This is a preprint of: "On the dynamics of the ACDM model of the universe", Jaume Llibre, Clàudia Valls, *Z. Angew. Math. Phys.*, vol. 76(206), 15 pages, 2025.
DOI: [10.1007/s00033-025-02592-w]

ON THE DYNAMICS OF THE ΛCDM MODEL OF THE UNIVERSE

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. In this paper we prove that the 3-dimensional ΛCDM model of the universe depending on the parameters $a, b \in [1, 4]$ is completely integrable. Moreover, when $a = b$ we describe the dynamics of all orbits in the positive octant of $\mathbb{R}^3$.

1. Introduction and statement of the main result

In this paper we improve the knowledge on the dynamics of the standard cosmological model of the universe, known as the ΛCDM model.

In the paper of Branković and Mijajlović [3] is studied the ΛCDM model under the hypotheses that the matter content of the universe is formed by barotropic perfect fluids without mutual interactions. More precisely, these authors showed that starting with the well known Friedmann equations [5] and doing the following assumptions:

- the average density of the matter fluid and the average density of the radiation fluid are barotropic perfect fluids without mutual interactions,
- the density of the matter content of the universe is the sum of the average densities of the matter and of the radiation,
- the pressure of the matter content of the universe is the sum of the pressures of the matter and of the radiation, and
- the spatial curvature density is less than or equal to zero,

then from equation (33) of [3] the Friedmann equations can be written as

$$\begin{aligned} \dot{x} &= -x\left(-a + \frac{b}{y} - \frac{2}{z}\right) - a, \\ \dot{y} &= -y\left(-b + \frac{a}{x} - \frac{2}{z}\right) - b, \\ \dot{z} &= -z\left(2 + \frac{a}{x} + \frac{b}{y}\right) + 2. \end{aligned} \tag{1}$$

In the differential system (1) the variables x, y and z denote the inverses of the matter density, the radiation density and the cosmological constant density, respectively. Moreover $a = 1 + 3\omega_m$ and $b = 1 + 3\omega_r$ being ω_m and ω_r the state parameters of

2010 *Mathematics Subject Classification.* Primary 34C07.
Key words and phrases. ΛCDM model, complete integrability, phase portrait.

the matter fluid and of the radiation fluid respectively. It is assumed that $x, y, z > 0$ and ω_m and ω_r are constant in $[0, 1]$, because these hypotheses are consistent with the macroscopic physics, see for details [1]. For additional information on the ΛCDM model see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the good survey in the wikipedia page https://en.wikipedia.org/wiki/Lambda-CDM_model/

Let U be an open and dense subset of $\mathbb{R}^3$. We say that a C^1 function $H\colon U \to \mathbb{R}$ non–constant in any open subset of U is a *first integral* of the differential system (1) if $H(x, y, z)$ is constant, for all the values of t for which the solution $(x(t), y(t), z(t))$ of system (1) is defined and contained in U. Let

$$X = P_1(x, y, z)\frac{\partial}{\partial x} + P_2(x, y, z)\frac{\partial}{\partial y} + P_3(x, y, z)\frac{\partial}{\partial z}$$

be the vector field associated to system (1), i.e.

$$\begin{aligned} P_1(x, y, z) &= -x\left(-a + \frac{b}{y} - \frac{2}{z}\right) - a, \\ P_2(x, y, z) &= -y\left(-b + \frac{a}{x} - \frac{2}{z}\right) - b, \\ P_3(x, y, z) &= -z\left(2 + \frac{a}{x} + \frac{b}{y}\right) + 2. \end{aligned}$$

Clearly a C^1 function $H\colon U \to \mathbb{R}$ in any open and dense subset U of $\mathbb{R}^3$ is a first integral of the differential system (1) if and only if $XH = 0$ in U.

Let $H\colon U_1 \to \mathbb{R}$ and $F\colon U_2 \to \mathbb{R}$ be two first integrals of the differential system (1), we say that they are *independent* in $U_1 \cap U_2$ if their gradients are independent in all the points $U_1 \cap U_2$ except perhaps in a zero Lebesgue measure set. The differential system (1) is *completely integrable* in $U_1 \cap U_2$ if it has two independent first integrals defined in $U_1 \cap U_2$. In this case the orbits of the differential system (1) are contained in the curves $\{H(x, y, z) = h\} \cap \{F(x, y, z) = f\}$, when h and f vary in $\mathbb{R}$.

Our first main result is:

Theorem 1. *For all values of the parameters a and b the differential system* (1) *is completely integrable.*

(i) *If $b \neq -2$ then the two independent first integrals are*

$$\begin{aligned} H(x, y, z) &= x^{2+b}y^{-2-a}z^{a-b}, \\ F(x, y, z) &= y^{-\frac{a+b+2}{b+2}}z^{\frac{a-b-2}{b+2}}(xyz - xy - xz - yz). \end{aligned}$$

(ii) *If $b = -2$ and $a \neq -2$, then the two independent first integrals are*

$$\begin{aligned} H(x, y, z) &= yz^{-1}, \\ F(x, y, z) &= x^{-a}z^{-a-4}(xyz - xy - xz - yz)^{a+2}. \end{aligned}$$

(iii) *If $a = b = -2$ then the two independent first integrals are*

$$H(x, y, z) = yz^{-1},$$
$$F(x, y, z) = xz^{-1}.$$

The proof of Theorem 1 is given in section 2.

Using the first integral $H(x, y, z)$ we can study the dynamics of the differential system (1) in every invariant surface $H(x, y, z) = h$ varying $h \in \mathbb{R}$. We do this study with all details in the particular case $a = b$, with $b \in [1, 4]$ for physical reasons. But the tools used for doing this study of the dynamics in the particular case $a = b$ can be used for all the values of a and b.

Note that on the first octant we can make the rescaling of the time of the form $dt = xyz\, d\tau$ then the differential system (1) becomes the polynomial differential system

$$\begin{aligned} x' &= x(2xy - bxz - ayz + axyz), \\ y' &= y(2xy - bxz - ayz + bxyz), \\ z' &= -z(-2xy + bxz + ayz + 2xyz), \end{aligned} \tag{2}$$

where the prime denotes derivative with respect to the new time τ.

For $a = b \in [1, 4]$ the restriction of $H = h$ yields

$$x = h^{1/(b+2)} y,$$

and so system (1) becomes

$$\begin{aligned} y' &= y^2(2ky - bkz - bz + bkyz), \\ z' &= -yz(-2ky + bkz + bz + 2kyz), \end{aligned} \tag{3}$$

where $k = h^{1/(b+2)}$. We note that $k < 0$ if and only if $a = b = 1, 3$, while $k \geq 0$ for all $a = b \in [1, 4]$.

The following theorem describes the dynamics of the differential system (3) for each value of $a = b \in [1, 4]$ and $k \in \mathbb{R}$ (because $h \in \mathbb{R}$). We need the notion of *Poincaré disc*.

Roughly speaking the *Poincaré disc* $\mathbb{D}^2$ is the unit closed disc centered at the origin of coordinates whose interior is diffeomorphic with $\mathbb{R}^2$ and whose boundary, the circle $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Note that we can go to the infinity in the plane $\mathbb{R}^2$ in as many as directions as points has the circle $\mathbb{S}^1$.

Any polynomial differential system can be extended analytically to the whole Poincaré disc, being the circle of the infinity invariant by this extended flow. This extension is called the *Poincaré compactification*, see for details [4, Chapter 5]. Two vector fields in the Poincaré disc are called *topologically equivalent* if there exists a homeomorphism sending orbits of the one vector field to orbits of the other vector field reversing or maintaining the orientation of the orbits. See again [4, Chapter 1] for more details.

Theorem 2. *For any $a = b \in [1, 4]$ the global phase portrait of system* (3) *in the Poincaré disc is topologically equivalent to one of:*

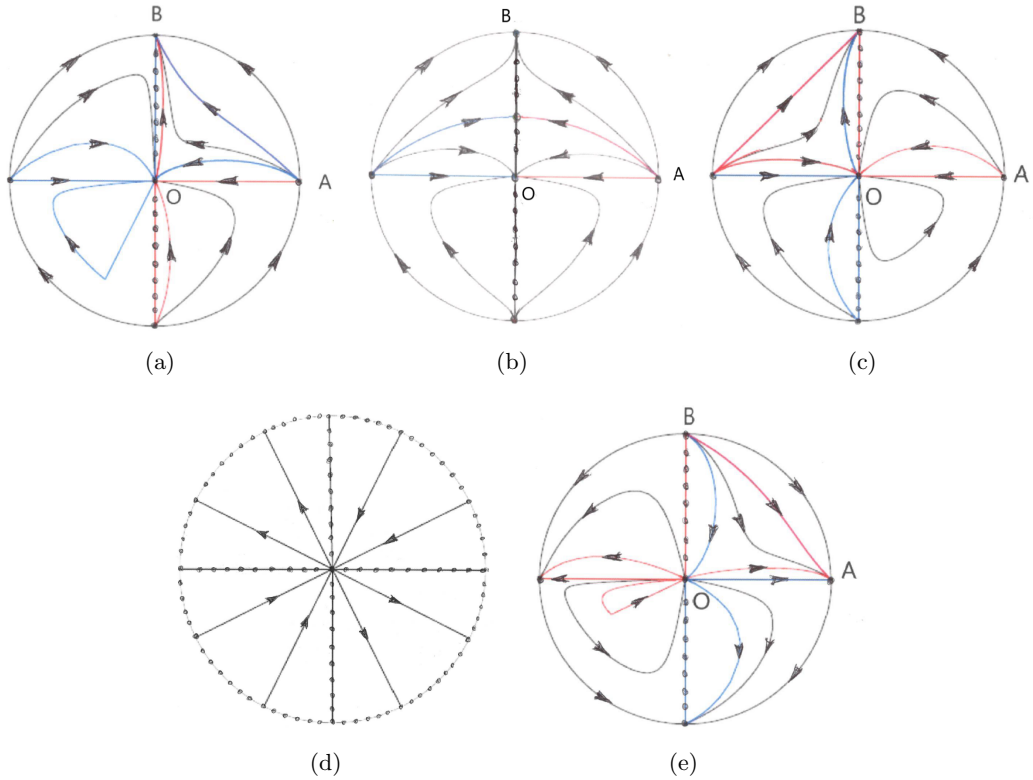


FIGURE 1. The phase portraits of the differential system (3). (a) For $k < -1$. (b) For $k = -1$. (c) For $k \in (-1, 0)$. (d) For $k = 0$. (e) For $k > 0$.

(i) *Figure 1(a) if* $k < -1$*;*
(ii) *Figure 1(b) if* $k = -1$*;*
(iii) *Figure 1(c) if* $k \in (-1, 0)$*;*
(iv) *Figure 1(d) if* $k = 0$*;*
(v) *Figure 1(e) if* $k > 0$.

The proof of Theorem 2 is given in section 3.

In what follows and for physical reasons we describe the dynamics of the orbits of the ΛCDM model (1) with $a = b \in [1, 4]$ in the positive octant of $\mathbb{R}^3$ according with the different values of k.

Assume $k < -1$ and $a = b = 1, 3$. Then from Figure 1(a) there are orbits in $H = k^{b+2}$

- starting at infinity in the point A and ending in the finite point O;
- starting at infinity in the point A and ending at infinity in the point B; and
- starting at the finite point O and ending at infinity in the point B.

Assume $k = -1$ and $a = b = 1, 3$. Then from Figure 1(b) there are orbits in $H = k^{b+2}$

- starting at infinity in the point A and ending in the finite point O;
- starting at infinity in the point A and ending at any point of the positive z-axis; and
- starting at infinity in the point A and ending at infinity in the point B.

Assume $k \in (-1, 0)$ and $a = b = 1, 3$. Then from Figure 1(c) all the orbits in $H = k^{b+2}$ start at infinity in the point A and end at the finite point O.

Assume $k = 0$ and $a = b \in [1, 4]$. Then from Figure 1(d) all the orbits in $H = 0$ are straight lines with positive slope starting at infinity and ending in the finite point O.

Assume $k > 0$ and $a = b \in [1, 4]$. Then from Figure 1(e) there are orbits in $H = k^{b+2}$

- starting at infinity in the point B and ending at the finite point O;
- starting at infinity in the point B and ending at infinity in the point A; and
- starting at the finite point O and ending at infinity in the point A.

2. Proof of Theorem 1

First we need some preliminary notions and results.

2.1. **Preliminaries for proving Theorem 1.** The results used in this section are presented for the differential system (1) in the first octant $R = \{x > 0, y > 0, z > 0\}$ but they are valid for any differential system in $\mathbb{R}^n$. The proof of Theorem 1 needs to recall some classic results on the Darboux theory of integrability.

Theorem 1 follows directly from the next theorem because a rescaling of the time does not change the first integrals of a differential system.

Theorem 3. *For all values of the parameters a and b the differential system* (2) *is completely integrable.*

(i) *If $b \neq -2$ then the two independent first integrals are the ones in the statement (i) of Theorem 1.*
(ii) *If $b = -2$ and $a \neq -2$, then the two independent first integrals are the ones in the statement (ii) of Theorem 1.*
(iii) *If $a = b = -2$ then the two independent first integrals are the ones in the statement (iii) of Theorem 1.*

For polynomial vector fields (2) one of the best tools to look for first integrals is the *Darboux theory of integrability*. Now we shall introduce the basic notions of this theory. Let $\mathbb{R}[x, y, z]$ be the ring of polynomials in the variables x, y, z with coefficients in $\mathbb{R}$. We say that $g = 0$ with $g \in \mathbb{R}[x, y, z]$ is an *invariant algebraic surface* of the vector field X if there exists a polynomial $K \in \mathbb{R}[x, y, z]$ such that $Xg = Kg$. The polynomial $K = K(x, y, z)$ is called the *cofactor* of g. It is easy to see that since system (2) is a polynomial differential system of degree three, every cofactor has degree at most two.

Of course, an invariant algebraic surface $g = 0$ with degree equal to one is an invariant plane.

Since on the points of an invariant algebraic surface $g = 0$ the gradient $(\partial g/\partial x, \partial g/\partial y, \partial g/\partial z)$ of the surface is orthogonal to the vector field R because $Xg = Kg$, the vector field X is tangent to the surface $g = 0$. Hence, if an orbit of the vector field X has a point on the surface $g = 0$ the whole orbit is contained in the surface $g = 0$. This justifies the name of invariant algebraic surface for $g = 0$.

In the proof of Theorem 3 we will use the following well known result of the Darboux theory of integrability; see for instance [4, Chapter 8], where it is proved for polynomial vector fields in $\mathbb{R}^2$, but the same proof works for polynomial vector fields in $\mathbb{R}^n$, and in particular for the vector field associated to the differential system (2).

Theorem 4. *Assume that the polynomial vector field X defined in $\mathbb{R}^3$ admits p invariant algebraic surfaces $g_i = 0$ with cofactors K_i for $i = 1, \ldots, p$. If there exist $\lambda_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i = 0$, then the function $|g_1|^{\lambda_1} \cdots |g_p|^{\lambda_p}$ is a first integral of the vector field X.*

2.2. **Proof of Theorem 3.** The differential system (2) has three invariant planes $g_i = 0$ for $i = 1, 2, 3$ and an invariant surface $g_4 = 0$ with cofactors K_i for $i = 1, 2, 3, 4$, namely

$$\begin{aligned} &g_1 = x, \quad K_1 = 2xy - bxz - ayz + axyz,\\ &g_2 = y, \quad K_2 = 2xy - bxz - ayz + bxyz,\\ &g_3 = z, \quad K_3 = 2xy - bxz - ayz - 2xyz,\\ &g_4 = xyz - xy - xz - yz, \quad K_4 = 4xy - 2bxz - 2ayz - 2xyz + axyz + bxyz. \end{aligned}$$

We consider two different cases.

Case 1: $b + 2 \neq 0$. Since $\sum_{i=1}^{4} \lambda_i K_i = 0$ for $\lambda_1 = b + 2$, $\lambda_2 = -a - 2$, $\lambda_3 = a - b$, and $\lambda_4 = 0$, by Theorem 4 we obtain that

$$\tilde{H}(x, y, z) = x^{\lambda_1} y^{\lambda_2} y^{\lambda_3} = x^{b+2} y^{-a-2} z^{a-b}$$

is a first integral of system (2).

On the other hand since $\sum_{i=1}^{4} \lambda_i K_i = 0$ for $\lambda_1 = 0$, $\lambda_2 = -(2 + a + b)/(2 + b)$, $\lambda_3 = (a - b - 2)/(2 + b)$, and $\lambda_4 = 1$, by Theorem 4 we obtain that

$$F(x, y, z) = y^{\lambda_2} z^{\lambda_3} (xyz - xy - xz - yz)^{\lambda_4} = y^{-\frac{2+a+b}{2+b}} z^{\frac{a-b-2}{2+b}} (xyz - xy - xz - yz)$$

is a first integral of system (2). Moreover, it is clear that H and F are independent. This proves statement (i).

Case 2: $b + 2 = 0$. In this case system (2) becomes

$$\begin{aligned} x' &= x(2xy + 2xz - ayz + axyz),\\ y' &= y(2xy + 2xz - ayz - 2xyz),\\ z' &= -z(-2xy - 2xz + ayz + 2xyz). \end{aligned} \tag{4}$$

Since $\sum_{i=1}^{4} \lambda_i K_i = 0$ for $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$, and $\lambda_4 = 0$, by Theorem 4 we obtain that

$$H(x, y, z) = \frac{y}{z}$$

is a first integral of system (4).

Now we consider two subcases.

Subcase 2.1*:* $a + 2 \neq 0$. Since $\sum_{i=1}^{4} \lambda_i K_i = 0$ for $\lambda_1 = -a$, $\lambda_2 = 0$, $\lambda_3 = -a - 4$, and $\lambda_4 = a + 2$, by Theorem 4 we obtain that

$$F(x, y, z) = x^{-a} z^{-a-4} (xyz - xy - xz - yz)^{a+2}$$

is a first integral of system (2). Moreover, it is clear that H and F are independent. This completes the proof of statement (ii).

Subcase 2.2*:* $a + 2 = 0$. Since $\sum_{i=1}^{4} \lambda_i K_i = 0$ for $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -1$, and $\lambda_4 = 0$, by Theorem 4 we obtain that

$$F(x, y, z) = \frac{x}{z}$$

is a first integral of system (2). Moreover, it is clear that H and F are independent. So statement (iii) is proved.

3. Proof of Theorem 2

3.1. **Proof of statement (iv) of Theorem 2.** System (3) with $k = 0$ becomes

$$y' = -by^2 z, \quad z' = -byz^2. \tag{5}$$

We introduce the new time $d\tau = ds/(byz)$ and in the new variable s system (5) becomes

$$\dot{y} = -y, \quad \dot{z} = -z, \tag{6}$$

where now the dot means derivative in the new time s. System (6) is integrable and has the rational first integral

$$R(y, z) = \frac{y}{z}.$$

So all the orbits are straight lines and the infinity is full of equilibrium points corresponding to the endpoints of these straight lines at infinity. See its phase portrait in Figure 6(d). To pass from system (6) to (5) we multiply by byz and so, the global phase portrait of system (3) when $k = 0$ in the Poincaré disc is given in Figure 1(d) that corresponds to convert the lines $y = 0$ and $z = 0$ in a line filled with equilibria, and to reverse the direction of the orbits on the region $yz < 0$.

3.2. **Proof of statement (ii) of Theorem 2.** System (3) with $k=-1$ becomes

$$y' = -y^3(2+bz), \quad z' = 2y^2 z(z-1). \tag{7}$$

We introduce the new time $d\tau = ds/y^2$, and in the new variable s system (7) becomes

$$\dot{y} = -y(2+bz), \quad \dot{z} = 2z(z-1), \tag{8}$$

where the dot means derivative in the new time s. We study system (8) in the Poincaré disc and obtain its global phase portrait in Figure 6(b). After that and taking into account that to pass from system (8) to system (7) we multiply by y^2 we will obtain the phase portrait of system (3) with $k=-1$ in Figure 1(b) from the phase portrait of system (8) converting the line $y=0$ in a line full of equilibrium points. So it is sufficient to study the global phase portrait of system (8).

System (8) has two equilibrium points: $(0,0)$ and $(0,1)$. The origin is a hyperbolic stable node because the eigenvalues of the Jacobian matrix at the origin are -2 and -2. Moreover the equilibrium point $(0,1)$ is a hyperbolic saddle because the eigenvalues of the Jacobian matrix at that point are $-(b+2)$ and 2.

Now we study the behaviour at the infinite equilibrium points of the local charts U_1 and U_2 (in the local chart U_2 it is only necessary to study the origin).

On the local chart U_1 system (8) becomes

$$\dot{u} = (b+2)u^2, \quad \dot{v} = v(bu+2v). \tag{9}$$

The unique equilibrium point on the straight line $v=0$ is the origin. The Jacobian matrix at that point is the zero matrix and so the origin is linearly zero. We must do a blow-up. The characteristic directions are the solutions of $2uv(u-v)=0$. Since $u=0$ is a characteristic direction if we do a vertical blow up in general we cannot control the orbits of the local phase portrait at the origin of system (9) that arrive or exit tangent to $u=0$. So before doing a vertical blow we translate the direction $u=0$ to the direction $u=v$ doing the change of variables

$$(u,v) \to (u_1, v_1), \quad \text{where } u_1 = u+v \text{ and } v_1 = v. \tag{10}$$

In this new variables system (9) writes

$$\dot{u}_1 = (2+b)u_1^2 - (b+4)u_1v_1 + 4v_1^2, \quad \dot{v}_1 = v_1(bu_1 + (2-b)v_1). \tag{11}$$

We do the vertical blow up

$$(u_1, v_1) \to (u_2, v_2), \quad \text{where } u_2 = u_1 \text{ and } v_2 = v_1/u_1, \tag{12}$$

Therefore system (11) becomes

$$\dot{u}_2 = u_2^2(b+2-(b+4)v_2 + 4v_2^2), \quad \dot{v}_2 = 2u_2v_2(1-2v_2)(v_2-1). \tag{13}$$

Rescaling the time $dt = u_2\, dt_1$ system (13) in the new time becomes

$$\dot{u}_2 = u_2(b+2-(b+4)v_2 + 4v_2^2), \quad \dot{v}_2 = 2v_2(1-2v_2)(v_2-1). \tag{14}$$

The unique three equilibrium points of system (14) on the straight line $u_2=0$ are $(0,0)$, $(0,1/2)$ and $(0,1)$. The equilibrium point $(0,0)$ is a hyperbolic saddle (the eigenvalues of the Jacobian matrix at that point are -2 and $b+2$), the equilibrium point $(0,1/2)$ is an unstable hyperbolic node (the eigenvalues of the Jacobian matrix

at that point are 1 and $(b+2)/2$), and the equilibrium point $(0,1)$ is a hyperbolic saddle (the eigenvalues of the Jacobian matrix at that point are -2 and 2).

Therefore the local phase portrait near the straight line $u_2 = 0$ for system (14) is topologically equivalent to the one of Figure 4(a) reversing the orientation of the orbits.

Now going back through the rescaling of the time $t_1 \mapsto t$ the phase portrait of Figure 4(a) provides the local phase portrait near the straight line $u_2 = 0$ of system (13) given in Figure 4(b) reversing the orientation of the orbits.

Going back through the change of variables (12) and taking into account that $\dot{u}_1|_{u_1=0} = 4v_1^2$ the local phase portrait of Figure 4(b) provides the local phase portrait at the origin of system (11) that is topologically equivalent to the one of Figure 4(c) reversing the orientation of the orbits.

Finally undoing the change of variables (10) the phase portrait of Figure 4(c) provides the local phase portrait at the origin of the local chart U_1 that is topologically equivalent to the one given in Figure 4(d) reversing the orientation of the orbits. Hence the local phase portrait at the origin of the local chart U_1 is formed by two hyperbolic sectors on the second and fourth quadrant separated by parabolic sectors located at the first and third quadrant.

On the local chart U_2 system (8) becomes

$$\dot{u} = -(b+2)u, \quad \dot{v} = 2v(v-1).$$

The origin is an equilibrium point and the Jacobian matrix evaluated at the origin of the local chart U_2 has the two eigenvalues $-(2+b)$ and -2, and so it is a hyperbolic stable node.

Using the Poincaré–Bendixson theorem (see [4, Corollary 1.30]) together with the fact that the straight lines $y = 0$ and $z = 0$ are invariant, the phase portraits of system (8) in the Poincaré disc are all topologically equivalent to the one of Figure 6(b). Taking into account the rescaling of the time $d\tau = ds/y^2$ we obtain for system (3) with $k = -1$ the phase portrait of Figure 1(b).

This completes the proof of statement (ii) of Theorem 2.

3.3. **Proof of statements (i), (iii) and (v) of Theorem 2.** We introduce the new time $d\tau = ds/y$ in system (3) and with this new time s, then system (3) becomes

$$\begin{aligned} y' &= y(2ky - bkz - bz + bkyz), \\ z' &= -z(-2ky + bkz + bz + 2kyz), \end{aligned} \tag{15}$$

where the dot means derivative in the new time s. We study system (15) in the Poincaré disc and obtain its global phase portraits. After that, taking into account that to pass from system (15) to system (3) we multiply by y and we will obtain the phase portraits of system (3) with $k < -1$ (respectively, $k \in (-1,0)$ and $k > 0$) in Figure 1(a) (respectively, Figure 1(c) and Figure 1(e)) from the phase portraits of system (15) given in Figure 6(a) (respectively, Figure 6(c) and Figure 6(e)) converting

the line $y = 0$ in a line full of equilibrium points and changing the direction of the orbits on the region $y < 0$. So it is sufficient to study the global phase portraits of system (15) given in Figures 6(a), 6(c) and 6(e).

3.3.1. *Finite equilibrium points.* Taking into account that $k \notin \{-1, 0\}$ and $a = b > 0$, we readily have that the unique finite equilibrium point is the origin that is linearly zero because the Jacobian matrix at that point is the zero matrix. So we need to do a blow-up process.

Since the equation providing the characteristic directions is the identity zero equation, we do the vertical blow-up

$$(y, z) \to (y_1, z_1), \quad \text{where } y_1 = y \text{ and } z_1 = z/y, \tag{16}$$

and in these new variables system (15) becomes

$$\begin{aligned} \dot{y}_1 &= y_1^2(2k - b(k+1)z_1 + bky_1z_1), \\ \dot{z}_1 &= -(2+b)ky_1^2z_1^2. \end{aligned} \tag{17}$$

We rescale the independent variable as follows $ds = y_1^2 ds_1$ in order to eliminate the common factor y_1^2 between $\dot{y}_1$ and $\dot{z}_1$, and we obtain the system

$$\begin{aligned} \dot{y}_1 &= 2k - b(k+1)z_1 + bky_1z_1, \\ \dot{z}_1 &= -(2+b)kz_1^2. \end{aligned} \tag{18}$$

This system has no equilibrium points on the straight line $y_1 = 0$. Assume first $k > 0$. Therefore the local phase portrait near the straight line $y_1 = 0$ for system (18) is topologically equivalent to the one of Figure 2(a).

Undoing the rescaling $s_1 \mapsto s$ from Figure 3(a) we obtain the local phase portrait near the straight line $y_1 = 0$ for system (17), that is topologically equivalent to the one of Figure 2(b).

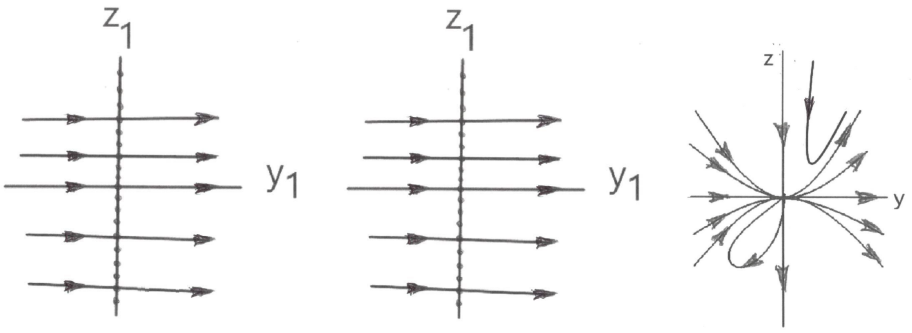
Going back through the change of variables (16) and taking into account that the z-axis is invariant and that $\dot{z}|_{y=0} = -b(k+1)z^2$, the phase portrait of Figure 2(b) provides the local phase portrait at the origin of system (15) that is given in Figure 2(c). As we see, the origin of coordinates is formed by one hyperbolic sector and one elliptic sector separated by parabolic sectors.

Proceeding in a similar way the local phase portraits at the origin of coordinates of system (15) when $k < -1$ is the one of Figure 2(c) but reversing the orientation of the orbits and when $k \in (-1, 0)$ is the one of Figure 3.

3.3.2. *Local chart U_1.* On the local chart U_1 system (15) becomes

$$\dot{u} = -(2+b)ku^2, \quad \dot{v} = v(-bku - 2kv + b(k+1)uv). \tag{19}$$

The unique equilibrium point on the straight line $v = 0$ is the origin. All the entries of the Jacobian matrix at that point are zero and so the origin is linearly zero. We must do a blow-up. The characteristic directions are the solutions of $2kuv(u - v) = 0$. Since



(a) (b) (c)

FIGURE 2. The blow up at the finite origin of coordinates for $k > 0$.

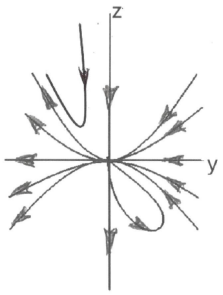


FIGURE 3. The local phase portrait at the finite origin of coordinates for $k \in (-1, 0)$.

$u = 0$ is a characteristic direction as explained above first we translate the direction $u = 0$ to the direction $u = v$ doing the change of variables

$$(u, v) \to (u_1, v_1), \quad \text{where } u_1 = u + v \text{ and } v_1 = v. \tag{20}$$

In this new variables system (9) writes

$$\begin{aligned} \dot{u}_1 &= -(b+2)ku_1^2 + (4+b)ku_1v_1 - 4kv_1^2 + b(1+k)u_1v_1^2 - b(1+k)v_1^3, \\ \dot{v}_1 &= -v_1((b+2)ku_1 - bkv_1 - b(1+k)u_1v_1 + b(1+k)v_1^2). \end{aligned} \tag{21}$$

We do a vertical blow up

$$(u_1, v_1) \to (u_2, v_2), \quad \text{where } u_2 = u_1 \text{ and } v_2 = v_1/u_1, \tag{22}$$

Therefore system (22) becomes

$$\begin{aligned} \dot{u}_2 &= -u_2^2((b+2)k - (4+b)kv_2 + 4kv_2^2 - b(k+1)u_2v_2^2 + b(k+1)u_2v_2^3), \\ \dot{v}_2 &= u_2v_2(v_2 - 1)(-2k + 4kv_2 - b(k+1)u_2v_2 + b(k+1)u_2v_2^2). \end{aligned} \tag{23}$$

We rescale the independent variable as follows $ds = u_2 ds_1$ in order to eliminate the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$, and we obtain the system

$$\begin{aligned} \dot{u}_2 &= -u_2((b+2)k - (4+b)kv_2 + 4kv_2^2 - b(k+1)u_2v_2^2 + b(k+1)u_2v_2^3), \\ \dot{v}_2 &= v_2(v_2-1)(-2k + 4kv_2 - b(k+1)u_2v_2 + b(k+1)u_2v_2^2). \end{aligned} \tag{24}$$

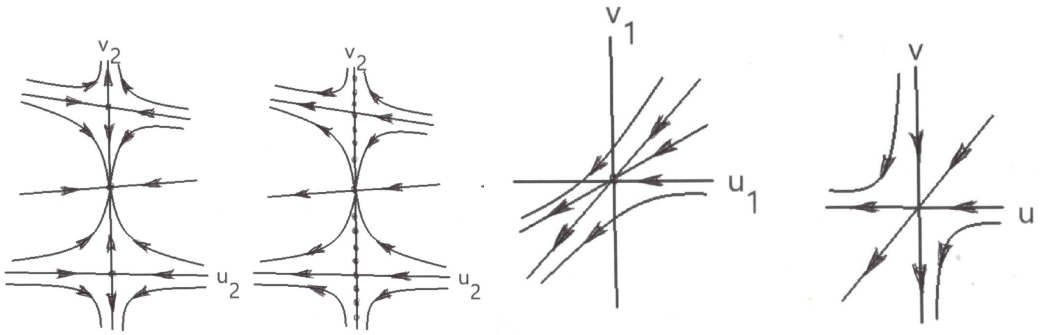
The equilibrium points of system (24) on the straight line $u_2 = 0$ are $(0,0)$, $(0,1/2)$ and $(0,1)$. The equilibrium point $(0,0)$ is a saddle (the eigenvalues of the Jacobian matrix at that point are $2k$ and $-(b+2)k$), the equilibrium point $(0,1/2)$ is a stable node if $k > 0$, otherwise it is an unstable node (the eigenvalues of the Jacobian matrix at that point are $-k$ and $-(b+2)k/2$), and the equilibrium point $(0,1)$ is a saddle (the eigenvalues of the Jacobian matrix at that point are $-2k$ and $2k$).

Assume $k > 0$. Therefore the local phase portrait near the straight line $u_2 = 0$ for system (24) is topologically equivalent to the one of Figure 4(a).

Undoing the rescaling $ds_1 = u_2 ds$ from Figure 4(a) we obtain the local phase portrait near the straight line $u_2 = 0$ for system (23), that is topologically equivalent to the one of Figure 4(b).

Now going back through the change of variables (22) the phase portrait of Figure 4(b) provides the local phase portrait at the origin of system (21) that is topologically equivalent to the one of Figure 4(c).

Going back through the change of variables (20) the phase portrait of Figure 4(c) provides the local phase portrait at the origin of system (19) that is topologically equivalent to the one of Figure 4(d). Hence the local phase portrait at the origin of the local chart U_1 is formed by two hyperbolic sectors separated by two parabolic sectors. When $k < 0$ proceeding in the same way we get Figure 4(d) reversing the orientation of the orbits.



(a) (b) (c) (d)

FIGURE 4. The blow up at the origin of the local chart U_1.

3.3.3. *Local chart U_2.* On the local chart U_2 system (15) becomes

$$\dot{u} = (2+b)ku^2, \quad \dot{v} = -v(-2ku - b(k+1)v + 2kuv) \tag{25}$$

The unique equilibrium point on the straight line $v = 0$ is the origin. All the entries of the Jacobian matrix at that point are zero and so the origin is linearly zero. We must do a blow-up. The characteristic directions are the solutions of $2kuv(u-v) = 0$. Since $u = 0$ is a characteristic direction as explained above first we translate the direction $u = 0$ to the direction $u = v$ doing the change of variables

$$(u, v) \to (u_1, v_1), \quad \text{where } u_1 = u + v \text{ and } v_1 = v. \tag{26}$$

In this new variables system (9) writes

$$\begin{aligned} \dot{u}_1 &= (b+2)ku_1^2 - 2(b+1)ku_1v_1 + b(1+2k)v_1^2 - 2ku_1v_1^2 + 2kv_1^3, \\ \dot{v}_1 &= v_1(2ku_1 + (b + (b-2)k)v_1 - 2ku_1v_1 + 2kv_1^2). \end{aligned} \tag{27}$$

We do a vertical blow up

$$(u_1, v_1) \to (u_2, v_2), \quad \text{where } u_2 = u_1 \text{ and } v_2 = v_1/u_1, \tag{28}$$

Therefore system (28) becomes

$$\begin{aligned} \dot{u}_2 &= u_2^2((b+2)k - 2k(b+1)v_2 + b(1+2k)v_2^2 - 2ku_2v_2^2 + 2ku_2v_2^3), \\ \dot{v}_2 &= -u_2(v_2-1)v_2(-bk + b(1+2k)v_2 - 2ku_2v_2 + 2ku_2v_2^2). \end{aligned} \tag{29}$$

We rescale the independent variable as follows $ds = u_2 ds_1$ in order to eliminate the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$, and we obtain the system

$$\begin{aligned} \dot{u}_2 &= -u_2((b+2)k - 2k(b+1)v_2 + b(1+2k)v_2^2 - 2ku_2v_2^2 + 2ku_2v_2^3), \\ \dot{v}_2 &= -(v_2-1)v_2(-bk + b(1+2k)v_2 - 2ku_2v_2 + 2ku_2v_2^2). \end{aligned} \tag{30}$$

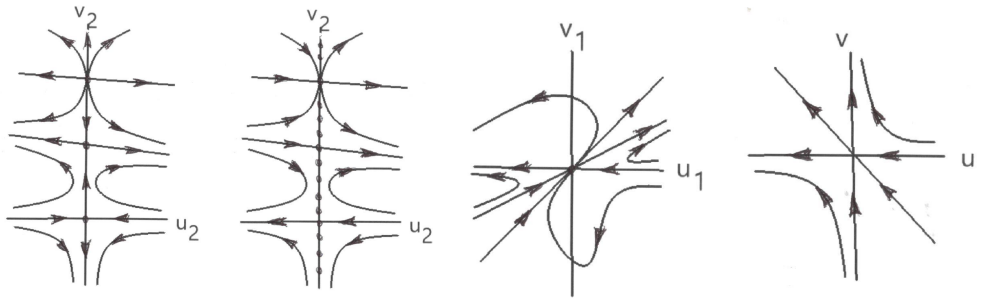
The equilibrium points of system (30) on the straight line $u_2 = 0$ are $(0, 0)$, $(0, 1)$ and $(0, k/(1+2k))$. The equilibrium point $(0, 0)$ is a saddle (the eigenvalues of the Jacobian matrix at that point are $-bk$ and $(2 + b)k$), the equilibrium point $(0, 1)$ is a saddle (the eigenvalues of the Jacobian matrix at that point are $-b(1 + k)$ and $b(1 + k)$), and the equilibrium point $(0, k/(1 + 2k))$ is a stable node if $k \in (-\infty, -1) \cup (-1/2, 0)$ and an unstable node if $k \in (-1, -1/2) \cup (0, \infty)$ (the eigenvalues of the Jacobian matrix at that point are $bk(1 + k)/(1 + 2k)$ and $(2 + b)k(1 + k)/(1 + 2k)$). Of course, when $k = -1/2$ this last point does not exist.

Assume $k \in (-1, -1/2)$. We do the blow-down process. In this case the local phase portrait near the straight line $u_2 = 0$ for system (30) is topologically equivalent to the one of Figure 5(a).

Undoing the rescaling $ds_1 = u_2 ds$ from Figure 5(a) we obtain the local phase portrait near the straight line $u_2 = 0$ for system (29), which is topologically equivalent to the one of Figure 5(b).

Going back through the change of variables (28) the phase portrait of Figure 5(b) provides the local phase portrait at the origin of system (27) which is topologically equivalent to the one of Figure 5(c).

Now going back through the change of variables (26) the phase portrait of Figure 5(c) provides the local phase portrait at the origin of system (25) which is topologically equivalent to the one of Figure 5(d). Hence the local phase portrait at the origin of the local chart U_2 is formed by two hyperbolic sectors separated by two parabolic sectors.



(a) (b) (c) (d)

FIGURE 5. The blow up at the origin of the local chart U_2.

Proceeding in a similar manner for the cases $k < -1$, $k \in (-1/2, 0)$, $k = -1/2$ and $k > 0$ we get that the local phase portrait of the origin of the local chart U_2 is the one given in: Figure 4(d) when $k < -1$, Figure 5(d) when $k \in [-1/2, 0)$ and Figure 4(d) reversing the orientation of the orbits when $k > 0$.

3.3.4. *Global phase portraits of system* (15). Taking into account the local behaviour of the origin of coordinates and the origins of the local charts U_1 and U_2 we need to distinguish between $k < -1$, $k \in (-1, 0)$ and $k > 0$. For all of them using that the straight lines $y = 0$ and $z = 0$ are invariant together with the Bendixson–Poincaré theorem we get that the phase portraits in the Poincaré disc are completely determined because we control all the separatrices (finite and infinite) where they born and where they die. So all phase portraits in the Poincaré disc of system (15) are topologically equivalent to the ones of Figure 6(a) when $k < -1$, Figure 6(c) when $k \in (-1, 0)$ and Figure 6(e) when $k > 0$.

To obtain the phase portraits of Figures 1(a), 1(c) and 1(e) we need to introduce in Figures 6(a), 6(c) and 6(e) the straight line $y = 0$ filled of equilibria and reverse the orientation of the orbits on the second and third quadrants.

ACKNOWLEDGEMENTS

The first author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Acadèmia de Ciències i Arts de Barcelona. The second author is partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

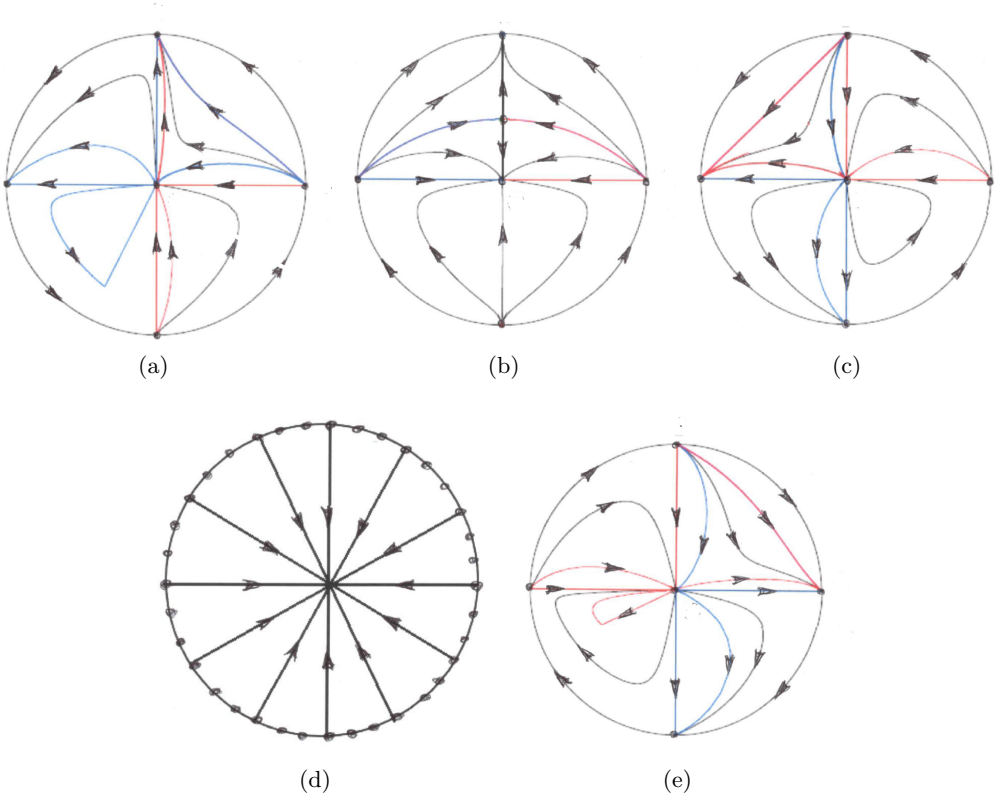


FIGURE 6. The phase portraits of the differential system (15). (a) For $k < -1$. (b) For $k = -1$. (c) For $k \in (-1, 0)$. (d) For $k = 0$. (e) For $k > 0$.

REFERENCES

[1] S. Bahamonde, C.G. Böhmer, S. Carloni, E.J. Copeland, W. Fang and N. Tamanini, Dynamical systems applied to cosmology: Dark energy and modified gravity. Physics Reports 775–777, 1–122 (2018).
[2] D. Branković, Cosmic time for multi–component universe. Serb. Astron. J. **201**, 15–23 (2020).
[3] D. Branković and Z. Mijajlović, Dynamics of the ΛCDM model of the universe from the aspect of the dynamical systems theory, (2022). DOI:10.48855/arXiv.2209.02605.
[4] F. Dumortier, J. Llibre, J.C. Artés, Qualitative Theory of Planar Differential Systems, in: UniversiText, Springer-Verlag, New York, 2006.
[5] A. Friedmann, Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes. Z. Physik **21**, 326–332 (1924).
[6] R. García–Salcedo, T. Gonzales, F.A. Horta–Rangel, I. Quiros, and D. Sanchez–Guzmán, Introduction to the application of the dynamical systems theory in the study of the dynamics of cosmological models of dark energy. Eur. J. Phys. **36**, 025008 (2015).
[7] M. Goliath and G.F.R. Ellis, Homogeneous cosmologies with a cosmological constant. Phys.Rev.D **60**, 023502 (1999).
[8] M.P. Hobson, G.P. Efstathiou, and A.N. Lasenby, General Relativity: An Introduction for Physicists. Cambridge University Press, New York (2006).

[9] A.R. Liddle, and D.H. Lyth, Cosmological Inflation and Large–Scale Structure. Cambridge University Press, Cambridge (2000).
[10] Z. Mijajlović, and D. Branković, Algebraic dependencies and representations of cosmological parameters. Publ. Astron. Obs. Belgrade **100**, 295–300 (2021).
[11] F. Niedermann, and M.S. Sloth, New early dark energy. Phys. Rev. D. **103**, L041303 (2021).
[12] J. Perez, A. Füzfa, T. Carletti, L. Mélot, and L. Guedezounme, The Jungle Universe: coupled cosmological models in a Lotka–Volterra framework. Gen. Relativ. Gravit. **46**, 1753 (2014).
[13] V. Poulin, T.L. Smith, T. Karwal, and M. Kamionkowski, Early Dark Energy can Resolve the Hubble Tension. Phys. Rev. Lett. **122**, 221301 (2019).
[14] J.P. Uzan, and R. Lehoucq, A dynamical study of the Friedmann equations. Eur. J. of Phys. **22**, 371–384 (2001).

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jaume.llibre@uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.tecnico.ulisboa.pt