

ON A CLASS OF RIGID POLYNOMIAL CENTERS

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ABSTRACT. When all the orbits of a planar differential system in a punctured neighborhood of a singular point are periodic, we say that the singular point is a *center*. The topological annulus formed only by all periodic orbits surrounding a center is called the *period annulus* of the center. When the periods of the periodic orbits of a period annulus are equal, the center is called *isochronous*. Moreover, an isochronous center is *rigid (or uniform)* if the angular velocity of its periodic orbits is constant.

It is well-known that the rigid polynomial centers at the origin of coordinates can be written as $x' = -y + xP(x, y)$, $y' = x + yP(x, y)$, where $P(x, y)$ is a polynomial. However, not all these systems have a center at the origin of coordinates, the origin can also be a focus. In this paper we characterize the rigid centers when $P(x, y)$ is a homogeneous polynomial of degree $n \geq 1$.

The phase portraits of the rigid polynomial homogeneous centers where $P(x, y)$ has degrees 1, 2, or 3 are well-known. Here we provide the phase portraits of the rigid polynomial homogeneous centers when $P(x, y)$ has degree four.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The notion of a center appeared already in the works of Poincaré [31] in 1881 and Dulac [12] in 1908. Informally the notion of an isochronous center goes back to Huygens, who in 1656 invented the pendulum clock, see [19, 30]. The study of isochronous centers has attracted the attention of many researchers along the time. Without trying to be exhaustive we cite some of them that deal with systems with arbitrary degree [3, 8, 18, 28, 36] and some other that deal with systems with fixed degree [3, 2, 4, 5, 6, 9, 10, 13, 14, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 29, 32, 33, 34, 35]. As far as the authors know the results that we obtain below in this paper have never appeared before in the literature.

We consider polynomial differential systems of the form

$$(1) \quad x' = -y + xP(x, y), \quad y' = x + yP(x, y),$$

where $P(x, y)$ is a homogeneous polynomial of degree $n \geq 1$. Note that the unique equilibrium point is the origin. Indeed, $x' = y' = 0$ implies $xx' + yy' = (x^2 + y^2)P(x, y) = 0$, that yields $x = y = 0$.

It is well known that the polynomial differential systems (1) can have at the origin a coordinates a weak focus or a center. In case it is a center, this center is rigid with

2010 *Mathematics Subject Classification.* 34C05.

Key words and phrases. Rigid center, global center, limit cycle.

angular velocity one, see for instance [7]. Our first result is to characterize the rigid centers of systems (1).

Given an integer number m we introduce the notation $m!! = m(m-2)(m-4) \cdots \delta$, with $\delta = 1$ if m is odd and $\delta = 2$ if m is even.

Theorem 1. *The rigid isochronous centers of the polynomial differential systems (1) where $P(x, y) = \sum_{r=0}^n a_r x^r y^{n-r}$ are*

- (a) *all of them when n is odd,*
- (b) *only systems (1) satisfying*

$$(2) \quad \sum_{r=0}^{n/2} a_{2r} A_{2r} = 0, \quad \text{where} \quad A_{2r} = \begin{cases} \frac{\pi(n-1)!!}{2^{n/2-1}(n/2)!} & \text{if } r = n/2, \\ \frac{\pi(2r-1)!!(n-3r-1)!!}{2^{r-1}r!n(n-2) \cdots (2r+2)} & \text{if } r < n/2, \end{cases}$$

when n is even.

The proof of Theorem 1 is given in section 2.

We recall that a center is *global* if its period annulus is $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Theorem 2. *Systems (1) have neither global centers nor limit cycles.*

The proof of Theorem 2 is given in section 3.

Roughly speaking the *Poincaré disc* is a closed unit disc whose interior has been identified with $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, is identified with the infinity of $\mathbb{R}^2$. In the plane $\mathbb{R}^2$ we can go to infinity in as many directions as points have the circle $\mathbb{S}^1$. See Chapter 5 of [11] for all necessary details for working in the Poincaré disc.

The phase portraits in the Poincaré disc of the rigid polynomial systems (1) for degrees 1, 2, 3 and 4 are well known, see [2, 4, 5, 20, 21, 27].

Theorem 3. *All phase portraits in the Poincaré disc of the differential systems (1) of degree 5 with a center at the origin of coordinates are topologically equivalent to one of the phase portraits in Figure 1.*

The proof of Theorem 3 is given in section 4. For example, Figures 1a), 1b) and 1c) are attained, respectively, when

$$\begin{aligned} P(x, y) &= \frac{147}{32}x^4 + \frac{35}{8}x^3y - \frac{585}{32}x^2y^2 + \frac{125}{16}xy^3 + \frac{3}{2}y^4, \\ P(x, y) &= \frac{-259}{32}x^4 + \frac{257}{64}x^3y + \frac{585}{32}x^2y^2 + \frac{89}{8}xy^3 + 2y^4, \\ P(x, y) &= -\frac{8}{5}x^4 + 2x^3y + \frac{9}{5}x^2y^2 - \frac{16}{5}xy^3 + y^4, \\ P(x, y) &= -x^4 + 2x^3y - 2xy^3 + y^4. \end{aligned}$$

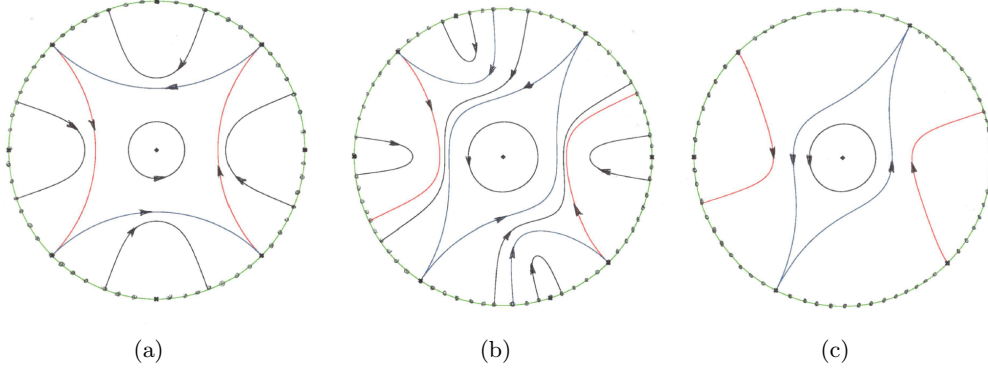


FIGURE 1. Phase portraits of system (1) in the Poincaré disc.

2. PROOF OF THEOREM 1

We first introduce polar coordinates by setting $x = r \cos \theta$, $y = r \sin \theta$ with $r \geq 0$ and $\theta \in [0, 2\pi]$. With this change of coordinates system (1) becomes

$$r' = r^{n+1} P(\cos \theta, \sin \theta), \quad \theta' = 1.$$

Taking θ as the new independent variable of this differential system we obtain the differential equation

$$\dot{r} = r^{n+1} P(\cos \theta, \sin \theta),$$

where the dot denotes derivative with respect to the variable θ . The solution $r(\theta; r_0)$ of this differential equation with initial condition $r(0; r_0) = r_0$ is

$$r(\theta; r_0) = \left(\frac{1}{r_0^n} - n \int_0^\theta P(\cos s, \sin s) ds \right)^{-1/n}.$$

In order that the origin of system (1) be a center we must have that

$$\int_0^{2\pi} P(\cos s, \sin s) ds = 0.$$

We need to compute such integral, that is,

$$\int_0^{2\pi} \sum_{r=0}^n a_r \cos^r s \sin^{n-r} s ds = \sum_{r=0}^n a_r \int_0^{2\pi} \cos^r s \sin^{n-r} s ds.$$

It follows from Formulae 2.511 and 2.512 of [15] (see also formulae (13) and (16) of the appendix where we have written them for completeness) that

$$\int_0^{2\pi} \cos^r s \sin^{n-r} s ds = 0$$

if r or $n - r$ are odd. So, if n is odd then either r or $n - r$ are odd and so the above integral is always zero. This concludes the proof of the theorem when n is odd.

When n is even it follows as before that the integral is zero if r is odd. On the other hand if r is even, since n is also even then $n - r$ is even and it follows from 2.5112 of [15] (see also formulae (11) with p even, equation (12), equation (14) with r even and (15) of the appendix) that if $r = 0$ or $r = n = 2m$, then

$$\int_0^{2\pi} \cos^r s \sin^{n-r} s \, ds = \frac{(2m-1)!!}{2^m m!} 2\pi,$$

and if $r \in \{1, \dots, n-2\}$ then (here $r = 2l$, $n = 2m$)

$$\begin{aligned} \int_0^{2\pi} \cos^r s \sin^{n-r} s \, ds &= \int_0^{2\pi} \cos^{2l} s \sin^{2m-2l} s \, ds \\ &= \frac{(2m-2l-1)!!}{2m(2m-2) \cdots (2l+2)} \int_0^{2\pi} \cos^{2l} s \, ds \\ &= \frac{(2m-2l-1)!!}{2m(2m-2) \cdots (2l+2)} \frac{(2l-1)!!}{2^l l!} 2\pi. \end{aligned}$$

Hence we get (2) (with m and l replaced respectively by $n/2$ and r) as we wanted to show. This completes the proof of the theorem.

3. PROOFS OF THEOREM 2

We compute the polynomial differential system in the local chart U_1 (for more details see Chapter 5 of [11]) and we get

$$(3) \quad u' = (1 + u^2)v^n, \quad v' = uv^{n+1} - vP(1, u).$$

Note that the line of the infinity $v = 0$ is filled with equilibria. Doing a rescaling of the time we eliminate the common factor v between u' and v' and we get the system

$$(4) \quad u' = (1 + u^2)v^{n-1}, \quad v' = uv^n - P(1, u).$$

Hence in all points $(u, 0)$ of the infinity in the local chart U_1 where $P(1, u) \neq 0$ it arrives or starts an orbit. So the origin can not be a global center.

All polynomial differential systems with an isochronous center have no limit cycles due to the fact that the period annulus of the center in some of its direction attains the infinity. Indeed, if the period annulus would not reach the infinity, then the boundary of the period annulus would be formed by a polycycle with some finite equilibrium point, and the period of all periodic orbits sufficiently close to the equilibrium point would tend to infinity contradicting the fact that the period was constant. Note that the boundary of the period annulus cannot be a periodic orbit γ because considering a Poincaré map defined in a transversal section to γ , this Poincaré map is an analytic function of one variable, that is the identity. So γ cannot be the last periodic orbit of the period annulus.

4. PROOF OF THEOREM 3

We consider the different forms that $P(x, y) = a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4$ of degree four can have. In order that the origin be a center from the proof of Theorem 1 we must have that

$$\int_0^{2\pi} P(\cos s, \sin s) ds = \frac{\pi}{4}(3a_{40} + a_{22} + 3a_{04}) = 0.$$

Then $a_{40} = -(a_{22} + 3a_{04})/3$, and so

$$(5) \quad P(x, y) = -\frac{1}{3}(a_{22} + 3a_{04})x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4.$$

In principle the polynomial $P(x, y)$ given in (5) could have:

- (i) Either four simple real roots, i.e. $P(x, y) = a_{04}(y - r_1x)(y - r_2x)(y - r_3x)(y - r_4x)$ with $r_1 < r_2 < r_3 < r_4$;
- (ii) or one double real root and two simple real roots, i.e. $P(x, y) = a_{04}(y - r_1x)^2(y - r_2x)(y - r_3x)$ with $r_1 \neq r_2$, $r_2 \neq r_3$ and $r_1 \neq r_3$;
- (iii) or one triple real root and one simple real root, i.e. $P(x, y) = a_{04}(y - r_1x)^3(y - r_2x)$ with $r_1 \neq r_2$;
- (iv) or one quadruple real root, i.e. $P(x, y) = a_{04}(y - r_1x)^4$;
- (v) or two double real roots, i.e. $P(x, y) = a_{04}(y - r_1x)^2(y - r_2x)^2$ with $r_1 < r_2$;
- (vi) or two simple real roots and one pair of complex conjugate roots, i.e. $P(x, y) = a_{04}(y - r_1x)(y - r_2x)((x - ay)^2 + b^2y^2)$ with $r_1 < r_2$ and $b \neq 0$;
- (vii) or one double real root and one pair of complex conjugate roots, i.e. $P(x, y) = a_{04}(y - r_1x)^2((x - ay)^2 + b^2y^2)$ with $b \neq 0$;
- (viii) or two pairs of complex conjugate roots, i.e. $P(x, y) = a_{04}((x - ay)^2 + b^2y^2)((x - cy)^2 + d^2y^2)$ with $bd \neq 0$;
- (ix) or one double pair of complex conjugate roots, i.e. $P(x, y) = a_{04}((x - ay)^2 + b^2y^2)^2$ with $b \neq 0$.

We will see that only cases (i), (ii), (iii) and (vi) are possible. This is done in the following lemma.

Lemma 4. *The polynomial $P(x, y)$ given in (5) can have:*

- (I) *either four simple real roots and $P(x, y)$ have the two forms*

$$(I.1) \quad P(x, y) = a_{04}P_1(x, y)/(r_1 + r_2 + r_3 + 3r_1r_2r_3) \text{ with}$$

$$P_1(x, y) = (r_1x - y)(r_2x - y)(r_3x - y)(3x + r_1r_2x + r_1r_3x + r_2r_3x + r_1y + r_2y + r_3y + 3r_1r_2r_3y)$$

$$\text{and } r_1 + r_2 + r_3 + 3r_1r_2r_3 \neq 0;$$

$$(I.2) \quad P(x, y) = -a_{04}P_2(x, y)/(3r_1^2 - 1) \text{ with}$$

$$P_2(x, y) = (r_1x - y)(r_4x - y)((r_1^2 - 3)x^2 - 8r_1xy - (3r_1^2 - 1)y^2)$$

$$\text{and } r_1 \neq \pm 1/\sqrt{3}.$$

- (II) *or one double real root and two simple real roots and $P(x, y)$ has the form $P(x, y) = a_{04}P_3(x, y)/(2r_1 + r_2 + 3r_1^2r_2)$ with*

$$P_3(x, y) = (r_1x - y)^2(r_2x - y)(3x + r_1^2x + 2r_1r_2x + 2r_1y + r_2y + 3r_1^2r_2y)$$

and $2r_1 + r_2 + 3r_1^2r_2 \neq 0$.

- (III) *or one triple real root and one simple real root and $P(x, y)$ has the form $P(x, y) = -a_{04}(r_1x - y)^3(x + r_1y)/r_1$ with $r_1 \neq 0$;*

- (IV) *or two simple real roots and one pair of complex conjugate roots and $P(x, y)$ has the form $P(x, y) = -a_{04}P_4(x, y)/(a + 2r_1)$ with*

$$P_4(x, y) = (r_1x - y)(2x + ar_1x + ay + 2r_1y)(x^2 - 2axy + y^2)$$

with $a^2 < 1$ and $a + 2r_1 \neq 0$.

Proof. Assume that P has the form in (i). Then equating P in (5) with the expression of P in case (i) we see that there are the two possible families of P listed in cases (I.1) and (I.2) in the statement of the lemma.

Assume that P has the form in (ii). Then equating P in (5) with the formula of P in case (ii) we see that P must be of the form in case (II) given in the statement of the lemma.

Assume that P has the form in (iii). Then equating P in (5) with the formula of P in case (iii) we see that P must be of the form in case (III) given in the statement of the lemma.

Assume that P has the form in (iv). Then equating P in (5) with the formula of P in case (iv) we see that either $P = 0$, or r_1 , must be complex and so this case is not possible.

Assume that P has the form in (v). Then equating P in (5) with the formula of P in case (v) we see that either $P = 0$ or either r_1 or r_2 must be complex and so this case is not possible.

Assume that P has the form in (vi). Then equating P in (5) with the formula of P in case (vi) we see that P must be of the form in case (IV) given in the statement of the lemma.

Assume that P has the form in (vii). Then equating P in (5) with the formula of P in case (vii) we see that either $P = 0$ or P has no complex roots. So this case is not possible.

Assume that P has the form in (viii). Then equating P in (5) with the formula of P in case (viii) we see that either $P = 0$ or P has real roots, so this case is not possible.

Assume that P has the form in (ix). Then equating P in (5) with the formula of P in case (iv) we see that either $P = 0$ or P is a complex polynomial. So this case is not possible. This concludes the proof of the lemma. $\square$

Now we will compute the phase portraits of system (1) in the Poincaré disc. To do so, and since we know that in the finite part of the Poincaré disc there is a unique

isochronous center at the origin of coordinates, we need to study the infinity. We already know that the infinity is filled with equilibria, and that after removing the common factor v between u' and v' in system (3) the equilibrium points $(u, 0)$ at the local chart U_1 are the possible roots of $P(1, u)$ (see system (4)). Moreover, to complete the study of the infinite equilibrium points we must study the origin of the local chart U_2 .

Lemma 5. *The origin of the local chart U_2 of system (1) is never a singular point.*

Proof. In the local chart U_2 system (1) becomes

$$u' = -(1 + u^2)v^4, \quad v' = -uv^5 - v\left(a_{04} + a_{13}u + 3a_{22}u^2 - a_{04}u^4 + \frac{a_{22}}{3}u^4\right).$$

There is a line of equilibria. By removing the common factor between u' and v' with a rescaling of the time we get that the above system becomes

$$u' = -(1 + u^2)v^3, \quad v' = -uv^4 - (a_{04} + a_{13}u + 3a_{22}u^2 - a_{04}u^4 + \frac{a_{22}}{3}u^4).$$

Note that this system on $(u, v) = (0, 0)$ is different from zero because $a_{04} \neq 0$ and so there are no equilibrium points in the local chart U_2 after removing the line of singularities. $\square$

In view of Lemma 5, in order to study the phase portraits of system (1) with $P(x, y)$ homogeneous of degree 4 and with a center at the origin we just need to study the infinite equilibria of the local chart U_1 . We will do it for each of the cases in (I)–(IV) given in Lemma 4 in separate lemmas.

Family (I.1). We rewrite family (I.1) as

$$P(x, y) = a(r_1x - y)(r_2x - y)(r_3x - y)(3x + r_1r_2x + r_1r_3x + r_2r_3x \\ + r_1y + r_2y + r_3y + 3r_1r_2r_3y),$$

with $a \neq 0$ and $r_1 + r_2 + r_3 + 3r_1r_2r_3 \neq 0$. In that case the local chart U_1 after removing the common factor v with a convenient rescale of time we get

$$u' = (1 + u^2)v^3, \\ v' = -3ar_1r_2r_3 - ar_1^2r_2^2r_3 - ar_1^2r_2r_3^2 - ar_1r_2^2r_3^2 + 3ar_1r_2u + ar_1^2r_2^2u + 3ar_1r_3u \\ + 3ar_2r_3u + ar_1^2r_2r_3u + ar_1r_2^2r_3u + ar_1^2r_3^2u + ar_1r_2r_3^2u + ar_2^2r_3^2u - 3ar_1^2r_2^2r_3^2u \\ - 3ar_1u^2 - 3ar_2u^2 - 3ar_3u^2 + 3ar_1^2r_2^2r_3u^2 + 3ar_1^2r_2r_3^2u^2 + 3ar_1r_2^2r_3^2u^2 + 3au^3 \\ - ar_1^2u^3 - ar_1r_2u^3 - ar_2^2u^3 - ar_1r_3u^3 - ar_2r_3u^3 - 3ar_1^2r_2r_3u^3 - 3ar_1r_2^2r_3u^3 \\ - ar_3^2u^3 - 3ar_1r_2r_3^2u^3 + ar_1u^4 + ar_2u^4 + ar_3u^4 + 3ar_1r_2r_3u^4 + uv^4.$$

Computing the solutions of $v'|_{v=0}$ we get that

$$u = r_1, \quad u = r_2, \quad u = r_3, \quad u = r_4 = -\frac{3 + r_1r_2 + r_1r_3 + r_2r_3}{r_1 + r_2 + r_3 + 3r_1r_2r_3}.$$

Computing the Jacobian matrix of the differential system at each of the equilibrium points $(r_i, 0)$ for $i = 1, 2, 3, 4$, we get that all of them are nilpotent. Applying Theorem 3.5 in [11] and using the notation

$$\begin{aligned} A &= a(r_1 - r_2)(r_1 - r_3)(3 + r_1^2 + 2r_1r_2 + 2r_1r_3 + r_2r_3 + 3r_1^2r_2r_3), \\ B &= a(r_2 - r_1)(r_2 - r_3)(3 + 2r_1r_2 + r_2^2 + r_1r_3 + 2r_2r_3 + 3r_1r_2^2r_3), \\ C &= a(r_3 - r_1)(r_3 - r_2)(3 + r_1r_2 + 2r_1r_3 + 2r_2r_3 + r_3^2 + 3r_1r_2r_3^2), \\ D_1 &= 9 + r_1^2 + 8r_1r_2 + r_2^2 + r_1^2r_2^2 + 8r_1r_3 + 8r_2r_3 + 8r_1^2r_2r_3 + 8r_1r_2^2r_3 + r_3^2 \\ &\quad + r_1^2r_3^2 + 8r_1r_2r_3^2 + r_2^2r_3^2 + 9r_1^2r_2^2r_3^2, \\ D_2 &= a(3 + r_1^2 + 2r_1r_2 + 2r_1r_3 + r_2r_3 + 3r_1^2r_2r_3)(3 + 2r_1r_2 + r_2^2 + r_1r_3 + 2r_2r_3 \\ &\quad + 3r_1r_2^2r_3)(3 + r_1r_2 + 2r_1r_3 + 2r_2r_3 + r_3^2 + 3r_1r_2r_3^2), \end{aligned}$$

we conclude that:

- $(r_1, 0)$ is a nilpotent center or focus if $A < 0$ and it is a nilpotent saddle if $A > 0$.
- $(r_2, 0)$ is a nilpotent center or focus if $B < 0$ and it is a nilpotent saddle if $B > 0$.
- $(r_3, 0)$ is a nilpotent center of focus if $C < 0$ and it is a nilpotent saddle if $C > 0$.
- $(r_4, 0)$ is a nilpotent center of focus if $D_1/D_2 > 0$ and it is a nilpotent saddle if $D_1/D_2 < 0$.

Computing all the possibilities for the signs of A , B , C , D_1 and D_2 together with the possible relative positions between r_1 , r_2 , r_3 and r_4 . Then we get that all global phase portraits in the Poincaré disc are topologically equivalent to the ones of (a), (b) or (c) in Figure 1.

Family (I.2). We rewrite family (I.2) as

$$P(x, y) = a(r_1x - y)(r_4x - y)((r_1^2 - 3)x^2 - 8r_1xy - (3r_1^2 - 1)y^2)$$

with $a \neq 0$ and $3r_1^2 - 1 \neq 0$. In that case the local chart U_1 after removing the common factor v with a convenient rescale of time we get

$$\begin{aligned} u' &= (1 + u^2)v^3, \\ v' &= 3ar_1r_4 - 10ar_1^3r_4 + 3ar_1^5r_4 - 3ar_1u + 10ar_1^3u - 3ar_1^5u - 3ar_4u + 18ar_1^2r_4u \\ &\quad - 27ar_1^4r_4u + 3au^2 - 18ar_1^2u^2 + 27ar_1^4u^2 - 9ar_1r_4u^2 + 30ar_1^3r_4u^2 - 9ar_1^5r_4u^2 \\ &\quad + 9ar_1u^3 - 30ar_1^3u^3 + 9ar_1^5u^3 + ar_4u^3 - 6ar_1^2r_4u^3 + 9ar_1^4r_4u^3 - au^4 + 6ar_1^2u^4 \\ &\quad - 9ar_1^4u^4 + uv^4. \end{aligned}$$

Computing the solutions of $v'|_{v=0}$ we get that

$$u = r_1, \quad u = r_2 = \frac{4r_1 + \sqrt{3}(1 + r_1^2)}{1 - 3r_1^2}, \quad u = r_3 = \frac{4r_1 - \sqrt{3}(1 + r_1^2)}{1 - 3r_1^2}, \quad u = r_4.$$

Computing the Jacobian matrix of the differential system at each of the equilibrium points $(r_i, 0)$ for $i = 1, 2, 3, 4$ we get that all of them are nilpotent. Applying Theorem 3.5 in [11] and using the notation

$$\begin{aligned} A &= 3a(1 + r_1^2)(1 - 3r_1^2)(r_1 - r_4), \\ B &= a(3r_1^2 - 1)(r_4 - r_1)(3 - r_1^2 + 8r_1r_4 - r_4^2 + 3r_1^2r_4^2), \\ C &= \frac{1 + 2\sqrt{3}r_1 + 3r_1^2}{a(3r_1^2 - 1)(\sqrt{3} + 7r_1 + 5\sqrt{3}r_1^2 + 3r_1^3 - r_4 - \sqrt{3}r_1r_4 + 3r_1^2r_4 + 3\sqrt{3}r_1^3r_4)}, \\ D &= \frac{1 - 2\sqrt{3}r_1 + 3r_1^2}{a(1 - 3r_1^2)(\sqrt{3} - 7r_1 + 5\sqrt{3}r_1^2 - 3r_1^3 + r_4 - \sqrt{3}r_1r_4 - 3r_1^2r_4 + 3\sqrt{3}r_1^3r_4)}, \end{aligned}$$

we conclude that (the numerator and denominator of C and D are never zero because $3r^2 - 1 \neq 0$)

- $(r_1, 0)$ is a nilpotent center or focus if $A < 0$ and it is a nilpotent saddle if $A > 0$.
- $(r_4, 0)$ is a nilpotent center or focus if $B < 0$ and it is a nilpotent saddle if $B > 0$.
- $(r_2, 0)$ is a nilpotent center of focus if $C < 0$ and it is a nilpotent saddle if $C > 0$.
- $(r_3, 0)$ is a nilpotent center of focus if $D < 0$ and it is a nilpotent saddle if $D > 0$.

Computing all the possibilities for the signs of A , B , C and D together with the possible relative positions between r_1 , r_2 , r_3 and r_4 we get that all global phase portraits in the Poincaré disc are topologically equivalent to the one in Figure 1(b).

Family (II). We rewrite family (II) as

$$P(x, y) = a(r_1x - y)^2(r_2x - y)(3x + r_1^2x + 2r_1r_2x + 2r_1y + r_2y + 3r_1^2r_2y)$$

with $a \neq 0$ and $2r_1 + r_2 + 3r_1^2r_2 \neq 0$. In that case the local chart U_1 after removing the common factor v with a convenient rescale of time we get

$$\begin{aligned} u' &= (1 + u^2)v^3, \\ (6) \quad v' &= 2ar_1 + ar_2 + 3ar_1^2r_2 + 3au - 3ar_1^2u - 2ar_1r_2u - 6ar_1^3r_2u - ar_2^2u - 3ar_1^2r_2^2u \\ &\quad - 6ar_1u^2 - 3ar_2u^2 + 3ar_1^4r_2u^2 + 6ar_1^3r_2^2u^2 + 3ar_1^2u^3 + ar_1^4u^3 + 6ar_1r_2u^3 \\ &\quad + 2ar_1^3r_2u^3 + 3ar_1^2r_2^2u^3 - 3ar_1^4r_2^2u^3 - 3ar_1^2r_2u^4 - ar_1^4r_2u^4 - 2ar_1^3r_2^2u^4 - uv^4. \end{aligned}$$

Computing the solutions of $v'|_{v=0}$ we get that

$$u = r_1, \quad u = r_2, \quad u = r_3 = -\frac{3 + r_1^2 + 2r_1r_2}{r_2 + r_1(2 + 3r_1r_2)}.$$

Computing the Jacobian matrix of the differential system at each of the equilibrium points $(r_i, 0)$ for $i = 1, 2, 3$, we get that $(r_2, 0)$ and $(r_3, 0)$ are nilpotent and $(r_1, 0)$ is

linearly zero. Applying Theorem 3.5 in [11] and taking the notation

$$\begin{aligned} A &= a(-r_1 + r_2)^2(3 + r_1^2 + 4r_1r_2 + r_2^2 + 3r_1^2r_2^2), \\ B &= -\frac{9 + r_1^2 + 16r_1r_2 + r_2^2 + 9r_1^2r_2^2}{9a(1 + r_1^2)(1 + r_1r_2)^2(3 + r_1^2 + 4r_1r_2 + r_2^2 + 3r_1^2r_2^2)}, \\ B_1 &= -a, \end{aligned}$$

we conclude that (the numerator of B is never zero because $2r_1 + r_2 + 3r_1^2r_2 \neq 0$)

- $(r_2, 0)$ is a nilpotent center or focus if $A < 0$ and it is a nilpotent saddle if $A > 0$.
- $(r_3, 0)$ with $r_2 \neq -(3 + r_1^2)/(2r_1)$ is a nilpotent center or focus if $B < 0$ and it is a nilpotent saddle if $B > 0$.
- $(r_3, 0)$ with $r_2 = -(3 + r_1^2)/(2r_1)$ is a nilpotent center or focus if $a < 0$ and it is a nilpotent saddle if $a > 0$.

Now we study the equilibrium point $(r_1, 0)$. We first translate it to the origin of coordinates by setting $(u, v) \mapsto (r_1 + u_1, v_1)$ and we get

$$\begin{aligned} \dot{u}_1 &= (1 + r_1^2 + 2r_1u_1 + u_1^2)v_1^3, \\ \dot{v}_1 &= 3ar_1u_1^2 + 3ar_1^3u_1^2 - 3ar_2u_1^2 + 3ar_1^4r_2u_1^2 - 3ar_1r_2^2u_1^2 - 3ar_1^3r_2^2u_1^2 + 3au_1^3 \\ &\quad + 5ar_1^2u_1^3 + 2ar_1r_2u_1^3 + 6ar_1^3r_2u_1^3 - ar_2^2u_1^3 - 3ar_1^2r_2^2u_1^3 + 2ar_1u_1^4 + ar_2u_1^4 \\ &\quad + 3ar_1^2r_2u_1^4 + r_1v_1^4 + u_1v_1^4. \end{aligned} \tag{7}$$

Note that $u_1 = 0$ is a characteristic direction, so we must do a twist before we apply the vertical blow-up, that is, we do the change of variables $(u_1, v_1) \mapsto (u_2 - v_2, v_2)$ and we get

$$\begin{aligned} \dot{u}_2 &= ar_1u_2^2 + 3ar_1^3u_2^2 - 3ar_2u_2^2 + 3ar_1^4r_2u_2^2 - 3ar_1r_2^2u_2^2 - 3ar_1^3r_2^2u_2^2 + 3au_2^3 \\ &\quad + 5ar_1^2u_2^3 + 2ar_1r_2u_2^3 + 6ar_1^3r_2u_2^3 - ar_2^2u_2^3 - 3ar_1^2r_2^2u_2^3 + 2ar_1u_2^4 + ar_2u_2^4 \\ &\quad + 3ar_1^2r_2u_2^4 - 6ar_1u_2v_2 - 6ar_1^3u_2v_2 + 6ar_2u_2v_2 - 6ar_1^4r_2u_2v_2 + 6ar_1r_2^2u_2v_2 \\ &\quad + 6ar_1^3r_2^2u_2v_2 - 9au_2^2v_2 - 15ar_1^2u_2^2v_2 - 6ar_1r_2u_2^2v_2 - 18ar_1^3r_2u_2^2v_2 + 3ar_2^2u_2^2v_2 \\ &\quad + 9ar_1^2r_2^2u_2^2v_2 - 8ar_1u_2^3v_2 - 4ar_2u_2^3v_2 - 12ar_1^2r_2u_2^3v_2 + 3ar_1v_2^2 + 3ar_1^3v_2^2 \\ &\quad - 3ar_2v_2^2 + 3ar_1^4r_2v_2^2 - 3ar_1r_2^2v_2^2 - 3ar_1^3r_2^2v_2^2 + 9au_2v_2^2 + 15ar_1^2u_2v_2^2 + 6ar_1r_2u_2v_2^2 \\ &\quad + 18ar_1^3r_2u_2v_2^2 - 3ar_2^2u_2v_2^2 - 9ar_1^2r_2u_2v_2^2 + 12ar_1u_2^2v_2^2 + 6ar_2u_2^2v_2^2 + 18ar_1^2r_2u_2^2v_2^2 \\ &\quad + v_2^3 - 3av_2^3 + r_1^2v_2^3 - 5ar_1^2v_2^3 - 2ar_1r_2v_2^3 - 6ar_1^3r_2v_2^3 + ar_2^2v_2^3 + 3ar_1^2r_2v_2^3 \\ &\quad + 2r_1u_2v_2^3 - 8ar_1u_2v_2^3 - 4ar_2u_2v_2^3 - 12ar_1^2r_2u_2v_2^3 + u_2^2v_2^3 - r_1v_2^4 + 2ar_1v_2^4 \\ &\quad + ar_2v_2^4 + 3ar_1^2r_2v_2^4 - u_2v_2^4, \end{aligned} \tag{8}$$

$$\begin{aligned}
\dot{v}_2 = & 3ar_1u_2^2 + 3ar_1^3u_2^2 - 3ar_2u_2^2 + 3ar_1^4r_2u_2^2 - 3ar_1r_2^2u_2^2 - 3ar_1^3r_2^2u_2^2 + 3au_3^2 \\
& + 5ar_1^2u_2^3 + 2ar_1r_2u_2^3 + 6ar_1^3r_2u_2^3 - ar_2^2u_2^3 - 3ar_1^2r_2^2u_2^3 + 2ar_1u_2^4 + ar_2u_2^4 \\
& + 3ar_1^2r_2u_2^4 - 6ar_1u_2v_2 - 6ar_1^3u_2v_2 + 6ar_2u_2v_2 - 6ar_1^4r_2u_2v_2 + 6ar_1r_2^2u_2v_2 \\
& + 6ar_1^3r_2^2u_2v_2 - 9au_2^2v_2 - 15ar_1^2u_2^2v_2 - 6ar_1r_2u_2^2v_2 - 18ar_1^3r_2u_2^2v_2 + 3ar_2^2u_2^2v_2 \\
& + 9ar_1^2r_2^2u_2^2v_2 - 8ar_1u_2^3v_2 - 4ar_2u_2^3v_2 - 12ar_1^2r_2u_2^3v_2 + 3ar_1v_2^2 + 3ar_1^3v_2^2 \\
& - 3ar_2v_2^2 + 3ar_1^4r_2v_2^2 - 3ar_1r_2^2v_2^2 - 3ar_1^3r_2^2v_2^2 + 9au_2v_2^2 + 15ar_1^2u_2v_2^2 \\
& + 6ar_1r_2u_2v_2^2 + 18ar_1^3r_2u_2v_2^2 - 3ar_2^2u_2v_2^2 - 9ar_1^2r_2^2u_2v_2^2 + 12ar_1u_2^2v_2^2 + 6ar_2u_2^2v_2^2 \\
& + 18ar_1^2r_2u_2^2v_2^2 - 3av_2^3 - 5ar_1^2v_2^3 - 2ar_1r_2v_2^3 - 6ar_1^3r_2v_2^3 + ar_2^2v_2^3 + 3ar_1^2r_2^2v_2^3 \\
& - 8ar_1u_2v_2^3 - 4ar_2u_2v_2^3 - 12ar_1^2r_2u_2v_2^3 + r_1v_2^4 + 2ar_1v_2^4 + ar_2v_2^4 + 3ar_1^2r_2v_2^4 \\
& + u_2v_2^4 - v_2^5.
\end{aligned}$$

We do the vertical blow-up $(u_2, v_2) = (u_3, u_3v_3)$ and after the rescaling of time $dt_1 = u_3dt$ (that is, we eliminate the common factor u_3 between $\dot{u}_3$ and $\dot{v}_3$) we get the system

$$\begin{aligned}
\dot{u}_3 = & u_3(3ar_1 + 3ar_1^3 - 3ar_2 + 3ar_1^4r_2 - 3ar_1r_2^2 - 3ar_1^3r_2^2 + 3au_3 + 5ar_1^2u_3 \\
& + 2ar_1r_2u_3 + 6ar_1^3r_2u_3 - ar_2^2u_3 - 3ar_1^2r_2^2u_3 + 2ar_1u_3^2 + ar_2u_3^2 + 3ar_1^2r_2u_3^2 \\
& - 6ar_1v_3 - 6ar_1^3v_3 + 6ar_2v_3 - 6ar_1^4r_2v_3 + 6ar_1r_2^2v_3 + 6ar_1^3r_2^2v_3 - 9au_3v_3 \\
& - 15ar_1^2u_3v_3 - 6ar_1r_2u_3v_3 - 18ar_1^3r_2u_3v_3 + 3ar_2^2u_3v_3 + 9ar_1^2r_2^2u_3v_3 - 8ar_1u_3^2v_3 \\
& - 4ar_2u_3^2v_3 - 12ar_1^2r_2u_3^2v_3 + 3ar_1v_3^2 + 3ar_1^3v_3^2 - 3ar_2v_3^2 + 3ar_1^4r_2v_3^2 - 3ar_1r_2^2v_3^2 \\
& - 3ar_1^3r_2^2v_3^2 + 9au_3v_3^2 + 15ar_1^2u_3v_3^2 + 6ar_1r_2u_3v_3^2 + 18ar_1^3r_2u_3v_3^2 - 3ar_2^2u_3v_3^2 \\
& - 9ar_1^2r_2^2u_3v_3^2 + 12ar_1u_3^2v_3^2 + 6ar_2u_3^2v_3^2 + 18ar_1^2r_2u_3^2v_3^2 + u_3v_3^3 - 3au_3v_3^3 + r_1^2u_3v_3^3 \\
& - 5ar_1^2u_3v_3^3 - 2ar_1r_2u_3v_3^3 - 6ar_1^3r_2u_3v_3^3 + ar_2^2u_3v_3^3 + 3ar_1^2r_2^2u_3v_3^3 + 2r_1u_3^2v_3^3 \\
& - 8ar_1u_3^2v_3^3 - 4ar_2u_3^2v_3^3 - 12ar_1^2r_2u_3^2v_3^3 + u_3^3v_3^3 - r_1u_3^2v_3^4 + 2ar_1u_3^2v_3^4 + ar_2u_3^2v_3^4 \\
& + 3ar_1^2r_2u_3^2v_3^4 - u_3^3v_3^4), \\
\dot{v}_3 = & -(-3ar_1 - 3ar_1^3 + 3ar_2 - 3ar_1^4r_2 + 3ar_1r_2^2 + 3ar_1^3r_2^2 - 3au_3 - 5ar_1^2u_3 \\
& - 2ar_1r_2u_3 - 6ar_1^3r_2u_3 + ar_2^2u_3 + 3ar_1^2r_2^2u_3 - 2ar_1u_3^2 - ar_2u_3^2 - 3ar_1^2r_2u_3^2 \\
& + 9ar_1v_3 + 9ar_1^3v_3 - 9ar_2v_3 + 9ar_1^4r_2v_3 - 9ar_1r_2^2v_3 - 9ar_1^3r_2^2v_3 + 12au_3v_3 \\
& + 20ar_1^2u_3v_3 + 8ar_1r_2u_3v_3 + 24ar_1^3r_2u_3v_3 - 4ar_2^2u_3v_3 - 12ar_1^2r_2^2u_3v_3 + 10ar_1u_3^2v_3 \\
& + 5ar_2u_3^2v_3 + 15ar_1^2r_2u_3^2v_3 - 9ar_1v_3^2 - 9ar_1^3v_3^2 + 9ar_2v_3^2 - 9ar_1^4r_2v_3^2 + 9ar_1r_2^2v_3^2 \\
& + 9ar_1^3r_2^2v_3^2 - 18au_3v_3^2 - 30ar_1^2u_3v_3^2 - 12ar_1r_2u_3v_3^2 - 36ar_1^3r_2u_3v_3^2 + 6ar_2^2u_3v_3^2 \\
& + 18ar_1^2r_2^2u_3v_3^2 - 20ar_1u_3^2v_3^2 - 10ar_2u_3^2v_3^2 - 30ar_1^2r_2u_3^2v_3^2 + 3ar_1v_3^3 + 3ar_1^3v_3^3 \\
& - 3ar_2v_3^3 + 3ar_1^4r_2v_3^3 - 3ar_1r_2^2v_3^3 - 3ar_1^3r_2^2v_3^3 + 12au_3v_3^3 + 20ar_1^2u_3v_3^3 + 8ar_1r_2u_3v_3^3 \\
& + 24ar_1^3r_2u_3v_3^3 - 4ar_2^2u_3v_3^3 - 12ar_1^2r_2^2u_3v_3^3 + 20ar_1u_3^2v_3^3 + 10ar_2u_3^2v_3^3 \\
& + 30ar_1^2r_2u_3^2v_3^3 + u_3v_3^4 - 3au_3v_3^4 + r_1^2u_3v_3^4 - 5ar_1^2u_3v_3^4 - 2ar_1r_2u_3v_3^4 - 6ar_1^3r_2u_3v_3^4 \\
& + ar_2^2u_3v_3^4 + 3ar_1^2r_2^2u_3v_3^4 + r_1u_3^2v_3^4 - 10ar_1u_3^2v_3^4 - 5ar_2u_3^2v_3^4 - 15ar_1^2r_2u_3^2v_3^4 \\
& - r_1u_3^2v_3^5 + 2ar_1u_3^2v_3^5 + ar_2u_3^2v_3^5 + 3ar_1^2r_2u_3^2v_3^5)
\end{aligned}
\tag{9}$$

The unique equilibrium point of system (9) on the straight line $u_3 = 0$ is $v_3 = 1$. Computing the eigenvalues of the Jacobian matrix of system (9) at the point $(0, 1)$ we

see that it is a nilpotent saddle. The local phase portrait near the straight line $u_3 = 0$ for system (9) is topologically equivalent to the one of Figure 2(a).

Undoing the rescaling $dt_1 = u_3 dt$ we get the phase portrait of Figure 2(b). Going back through the changes of variables from the phase portrait of Figure 2(b), we obtain the local phase portrait at the origin of system (4) which is topologically equivalent to the one of Figure 2(c).

Going back through the changes of variables from the phase portrait of Figure 2(c), we obtain the local phase portrait at the origin of system (7) which is topologically equivalent to the one of Figure 2(d). Finally, going back through the changes of variables from the phase portrait of Figure 2(d), we obtain the local phase portrait of the point $(r_1, 0)$ of the local chart U_1 which is topologically equivalent to Figure 2(e) that is, to two hyperbolic sectors.

Computing all the possibilities for the signs of A , B and B_1 together with the possible relative positions between r_1 , r_2 and r_3 we get that all global phase portraits in the Poincaré disc are topologically equivalent to the one in Figure 1(c).

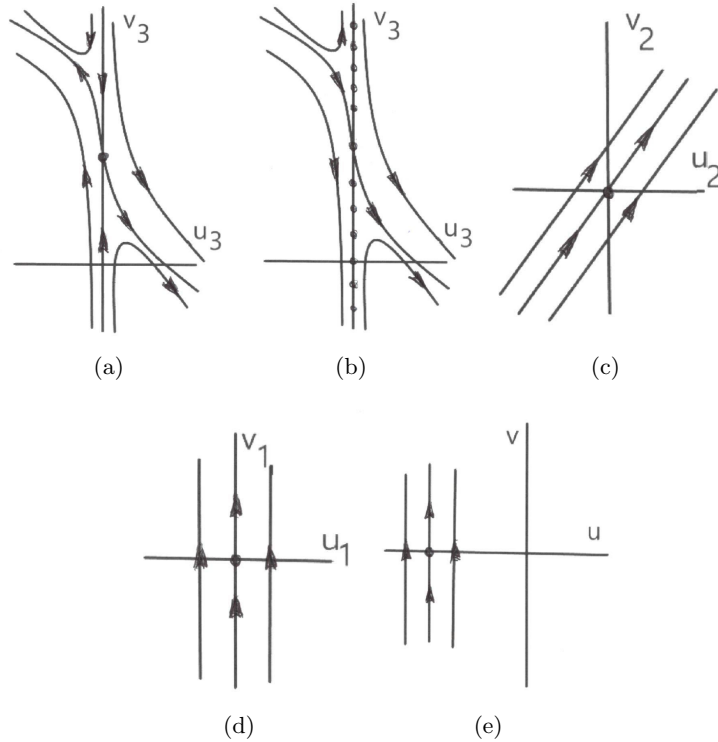


FIGURE 2. The blow up of the equilibrium $(r_1, 0)$ of the differential system (6).

Family (III). We rewrite family (III) as

$$P(x, y) = a(rx - y)^3(x + ry),$$

with $a \neq 0$ and $r \neq 0$. In that case the differential system in the local chart U_1 , after removing the common factor v with a convenient rescale of time is

$$\begin{aligned} u' &= (1 + u^2)v^3, \\ v' &= -ar^3 + 3ar^2u - ar^4u - 3aru^2 + 3ar^3u^2 + au^3 - 3ar^2u^3 + aru^4 + uv^4. \end{aligned}$$

Computing the solutions of $v'|_{v=0}$ we obtain

$$u = -\frac{1}{r}, \quad u = r$$

Then from the Jacobian matrix of the differential system we obtain that $(-1/r, 0)$ is nilpotent and $(r, 0)$ is linearly zero. Applying Theorem 3.5 in [11] we conclude that the equilibrium point $(-1/r, 0)$ is a nilpotent focus or center if $a > 0$ and a nilpotent saddle if $a < 0$.

Now we study the local phase portrait of the equilibrium point $(r, 0)$. We translate it to the origin of coordinates by performing the change $(u, v) \mapsto (u_1 + r_1, v_1)$. Doing so we obtain

$$\dot{u}_1 = (1 + r^2 + 2ru_1 + u_1^2)v_1^3, \quad \dot{v}_1 = au_1^3 + ar^2u_1^3 + aru_1^4 + rv_1^4 + u_1v_1^4.$$

We do a vertical blow-up by setting $(u_1, v_1) \mapsto (u_2, u_2v_2)$ and we get

$$\dot{u}_2 = u_2^3(1 + r^2 + 2ru_2 + u_2^2)v_2^3, \quad \dot{v}_2 = -u_2^2(1 + r^2 + ru_2)(v_2^4 - a).$$

Eliminating the common factor u_2^2 between $\dot{u}_2$ and $\dot{v}_2$ by rescaling the time we obtain the system

$$(10) \quad \dot{u}_2 = u_2(1 + r^2 + 2ru_2 + u_2^2)v_2^3, \quad \dot{v}_2 = -(1 + r^2 + ru_2)(v_2^4 - a).$$

If $a < 0$ there are no equilibrium points of equation (10) on the straight line $u_2 = 0$. If $a > 0$ there are two equilibrium points of system (10) on the straight line $u_2 = 0$ which are $(0, \pm a^{1/4})$. Computing the eigenvalues of the Jacobian matrix of system 10 at these points we conclude that they are hyperbolic saddles. Going back through the changes of variables and rescaling of the time we get that the equilibrium point $(r, 0)$ is a center or focus if $a < 0$, and a saddle if $a > 0$.

Computing all the possibilities for the signs of a together with the possible relative positions between r and $-1/r$ we get that all the global phase portraits in the Poincaré disc are topologically equivalent to the one in Figure 1(c).

Family (IV). We rewrite family (IV) as

$$P(x, y) = b(r_1x - y)(2x + ar_1x + ay + 2r_1y)(x^2 - 2axy + y^2),$$

with $b \neq 0$, $a^2 < 1$ and $a + 2r_1 \neq 0$. In that case the differential system in the local chart U_1 after removing the common factor v with a convenient rescale of time becomes

$$\begin{aligned} u' &= (1 + u^2)v^3, \\ v' &= -2br_1 - abr_1^2 + 2bu + 4abr_1u - 2br_1^2u + 2a^2br_1^2u - 3abu^2 + 3abr_1^2u^2 + 2bu^3 \\ &\quad - 2a^2bu^3 - 4abr_1u^3 - 2br_1^2u^3 + abu^4 + 2br_1u^4 + uv^4. \end{aligned}$$

Computing the solutions of $v'|_{v=0}$ we get that

$$u = r_1, \quad u = r_2 = -\frac{2 + ar_1}{a + 2r_1}.$$

Then from the Jacobian matrix at each of these equilibria we get that they are nilpotent. Applying Theorem 3.5 in [11] and taking the notation

$$\begin{aligned} A &= 2b(1 - 2ar_1 + r_1^2)(1 + ar_1 + r_1^2), \\ B &= \frac{4 + a^2 + 8ar_1 + 4r_1^2 + a^2r_1^2}{2b(1 + ar_1 + r_1^2)(4 + 5a^2 + 16ar_1 + 2a^3r_1 + 4r_1^2 + 5a^2r_1^2)}, \end{aligned}$$

we conclude that (the numerator of B is never zero because $a + 2r_1 \neq 0$)

- $(r_1, 0)$ is a nilpotent center or focus if $A < 0$ and it is a nilpotent saddle if $A > 0$.
- $(r_2, 0)$ is a nilpotent center or focus if $B < 0$ and it is a nilpotent saddle if $B > 0$.

Computing all the possibilities for the signs of A and B together with the possible relative positions between r_1 and r_2 we get that all global phase portraits in the Poincaré disc are topologically equivalent to the one in Figure 1(c).

APPENDIX: FORMULAE IN [15]

$$\begin{aligned} &\int \sin^p s \cos^{2q} s \, ds \\ (11) \quad &= \frac{\sin^{p+1} s}{2q + p} \left(\cos^{2q-1} s + \sum_{k=1}^{q-1} \frac{(2q-1)(2q-3) \cdots (2q-2k+1)}{(2q+p-2)(2q+p-4) \cdots (2q+p-2k)} \cos^{2q-2k-1} s \right) \\ &\quad + \frac{(2q-1)!!}{(2q+p)(2q+p-2) \cdots (p+2)} \int \sin^p s \, ds \end{aligned}$$

$$\begin{aligned} &\int \sin^{2p} s \, ds = -\frac{\cos s}{2p} \left(\sin^{2p-1} s + \sum_{k=1}^{p-1} \frac{(2p-1)(2p-3) \cdots (2p-2k+1)}{2^k(p-1)(p-2) \cdots (p-k)} \sin^{2p-2k-1} s \right) \\ (12) \quad &\quad + \frac{(2p-1)!!}{2^p p!} s \end{aligned}$$

$$(13) \quad \int \sin^p s \cos^{2q+1} s \, ds = \frac{\sin^{p+1} s}{2q+p+1} \left(\cos^{2q} s + \sum_{k=1}^q \frac{2^k q(q-1) \cdots (q-k+1)}{(2q+p-1)(2q+p-3) \cdots (2q+p-2k+1)} \cos^{2q-2k} s \right)$$

$$(14) \quad \int \cos^p s \sin^{2q} s \, ds = -\frac{\cos^{p+1} s}{2q+p} \left(\sin^{2q-1} s + \sum_{k=1}^{q-1} \frac{(2q-1)(2q-3) \cdots (2q-2k+1)}{(2q+p-2)(2q+p-4) \cdots (2q+p-2k)} \sin^{2q-2k-1} s \right) \\ + \frac{(2q-1)!!}{(2q+p)(2q+p-2) \cdots (p+2)} \int \cos^p s \, ds$$

$$(15) \quad \int \cos^{2p} s \, ds = \frac{\sin s}{2p} \left(\cos^{2p-1} s + \sum_{k=1}^{p-1} \frac{(2p-1)(2p-3) \cdots (2p-2k+1)}{2^k (p-1)(p-2) \cdots (p-k)} \cos^{2p-2k-1} s \right) \\ + \frac{(2p-1)!!}{2^p p!} s$$

$$(16) \quad \int \cos^p s \sin^{2q+1} s \, ds = -\frac{\cos^{p+1} s}{2q+p+1} \left(\sin^{2q} s + \sum_{k=1}^q \frac{2^k q(q-1) \cdots (q-k+1)}{(2q+p-1)(2q+p-3) \cdots (2q+p-2k+1)} \sin^{2q-2k} s \right)$$

FUNDING

The first author is partially supported by the Agencia Estatal de Investigación grant PID2022-136613NB-I00, the AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Reial Acadèmia de Ciències i Arts de Barcelona. The second author is partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

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