hold up to the IBPCAD domains when $L = -\Delta$, and for certain other specific L , the estimates hold *even for domains with fractal boundaries* (see for the existence of such an operator in the complement of the 4-corner Cantor set).

We emphasize that the constant C in the estimates (eq.efn1) and (eq.efn2) does not depend on $|\Omega|$, nor on the eigenvalue E . There is a vast literature studying the geometry of eigenfunctions of $-\Delta$ and estimates of eigenfunctions near the boundary; for a few related results, see for instance, and the survey .

The regularity problem for Dahlberg-Kenig-Pipher operators

We now pass to an important application of Theorem 1. We will see that Theorem 1 helps us solve the regularity problem for certain elliptic operators whose gradient satisfies a Carleson measure condition. Denote

$$\|\text{rm osc}\|_{\{B(x,\delta(x)/2)\}(A)} = \|\sup_{y,z \in B(x,\delta(x)/2)} |A(y) - A(z)|\}.$$

Definition 1 (DKP and DPR conditions). We say that a real matrix function A in $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, satisfies the *K-Dahlberg-Kenig-Pipher condition* (K -DKP) in Ω if $A \in \text{Lip}_{\text{loc}}(\Omega)$ and

$$d\mu(x) := \|\Omega\| \text{ess sup} \{ \|\delta\Omega(x)\| \|\nabla A(z)\|^2 : z \in B(x, \delta\Omega(x)/2) \} \, dm(x)$$

is a K -Carleson measure with respect to $\mathbb{P}|_{\partial\Omega}$. That is, for any ball B centered in $\partial\Omega$,

$$\mu(B \cap \Omega) \leq K |\mathbb{P}|^n(B \cap \partial\Omega).$$

We say that A satisfies the *τ -Dindoš-Pipher-Rule condition* (τ -DPR) in Ω if the measure

$$\|\text{d}\mu(x) = \{ \|\text{mathds 1}\| \|\Omega(x)\| \|\delta\Omega(x)\| \|\Omega(x)\|^{-1} \} \|\text{rm osc}\|_{\{B(x,\delta\Omega(x)/2)\}(A)^2, \text{dm}(x)}\}$$

is a τ -Carleson measure with respect to $\mathbb{P}|_{\partial\Omega}$.

We will say that A is a DKP (DPR) matrix and that $L = -\text{div } A \nabla$ is a DKP (DPR) operator if A satisfies the K -DKP (τ -DPR) condition for some $K > 0$ ($\tau > 0$). It is easy to see that if A satisfies the K -DKP condition, then it also satisfies the τ -DPR condition, with $\tau \leq K$. The DKP operators first arose in the literature as pullbacks of the Laplacian under a mapping of Dahlberg-Kenig-Stein . The well-posedness of the L^p -Dirichlet problem for DKP operators in Lipschitz domains was shown in , and their result carries over to chord-arc domains, and more generally to domains with interior big pieces of chord-arc domains considered in (see Section 2.1.2 for the definition), by the methods of (see for instance the remarks following Theorem 1.1 in).

The question of establishing that $(R_p L)$ is solvable for some $p > 1$ when L is a DKP operator has remained open since 2001, even when Ω is the unit ball. The closest result in the published literature is from , where the authors show that for any $p \in (1, \infty)$, $(R_p L)$ is solvable if Ω is a Lipschitz domain *with small enough Lipschitz constant* ℓ and if A satisfies the τ -DPR condition *with small enough* $\tau > 0$, depending on p .

The following theorem is our third main result, which establishes that $(D_p L^*) \sqsubset (R_p L)$ for DKP operators on bounded Corkscrew domains with uniformly n -rectifiable boundaries (see Section 2.1.2 for the definition of uniform n -rectifiability). This solves the question posed in the previous paragraph.

Theorem 1 ($(D_p L^*) \sqsubset (R_p L)$ for DKP operators). *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain satisfying the corkscrew condition and with uniformly n -rectifiable boundary. Let $p \in (1, \infty)$, p' its Hölder conjugate, and $L = -\text{div } A \nabla$, where A is a DKP matrix in Ω . Suppose that $(D_p L^*)$ is solvable in Ω . Then $(R_p L)$ is solvable in Ω , and the $(R_p L)$ constant depends only on p , n , λ , the corkscrew constant, the uniform n -rectifiability constants, the DKP constant, and the $(D_p L^*)$ constant.*

Remark 1. In the case of the Laplacian, a result analogous to Theorem 1 was recently obtained in by two of us for $p \in (1, 2 + \varepsilon_0)$, but the proof relied heavily on the L^2 boundedness of the Riesz transform and the double layer potential on uniformly n -rectifiable sets, while this theory is not available for layer potentials associated to DKP operators. Because of this, our proof of Theorem 1 must by necessity be substantially different from the argument in . In fact, our proof of Theorem 1 gives an alternative argument to and improves upon it, since our result holds for $p \in (1, \infty)$.

As mentioned before, for any DKP matrix A there exists $p > 1$ so that $(D_p L^*)$ is solvable in Ω whenever Ω satisfies the corkscrew condition and the IBPCAD condition, and its boundary $\partial\Omega$ is n -Ahlfors regular ; and moreover, if Ω satisfies the IBPCAD condition and $\partial\Omega$ is n -Ahlfors regular, then $\partial\Omega$ is uniformly n -rectifiable (see Section 2.1.2 for the definitions). Hence we have the following corollary.

Corollary 1 ((R_p^L) for DKP operators in IBPCAD domains). *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain satisfying the corkscrew condition and the IBPCAD condition, and assume that $\partial\Omega$ is n -Ahlfors regular. Let $L = -\operatorname{div} A \nabla$ where A is a DKP matrix in Ω . Then there exists $p > 1$ such that (R_p^L) is solvable in Ω .*

Corollary 1 fully answers the question of the solvability of (R_p^L) for some $p > 1$ when L is a DKP operator on a bounded rough domain, but we mention again that this problem had been open even when Ω is the unit ball. If, in addition to the hypothesis of Corollary 1, we have that Ω satisfies the Harnack chain condition, then Ω is a chord-arc domain (see Section 2.1.2 for definitions), and in this case we can use the theory of Carleson perturbations for the regularity problem to deduce the following improvement to Corollary 1.

Corollary 1 ((R_p^L) for DPR operators). *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded chord-arc domain, and let $L = -\operatorname{div} A \nabla$ where A is a DPR matrix in Ω . Then there exists $p > 1$ such that (R_p^L) is solvable in Ω .*

Let us now give a very quick overview on the method of proof of Theorem 1, which is presented in Sections 4–7. First, as a main innovation, we use the duality between non-tangential maximal functions and Carleson functionals (Proposition 1) to reduce the regularity problem to the study of the conormal derivative of solutions to Poisson–Dirichlet problems. Indeed, for a solution w to the Poisson–Dirichlet problem ((eq:poisson)) with $H \equiv 0$, we show in Proposition 1 that the variational conormal derivative $\delta_{v_\varphi} w$ (see Section 4 for the definition) is a bounded linear functional on the Hajlasz–Sobolev space $W^{1,p}(\partial\Omega)$, and the new estimate ((eq:boundconormal)) is the main ingredient in our proof of Theorem 1, since it replaces a 1-sided Rellich estimate.

Given Proposition 1, Theorem 1 follows rather easily from the duality between $\square$ and $\square$ and the L^p -solvability of the Poisson–Dirichlet problem, as shown early in Section 4. However, the main estimate ((eq:boundconormal)) of Proposition 1 requires a completely new argument, even if one were to cut out the sharp geometric considerations.

To prove Proposition 1, first we conceive of an appropriate corona decomposition which is able to invoke the results of as “black boxes” on each corona subdomain. With the corona decomposition at hand, we construct an “almost L -elliptic” extension of Lipschitz boundary data. Both the corona decomposition and the construction of the almost L -elliptic extension are based on analogous ideas from where the regularity problem for the Laplace equation in uniformly n -rectifiable domains was solved. Nevertheless, the corona decomposition in the present manuscript requires highly non-trivial modifications from the one in to ensure that the hypotheses of are satisfied by L on each corona subdomain.

The next step in the proof is to show Proposition 1, which is an L^p estimate on the modified non-tangential maximal function of the gradient of the almost L -elliptic extension. This estimate is interesting in its own right, since it illustrates the naturality of the almost L -elliptic extension, and it is new even for the Laplacian. Indeed, Proposition 1 together with imply that the almost harmonic extension is a correct analogue of the Váropoulos extension for one “smoothness level” up. However, the proof is very delicate, and the tools which we use to achieve it are quite sharp, such as the localization theorem for the regularity problem, and the dyadic Carleson embedding results that we also invoked in the proof of Theorem 1.

The last step in the proof of Proposition 1 is to show the estimate

$$|\int_{\partial\Omega} A \nabla w \nabla v_\varphi dm| \lesssim [\|g\|_{L^p(\partial\Omega)} + \|\square_2(F)\|_{L^p(\partial\Omega)}] \|\nabla H\|_{L^p(\partial\Omega)},$$

where $F \in \mathcal{F}(\mathcal{C}_{\{2,p\}})$, $\varphi \in \operatorname{Lip}(\partial\Omega)$, w is the unique weak solution to the problem ((eq:poisson)) with $H = 0$, and v_φ is the almost L^* -elliptic extension of φ . To prove ((eq:test1)), we combine the L^p -solvability of the Poisson–Dirichlet problem (it is only here where we use that (D_p^L) is solvable in Ω) with the $\square$ - $\square$ duality and the heuristic that $L^* v_\varphi$ should belong to $\mathcal{F}(\mathcal{C}_{\{2^*,p\}})$. Related estimates were shown in Sections 4 and 5 of , although our argument for ((eq:test1)) is different and does not follow from the arguments in ; instead, we appeal substantially to the proof of Proposition 1.

Our manuscript is the first paper that considers the natural connections between the $\square$ - $\square$ duality, the L^p -solvability of the Poisson problem, and the duality between (D_p^L) and (R_p^L) . It is the new marriage of these ideas that affords us the solution of the regularity problem for DKP operators in such large geometric generality.

Historical remarks

Let us provide some historical context to our investigation. For elliptic operators $L = -\operatorname{div} A \nabla$, solvability of (D_p^L) and (R_p^L) in rough domains has garnered a lot of attention in the last several decades. For a historical overview of the case $L = -\Delta$ in rough domains, we refer the reader to the introduction of . Hence we shall focus now on the rich history related to the boundary value problems for the operators L with rough (real) coefficients. The problems are always homogeneous, unless stated otherwise.

Early history of the homogeneous problems with singular data

In 1924, the celebrated result of Wiener characterized the domains for which the continuous Dirichlet problem for the Laplacian is solvable (now known as *Wiener regular domains*). At the end of the 1950s, the De Giorgi–Nash–Moser theory opened the door for the systematic study of boundary value problems for elliptic operators $L = -\operatorname{div} A \nabla$ whose matrices A are merely bounded and elliptic, and only a few years later, Littman, Stampacchia, and Weinberger proved that the continuous Dirichlet problem for L with merely bounded and elliptic coefficients is well-posed on a domain if and only if the domain is Wiener regular. In other words, the *continuous* Dirichlet problem is well-posed for the Laplacian if and only if it is well-posed for L .

Near the end of the 1970s, Dahlberg proved that (D_2^{-A}) is solvable in Lipschitz domains , and it was then understood that in Lipschitz domains, the solvability of (D_p^L) for some $p > 1$ is equivalent to quantitative absolute continuity of the L -elliptic measure ω_L (namely, that $\omega_L \in A_\infty(\sigma)$ in a local sense). Despite the results of Caffarelli, Fabes, Mortola, and Salsa , who proved that non-negative solutions u to $Lu = 0$ in the unit ball have nontangential limits at ω_L -a.e. point in the boundary for symmetric operators L with merely bounded and elliptic coefficients, examples were soon found by Caffarelli, Fabes, Kenig , and independently by Modica and Mortola , of elliptic measures which were mutually

singular with respect to the surface measure on smooth domains, in contrast to the situation of the continuous Dirichlet problem described earlier.

In the positive direction, for symmetric operators L with smooth and bounded coefficients, Jerison and Kenig solved (D_2^L) in Lipschitz domains, relying on a so-called Rellich identity, available only in the symmetric case. On C^1 domains and for symmetric, globally continuous operators, the solvability of (D_p^L) for any $p > 1$ became completely understood in the work of Fabes, Jerison, and Kenig, who characterized the symmetric operators L for which $\omega_L \ll \sigma$ in terms of a Dini condition; this condition also guaranteed that (D_p^L) is solvable for any $p > 1$.

Although had completely characterized the case of symmetric, globally continuous operators on C^1 domains, the characterization remained open for non-symmetric elliptic operators with rough coefficients, and perhaps more importantly, it was hard to verify the Dini-type condition of for certain important operators that arose in practice, except for t -independent operators, which we will briefly discuss a bit later.

Dahlberg's conjectures

As mentioned in , Dahlberg posed two conjectures in 1984. The first conjecture concerned whether the LP -solvability of the Dirichlet problem was stable under certain perturbations of the coefficients. More precisely, if $L_0 = -\operatorname{div} A_0 \nabla$ and $L = -\operatorname{div} A \nabla$ are two elliptic operators, such that $(D_p^{L_0})$ is solvable in the half-space $\mathbb{L}^{n+1}$ and

$$\int_{\mathbb{A}_0(y)^{n/2}} \left| \frac{\int_{\partial \mathbb{B}(x, \delta(x)/2)} |A(y) - A_0(y)|^2}{\delta(x)} dx \right| dy < \infty$$

does it follow that (D_q^L) is solvable in $\mathbb{L}^{n+1}$, for some $q > 1$ possibly larger than p ? This question was resolved in the affirmative by Fefferman, Kenig, and Pipher in . The condition (eq.fkp) has come to be known as the *Fefferman-Kenig-Pipher* condition (FKP), and its applicability in rough domains has continued to be heavily studied in recent years . For a more thorough review of these results, see or . Let us also remark that the FKP condition is known to also preserve the solvability of (R_p^L) (with some q perhaps larger than p); this was shown for symmetric operators on the ball by Kenig and Pipher in , and it has been recently extended to the case of rough domains with Harnack chains and non-symmetric operators by Dai, Feneuil and Mayboroda in .

Dahlberg's second conjecture asked whether for some $p > 1$, (D_p^L) is solvable in the half-space when L is a DKP operator (see Definition 1). As mentioned in Section 1.1.2, this question was resolved in the affirmative by Kenig and Pipher in . Dindoš, Petermichl, and Pipher later proved in that for any $p > 1$, if A is a (possibly non-symmetric) K -DKP matrix with small enough constant $K = K(p) > 0$ and Ω is a Lipschitz domain with small enough Lipschitz constant $\ell = \ell(p) > 0$, then (D_p^L) is solvable in Ω ; under the same assumptions, (R_p^L) is also solvable in Ω , but this was shown a decade later by Dindoš, Pipher, and Rule . Via the theory of FKP perturbations described above, the respective results of were also generalized in the respective articles to the τ -DPR operators with $\tau = \tau(p) > 0$ small enough.

Further results

Let us return to the case of t -independent real operators. Although the symmetric t -independent case was already understood by the work of , the first non-trivial positive result for non-symmetric t -independent operators on the half-space $\mathbb{L}^{n+1}$ was obtained by Kenig, Koch, Pipher, and Toro when $n = 1$. There, the authors showed that there exists $p > 1$ such that (D_p^L) is solvable, and unlike the symmetric t -independent case, p may not be made precise. Later, Kenig and Rule showed in that $(D_p^{L^*}) \square (R_p^L)$ for the t -independent operators when $n = 1$, and thus it followed that (R_p^L) was solvable for some $p > 1$. These results of and were generalized to $n \geq 2$ by Hofmann, Kenig, Mayboroda, and Pipher in and , where the solution to the Kato problem played a role in the proof. We remark that there is an extensive literature regarding solvability of boundary value problems in the complex, t -independent case, which we do not review. For a recent review in this direction, see .

We now comment on the solvability of the boundary value problems for DKP operators in rough domains beyond the Lipschitz setting. As mentioned in Section 1.1.2, it has been known from that if L is a DKP operator and Ω satisfies the corkscrew condition and the IBPCAD condition, then there exists $p > 1$ so that (D_p^L) is solvable in Ω . However, the solvability of $(R_p^{-\mathcal{A}})$ beyond the Lipschitz setting remained an open problem for over 30 years until the recent work of the first and third author of this manuscript , where it was shown that $(D_p^{-\mathcal{A}}) \Leftrightarrow (R_p^{-\mathcal{A}})$ whenever the bounded domain Ω satisfies the corkscrew condition and has n -Ahlfors regular boundary, extending the duality between the Dirichlet problem and regularity problem for the Laplacian shown by Jerison and Kenig in the Lipschitz setting .

We turn to a few words on the duality between $(D_p^{L^*})$ and (R_p^L) . In the case that Ω is a star-like Lipschitz domain, Kenig and Pipher showed in that $(R_p^L) \square (D_p^{L^*})$, but the converse direction has remained an open problem. Shen proved in that if $(D_p^{L^*})$ is solvable and (R_q^L) is solvable for some $q > 1$, then (R_p^L) is solvable; later this result was extended by Dindoš and Kirsch . Complex analogues of the duality result of are hopeless in the full generality, as was shown by Mayboroda in ; however, Hofmann, Kenig, Mayboroda, and Pipher proved $(D_p^{L^*}) \Leftrightarrow (R_p^L)$ in the range $p \in (1, 2+\varepsilon)$ for elliptic equations with bounded t -independent coefficients under the assumption that De Giorgi-Nash-Moser estimates hold, and Auscher and one of us proved the equivalence for $p \in (p_0, 2]$ for elliptic systems with t -independent complex coefficients assuming De Giorgi-Nash-Moser estimates, for some $p_0 < 1$ determined by the exponent in the assumed interior Hölder condition.

Connections to the literature on the Poisson problem

So far, we have discussed results only for the homogeneous problem (eq.dirichlet) . As remarked in Section 1.1.1, estimates controlling the left-hand side of (eq.ntmaxn) have never appeared explicitly for the Poisson-Dirichlet problem (eq.poisson) in a domain $\Omega \subseteq \mathbb{R}^{n+1}$. Nevertheless, related estimates for the inhomogeneous case have been studied

$$\|w\|_{L_t^p} := \|(I-\Delta)^{\theta/2} w\|_{L^p(\mathbb{R}^{n+1})}$$

$$\|w\|_{L^p_\theta} := \|(I-\Delta)^{\theta/2}w\|_{L^p(\mathbb{R}^{n+1})}$$

$$\text{hwl}_{L_\theta^p(\Omega)} \leq \text{CHH}_{L_{\theta-2^p}(\Omega)}, \quad \text{for each } H \in L_{\theta-2^p}(\Omega).$$

The estimates (eq:estp) are linked to the solvability of the homogeneous Dirichlet problem (eq:dirichlet) with boundary data in Besov spaces $\mathcal{S}'^s(\mathbb{R}^n)$. These boundary value problems with fractional data constitute a scale between the homogeneous Dirichlet problem at one end, and the homogeneous regularity problem at the other end. For $p > 1$ and $\theta \in (0, 1)$, define $Z_p(\theta)$ as the space of functions $F \in L_{\text{loc}}^{p,2}(\Omega)$ for which

[illegible]
$$\frac{\partial}{\partial \theta} \left(\frac{1}{F(\theta)} \right) = - \frac{f'(\theta)}{f^2(\theta)}$$

We now turn to results in the literature which are close to (eq.ntmaxn) and (eq.estregl). Some closely related estimates for the Poisson problem appeared in . In the case that $\Omega = \mathbb{R}^{n+1}_+$, $n \geq 1$, if L and L^* are complex bounded elliptic operators with t -independent coefficients satisfying De Giorgi-Nash-Moser bounds, it was shown in that if $L\varphi^{t+1}$ is the operator whose integral kernel is the fundamental solution of L in $\mathbb{R}^{n+1}_+$, then

[illegible]

Related results

As we were finishing the first version of this manuscript, we learned that M. Dindoš, S. Hofmann, and J. Pipher had simultaneously and independently obtained the conclusion of Corollary 1 in unbounded Lipschitz graph domains in $\mathbb{P}^{n+1}$, $n \geq 1$, when L is a DPR operator, via a different method. Both the first version of this manuscript and the first version of the paper of M. Dindoš, S. Hofmann, and J. Pipher were posted on arXiv.org on July 21, 2022. Their proof of their analogue of Corollary 1 is significantly shorter than ours, owing to the more restrictive geometric setting that they study. Indeed, they perform a clever reduction of the problem to the solvability of $(R_q^{L_0})$ for some $q > 1$, where L_0 is a block form operator, but this approach does not work in more general domains since it necessitates the existence of a uniform “preferred direction” at the boundary, as occurs in a domain above a Lipschitz graph. The passage from Lipschitz graph domains to uniformly n -rectifiable domains, or even to chord-arc domains, is highly non-trivial, and requires a novel approach via the Poisson-Dirichlet problem. Even in the case of the Laplacian, the same problem had remained open for over 30 years until its solution in [1], illustrating that the geometry of the domain creates significant difficulties which are hard to overcome.

Outline

The rest of the paper is organized as follows. In Section 2, we provide basic definitions and lemmas. In Section 3, we prove Theorems 1 and 1 on the L^p -solvability of the Poisson problem. In Section 4, we give an overview of the structure of the proof of Theorem 1 on the regularity problem for DKP operators. In Section 5, we construct the corona decomposition tailored to exploit the results of on each corona subdomain. In Section 6, we define the almost L -elliptic extension based on the corona decomposition of Section 5. Then, in Section 7 we finish the proof of Theorem 1 by proving an estimate on the conormal derivative of solutions to Poisson problems (see Proposition 1). Finally, in Appendix 8 we provide proofs for some lemmas of Section 2.

Preliminaries

We write $a \lesssim b$ if there exists a constant $C > 0$ so that $a \leq Cb$ and $a \lesssim_p b$ if C depends on the parameter t . We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_p b$ similarly.

All measures in this paper are assumed to be Radon measures. We denote by $m = m_{n+1}$ the Lebesgue measure on $\mathbb{R}^{n+1}$. Given a measure μ on $\mathbb{R}^{n+1}$, the *Hardy-Littlewood maximal function* of a measurable function f on $\text{supp} \mu$ is defined as

$$M\mu(x) := \sup_{B \ni x} \frac{1}{m(B)} \int_B d\mu, \quad x \in \text{supp} \mu,$$

where the supremum is taken over all possible balls B centered at $\text{supp} \mu$.

Given $q \in (0, \infty]$, a set $U \subset \mathbb{R}^{n+1}$, a measure μ on $\mathbb{R}^{n+1}$, and a function $H \in L^q(U, d\mu)$, denote the $L^q(\mu)$ mean of H over U as

$$M_{q,\mu}(H) := \left(\frac{1}{\mu(U)} \int_U |H(x)|^q d\mu \right)^{1/q}.$$

If μ is the Lebesgue measure, we write $m_{q,U} = m_{q,U,\mu}$. If $q = 1$, we write $m_{U,\mu} = m_{q,U,\mu}$.

Geometric preliminaries

Lipschitz domains

We say that $Z \subset \mathbb{R}^{n+1}$ is an ℓ -cylinder of diameter d if there is a coordinate system $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ such that $Z = \{(x, t) : |x| \leq d, -2\ell d \leq t \leq 2\ell d\}$. Also, for all $s > 0$, we denote $sZ = \{(x, t) : |x| \leq sd, -2\ell sd \leq t \leq 2\ell sd\}$.

We say that Ω is a *Lipschitz domain* with Lipschitz character (ℓ, N, C_0) if there is $r_0 > 0$ and at most N ℓ -cylinders $Z_j, j = \overline{1, N}$ of diameter d with $C_0^{-1}r_0 \leq d \leq C_0r_0$ such that²

- $8Z_j \cap \partial\Omega$ is the graph of a Lipschitz function ϕ_j with $\|\nabla \phi_j\|_\infty \leq \ell$, $\phi_j(0) = 0$,
- $\partial\Omega = \cup_j (Z_j \cap \partial\Omega)$,
- We have that $8Z_j \cap \Omega = \{(x, t) \in 8Z_j : \phi_j(x) > t\}$.

We also say that Ω is a Lipschitz domain with Lipschitz constant ℓ .

Quantitative conditions on the geometry of domains

A measure μ in $\mathbb{R}^{n+1}$ is called *n-Ahlfors regular* if there exists some constant $C_0 > 0$ such that

$$C_0^{-1}r^n \leq \mu(B(x, r)) \leq C_0r^n \quad \text{for all } x \in \text{supp} \mu \text{ and } 0 < r \leq \text{diam}(\text{supp} \mu).$$

A measure μ is *uniformly n-rectifiable* if it is *n-Ahlfors regular* and there exist constants $\theta, M > 0$ such that for each $x \in \text{supp} \mu$ and each $r \in (0, \text{diam}(\text{supp} \mu)]$, there is a Lipschitz mapping g from the n -dimensional ball $B_H(0, R)$ to $\mathbb{R}^{n+1}$ with $\text{Lip}(g) \leq M$ and satisfying the bound $\mu(B(x, r) \cap g(B_H(0, r))) \geq \theta r^n$. A set $E \subset \mathbb{R}^{n+1}$ is *n-Ahlfors regular* if $\mathbb{P}|_E$ is *n-Ahlfors regular*, where $\mathbb{P}$ is the n -dimensional Hausdorff measure, which we assume to be normalized so that it coincides with m_n in $\mathbb{R}^n$. Also, E is *uniformly n-rectifiable* if $\mathbb{P}|_E$ is *uniformly n-rectifiable*. The notion of uniform rectifiability was introduced in and , and it should be considered a quantification of rectifiability.

By a *domain* we mean a connected open set. In this paper, Ω is always a domain in $\mathbb{R}^{n+1}$ with $n \geq 2$. As mentioned in the introduction, we denote the restriction of the *n*-Hausdorff measure to $\partial\Omega$ by σ , and we call it the *surface measure* on $\partial\Omega$.

We say that Ω satisfies the *corkscrew condition* if there exists $c > 0$ such that for each $x \in \partial\Omega$ and every $r \in (0, 2\text{diam} \Omega)$, there exists a ball $B \subset B(x, r) \cap \Omega$ so that $r(B) \geq cr$.

Given two points $x, x' \in \Omega$, and a pair of numbers $M, N \geq 1$, an (M, N) -Harnack chain connecting x to x' , is a chain of open balls $B_1, \dots, B_N \subset \Omega$, with $x \in B_1, x' \in B_N, B_k \cap B_{k+1} \neq \emptyset$ and $M^{-1} \text{diam} B_k \leq \text{dist}(B_k, \partial\Omega) \leq M \text{diam} B_k$. We say that Ω satisfies the *Harnack chain condition* if there is a uniform constant M such that for any $x, x' \in \Omega$, there exists an (M, N) -Harnack chain connecting them, with N depending only on M and on $|x - x'| / \min\{\delta(x), \delta(x')\}$.

We say that Ω is a *chord-arc domain* if Ω satisfies the Harnack chain condition, if both Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the corkscrew condition, and if $\partial\Omega$ is *n-Ahlfors regular*.

We say that Ω has *Interior Big Pieces of Chord-Arc Domains*, or that Ω satisfies the IBPCAD condition, if there exist positive constants η and C , and $N \geq 2$, such that for every $x \in \Omega$, with $\delta_\Omega(x) < \text{diam}(\partial\Omega)$, there is a chord-arc domain $\Omega_x \subset \Omega$ satisfying

- $x \in \Omega_x$,
- $\text{dist}(x, \partial\Omega_x) \geq \eta \delta_\Omega(x)$,
- $\text{diam}(\Omega_x) \leq C \delta_\Omega(x)$.

- $\sigma(\partial\Omega_x\cap B(x,N\delta_G(x)))\geq \eta\sigma(B(x,N\delta_G(x)))=\eta\delta_G(x)^n$.
- The chord-arc constants of the domains Ω_x are uniform in x .

Dyadic lattices and the β numbers

Given an n -Ahlfors regular measure μ in $\mathbb{R}^{n+1}$, we consider the dyadic lattice $\square_\mu$ of “cubes” built by David and Semmes . The properties satisfied by $\square_\mu$ are the following. Assume first, for simplicity, that $\text{diam}(\text{supp}\mu)=\infty$. Then for each $j\in\mathbb{Z}$ there exists a family $\square_{\mu,j}$ of Borel subsets of $\text{supp}\mu$ (the dyadic cubes of the j -th generation) such that:

- each $\square_{\mu,j}$ is a partition of $\text{supp}\mu$, i.e. $\text{supp}\mu=\cup Q\in\square_{\mu,j}Q$ and $Q\cap Q'=\emptyset$ whenever $Q,Q'\in\square_{\mu,j}$ and $Q\neq Q'$;
- if $Q\in\square_{\mu,j}$ and $Q'\in\square_{\mu,k}$ with $k\leq j$, then either $Q\subset Q'$ or $Q\cap Q'=\emptyset$;
- if for all $j\in\mathbb{Z}$ and $Q\in\square_{\mu,j}$ we have $2^{-j}\leq\text{diam}(Q)\leq 2^{-j}$ and $\mu(Q)\approx 2^{-jn}$.
- there exists $C>0$ such that, for all $j\in\mathbb{Z}$, $Q\in\square_{\mu,j}$ and $0<\tau<1$,

$$\mu(\{x\in Q:\text{dist}(x,\text{supp}\mu\cap Q)\leq \tau 2^{-j}\})+\mu(\{x\in\text{supp}\mu\setminus Q:\text{dist}(x,Q)\leq \tau 2^{-j}\})\leq C\tau^1C_2^{-jn}.$$

This property is known as the *thin boundary condition*, and it implies the existence of a point $x_Q\in Q$ (the *center* of Q) such that $\text{dist}(x_Q,\text{supp}\mu\cap Q)\geq 2^{-j}$ (see).

We set $\square_\mu:=\cup_j\in\mathbb{Z}\square_{\mu,j}$. If $\text{diam}(\text{supp}\mu)<\infty$, the families $\square_{\mu,j}$ are only defined for $j\geq j_0$, with $2^{-j_0}\approx\text{diam}(\text{supp}\mu)$, and the same properties above hold for $\square_\mu:=\cup_{j\geq j_0}\square_{\mu,j}$.

Given a cube $Q\in\square_{\mu,j}$ we say that its side length is 2^{-j} and denote it by $\ell(Q)$. Notice that $\text{diam}Q\leq \ell(Q)$. We also denote

$$B(Q):=B(x_Q,c_1\ell(Q)),\qquad B_Q:=B(x_Q,\ell(Q))\supset Q,$$

where $c_1>0$ is chosen so that $B(Q)\cap\text{supp}\mu\subset Q$, for all $Q\in\square_\mu$. For $\lambda>1$, we write

$$\lambda Q=\{x\in\text{supp}\mu:\text{dist}(x,Q)\leq (\lambda-1)\ell(Q)\}.$$

The side length of a “true cube” $P\subset\mathbb{R}^{n+1}$ is also denoted by $\ell(P)$. On the other hand, given a ball $B\subset\mathbb{R}^{n+1}$, its radius is denoted by $r(B)$. For $\lambda>0$, λB is the ball concentric with B with radius $\lambda r(B)$.

Given $E\subset\mathbb{R}^{n+1}$, a ball B , and a hyperplane L , we denote

$$\beta_E(B,L)=\sup_{\{y\in E\cap B\}}\frac{\text{distance}\{\text{dist}\}(y,L)}{\sup_{\{y\in L\cap B\}}\frac{\text{distance}\{\text{dist}\}(x,E)}{r(B)}}.$$

We set $b\beta_E(B,L)=\inf_Lb\beta_E(x,r,L)$, where the infimum is taken over all hyperplanes $L\subset\mathbb{R}^{n+1}$. For $B=B(x,r)$, we also write $b\beta_E(x,r,L)=b\beta_E(B,L)$, and $b\beta_E(x,r)=b\beta_E(B)$.

For $p\geq 1$, a measure μ , a ball B , and a hyperplane L , we set

$$\beta_{\mu,p}(B,L)=\left(\frac{1}{r(B)^n}\int_B\left(\frac{\text{distance}\{\text{dist}\}(x,L)}{r(B)}\right)^p\text{d}\mu(x)\right)^{1/p}.$$

We define $\beta_{\mu,p}(B)=\inf_L\beta_{\mu,p}(B,L)$, where the infimum is taken over all hyperplanes L . For $B=B(x,r)$, we also write $\beta_{\mu,p}(x,r,L)=\beta_{\mu,p}(B,L)$, and $\beta_{\mu,p}(x,r)=\beta_{\mu,p}(B)$. For a given cube $Q\in\square_\mu$ we define:

$$\begin{aligned} \beta_{\mu,p}(Q,L)&= \beta_{\mu,p}(\lambda Q,L), & b\beta_{\mu,p}(Q,L)&= b\beta_{\mu,p}(Q), & b\beta_{\mu,p}(\lambda Q)&= b\beta_{\mu,p}(Q), \\ \beta_{\mu,p}(\lambda Q)&= \beta_{\mu,p}(Q), & b\beta_{\mu,p}(\lambda Q)&= b\beta_{\mu,p}(Q), & b\beta_{\mu,p}(Q)&= b\beta_{\mu,p}(\lambda Q), \end{aligned}$$

Also, we define similarly

$$b\beta_{\mu,p}(\lambda Q,L),\quad b\beta_{\mu,p}(\lambda Q,L),\quad b\beta_{\mu,p}(Q),\quad b\beta_{\mu,p}(\lambda Q),$$

by identifying these coefficients with the analogous ones in terms of B_Q . These coefficients are defined in the same way as $b\beta_{\text{supp}\mu}(B,L)$ and $b\beta_{\text{supp}\mu}(B)$, replacing again B by $Q\in\square_\mu$ or λQ .

The Hajlasz-Sobolev spaces

Let (Σ,σ) be a metric space with σ a doubling measure on Σ , which means that there is a uniform constant $C_\sigma\geq 1$ such that $\sigma(B(x,2r))\leq C_\sigma\sigma(B(x,r))$, for all $x\in\Sigma$ and $r>0$.

For a Borel function $f:\Sigma\rightarrow\mathbb{R}$, we say that a non-negative Borel function $g:\Sigma\rightarrow\mathbb{R}$ is a *Hajlasz upper gradient* of f if

$$|f(x)-f(y)|\leq |x-y|(g(x)+g(y))\qquad\text{for }\sigma\text{-a.e. }x,y\in\Sigma.$$

We denote the collection of all the Hajlasz upper gradients of f by $D(f)$. For $p\geq 1$, we denote by $\mathcal{W}^{1,p}(\Sigma)$ the space of Borel functions f which have a Hajlasz upper gradient in $L^p(\sigma)$, and we let $W^{1,p}(\Sigma)$ be the space of functions $f\in L^p(\sigma)$ which have a Hajlasz upper gradient in $L^p(\sigma)$; that is, $\mathcal{W}^{1,p}(\Sigma)=\mathcal{W}^{1,p}(\Sigma)$. We define the semi-norm

$$\|f\|_{\mathcal{W}^{1,p}(\Sigma)}=\left(\int_\Sigma|f|^p\text{d}\sigma+\inf_{g\in D(f)}\int_\Sigma|g|^p\text{d}\sigma\right)^{1/p}$$

and the scale-invariant norm

$$\|f\|_{W^{1,p}(\Sigma)}=\text{diam}(\Sigma)^{-1}\|f\|_{L^p(\Sigma)}+\inf_{g\in D(f)}\|g\|_{L^p(\Sigma)}.$$

For any metric space Σ , if $p\in(1,\infty)$, from the uniform convexity of $L^p(\sigma)$, one easily deduces that the infima in the definitions of $\|f\|_{W^{1,p}(\Sigma)}$ and $\|g\|_{W^{1,p}(\Sigma)}$ are uniquely attained. We denote by $\nabla_{H,p}f$ the function g which attains the infimum. The spaces $\mathcal{W}^{1,p}(\Sigma)$ and $W^{1,p}(\Sigma)$ were introduced in , and are known as the *Hajlasz-Sobolev spaces*.

Properties of the Carleson and non-tangential maximal operators

First we state the well-known change-of-aperture lemma for the non-tangential maximal functions.

Lemma 1. Fix $p > 1$, $\alpha, \beta > 0$, and $\hat{c}_1, \hat{c}_2 \in (0, 1/2]$. We have that

$$\begin{aligned} & \| \tilde{c}_1, q(H) \|_{LP(\partial\Omega)} \lesssim_{n,q,p} \| \tilde{c}_2, q(H) \|_{LP(\partial\Omega)}, \quad \text{for any } H \in L_{loc}^q(\Omega). \\ & (u) \text{Vert}_{L^p(\partial\Omega)} \approx_{\alpha, \beta, n, p} (u) \text{Vert}_{L^p(\partial\Omega)} \end{aligned}$$

The L^p norms of the Carleson functionals $\tilde{c}_{\hat{c},q}$ defined in ([eq:carlesonfn]) are equivalent under a change of the parameter $\hat{c}$, as the next lemma states. We defer its proof to Appendix 8.1.

Lemma 1. Let $\hat{c}_1, \hat{c}_2 \in (0, 1/2]$, $q \in [1, \infty)$, and $p > 1$. Then

$$\| \tilde{c}_1, q(H) \|_{LP(\partial\Omega)} \lesssim_{n,q,p} \| \tilde{c}_2, q(H) \|_{LP(\partial\Omega)}, \quad \text{for any } H \in L_{loc}^q(\Omega).$$

As remarked in the introduction, there is a duality between the operators $\tilde{c}$ and $\square$. The precise results which we shall use are stated in Proposition 1 below. When Ω is the half-space, these results were shown by Hytönen and Rosén in [10]. For the benefit of the reader, in Appendix 8.2, we give the proof of the proposition in our setting, by appealing to the discrete vector-valued model in [11].

Proposition 1 (Duality between $\square$ and $\tilde{c}$). Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be a domain satisfying the corkscrew condition and such that $\partial\Omega$ is n -Ahlfors regular. Suppose that either Ω is bounded, or that $\partial\Omega$ is unbounded. Let $p, q \in (1, \infty)$ and p', q' their Hölder conjugates. Then $\|\tilde{c}_{\hat{c},q}\|_{L^p(\partial\Omega)} = \|\square_{\hat{c},q'}\|_{L^{p'}(\partial\Omega)}$, and moreover,

$$\begin{aligned} & \| \tilde{c}_{\hat{c},q}(u) \|_{L^p(\partial\Omega)} \lesssim \| \square_{\hat{c},q'}(u) \|_{L^{p'}(\partial\Omega)} \\ & \| \square_{\hat{c},q'}(u) \|_{L^{p'}(\partial\Omega)} \lesssim \| \tilde{c}_{\hat{c},q}(u) \|_{L^p(\partial\Omega)} \\ & \| \tilde{c}_{\hat{c},q}(u) \|_{L^p(\partial\Omega)} \lesssim \| \square_{\hat{c},q'}(u) \|_{L^{p'}(\partial\Omega)} \\ & \| \square_{\hat{c},q'}(u) \|_{L^{p'}(\partial\Omega)} \lesssim \| \tilde{c}_{\hat{c},q}(u) \|_{L^p(\partial\Omega)} \end{aligned}$$

Remark 1. For any $p, q \in (1, \infty)$, the space $\|\tilde{c}_{\hat{c},q}\|_{L^p(\partial\Omega)}$ is not reflexive; indeed, we have that $\|\tilde{c}_{\hat{c},q}\|_{L^p(\partial\Omega)} \subsetneq \|\tilde{c}_{\hat{c},q}\|_{L^p(\partial\Omega)}$. See [12].

To obtain the $L^{p'}$ -solvability of the Poisson-Dirichlet problem with general data, we need the following approximation lemma. Its proof is deferred to Appendix 8.3.

Lemma 1. Fix $p, q \in (1, \infty)$ and $\hat{c} \in (0, 1/2]$. Then

- There exists a sequence of functions $\{\tilde{c}_{\hat{c},q,k}\}_{k \in \mathbb{N}}$ such that each H_k is compactly supported in Ω , $H_k \rightarrow H$ strongly in $L_{loc}^q(\Omega)$, $H_k \rightarrow H$ pointwise a.e. in Ω , and $H_k \rightarrow H$ strongly in $\|\tilde{c}_{\hat{c},q}\|_{L^p(\partial\Omega)}$ as $k \rightarrow \infty$.
- Suppose that H is compactly supported in Ω . Then there exists a family of functions $\{\tilde{c}_{\hat{c},q,\varepsilon}\}_{\varepsilon > 0}$ such that each H_ε is compactly supported in Ω , $H_\varepsilon \rightarrow H$ strongly in $L_{loc}^q(\Omega)$, and $\|\tilde{c}_{\hat{c},q,\varepsilon}\|_{L^p(\partial\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Whitney decompositions

Given an open set $U \subseteq \mathbb{R}^{n+1}$ and a number $\hat{c} \in (0, 1/2]$, let $k \in \mathbb{N}$ be the unique integer such that $2^{-k} \leq \hat{c}/4 < 2^{-k+1}$, and we say that a collection of closed cubes, $\square_{\hat{c}}(U)$, is a $\hat{c}$ -Whitney decomposition of U if the interiors of the cubes in $\square_{\hat{c}}(U) = \square_{\hat{c}}(U)$ are pairwise disjoint, $\bigcup_{I \in \square_{\hat{c}}(U)} I = U$, and moreover

$$2^{k+2} \text{diam } I \leq \text{dist}(I, \partial U) \leq 2^{k+5} \text{diam } I, \quad \text{for each } I \in \square_{\hat{c}}(U).$$

Such a $\hat{c}$ -Whitney decomposition always exists; this can be attained by dyadically subdividing $k+2$ times each Whitney cube from the standard decomposition.

Suppose that $\partial\Omega$ is n -Ahlfors regular and consider the dyadic lattice $\square_{\sigma}$ defined in Section 2.1.3. Then for each $I \in \square_{\sigma}(\Omega)$ there is some cube $Q \in \square_{\sigma}$ such that

$$\| \tilde{c}_{\hat{c},q}(u) \|_{L^p(\partial\Omega)} \lesssim \| \square_{\hat{c},q'}(u) \|_{L^{p'}(\partial\Omega)}$$

For any $I \in \square_{\sigma}(\Omega)$, we let $b(I) = b_{\hat{c}}(I)$ be the collection of all cubes $Q \in \square_{\sigma}$ which satisfy ([eq:bi]). If $Q \in b(I)$, we say that Q is a *boundary cube* of I . There is a uniformly bounded number of $Q \in b(I)$, depending only on n and the n -Ahlfors regularity of $\partial\Omega$. Conversely, given $Q \in \square_{\sigma}$ we let

$$w(Q) := \bigcup_{I \in \square_{\sigma}: Q \in b(I)} I.$$

It is easy to check that $w(Q)$ is made up of at most a uniformly bounded number of cubes I , but it may happen that $w(Q) = \emptyset$. Moreover, if $I \in \square_{\sigma}$ and $I \cap w(Q) \neq \emptyset$, then $I \subset w(Q)$ since the Whitney cubes are disjoint.

Given a Whitney cube $I \in \square_{\sigma}(U)$, we use the notation I^* for the slightly larger cube of same center as I and such that $\text{diam } I^* = (1+\theta) \text{diam } I$, where $\theta \in (0, 1/10)$ is a small fixed constant. We also define

$$w^*(Q) := \bigcup_{I \in \square_{\sigma}: Q \in b(I)^*} I.$$

Elliptic PDE preliminaries

Throughout this subsection, we assume only that Ω is a domain in $\mathbb{R}^{n+1}$, $n \geq 2$. We write $\mathcal{W}_2^{n+1} = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)\}$ and that $\mathcal{W}_2^{n+1} = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)\}$.

Recall that $C_c^\infty(\Omega)$ is the space of compactly supported smooth functions in Ω , and that for $p \in [1, \infty)$, $W^{1,p}(\Omega)$ is the Sobolev space of p -th integrable functions in Ω whose weak derivatives exist in Ω and are p -th integrable functions, while $W_0^{1,p}(\Omega)$ is the

completion of $C_c^\infty(\mathcal{Q})$ under the norm $\|u\|_{H^{1,p}(\mathcal{Q})} := \|u\|_{L^p(\mathcal{Q})} + \|\nabla u\|_{L^p(\mathcal{Q})}$. Moreover, $\mathcal{W}^{1,p}(\Omega)$ consists of the $L_{\text{loc}}^1(\mathcal{Q})$ functions whose weak gradient is p -th integrable over $\mathcal{Q}$, and we denote by $L^p(\mathcal{Q})$ the space of p -th integrable functions with compact support in $\mathcal{Q}$. We let $Y_0^{1,2}(\mathcal{Q})$ be the completion of $C_c^\infty(\mathcal{Q})$ under the norm $\|u\|_{Y_0^{1,2}(\mathcal{Q})} := \|u\|_{L^2(\mathcal{Q})} + \|\nabla u\|_{L^2(\mathcal{Q})}$.

We assume throughout that A is a real, not necessarily symmetric matrix satisfying [\(eq.elliptic\)](#). Define the elliptic operator L acting formally on real-valued functions u by
$$\operatorname{div}(\nabla u) = \sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i} \big(a_{ij} \frac{\partial u}{\partial x_j} \big).$$

We write A^T for the transpose of A , and $L^* = -\operatorname{div} A^T \nabla$.

Properties of weak solutions

Given $H \in L_{\text{loc}}^{2,\infty}(\mathcal{Q})$ and $F \in L_{\text{loc}}^2(\mathcal{Q})$, we say that a function $w \in W_{\text{loc}}^{1,2}(\mathcal{Q})$ solves $Lw = H - \operatorname{div} F$ in the weak sense, or that w is a weak solution of the equation $Lw = H - \operatorname{div} F$, if for any $\phi \in C_c^\infty(\mathcal{Q})$, we have that

$$\int_{\mathcal{Q}} A \nabla w \nabla \phi \, dm = \int_{\mathcal{Q}} H \phi + F \nabla \phi \, dm.$$

The proof of the following basic inequality which weak solutions verify is standard.

Lemma 1 (Caccioppoli inequality). *Let B be a ball in $\mathcal{Q}$ and let $w \in W_{\text{loc}}^{1,2}(B^*)$ be a weak solution to the equation $Lw = H - \operatorname{div} F$ in B^* , where $F \in L^2(B^*)$, $H \in L^{2,\infty}(B^*)$, and $\theta \in (0,1)$. Then*

$$w|_{B^*}^2 \leq \frac{1}{\theta} \frac{1}{r(B)^2} \int_{B^*} |w|^2 \, dm + \frac{1}{\theta} \Big(\int_{B^*} |H|^2 \, dm \Big)^{\frac{1}{2}} + \int_{B^*} |F|^2 \, dm.$$

The next lemma comprises the conclusions of the De Giorgi-Nash-Moser theory, a capstone result in the regularity of elliptic PDEs with rough coefficients.

Lemma 1. *Let B be a ball in $\mathcal{Q}$, $B^* = (1+\theta)B$ for some $\theta \in (0,1]$, and let $u \in W^{1,2}(B^*)$ be a weak solution to the equation $Lu = 0$ in B^* . Then there exists $\eta \in (0,1)$ so that u is Hölder continuous with exponent η in B^* . Moreover, for any $q > 0$,*

$$\begin{aligned} & \sup_{x \in B} |u(x)| \leq \frac{1}{\eta} \Big(\sup_{x \in B} |u(x)| + \int_{B^*} |u|^q \, dm \Big)^{\frac{1}{q}} + \frac{1}{\eta} \int_{B^*} |u|^q \, dm \\ & \text{and} \\ & \sup_{x \in B} |u(x)| \leq \frac{1}{\eta} \Big(\sup_{x \in B} |u(x)| + \int_{B^*} |u|^q \, dm \Big)^{\frac{1}{q}} + \frac{1}{\eta} \int_{B^*} |u|^q \, dm \end{aligned}$$

$$\begin{aligned} & \sup_{x \in B} |u(x)| \leq \frac{1}{\eta} \Big(\sup_{x \in B} |u(x)| + \int_{B^*} |u|^q \, dm \Big)^{\frac{1}{q}} + \frac{1}{\eta} \int_{B^*} |u|^q \, dm \\ & \text{and} \\ & \sup_{x \in B} |u(x)| \leq \frac{1}{\eta} \Big(\sup_{x \in B} |u(x)| + \int_{B^*} |u|^q \, dm \Big)^{\frac{1}{q}} + \frac{1}{\eta} \int_{B^*} |u|^q \, dm \end{aligned}$$

When A is Lipschitz, we have a control on the size of the gradient of the solution. The precise technical result which we shall use is the following; it is a consequence of the proof of .

Lemma 1. *Let B be a ball in $\mathcal{Q}$, $L = -\operatorname{div} A \nabla$, with A a Lipschitz matrix function verifying [\(eq.elliptic\)](#) on $2B$, and suppose that $\|\nabla A\|_{L^\infty(\mathbb{R}^n)} \leq C$. Let u solve $Lu = 0$ in $2B$. Then ∇u is continuous on $2B$, and*

$$\begin{aligned} & \sup_{x \in B} |\nabla u(x)| \leq \frac{1}{\eta} \Big(\sup_{x \in B} |\nabla u(x)| + \int_{2B} |\nabla u|^2 \, dm \Big)^{\frac{1}{2}} + \frac{1}{\eta} \int_{2B} |\nabla u|^2 \, dm \\ & \text{and} \\ & \sup_{x \in B} |\nabla u(x)| \leq \frac{1}{\eta} \Big(\sup_{x \in B} |\nabla u(x)| + \int_{2B} |\nabla u|^2 \, dm \Big)^{\frac{1}{2}} + \frac{1}{\eta} \int_{2B} |\nabla u|^2 \, dm \end{aligned}$$

Green's function and elliptic measure

We start with the definition and properties of the integral kernel to the solution operator L^{-1} , known as the Green's function.

Definition 1 (Green's function). Assume that $\mathcal{Q} \subset \mathbb{R}^{n+1}$ is an open, connected set, and let $L = -\operatorname{div} A \nabla$ where A is a real, strongly elliptic, not necessarily symmetric matrix of bounded measurable coefficients. There exists a unique non-negative function $G_L : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$, called *Green's function* for L , satisfying the following properties:

- For each $x, y \in \mathcal{Q}$, $G_L(x, y) \leq |x - y|^{1-\eta}$, and $G_L(x, y) > 0$ in $\mathcal{Q}$.
- For each $\hat{c} \in (0,1)$, $x \in \mathcal{Q}$ and any $y \in B(x, \hat{c}\delta(x))$, we have $|x - y|^{1-\eta} \leq \hat{c} G_L(x, y)$.
- For each $x \in \mathcal{Q}$, $\int_{\mathcal{Q}} G_L(x, \cdot) \, d\mu(\cdot) \in C(\overline{\mathcal{Q}} \setminus \{x\}) \cap W^{1,2}_0(\mathcal{Q})$ and $G_L(x, \cdot)|_{\partial \mathcal{Q}} \equiv 0$.
- For each $x \in \mathcal{Q}$, the identity $L G_L(\cdot, x) = \delta_x$ holds in the distributional sense; that is,
$$\int_{\mathcal{Q}} \Delta(y) \nabla_y G_L(y, x) \nabla \phi(y) \, dm(y) = \phi(x), \quad \text{for any } \phi \in C_c^\infty(\mathcal{Q}).$$
- If $H \in L^{\frac{n+1}{n+1-\theta}}(\mathcal{Q})$ and $F \in L^n(\mathcal{Q})$, we have that the function
$$w(x) = \int_{\mathcal{Q}} G_L(y, x) H(y) \, dm(y) + \int_{\mathcal{Q}} \nabla_y G_L(y, x) F(y) \, dm(y)$$
 solves the equation $L^* w = H - \operatorname{div} F$ in the weak sense in $\mathcal{Q}$, $w \in Y_0^{1,2}(\mathcal{Q})$, and $\|\nabla w\|_{L^\infty(\mathcal{Q})} \leq C \|H\|_{L^{\frac{n+1}{n+1-\theta}}(\mathcal{Q})} + \|\nabla F\|_{L^n(\mathcal{Q})}$.
- For each $x, y \in \mathcal{Q}$ with $x \neq y$,

$$G_L(x,y) = G_L^*(y,x).$$

vii. Define the operators L_Q^{-1} and $L_Q^{-1}\text{div}$ by

$$(L_Q^{-1}H)(x) := \int_Q G_L(x,y)H(y) \, dm(y), \quad (L_Q^{-1}\text{div}F)(x) := \int_Q \nabla_y G_L(x,y)F(y) \, dm(y), \quad x \in \Omega.$$

Then we have that L_Q^{-1} maps $L^2(\Omega)$ into $L^{2^*}(\Omega)$, and the operator $L_Q^{-1}\text{div}$ maps $L^2(\Omega)$ into $L^{2^*}(\Omega)$.

The construction of the Green's function in bounded domains for real non-symmetric elliptic matrices with bounded and measurable coefficients may be found in , while the unbounded case is shown in . The sharp representation formula ((eq.represent1)) for H and F in Lorentz spaces has been shown in .

To define the elliptic measure, we borrow the setting of . Assume that ∂Q is n -Ahlfors regular, in this setting, Ω is Wiener regular, so that the continuous Dirichlet problem for L is solvable in Ω . By the maximum principle and the Riesz Representation Theorem, there exists a family of probability measures $\{\omega_L^x\}_{x \in \Omega}$ on ∂Q so that for each $f \in C_c(\partial Q)$ and each $x \in \Omega$, the solution u to the continuous Dirichlet problem with data f satisfies that $u(x) = \int_{\partial Q} f(\zeta) \, d\omega_L^x(\zeta)$. We call ω_L^x the L -elliptic measure with pole at x .

We now turn to several well-known estimates that the L -elliptic measure verifies, see and . For a proof of the next lemma, see .

Lemma 1 (Bourgain's estimate). *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be open with n -Ahlfors regular boundary. Then there exists $c > 0$ depending only on n, λ , and the n -Ahlfors regularity constant, such that for any $\xi \in \partial\Omega$ and $r \in (0, \text{diam}(\partial\Omega)/2]$, we have that $\omega^x(B(\xi, 2r) \cap \partial\Omega) \geq c$, for all $x \in \Omega \cap B(\xi, r)$.*

Moreover, the preceding lemma implies the following result via a standard argument involving the maximum principle and the pointwise upper bound for the Green's function in Definition [def.green] (i). For a proof in the case that $L = -\Delta$, which is readily generalized to our setting, see .

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be open with n -Ahlfors regular boundary. Let $B = B(x_0, r)$ be a closed ball with $x_0 \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$. Then*

$$\int_{\Omega} \frac{1}{|x-y|^{n+1}} \, d\mu(y) \leq C \int_{\Omega} \frac{1}{|x-y|^{n+1}} \, d\mu(y) \quad \text{for all } x \in \Omega \cap B(x_0, r).$$

where the implicit constant depends only on n, λ , and the n -Ahlfors regularity constant.

The following generalization of the formula ((eq.represent1)) is well-known; for a detailed proof in our setting, see .

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open, connected set with n -Ahlfors regular boundary. For any $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$, we have that*

$$\Phi(x) = \int_{\partial\Omega} \Phi \, d\sigma + \int_{\Omega} \nabla_y G_L(x,y) \cdot \nabla \Phi(y) \, dm(y), \quad \text{for m-a.e. } x \in \Omega.$$

A boundary version of the Hölder continuity of solutions will prove useful; see .

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be open with n -Ahlfors regular boundary, and let $\xi \in \partial\Omega$, $r \in (0, \text{diam}(\partial\Omega))$. Suppose that u is a non-negative solution of $Lu = 0$ in $\Omega \cap B(\xi, 2r)$, which vanishes continuously on $B(\xi, 2r) \cap \partial\Omega$. Then there exist $\eta \in (0, 1)$ and $C \geq 1$, depending only on n, λ , and the n -Ahlfors regularity constant, so that*

$$\int_{B(\xi, r)} u \, d\mu \leq C \int_{B(\xi, 2r)} u \, d\mu \quad \text{for all } x \in \Omega \cap B(\xi, r).$$

Now, we record a well-known equivalence between the solvability of (D_p^L) and a weak- RH_p property of the Poisson kernel. Under our lax geometric assumptions, the proposition is essentially shown in for the case of the Laplacian. The direction (b) $\Rightarrow$ (a) in the proposition below is already shown in ; for completeness, we give a full proof of the direction (a) $\Rightarrow$ (b), but we defer it to Appendix 8.4.

Proposition 1. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be open with n -Ahlfors regular boundary and satisfying the corkscrew condition. Fix $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$. The following are equivalent.*

- (D_p^L) is solvable in Ω .
- The L -elliptic measure ω_L is absolutely continuous with respect to σ , and for every ball B centered at $\partial\Omega$ satisfying that $\text{diam } B \leq 2\text{diam}(\partial\Omega)$,

$$\int_B \left(\int_{\Omega} \frac{1}{|x-y|^{n+1}} \, d\mu(y) \right)^{p'} \, d\sigma(x) \leq C \int_B \left(\int_{\Omega} \frac{1}{|x-y|^{n+1}} \, d\mu(y) \right)^p \, d\sigma(x) \quad \text{for all } x \in \Omega \cap B.$$

- The L -elliptic measure ω_L is absolutely continuous with respect to σ and there is some $A > 1$ big enough such that, for every ball B centered in $\partial\Omega$ with $\text{diam}(B) \leq 2\text{diam}(\partial\Omega)$ and all $x \in AB$ such that $\text{dist}(x, \partial\Omega) \geq A^{-1}r(B)$, it holds that

$$\int_B \left(\int_{\Omega} \frac{1}{|x-y|^{n+1}} \, d\mu(y) \right)^{p'} \, d\sigma(x) \leq C \int_B \left(\int_{\Omega} \frac{1}{|x-y|^{n+1}} \, d\mu(y) \right)^p \, d\sigma(x) \quad \text{for all } x \in \Omega \cap B.$$

In Section 7, we will make use of the following powerful localization result for the regularity problem, shown in the proof of . Define

On the other hand, we have that $\operatorname{dist}(I, \xi) \geq \frac{1}{4} |y - \xi|$. By fixing $\beta \geq \alpha$ large enough that $\frac{1}{\beta} \leq \frac{1}{10}$, we thus guarantee that

$$\operatorname{dist}(Q, \xi) \geq \operatorname{dist}(I, \xi) - (\operatorname{diam} Q) \geq |y - \xi|/4 \geq \tau \delta(x)/4 \geq 2^{-3} \tau \delta(x).$$

Now let $m = m_Q \geq -3$ be the unique integer such that $2^m \tau \delta(x) \leq \operatorname{dist}(Q, \xi) < 2^{m+1} \tau \delta(x)$, and suppose that $\ell(Q) \geq \tilde{c} 2^m \tau \delta(x)$. Then

$$\begin{aligned} & \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \\ & \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \end{aligned}$$

which gives a contradiction if β is chosen large enough (depending on $\tilde{c}$). Thus we have that $\ell(Q) < \tilde{c} 2^m \tau \delta(x)$, so that $Q \in \mathfrak{F}^m$, finishing the proof of [\(eq.ine\)](#).

With the claim at hand, for each $m \geq -3$ we let $\{Q_j^m\}_j$ be the smallest family of cubes of uniform generation k with $2^{-k} \leq \tilde{c} 2^m \tau \delta(x) < 2^{-k+1}$ such that for each $Q \in \mathfrak{F}^m$, there exists Q_j^m with $Q \subseteq Q_j^m$. We write $\mathfrak{F}^m$ for the family of cubes contained in Q_j^m . We now claim that for any $m \geq -3$, j , and $Q \in \mathfrak{F}^m$, we have that $x \in Q \setminus 2^{14} B_Q$. Fix a large integer $m_0 \geq -3$ to be determined later. If $m \geq m_0$, the triangle inequality implies that

$$\begin{aligned} & \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \\ & \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \end{aligned}$$

We have thus shown that for any $m \geq -3$, any j , and any $Q \in \mathfrak{F}^m$, we have that $x \in Q \setminus 2^{14} B_Q$. Proceeding as in [\(eq.i2first1\)](#) and [\(eq.i21\)](#), and using the preceding claim to apply Lemma [1](#) and obtain the appropriate analogue of [\(eq.usesupper1\)](#), we deduce that

$$\begin{aligned} & \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \\ & \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \end{aligned}$$

Arguing as in the proof of [\(eq.i231\)](#), for fixed m and j we may show that

$$\begin{aligned} & \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \\ & \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \end{aligned}$$

where we used the n -Ahlfors regularity of $\partial\Omega$, the fact that $\ell(Q_j^m) \approx 2^m \tau \delta(x)$, and Proposition [1](#). It follows from [\(eq.i3211\)](#) and [\(eq.i3221\)](#) that

$$T_{32}(x) \leq (\sum_m \geq -3 \omega^x(\cup_{j \geq 1} 2^{16} B_{Q_j^m})) (\mathcal{M}_\sigma(\square_2(F)^{p_1'})(\xi))^{1/p_1'}.$$

Let $E = \cup_{j \geq 1} 2^{16} B_{Q_j^m}$. By choosing $\tilde{c}$ small enough, we may guarantee that $\operatorname{dist}(E, \xi) > 2^{m-2} \tau \delta(x)$, and therefore $\omega^1(E) = 0$ on $B(\xi, 2^{m-2} \tau \delta(x)) \cap \partial\Omega$. Moreover, for all m large enough such that $2^m \geq 8/\tau$, we have that $x \in B(\xi, 2^{m-2} \tau \delta(x))$, and for such m , since $v(x) = \omega^1(E)$ solves $Lv = 0$ in Ω , the boundary Hölder inequality, Lemma [1](#), implies that $\frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx$, so that

$$\sum_m \geq -3 \omega^x(E) \leq \sum_m \geq -3 2^{-mm} \leq C.$$

Using this result on [\(eq.i3231\)](#), it follows that

$$\sup_x \int_{\gamma(\xi)} T_{32}(x) \ell_{I'}(\partial\Omega, d(\xi)) \leq \ell(\mathcal{M}_\sigma(\square_2(F)^{p_1'}))^{1/p_1'} \ell_{I'}(\partial\Omega) \leq \ell(\square_2(F)) \ell_{I'}(\partial\Omega).$$

For T_{31} , let $\square_1$ be the collection of Whitney cubes $I \in \square_1$ such that $I \cap (\gamma\beta(\xi) \setminus (B^x \cup B_\beta)) \neq \emptyset$. It is easy to see that $\cup_{I \in \square_1} I \subset \gamma\beta(\xi) \setminus (B^x \cup B_\beta)$. For any $I \in \square_1$ and $y \in I^\#$, we distinguish two cases: either $\delta(y) \leq K\delta(x)$ or $\delta(y) > K\delta(x)$, for K large to be specified. If $\delta(y) \leq K\delta(x)$, then $|y - x| \geq 2^{-4} K^{-1} \delta(y)$. If $\delta(y) > K\delta(x)$, then

$$|y - x| \geq |y - \xi| - |x - \xi| \geq \delta(y) - (1 + \alpha)\delta(x) > \delta(y) - (1 + \alpha)K^{-1}\delta(y),$$

so if we choose K large enough that $\frac{1}{K} \geq \frac{1}{1 + \alpha}$, then we have that $|y - x| > \delta(y)/2$. Thus, in any case we have that $|y - x| \geq \delta(y)$, and similarly, $|y - \xi| \geq \delta(y)$ for all $z \in B(x, \delta(x)/32)$. With this in mind, we see that

$$\begin{aligned} & \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \\ & \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \end{aligned}$$

where we have used that $G(z, y) \leq |y - z|^{1-n} \leq \delta(y)^{1-n}$ for each $y \in \gamma\beta(\xi) \setminus (B^x \cup B_\beta)$ and each $z \in B(x, \delta(x)/32)$. From [\(eq.i311\)](#) it follows that

$$\begin{aligned} & \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \\ & \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \leq \frac{1}{\ell(Q)} \int_Q \frac{1}{|x - \xi|^{2n-1}} dx \end{aligned}$$

where in the last inequality we have used [\(eq.tentandcar\)](#).

Conclusion

From the estimates [\(eq.i1done\)](#), [\(eq.i2done\)](#), [\(eq.i32done\)](#) and [\(eq.i31done\)](#), we deduce that

$$\|\widetilde{V}(\widetilde{u}_N)\|_{L^2(\Omega)} \leq C \|\widetilde{u}_N\|_{L^2(\Omega)} + C \|\widetilde{u}_N\|_{L^2(\Omega)}$$

It remains only to estimate the last term in the right-hand side of [\(eq.break\)](#). However, this term is almost completely analogous to the term we have just bounded in [\(eq.w2done\)](#), thus let us only comment on the changes. First, as in [\(eq.split\)](#) we break up $\|\widetilde{u}_N\|_{L^2(\Omega)}$ into three analogous parts T_1' , T_2' , T_3' . The term T_1' is controlled in the same way as T_1 was controlled, except now we use the fact that $L^2(\Omega) \rightarrow L^2(\Omega)$. The term T_2' is controlled similarly as T_2 was controlled, but now we do not need to use the Caccioppoli inequality; instead, for instance in the analogue of [\(eq.i2first\)](#) we use the basic estimate

$$\int_{\Omega} |\nabla \widetilde{u}_N|^2 dx \leq C \int_{\Omega} |\widetilde{u}_N|^2 dx + C \int_{\Omega} |\widetilde{u}_N|^2 dx + C \int_{\Omega} |\widetilde{u}_N|^2 dx$$

and mimic the proof of [\(eq.i2done\)](#) to finish the estimate for T_2' . The term T_3' is split into two terms T_{31}' and T_{32}' analogously as in [\(eq.split2\)](#), and we bound each of these separately; the only difference with the detailed proofs above will be the use of [\(eq.change\)](#) instead of the Caccioppoli inequality at the very beginning. As such, we control all three terms T_1' , T_2' and T_3' in the expected ways, and obtain the estimate

$$\|\widetilde{u}_N\|_{L^2(\Omega)} \leq C \|\widetilde{u}_N\|_{L^2(\Omega)} + C \|\widetilde{u}_N\|_{L^2(\Omega)}$$

From [\(eq.break\)](#), [\(eq.w2done\)](#), and [\(eq.w3done\)](#), the desired bound [\(eq.ntmaxn\)](#) follows.

We now consider an extension of Theorem 1 to the case of $F \in L^p(\Omega)$ and H with $\Delta_\Omega H \in L^p(\Omega)$.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain satisfying the corkscrew condition and with n -Ahlfors regular boundary. Let $p > 1$, p' its Hölder conjugate, and $L = -\operatorname{div} A \nabla$. Assume that $(D_p L)$ is solvable in Ω . Let $g \in L^p(\partial\Omega)$, H such that $\Delta_\Omega H \in L^p(\Omega)$. Then there exists a weak solution $w \in W_{loc}^{1,2}(\Omega)$ to the problem [\(eq.poisson\)](#) satisfying the estimate [\(eq.ntmaxn\)](#). The equation $w = g$ on $\partial\Omega$ is understood as

$$\lim_{r \rightarrow 0} \int_{\partial B_r(x)} \nabla w \cdot \nu = g(x) \quad \text{for } x \in \partial\Omega$$

Proof. Since $(D_p L)$ is solvable in Ω , it is well-known that for any $g \in L^p(\partial\Omega)$, there exists a weak solution $u \in W_{loc}^{1,2}(\Omega) \cap C_{loc}^\infty(\Omega)$ to the equation $Lu = 0$ in Ω satisfying that $\lim_{r \rightarrow 0} \int_{\partial B_r(x)} \nabla u \cdot \nu = g(x)$, for σ -a.e. $x \in \partial\Omega$. Hence, by linearity, it is enough for us to consider the case $g = 0$. By Lemma 1, there exist sequences $\{F_k\}$, $\{h_k\}$ of compactly supported Lipschitz functions on Ω such that $F_k \rightarrow F$ strongly in $L^p(\Omega)$ and $h_k \rightarrow \delta_\Omega H$ strongly in $L^p(\Omega)$, as $k \rightarrow \infty$. Let w_k be the unique weak solution in $C(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$ to the equation $Lw_k = \delta_\Omega^{-1} h_k - \operatorname{div} F_k$. For any $k, \ell \in \mathbb{N}$, we have that

$$L(w_k - w_\ell) = \delta_\Omega^{-1} (h_k - h_\ell) - \operatorname{div} (F_k - F_\ell),$$

and $w_k - w_\ell \in W_0^{1,2}(\Omega)$. Therefore, since we have already shown the bound [\(eq.ntmaxn\)](#) for the case of compactly supported Lipschitz data, we see that

$$\|w_k - w_\ell\|_{L^p(\Omega)} \leq C \|h_k - h_\ell\|_{L^p(\Omega)} + C \|F_k - F_\ell\|_{L^p(\Omega)}$$

as $k \rightarrow \infty$. This shows that the sequence $\{w_k\}$ is strongly convergent in the Banach space $W^{1,2}(\Omega)$, defined in Section 1.1.1. Let $w = \lim_{k \rightarrow \infty} w_k$ be the limit of $\{w_k\}$ in $W^{1,2}(\Omega)$, and furthermore, note that $w_k \rightarrow w$ strongly in $L_{loc}^2(\Omega)$ as $k \rightarrow \infty$. Let us remark that since we also have that $F_k \rightarrow F$ strongly in $L_{loc}^2(\Omega)$ and that $h_k \rightarrow \delta_\Omega H$ strongly in $L_{loc}^2(\Omega)$, then by the Caccioppoli inequality (Lemma 1), we see that the sequence $\{\nabla(w_k - w_\ell)\}$ is strongly convergent in $L_{loc}^2(\Omega)$. Then it is an exercise to check that $w \in W_{loc}^{1,2}(\Omega)$ with $\nabla w \rightarrow \nabla w$ strongly in $L_{loc}^2(\Omega)$. From this fact, it easily follows that the equation $Lw = H - \operatorname{div} F$ holds in the weak sense in Ω .

Finally, we claim that

$$\lim_{r \rightarrow 0} \int_{\partial B_r(x)} \nabla w \cdot \nu = 0 \quad \text{for } x \in \partial\Omega$$

However, this follows easily since $\|\nabla w\|_{L^2(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$ (after passing to a subsequence $\{k_j\}$ pointwise σ -a.e. on $\partial\Omega$ as $k_j \rightarrow \infty$).

Proof of Theorem 1

Proof of (c) $\Rightarrow$ (a). We follow part of the argument for . Assume that (c) holds, and we show that (a) holds by proving that the assumptions in Proposition 1 (c) hold. To this end we prove that for B , A , and $x \in Q$ as in Proposition 1 (c),

$$\mathbb{S}\{\mathcal{M}\}(\sigma, \tau) = \sup_{0 \leq t \leq 1} \|\partial_t \Omega_t\|_{\mathcal{M}}(\sigma, \tau),$$

for any signed Radon measure τ .

$$\begin{aligned} & \mathbb{S} \begin{Bmatrix} \text{gathered} \\ \text{label} \{ \text{eq_compb} \} \end{Bmatrix} \omega \text{mega} \text{L}^{\wedge} \text{X} \{ \text{b_xi} \} \text{lessmin} \text{mathchoice} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{displaystyle} \{ \text{texstyle} \} \} \} \text{int} \} \mathbb{S} \text{hbox} \{ \mathbb{S} \text{texstyle} \} \} \text{kgern} . \text{5wd} \\ & \quad 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{texstyle} \{ \text{scriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptstyle} \} \} \\ & \quad \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{scriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \text{int} \{ \text{B} \{ \text{deltax} \} / 8 \} \} \omega \text{mega} \text{L}^{\wedge} \text{Z} \{ \text{dmz} \} \text{leq} \text{mathchoice} \\ & \quad \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{displaystyle} \{ \text{texstyle} \} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{texstyle} \} \} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{texstyle} \{ \text{scriptstyle} \} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptstyle} \} \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{scriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \} \} \} \text{kgern} . \text{5wd} 0 \} \} \} \\ & \quad \text{vint} \{ \text{B} \{ \text{deltax} \} / 8 \} \} \{ \text{partial} \Omega \} \text{varphi} \{ \text{B} \{ \text{xi} \} \} \text{d} \omega \text{mega} \text{L}^{\wedge} \text{Z} \{ \text{dmz} \} \} = - \\ & \quad \text{mathchoice} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{displaystyle} \{ \text{texstyle} \} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{texstyle} \} \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{texstyle} \{ \text{scriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptstyle} \} \} \} \text{kgern} . \text{5wd} \\ & \quad 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{scriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \} \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \text{vint} \{ \text{B} \{ \text{deltax} \} / 8 \} \} \Omega \text{mega} \text{A}^{\wedge} \text{T} \text{nabla} \text{a} + \text{G} \{ \text{L} \} \{ \text{zy} \} \text{nabla} \\ & \quad \text{varphi} \{ \text{B} \{ \text{xi} \} \} \text{d} \text{m} \{ \text{y} \} \text{d} \text{m} \{ \text{y} \} \text{d} \text{m} \{ \text{z} \} \text{lessmin} \text{frac} \{ 1 \} \text{int} \{ \text{2B} \{ \text{xi} \} \} \text{B} \{ \text{g} \} \text{mathchoice} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{displaystyle} \{ \text{texstyle} \} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{texstyle} \} \} \} \text{kgern} . \text{5wd} \\ & \quad 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{texstyle} \{ \text{scriptstyle} \} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptstyle} \} \} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{scriptstyle} \{ \text{scriptscriptstyle} \} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \} \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \text{vint} \{ \text{B} \{ \text{deltax} \} / 8 \} \} \text{nabla} \text{a} + \text{G} \{ \text{L} \} \{ \text{zy} \} \} \text{d} \text{B} \{ \text{g} \} \text{d} \text{m} \{ \text{y} \} \} \\ & \quad \text{lessmin} \text{int} \{ \text{CB} \{ \text{xi} \} \} \text{widetilde{mathcal{N}}} \} \cdot \text{2}^{\wedge} \{ \text{C} \{ \text{deltax} \} \} \} \text{B} \{ \text{g} \} \{ \text{mathds} \\ & \quad 1 \} \} \Omega \text{mega} \text{backslash B} \{ \text{B} \{ \text{xi} \} \} \text{d} \{ \text{deltax} \} / 4 \} \} \text{mathchoice} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{displaystyle} \{ \text{texstyle} \} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{texstyle} \} \} \} \text{kgern} . \text{5wd} \\ & \quad 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{texstyle} \{ \text{scriptstyle} \} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptstyle} \} \} \\ & \quad \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} 0=\text{hbox} \{ \mathbb{S} \text{scriptstyle} \{ \text{scriptscriptstyle} \} \} \} \text{int} \} \} \\ & \quad \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \} \} \} \text{kgern} . \text{5wd} 0 \} \} \{ \{ \text{setbox} \\ & \quad 0=\text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \{ \text{scriptscriptstyle} \} \} \text{int} \} \} \text{vcenter} \{ \text{hbox} \{ \mathbb{S} \text{scriptscriptstyle} \\ & \quad \} \} \} \text{kgern} . \text{5wd} 0 \} \} \text{vint} \{ \text{B} \{ \text{deltax} \} / 8 \} \} \text{nabla} \text{a} \cdot \text{2} \{ \text{G} \{ \text{L} \} \} \\ & \quad \{ \text{z} \cdot \text{cdot} \} \} \text{d} \text{m} \{ \text{z} \} \} \text{B} \{ \text{g} \} \cdot \text{d} \{ \text{sigma} \} \} \text{end} \{ \text{gathered} \} \mathbb{S} \end{aligned}$$

Taking the supremum over $0 < r \leq \text{dist}(x, \partial\Omega)/4$, we derive

$$\begin{aligned} & \mathbb{S}\mathbb{S}\{\mathcal{M}_1\} \cdot \{\text{\texttt{_sigma}}_0\} \cdot \{\text{\texttt{_omega}}^{\wedge} \text{\texttt{_L}}(\text{\texttt{_xi}})\} \cdot \text{\texttt{_lesssim}} \\ & \quad \{\mathcal{M}_1\} \cdot \{\mathcal{M}_1\} \cdot \{\text{\texttt{_sigma}}_0\} \cdot \{\text{\texttt{_mathfrak{m}}_{\text{aths}}}\} \\ & \quad \text{\texttt{_l}}_1 \cdot \{\text{\texttt{_B}}(x, \text{\texttt{_C}}^{\text{\texttt{_d}}})\} \cdot \text{\texttt{_widelite}}\{\text{\texttt{_mathcal{M}}}(N)\} \cdot \text{\texttt{_2}}^{\text{\texttt{_C}}}\{\text{\texttt{_delta}}(x)\} \cdot \text{\texttt{_Big}}\{\text{\texttt{_mathfrak{m}}_{\text{aths}}}\} \\ & \quad \cdot \{\text{\texttt{_Omega}}\backslash \text{\texttt{_B}}(x, \text{\texttt{_delta}}(x)/4)\} \cdot \text{\texttt{_mainchoice}} \{\{\text{\texttt{_setbox}} \\ & \quad 0=\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_displaystyle}}\}\text{\texttt{_textstyle}}\}\{\text{\texttt{_int}}\}\} \cdot \text{\texttt{_vcenter}}\{\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\}\text{\texttt{_kern}}-\text{\texttt{_5wd}} \\ & \quad 0\}\} \cdot \{\text{\texttt{_setbox}} 0=\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_scriptstyle}}\}\{\text{\texttt{_int}}\}\} \cdot \text{\texttt{_vcenter}}\{\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_scriptstyle}} \\ & \quad \}\}\text{\texttt{_kern}}-\text{\texttt{_5wd}} 0\}\} \cdot \{\text{\texttt{_setbox}} 0=\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_scriptscriptstyle}}\}\{\text{\texttt{_scriptscriptstyle}}\}\{\text{\texttt{_int}}\}\} \\ & \quad \cdot \text{\texttt{_vcenter}}\{\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_scriptscriptstyle}}\}\}\text{\texttt{_kern}}-\text{\texttt{_5wd}} 0\}\} \cdot \{\text{\texttt{_setbox}} \\ & \quad 0=\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_scriptscriptstyle}}\}\{\text{\texttt{_scriptscriptstyle}}\}\}\text{\texttt{_int}}\} \cdot \text{\texttt{_vcenter}}\{\text{\texttt{_hbox}}\{\text{\texttt{_S}}\}\text{\texttt{_scriptscriptstyle}} \\ & \quad \}\}\text{\texttt{_kern}}-\text{\texttt{_5wd}} 0\}\} \cdot \text{\texttt{_int}}\{\text{\texttt{_B}}(x, \text{\texttt{_delta}}(x)/8)\} \cdot \text{\texttt{_nabla}}_2 \cdot \text{\texttt{_G}}_L\{\text{\texttt{_Z}}(\text{\texttt{_z}}\cdot\text{\texttt{_cdot}})\text{\texttt{_dm}}\text{\texttt{_Z}}(\text{\texttt{_B}})\}\text{\texttt{_Big}}\} \\ & \quad (x), \mathbb{S}\mathbb{S} \end{aligned}$$

so that

$$\begin{gathered} \label{eq.fkf4} \mathcal{M}_{\sigma,0} \backslash \\ \omega^x \mathcal{L} \backslash \mathcal{L}^p(\Lambda B_{\sigma}) \backslash \end{gathered}$$

Let us begin. For each $Q \in \square_{\sigma}$ and a fixed $M \geq 2$ large to be chosen below, we let

$$\lambda_{\mathcal{Q}}(Q) = \ell(Q)^2 \sup \{ |\nabla A(x)|^2 : x \in B(x_Q, M\ell(Q)) \cap \Omega, \delta_{\mathcal{Q}}(x) \geq M^{-1}\ell(Q) \}.$$

Lemma 1. *Suppose that A satisfies the DKP condition in Ω . Then there exists $C(M) > 0$ such that*

$$\sum_{Q \in \square_{\sigma}} Q \subset \mathbb{R}^d \mathcal{Q}(Q) \sigma(Q) \leq C(M) \sigma(R), \quad \text{for any } R \in \square_{\sigma}$$

Proof. For $a > 0$, we denote

$$\Omega_a = \{x \in \Omega : \delta_{\mathcal{Q}}(x) \geq a\}.$$

Given $Q \in \square_{\sigma}$, applying the Besicovitch covering theorem we can cover $B(x_Q, M\ell(Q)) \cap \Omega_{M^{-1}\ell(Q)}$ by a family of balls B_i , $i \in I_Q$, centered in points $x_i \in B(x_Q, M\ell(Q)) \cap \Omega_{M^{-1}\ell(Q)}$, with radii $r_i = \delta_{\mathcal{Q}}(x_i)/8$, so that moreover the family $\{B_i\}_i \in I_Q$ has finite superposition. From the fact that $r_i \leq M\ell(Q)$ for each i and the finite superposition, it easily follows that $\#I_Q \leq M\delta$. Then we have

$$\begin{aligned} & \sum_{\substack{x \in B(z, \delta_{\mathcal{Q}}(\Omega_{M^{-1}\ell(Q)})) \\ \ell(Q) \leq \delta_{\mathcal{Q}}(x) \leq 2\ell(Q)}} \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \\ & \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \\ & \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \end{aligned}$$

where in the second inequality we took into account that $B_i \subset B(z, \delta_{\mathcal{Q}}(z)/2)$ for any $z \in B_i$. Therefore,

$$\begin{aligned} & \sum_{R \in \mathcal{R}} \lambda_{\mathcal{Q}}(R) \leq \sum_{R \in \mathcal{R}} \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \\ & \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \\ & \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \leq \sum_{i \in I_Q} \int_{B_i} |\nabla A(x)|^2 dx \end{aligned}$$

by the finite superposition of $B(x_Q, M\ell(Q)) \cap \Omega_{M^{-1}\ell(Q)}$, with $Q \in \square_{\sigma}$, $Q \subset R$, and the DKP condition. $\square$

The approximating Lipschitz graph

In this subsection we describe how to associate an approximating Lipschitz graph to a cube $R \in \square_{\sigma}$ assuming $b\beta_{\sigma}(k_1 R)$ to be small enough for some big constant $k_1 > 2$ (where we denoted $b\beta_{\sigma} \equiv b\beta_{\sigma, \text{supp } \sigma}$). The construction is based on the arguments in [1]. The first step consists in defining suitable stopping cubes.

Given $x \in \mathbb{R}^{n+1}$, we write $x = (x', x_{n+1})$. For a given cube $R \in \square_{\sigma}$ we denote by L_R a best approximating hyperplane for $b\beta_{\sigma}(k_1 R)$. We also assume, without loss of generality, that

$$\text{SSL}_R \perp \text{texp is the horizontal hyperplane } \{x_{n+1} = 0\}.$$

We let x_R^0 be the orthogonal projection of x_R on L_R (i.e., $x_R^0 = ((x_R)', 0)$). For a given $\gamma \in (0, 1/4)$ to be chosen below, we let

$$\text{SS} x_{-R}^+ = \text{big}((x_{-R}') \cdot (\text{frac } 14 - \gamma) r(\mathcal{B}(R)) \text{big}), \quad \text{quad } x_{-R}^- = \text{big}((x_{-R}') \cdot (-\text{frac } 14 + \gamma) r(\mathcal{B}(R)) \text{big}),$$

and we consider the balls

$$\text{SS} B^+(R) = B(x_{-R}^+, \text{frac } 14 r(\mathcal{B}(R))), \quad \text{quad } B^-(R) = B(x_{-R}^-, \text{frac } 14 r(\mathcal{B}(R))),$$

and the cylinders

$$\text{SS} C(R) = \text{big}(\{x \in \mathbb{R}^{n+1} : |x' - (x_{-R}')| \leq \text{frac } 14 \gamma \text{big}^{1/2} r(\mathcal{B}(R)), \{x_{n+1}\} \leq \text{frac } 14 \gamma \text{big}\}, \text{SS}$$

$$\text{SS} C^-(R) = \text{big}(\{x \in \mathbb{R}^{n+1} : |x' - (x_{-R}')| \leq \text{frac } 14 r(\mathcal{B}(R)), \{x_{n+1}\} \leq \gamma \text{big}\}, \text{SS}$$

It is easy to check that $C(R) \subset B^+(R) \cap B^-(R) \subset C(R)$. Also, in the case when $\text{dist}(x_R, L_R) \leq r(\mathcal{B}(R))/4$, say, we have

$$B^+(R) \cup B^-(R) \subset B(R).$$

We consider constants ε, δ , and θ such that $0 < \varepsilon \ll \delta \ll \gamma \leq 1/4$ and $0 < \theta \ll 1$ to be chosen later (depending on the corkscrew condition and the uniform rectifiability constants), $k_1 > 2$, and we denote by $\mathfrak{B}$ or $\mathfrak{B}(\varepsilon, \theta)$ the family of cubes $Q \in \square_{\sigma}$ such that either $b\beta_{\sigma}(k_1 Q) > \varepsilon$ or $\lambda_{\mathcal{Q}}(Q) > \theta$.

We consider now $R \in \square_{\sigma} \setminus \mathfrak{B}$. Notice that in this case ([eqinc77]) holds, and also

$$\text{SS} \text{label{eqinc78}} \text{big}(\{x \in R : |x' - (x_{-R}')| \leq \text{frac } 14 \gamma \text{big}^{1/2} r(\mathcal{B}(R)) \text{big}\} \subset C(R). \text{SS}$$

We let $\{\text{rm } \{\text{Stop}\}\}(R)$ be the family of maximal cubes $Q \in \square_{\sigma}(R)$ such that at least one of the following holds:

- $Q \cap B^+(R) \cap B^-(R) = \emptyset$.
- $Q \in \mathfrak{B}(\varepsilon, \theta)$, i.e., $b\beta_{\sigma}(k_1 Q) > \varepsilon$ or $\lambda_{\mathcal{Q}}(Q) > \theta$.
- $\angle(L_Q, L_R) > \delta$, where L_Q, L_R are best approximating hyperplanes for $\beta_{\sigma, \mathcal{Q}}(k_1 Q)$ and $\beta_{\sigma, \mathcal{Q}}(k_1 R)$, respectively, and $\angle(L_Q, L_R)$ denotes the angle between L_Q and L_R .
- $\sum P \in \square_{\sigma} : Q \subset P \subset \mathbb{R}^d \mathcal{Q}(P) \geq \theta$.

We denote by $\{\text{rm Tree}\}(R)$ the family of cubes in $\square_{\sigma}(R)$ which are not strictly contained in any cube from $\{\text{rm } \{\text{Stop}\}\}(R)$. We also consider the function

$$\text{SS} d_R(x) = \inf_{Q \in \{\text{rm Tree}\}(R)} \text{big}(\text{operatorname{dist}}(x, Q) + \text{mathop{mathrm}{diam}}(Q) \text{big}). \text{SS}$$

Notice that d_R is 1-Lipschitz. Assuming k_1 big enough (but independent of ε and δ) and arguing as in the proof of [1], the following holds:

Lemma 1. *Denote by Π_R the orthogonal projection on L_R . There is a Lipschitz function $A : L_R \rightarrow L_R^{\perp}$ with slope at most $C\delta$ such that*

Suppose that $U_R \subset \Omega$. We denote $B^0(R) = B(x_R^0, r(B(R)))$ and we let Γ_R^+ be the Lipschitz graph of the function $B^0(R) \cap L_R \ni x' \mapsto A(x') + \delta D R(x')$. Notice that this is a Lipschitz function with slope at most $C\delta < 1$ (assuming δ small enough). Then we define

$$\underline{\Omega}_R^+ = \{x = (x', x_{n+1}) \in B^+(R) : x_{n+1} > A(x') + \delta D_R(x')\}.$$

In case that $V_R \subset \Omega$, we define Γ_R^- and Ω_R^- analogously, replacing the above function $A(x') + \delta D_R(x')$ by $A(x') - \delta D_R(x')$. If $\mathbb{V}_R \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$, then we set $\Omega_R^- = \emptyset$. In any case, we define

$$\Omega_R = \Omega_R^+ \cup \Omega_R^-.$$

From Lemma 1 and the assumption that $C_1 C_2 \varepsilon \ll \delta$, it is immediate to check that

$$\frac{\partial R(x)}{\partial \Omega} \geq 0 \text{ for all } x \in \Omega \quad \text{R.S.}$$

For a given $a > 1$, we say that two cubes Q, Q' are a -close if

$$a^{-1}\ell(Q) \leq \ell(Q') \leq a\ell(Q) \text{ and } \text{dist}(Q, Q') \leq a(\ell(Q) + \ell(Q')).$$

We say that $Q \in \square_\sigma$ is a -close to $\text{Tree}(R)$ if there exists some $Q' \in \text{Tree}(R)$ such that Q and Q' are a -close. For $1 < a^* < a^{**}$ to be fixed below, we define the augmented trees

$$\mathcal{T}^*(R) = \{Q \in \mathcal{D} \mid \mu: \text{text}\{QS \text{ is } \mathcal{T}^*\text{-close to } \mathcal{T}^*(R)\}\},$$

$$\mathbb{S}^{\mathrm{Tree}}(\mathcal{R}) = \{Q \in \mathcal{D} : \mu : \text{\textit{SQS is }} \mathcal{S}^{\mathrm{Tree}}\text{-close to } \mathcal{R}\}.$$

Obviously, $\text{Tree}(R) \subset \text{Tree}^*(R) \subset \text{Tree}^{**}(R)$. Notice also that the families of cubes from $\text{Tree}^*(R)$ or $\text{Tree}^{**}(R)$ may not be trees.

We now let $\square(U)$ be a $(1/4)$ -Whitney decomposition of Ω (see Section 2.1.6).

Lemma 1. *Assuming $a^* > 1$ to be big enough, we have*

$$\overline{\Omega_R} \cap \Omega \subset \bigcup_{Q \in \text{Tree}^*(R)} w(Q).$$

Recall that $w(Q)$ is the Whitney region associated with Q (see Section 2.1.6). Notice that if $Q_R^- \neq \emptyset$, it may happen that $w(Q)$ is the union of some Whitney cubes contained in Q_R^+ and others in Q_R^- , for example.

Proof. Let $P \in \square(\Omega)$ be such that $\mathbb{P}(\text{cap } \overline{\Omega}, R) \leq \epsilon$ and fix $Q \in b(P)$. It suffices to show that $Q \in \{\text{rm Tree}\}^*(R)$ if a^* is taken big enough. To this end, we show that there exists some $Q' \in \{\text{rm Tree}\}(R)$ which is a^* -close to Q .

Notice first that $\ell(P) = C_5(\ell(R))$ for some fixed constant C_5 , because P intersects $\mathcal{S}(\text{overline}(\Omega_{\text{RS}}) \cap \mathcal{R})$ and $\mathcal{S}(\text{overline}(\Omega_{\text{RS}}) \cap \mathcal{R}) \cap \mathcal{P}(\text{cub}(\text{overline}(\Omega_{\text{RS}}) \cap \mathcal{R}))$. Then by [Lemma 3.1](#) we have $\mathcal{S}(\text{frac}(\text{def}(\mathcal{L}_2), \mathcal{L}_2) \cap \mathcal{R}) \cap \mathcal{Q} \leq \ell(\text{oper}(\text{mate}(\text{dist}(\mathcal{L}_2, \mathcal{P}), \text{partial}(\Omega_{\text{RS}})) \cap \mathcal{Q})) \approx \ell(\mathcal{P} \cap \mathcal{Q})$, for $\mathcal{Q}$ as above. Thus, $d_R(\mathcal{P}, \mathcal{Q}) \leq D_R(\mathcal{P}, \mathcal{Q}) \leq D_R(\mathcal{L}_2) + C\ell(\mathcal{Q}) \leq \sigma^{-1}(\ell(\mathcal{Q}))$. From the definition of d_R we infer that there exists some cube $\mathcal{Q}' \cap (\text{rm Tree}) \cap \mathcal{R}$ such that $(\ell(\mathcal{Q}') + \text{dist}(\mathcal{Q}', \mathcal{Q})) \leq C\sigma^{-1}(\ell(\mathcal{Q}))$.

If $\ell(Q) \geq C_5^{-1}\ell(R)$, we let $Q' = Q$. Otherwise, we let Q' be an ancestor of Q belonging to $\mathcal{S}(\text{rm Tree})(R)\mathcal{S}$ and satisfying $C_5^{-1}\ell(Q) \leq \ell(Q') < 2C_5^{-1}\ell(Q)$. The above condition $\ell(Q) = \ell(P) \leq C_5\ell(R)$ ensures the existence of Q' . Then, in any case, it follows that Q' is a^* -close to Q , for a^* big enough depending on δ . $\square$

In the rest of the lemmas in this subsection, we assume, without loss of generality, that $\Omega_R^+ \subset U_R \subset \Omega$.

Lemma 1. *If $SQ \text{ in } \{ \text{rm Tree} \}(R)$, then $\text{dist}(w(Q), \Omega_R^+) \leq C\ell(Q)$. Also, if $\Omega_R^- \neq \emptyset$, $\text{dist}(w(Q), \Omega_R^-) \leq C\ell(Q)$.*

Proof. We will prove the first statement. The second one follows by the same arguments. It is clear that $\text{dist}(w(Q), \Omega_R^+) \leq C\ell(R)$, and so the statement above holds if $\ell(Q) \geq \ell(R)$.

So we may assume that $\ell(Q) \leq c_1 \ell(R)$ for some small c_1 to be fixed below. By construction, the parent $\tilde{Q}$ of Q satisfies $\tilde{Q} \cap B^+(R) \cap B^-(R) \neq \emptyset$. Thus there exists $z \in B^+(R) \cap B^-(R)$ such that $|z - x_Q| \leq \ell(Q)$. Clearly, it holds $\text{dist}(z, \partial\Omega) \leq \ell(Q)$. On the other hand, we consider the point $x = (z', z'' + 1 + 2\ell(Q))$, so that

$$\text{dist}(x, \partial\Omega \cup \Gamma_R^+) \geq \ell(Q).$$

By definition of d_R and D_R , $D_R(xQ) \leq d_R(xQ) \leq \ell(Q)$. Hence, $D_R(x) \leq D_R(xQ) + C\ell(Q) \leq \ell(Q) \leq \delta_{\mathcal{A}}(x)$. Assuming δ small enough, we deduce that $10\delta D_R(x) \leq \text{dist}(x, \partial Q) \leq C\ell(Q) \leq c_1 C\ell(R)$. By the definition of R_R^+ , it is easy to check that this implies that $x \in R_R^+$ if c_1 is small enough. Indeed, since $z \in B^+(R) \cap B^-(R)$, by [\[Lec99, 771\]](#) and the last estimate, $A(x') + \delta D_R(x') \leq x_{n+1} + \ll c_1 \ell(R)$. $\square$

We denote $\text{Tree}^{**}(\mathcal{R}) := \{Q \in \text{Tree}^{**}(\mathcal{R}) : w(Q) \not\subseteq \Omega_{\mathcal{R}}\}$.

Lemma 1. *For all $S \in \square_{\mathcal{G}}$ we have*

$$\mathcal{S} \sum_{Q \in \text{partialTree}^{\wedge^{***}}(R): Q \subseteq S} \text{sigma}(Q) \leq \text{sigma}(S).$$

Proof. We will prove the following:

Claim 1. For each $Q \in \text{partial-Tree}^{***}(R)$ there exists some cube $P = P(Q)$

$$\begin{aligned} & \mathbb{S} \{ \text{begin} \{ \text{gathered} \} \text{label} \{ \text{eq} \} \text{limit} \} \text{T} \cdot \{ 1211, \text{R} \} \text{p} \{ \text{leq} \} \text{sum} \cdot \{ \text{j} \} \text{in} \{ \text{mathcal{M}} \} \{ \text{W} \} \\ & \{ \Omega \text{mega_R} \} \text{J} \{ \text{cap} \} \text{partial} \Omega \text{mega_k} \text{neq} \{ \text{varnothing} \} \text{int} \cdot \{ \text{j} \} \{ \text{cap} \} \text{partial} \{ \Omega \text{mega_k} \} \{ \text{nabla} \\ & \text{v_R} \} \text{p} \{ \text{d} \} \{ \text{mathcal{M}} \} \{ \text{h} \} \{ \text{leq} \} \text{sum} \cdot \{ \text{j} \} \text{in} \{ \text{mathcal{M}} \} \{ \text{W} \} \\ & \{ \Omega \text{mega_R} \} \text{J} \{ \text{cap} \} \text{partial} \Omega \text{mega_k} \text{neq} \{ \text{varnothing} \} \text{Vert} \text{nabla} \text{v_R} \text{Vert} \{ \text{L}^{\infty} \} \{ \text{infty} \} \} \\ & \{ \text{j} \} \} \text{p} \{ \text{ell} \} \text{J} \{ \text{n} \} \{ \text{lessim} \} \text{sum} \cdot \{ \text{j} \} \text{in} \{ \text{mathcal{M}} \} \{ \text{W} \} \\ & \{ \Omega \text{mega_R} \} \text{J} \{ \text{cap} \} \text{partial} \Omega \text{mega_k} \text{neq} \{ \text{varnothing} \} \text{big} \{ \text{mathchoice} \} \{ \{ \text{setbox} \\ & 0 = \text{hbox} \{ \text{displaystyle} \} \{ \text{textstyle} \} \} \{ \text{int} \} \} \{ \text{center} \} \{ \text{hbox} \{ \text{S} \{ \text{t} \text{extstyle} \} \} \} \} \text{kern} \cdot 5 \text{wd} \\ & 0 \} \} \{ \{ \text{setbox} \} 0 = \text{hbox} \{ \text{S} \{ \text{t} \text{extstyle} \} \{ \text{scriptstyle} \} \} \{ \text{int} \} \} \{ \text{center} \} \{ \text{hbox} \{ \text{S} \{ \text{criptstyle} \} \\ & \} \} \text{kern} \cdot 5 \text{wd} 0 \} \} \{ \{ \text{setbox} \} 0 = \text{hbox} \{ \text{S} \{ \text{criptstyle} \} \{ \text{scriptscriptstyle} \} \} \{ \text{int} \} \} \{ \text{center} \} \{ \text{hbox} \{ \text{S} \{ \text{criptscriptstyle} \} \} \} \text{kern} \cdot 5 \text{wd} 0 \} \} \{ \{ \text{setbox} \\ & 0 = \text{hbox} \{ \text{S} \{ \text{criptscriptstyle} \} \{ \text{scriptscriptstyle} \} \} \{ \text{int} \} \} \{ \text{center} \} \{ \text{hbox} \{ \text{S} \{ \text{criptscriptstyle} \\ & \} \} \text{kern} \cdot 5 \text{wd} 0 \} \} \} \{ \text{int} \} \} \{ \text{J} \} \} \{ \text{nabla} \text{v_R} \} \text{d} \{ \text{m} \} \{ \text{big} \} \{ \frac{\text{p} \{ 2 \} \text{ell} \{ \text{J} \} \text{n}}{\text{lessim} \text{sum} \cdot \{ \text{j} \} \text{in} \{ \text{mathcal{M}} \} \{ \text{W} \}} \} \\ & \{ \Omega \text{mega_R} \} \text{J} \{ \text{cap} \} \text{partial} \Omega \text{mega_k} \text{neq} \{ \text{varnothing} \} \text{big} \{ \text{inf} \cdot \{ \text{zeta} \} \text{in} \{ \text{hat} \text{b_} \{ \Omega \text{mega_R} \} \\ & \{ \text{j} \} \} \{ \text{widetilde} \{ \text{mathcal{M}} \} \{ \text{N} \} \} \cdot 2^{\wedge} \{ \Omega \text{mega_R} \} \{ \text{nabla} \text{v_R} \} \{ \text{zeta} \} \} \text{big} \{ \text{p} \{ \text{ell} \} \text{J} \{ \text{n} \} \\ & \{ \text{lessim} \} \text{sum} \cdot \{ \text{j} \} \{ \text{Pin} \{ \text{mathcal{M}} \} \{ \text{G} \} \} \} \{ \text{big} \{ \text{inf} \cdot \{ \text{zeta} \} \} \text{big} \{ \text{inf} \cdot \{ \text{zeta} \} \} \text{big} \{ \text{p} \{ \text{ell} \} \{ \text{P} \} \} \} \\ & \{ \text{widetilde} \{ \text{mathcal{M}} \} \{ \text{N} \} \} \cdot 2^{\wedge} \{ \Omega \text{mega_R} \} \{ \text{nabla} \text{v_R} \} \{ \text{zeta} \} \} \text{big} \{ \text{p} \{ \text{ell} \} \{ \text{P} \} \} \} \\ & \text{end} \{ \text{gathered} \} \mathbb{S} \end{aligned}$$
$$\sum_{P \in \square_1 : P \subset S} \ell(P)^n \lesssim \ell(S)^n.$$
$$T_{\left(\left[1211, R \right] \right)^{\text{plessim}} \text{int}_{\left\{ \text{partial} \Omega_R \right\}} \left\{ \text{mathcal{M}} \right\} \left\{ M \right\} \left\{ \text{mathcal{H}} \right\} \left\{ N \right\} \right\} 2^{\left\{ \Omega_R \right\}} \left(\text{nabla}_v R \right)^{\text{p}, d \text{mathcal{H}}} \left\{ \text{mathcal{H}} \right\} \left\{ \text{nlessim} \left\{ \text{Vert} \left(\text{widetilde{mathcal{H}}} \left\{ \text{mathcal{H}} \right\} \right) \right\} 2^{\left\{ \Omega_R \right\}} \right) \left(\text{nabla}_v R \right) \left\{ \text{L}^{\text{p}} \left(\text{partial} \Omega_R \right) \right\} \left\{ \text{plessim} \left\{ \text{Vert} \left(\text{nabla}_v \left\{ \text{t} R \right\} \right) \right\} \left\{ \text{widetilde{mathcal{H}}} \left\{ \text{mathcal{H}} \right\} \right\} \right\} \left\{ \text{L}^{\text{p}} \left(\text{partial} \Omega_R \right) \right\}, \mathbb{S}$$
[illegible]

Recall that $\mathcal{T}_{722,n}$ has n as defined in (eqa:trick1); it remains to bound $\mathcal{T}_{722}$. Recall that for each $\mathcal{S} \subseteq \mathcal{I} \setminus \text{rm}(\text{Top})\mathcal{S}$, we denote by $\mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}$ the family of Whitney cubes $I \in \sqcup(Q)$ such that $I' = (+1)\partial I \subset \mathcal{Q}_R$ and we denote by $\mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS}$ all the cubes $\mathcal{S} \cap \text{mathcal{I}nnermathcal{M}athcal{W}}(\text{W}) \setminus \text{rm}(\Omega_{\text{W}}) \setminus \text{backslash} \setminus \text{rm}(\text{Ins}) \setminus \text{rm}(\text{RS})$ which intersect $\mathcal{Q}_R$. For each $\mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}$, let $\eta_{\mathcal{Q}_R} \in C_c^\infty(\mathbb{R})$ be such that $\eta_{\mathcal{Q}_R} \equiv 1$ on I and $\mathcal{S} \cap \text{mathcal{I}nnermathcal{M}athcal{W}}(\text{W}) \setminus \text{rm}(\Omega_{\text{W}}) \setminus \text{backslash} \setminus \text{rm}(\text{Ins}) \setminus \text{rm}(\text{RS})$. Set $\mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS} = \sum_{\mathcal{S} \in \mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}} \mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS}$, and note that $\eta_{\mathcal{Q}_R} > 0$ on $\bigcup_{\mathcal{S} \in \mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}} \mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS}$. For each $\mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}$, let $\mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS} = \mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS}$, and define $\mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS} = \sum_{\mathcal{S} \in \mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}} \mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS}$. Then $\eta_{\mathcal{Q}_R} \in C_c^\infty(\mathbb{R})$, $0 \leq \eta_{\mathcal{Q}_R} \leq 1$, $\eta_{\mathcal{Q}_R} \equiv 1$ on $\bigcup_{\mathcal{S} \in \mathcal{S} \setminus \text{rm}(\text{Ins}) \setminus \text{RS}} \mathcal{S} \setminus \text{rm}(\text{Bdry}) \setminus \text{RS}$, and

Moreover, $\eta_R \equiv 0$ on $\partial\Omega_R \setminus \partial\Omega$. Now note that

```

$$\begin{aligned}
&\text{begin}\{\text{gathered}\}\text{'label}\{\text{eq.tricky}\}\int\limits_{\text{'Omega\_R}}\text{Anabla}\text{'v\_F}\text{na} \\
&\quad \text{v\_R,dm}=\int\limits_{\text{'Omega\_R}}\text{Anabla}\text{'w\_F}\text{eta}\{\text{'R}\}\text{inabla} \\
&\quad \text{v\_R,dm}+\int\limits_{\text{'Omega\_R}}\text{Anabla}\text{'w\_F}\{\text{1-eta}\}\{\text{'R}\}\text{inabla}\text{'v\_R,dm}\} \\
&\quad =\int\limits_{\text{'Omega\_R}}\text{A}\text{'T}\text{nabla} \\
&\quad \text{'v\_R}\text{inabla}\text{'w\_F}\text{eta}\{\text{'R}\}\text{,dm}+\int\limits_{\text{'Omega\_R}}\text{Anabla}\text{'w\_F}\{\text{1-eta}\}\{\text{'R}\}\text{inabla} \\
&\quad \text{v\_R,dm}=\int\limits_{\text{'Omega\_R}}\text{Anabla}\text{'w\_F}\{\text{1-eta}\}\{\text{'R}\}\text{inabla}\text{'v\_R,dm}\text{'end}\text{gathered}$$
\end{aligned}

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where we used that $L^* v_R = 0$ in Ω_R and that $w_{F|R} \in W_0^{1,2}(\Omega_R)$. Next, we see that

[illegible]

where we have used (eq. 4.1), Lemma 1, Lemma 1. Note that the term $\sum_{R \in \text{Top}} T_{1222, R}$ may be controlled in exactly the same way which we controlled the term $T_{\gamma\gamma}$ in Section 7.1.4. Hence, we have that

$$\frac{\sum_{R \in \text{Top}} T_{1222,R} \cdot \text{lessim}(\text{Ver}(\text{mathscr{C}}_2(F))\text{Ver}(L^{\{p\}}(\partial \Omega)))}{\text{Ver}(\nabla H_p) \varphi(\text{Ver}(L^p(\partial \Omega)))}$$

[illegible]

$$\begin{aligned} & \text{\$Slabel{eq.claim2}} \int_{\text{int}} \langle \text{Omega} \rangle \text{cap} \langle \text{B} \rangle \text{v} \rangle \rangle \langle \text{B} \rangle \langle \text{mathchoice} \{ \langle \text{setbox} \\ & \quad 0=\text{hbox} \langle \text{\$displaystyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \langle \text{hbox} \langle \text{\$textstyle} \langle \text{\$style} \rangle \langle \text{ker} \rangle \cdot \text{\$wd} \\ & \quad 0 \rangle \} \rangle \langle \text{setbox} 0=\text{hbox} \langle \text{\$textstyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{vcenter} \langle \text{hbox} \langle \text{\$scriptstyle} \\ & \quad \text{\$} \rangle \langle \text{ker} \rangle \cdot \text{\$wd} 0 \rangle \} \rangle \langle \text{setbox} 0=\text{hbox} \langle \text{\$scriptstyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \\ & \quad \langle \text{vcenter} \langle \text{hbox} \langle \text{\$scriptstyle} \rangle \rangle \langle \text{ker} \rangle \cdot \text{\$wd} 0 \rangle \} \rangle \langle \text{setbox} \\ & \quad 0=\text{hbox} \langle \text{\$scriptstyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \langle \text{vcenter} \langle \text{hbox} \langle \text{\$scriptstyle} \\ & \quad \text{\$} \rangle \langle \text{ker} \rangle \cdot \text{\$wd} 0 \rangle \} \rangle \int_{\text{int}} \langle \text{B} \rangle \langle \text{x} \rangle \langle \text{hat} \\ & \quad \text{c} \rangle \langle \text{1} \rangle \langle \text{delta} \langle \text{x} \rangle \rangle \rangle \langle \text{H} \rangle \langle \text{q} \rangle \langle \text{dm} \rangle \langle \text{B} \rangle \rangle \langle \text{1} \rangle \langle \text{frac} \rangle \langle \text{1} \rangle \langle \text{dm} \rangle \langle \text{x} \rangle \langle \text{leq} \rangle \int_{\text{int}} \langle \text{Omega} \rangle \text{cap} \\ & \quad \langle \text{B} \rangle \langle \text{x} \rangle \langle \text{2r} \rangle \rangle \langle \text{mathchoice} \{ \langle \text{setbox} 0=\text{hbox} \langle \text{\$displaystyle} \langle \text{textstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \\ & \quad \langle \text{vcenter} \langle \text{hbox} \langle \text{\$textstyle} \rangle \rangle \langle \text{ker} \rangle \cdot \text{\$wd} 0 \rangle \} \rangle \langle \text{setbox} \\ & \quad 0=\text{hbox} \langle \text{\$textstyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \langle \text{vcenter} \langle \text{hbox} \langle \text{\$scriptstyle} \rangle \rangle \langle \text{ker} \rangle \cdot \text{\$wd} \\ & \quad 0 \rangle \} \rangle \langle \text{setbox} 0=\text{hbox} \langle \text{\$scriptstyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \\ & \quad \langle \text{vcenter} \langle \text{hbox} \langle \text{\$scriptstyle} \rangle \rangle \langle \text{ker} \rangle \cdot \text{\$wd} 0 \rangle \} \rangle \langle \text{setbox} \\ & \quad 0=\text{hbox} \langle \text{\$scriptstyle} \langle \text{scriptstyle} \rangle \langle \text{int} \rangle \} \langle \text{int} \rangle \} \langle \text{vcenter} \langle \text{hbox} \langle \text{\$scriptstyle} \\ & \quad \text{\$} \rangle \langle \text{ker} \rangle \cdot \text{\$wd} 0 \rangle \} \rangle \int_{\text{int}} \langle \text{B} \rangle \langle \text{y} \rangle \langle \text{hat} \text{c} \rangle \langle \text{1} \rangle \langle \text{delta} \langle \text{y} \rangle \rangle \rangle \langle \text{H} \rangle \langle \text{q} \rangle \langle \text{dm} \rangle \langle \text{B} \rangle \rangle \langle \text{1} \rangle \langle \text{frac} \rangle \langle \text{1} \rangle \langle \text{dm} \rangle \langle \text{y} \rangle \rangle \text{\$S} \end{aligned}$$

Using [\(eq.claim2\)](#) in [\(eq.cq\)](#) immediately yields the estimate $\square_{\text{c}_2, q} \mathcal{H}(\mathcal{C}) \leq \square_{\text{c}_1, q} \mathcal{H}(\mathcal{C})$, from which the desired bound follows. Thus it remains only to prove [\(eq.claim2\)](#), which we do via a change of variables. First, define the map $F_1: \mathbf{p}^1 + 1 \rightarrow \mathbf{p}^1 + 1$

[illegible]
$$J_j(x) = 1 + \frac{c_2}{\|\nabla \Delta_\Omega(x)\|} \frac{1}{2}, \quad \text{for } x \in \Omega,$$

Let us define $F_j^{-1} : F_j(\Omega \cap B(\xi, r)) \rightarrow \Omega \cap B(\xi, r)$ the inverse map in $\Omega \cap B(\xi, r)$, so that $F_j(F_j^{-1}(y)) = y$ for each $y \in F_j(\Omega \cap B(\xi, r))$. We observe that

[illegible]

Proof of Proposition 1

Before we get started though, let us record a technical lemma, which is Lemma 3.5 in when $\Omega = \mathbb{I}_+^{n+1}$. This lemma is what essentially allows us to obtain “change of aperture” results, and it is needed in the proofs of Lemmas 1 and 1. Its proof in our case is very similar to that of , and thus we omit it.

$$0 < (\sup_{\zeta \in E} |\zeta - \xi|)^n \leq C_2 \sigma(E).$$

The discrete model of Hytönen and Rosén

$$\square_\sigma^q(u)(\zeta) := \sup_{Q \in \square_\sigma : \zeta \in Q} m(w^*(Q))^{-1/q} \|u_Q\|_{L^q(w^*(Q))}, \quad \zeta \in \partial\Omega.$$

For a sequence $(H) = (H_Q)_{Q \in \square_\sigma}$ where $H_Q \in L^1(w^*(Q))$, we define the discrete Carleson functional (acting on sequences (H)) by

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

Let $p \in [1, \infty)$ and p' its Hölder conjugate. Then ()

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

Passing from functions to sequences and viceversa

In order to use the estimates (eq.hr1)-(eq.hr3) to prove Proposition 1, we need to pass from the discrete model to our setting. As a first step, we need a natural way to identify sequences $(u_Q)_Q \in \ell^p$ with functions $u \in L^p(\Omega)$.

Given $u \in L^p(\Omega)$, we define a sequence $S(u) = (u_Q)_Q \in \ell^p$ as follows: for each $Q \in \mathcal{Q}$, let $u_Q = \int_Q u(x) dx$. Given a sequence $(u_Q)_Q \in \ell^p$ with $u_Q \in L^p(Q)$, we construct a function $F(u) \in L^p(\Omega)$ as follows:

$$F(u)(x) = \sum_{Q \in \mathcal{Q}} \frac{u_Q}{|Q|} \chi_Q(x).$$

Note that $1 \leq \sum_Q |Q| \leq C$ for every $x \in \Omega$. Moreover, for any function $u \in L^p(\Omega)$, we have that $F(S(u)) = u$ pointwise in Ω , but $S \circ F$ is not the identity on sequences.

Let us now see that with these maps, we are able to extend the estimates (eq.hr1)-(eq.hr3) to dyadic versions of ℓ^p and ℓ^q acting on functions. Let

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

It is easy to see then that

$$\| \mathscr{C}_- \{q\}^s(H)(x) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} H(x) dx \right| \right)^{\frac{1}{p'}}.$$

pointwise on $\partial\Omega$, for any $u \in L^p(\Omega)$, $H \in L^p(\Omega)$. The behavior of $F(u)$ is a bit more subtle in our setting; we record it in the following lemma.

Lemma 1. Let $p \in [1, \infty)$, $q \in [1, \infty]$, and p', q' the Hölder conjugates. Suppose that either Ω is bounded, or $\partial\Omega$ is unbounded. Then

$$\| \mathscr{C}_- \{q\}^s(F(u))(x) \|_{\mathcal{H}} \approx \| \mathscr{C}_- \{q\}^s(u)(x) \|_{\mathcal{H}}, \quad \text{for each } (u) = (u_Q)_Q \in \ell^p$$

and

$$\| \mathscr{C}_- \{q\}^d(F(H))(x) \|_{\mathcal{H}} \approx \| \mathscr{C}_- \{q\}^d(H)(x) \|_{\mathcal{H}}, \quad \text{for each } (H)_Q \in \ell^q$$

Proof. **Proof of (eq.ns1).** Let us prove the direction $\geq$ first; in fact, we show that

$$\| \mathscr{C}_- \{q\}^s(u) \|_{\mathcal{H}} \leq \| \mathscr{C}_- \{q\}^d(F(u)) \|_{\mathcal{H}} \quad \text{pointwise on } \partial\Omega.$$

Without loss of generality, we may assume that $u_Q \geq 0$ for each $Q \in \mathcal{Q}$, and then, by the definition of $F(u)$,

$$u_Q(x) \leq F(u)(x), \quad \text{for all } x \in \Omega.$$

Fix $\xi \in \partial\Omega$ and $Q \in \mathcal{Q}$ such that $\xi \in Q$. Then

$$\| \mathscr{C}_- \{q\}^s(u) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} u(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} u(x) dx \right| \right)^{\frac{1}{p'}}.$$

so that (eq.nsp1) follows.

We now consider the direction $\leq$ in (eq.ns1). We will show that the hypotheses of Lemma 1 are verified. Without loss of generality, we may again assume that $u_Q \geq 0$ for each $Q \in \mathcal{Q}$. Fix $\xi \in \partial\Omega$ and $\tau > 0$, and assume that $\| \mathscr{C}_- \{q\}^d(F(u))(\xi) \|_{\mathcal{H}} > \tau$. Then there exists $Q \in \mathcal{Q}$ with $\xi \in Q$ so that

$$\| \mathscr{C}_- \{q\}^d(F(u))(\xi) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} F(u)(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} F(u)(x) dx \right| \right)^{\frac{1}{p'}}.$$

where we used the Minkowski inequality and the definition of F . Let $\mathcal{F}_Q$ be the family of all $S \in \mathcal{Q}$ such that $w^*(S) \cap w^*(Q) \neq \emptyset$. If $S \in \mathcal{F}_Q$, then by standard geometric arguments, we may see that $\ell(S) \approx \ell(Q)$ and $\text{dist}(S, Q) \leq \ell(Q)$; these facts imply that $\text{card } \mathcal{F}_Q \approx 1$ and that $m(w^*(S)) \approx m(w^*(Q))$ for all $S \in \mathcal{F}_Q$. With these observations in mind, from (eq.ns2) we deduce that there exists $\tilde{S} \in \mathcal{F}_Q$ so that

$$\| \mathscr{C}_- \{q\}^d(F(u))(\xi) \|_{\mathcal{H}} \approx \| \mathscr{C}_- \{q\}^d(F(u))(\xi) \|_{\mathcal{H}} = \sup_{Q \in \mathcal{Q}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} F(u)(x) dx \right| \right)^{\frac{1}{p}} \left(\sum_{Q' \in \mathcal{Q}} \left| \int_{Q'} F(u)(x) dx \right| \right)^{\frac{1}{p'}}.$$

In other words, there exists $c_1 > 0$ so that $\| \mathscr{C}_- \{q\}^d(u) \|_{\mathcal{H}} \geq c_1 \tau$ on $\tilde{S}$. On the other hand, since

$$\sup_{\xi \in \partial\Omega} \| \mathscr{C}_- \{q\}^d(u) \|_{\mathcal{H}} \leq \text{diam } S + \text{dist}(S, Q) + \text{diam } Q \leq \ell(S) + \ell(Q) \approx \ell(S),$$

$$\mathbb{B}(\mathcal{M}^{\text{choice}}_{\text{setbox } 0=\text{hbox}\{ \$ \displaystyle \{\text{textstyle}\} \{\text{int}\} \$}$$

We turn to (ii). Fix $H \in \mathcal{C}^1(\mathbb{R}^n)$ such that H is compactly supported in Ω , and fix

residence at the Hausdorff Research Institute in Spring 2022 during the program "Interactions between geometric measure theory, singular integrals, and PDEs".⁴²

2. We could also say that Q has character (l, N, r_0, C_0) . However, notice that, since ∂Q is covered by at most N cylinders of diameter comparable to r_0 , we have $r_0 \geq \text{diam}(\partial Q)/N$. So $r_0/\text{diam}(\partial Q)$ depends on N .⁴³
3. The reason for the choice of the subscript 2 in T_2 will be apparent in Section 7.2.⁴⁴