

THE EASIEST POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$ HAVING AN INVARIANT HYPERBOLOID

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This paper answers the following two questions: *What are the easiest polynomial differential systems in $\mathbb{R}^3$ having an invariant hyperboloid of one sheet, or an invariant hyperboloid of two sheets?*, and *for such polynomial differential systems what are their phase portraits on such an invariant hyperboloids?* To solve these questions a method based on first integrals, symmetry, analysis of the nature of equilibrium points, and invariant algebraic surfaces is employed.

Keywords: Polynomial differential system in $\mathbb{R}^3$; invariant hyperboloids; phase portrait.

1. Introduction and statement of the main results

Let S be a set. When every orbit of a differential system having a point in the set S is completely contained in S , we say that the set S is *invariant* for such a differential system. The set S can be a curve for a planar differential system, or a surface for a differential system in the space, ... Darboux in [6] was one of first mathematicians in studying the invariant sets of the differential systems. He used the existence of invariant sets for studying the integrability of the differential systems.

After Darboux many authors have studied different differential systems having many distinct kinds of invariant sets, including invariant circles [1; 19; 24], invariant spheres [2; 3; 9; 17; 22], and invariant tori [25; 26; 27], ... In this paper, we aim to study the simplest polynomial differential systems in $\mathbb{R}^3$ that have invariant hyperboloids of one sheet and two sheets. The hyperboloid surface was chosen for its intricate geometric properties, which facilitate the analysis of complex dynamics and topological behaviors that are not observable on more straightforward surfaces. This selection broadens the scope and significance of the study, aligning it with the investigation of systems with more advanced invariant sets.

A polynomial differential system in $\mathbb{R}^3$ is a differential system of the form

$$\begin{aligned}\dot{x} &= P(x, y, z), \\ \dot{y} &= Q(x, y, z), \\ \dot{z} &= R(x, y, z),\end{aligned}\tag{1}$$

where P , Q and R are real polynomials in the variables x , y and z , and the dot denotes derivative with

respect to the time t . The degree of the polynomial differential system (1) is the maximum of the degrees of the polynomials P , Q and R .

In $\mathbb{R}^3$ we consider the hyperboloid of two sheets, $x^2 + y^2 - z^2 = -1$, and the hyperboloid of one sheet, $x^2 + y^2 - z^2 = 1$. We say that a hyperboloid is *invariant* under the flow of the differential system (1) if for every orbit $(x(t), y(t), z(t))$ of the differential system (1) having a point on that hyperboloid the whole orbit is contained in the hyperboloid.

Two natural questions about the invariant hyperboloid of one or two sheets of a polynomial differential system (1) are: *What are the easiest polynomial differential systems (1) in $\mathbb{R}^3$ having such an invariant hyperboloid*, and *for such polynomial differential systems what are their phase portraits on the invariant hyperboloid*? The objective of this paper is to give an answer to these two questions.

Let U be an open and dense set in $\mathbb{R}^3$. We recall that a C^1 function $H : U \rightarrow \mathbb{R}$ which is non-locally constant is a *first integral* of the differential system (1) if H is constant on all the solutions $(x(t), y(t), z(t))$ contained in U . In other words, on the solution $(x(t), y(t), z(t))$ we have that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R = 0.$$

Let $f(x, y, z)$ be a real polynomial. We say that the algebraic surface $f(x, y, z) = 0$ is *invariant* for the polynomial differential system (1) if there exists a real polynomial $k(x, y, z)$ satisfying the equality

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = kf. \quad (2)$$

The polynomial k is called the *cofactor* of the invariant algebraic surface $f(x, y, z) = 0$, and from (2) it follows that the degree of the polynomial k is at most the degree of the polynomial differential system (1) minus one.

From (2) it follows that the gradient $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ of f is orthogonal to the vector field (P, Q, R) of the polynomial differential system (1) on the points of an invariant algebraic surface $f(x, y, z) = 0$. Therefore the vector field (P, Q, R) is contained in the tangent plane to the surface $f(x, y, z) = 0$ at every point of the invariant algebraic surface $f(x, y, z) = 0$. Hence the surface $f(x, y, z) = 0$ is formed by orbits of the vector field (P, Q, R) , in other words if an orbit has a point on the invariant surface $f(x, y, z) = 0$, then the whole orbit is contained in it. This justifies the name “invariant algebraic surface” for the surface $f(x, y, z) = 0$. For more details on first integrals and invariant algebraic surfaces, see [8; 12; 22; 23].

A *phase portrait* of a differential system on an invariant hyperboloid is a geometric representation of the orbits of the differential system in the hyperboloid. The best qualitative result for a differential system is to provide its *phase portrait*, i.e. the decomposition of the domain of definition of the differential system as union of all its orbits.

Two phase portraits on an invariant hyperboloid are *topologically equivalent* if there is a homeomorphism of the hyperboloid that sends orbits of the first phase portrait onto orbits of the second phase portrait, either preserving or reversing the orientation of all the orbits.

Certainly, the easiest polynomial differential systems in $\mathbb{R}^3$ are the linear ones:

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3z, \\ \dot{y} &= b_0 + b_1x + b_2y + b_3z, \\ \dot{z} &= c_0 + c_1x + c_2y + c_3z, \end{aligned} \quad (3)$$

i.e. the polynomial differential systems of degree one. Furthermore, when considering polynomial differential systems of degree higher than one some among the easiest ones are

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3z, \\ \dot{y} &= b_0 + b_1x + b_2y + b_3z, \\ \dot{z} &= c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2. \end{aligned} \quad (4)$$

We recall that the *Poincaré ball* is a unit ball in $\mathbb{R}^3$ centered at the origin of coordinates whose interior has been identified with the whole space $\mathbb{R}^3$, and its boundary the 2-dimensional sphere $\mathbb{S}^2$ is identified with the infinity of the space $\mathbb{R}^3$. Note that in $\mathbb{R}^3$ we can go or come from the infinity in as many directions as points have the sphere $\mathbb{S}^2$. For more details on the Poincaré ball see subsection 2.2.

Let X be the hyperboloid of one sheet or two sheets, or the Poincaré ball. Then we say that two differential systems defined on X are *topologically equivalent* if there is a homeomorphism in X that send the orbits of one system onto the orbits of the other system, preserving or reversing the sense of all the orbits.

In the next three theorems we characterize the linear differential systems (3) and the polynomial differential systems (4) having an invariant hyperboloid, and we describe the dynamics of these differential systems on these invariant hyperboloids.

Theorem 1. *The linear differential systems (3) in $\mathbb{R}^3$ for which the hyperboloid of one or two sheets ($x^2 + y^2 - z^2 = -1$ or $x^2 + y^2 - z^2 = 1$ respectively) is invariant are*

$$\begin{aligned}\dot{x} &= a_2y + a_3z, \\ \dot{y} &= -a_2x + b_3z, \\ \dot{z} &= a_3x + b_3y.\end{aligned}\tag{5}$$

These differential systems have the first integral $H = x^2 + y^2 - z^2$, so they have simultaneously both types of invariant hyperboloids. Moreover, on each one of these hyperboloids $x^2 + y^2 - z^2 = h$ with $h \neq 0$ the phase portrait of the differential systems (5) has two equilibrium points, one equilibrium point or no equilibrium points. The orbits reveal the shapes of ellipses, hyperbolas, and parabolas. Thus, these equilibrium points are either centers or saddles on the invariant hyperboloid. The distinct topological phase portraits exhibited for the differential systems (5) are topologically equivalent to the ones illustrated in Figures 1, 2 and 3.

Theorem 2. *The quadratic polynomial differential systems (4) in $\mathbb{R}^3$ for which the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ is invariant are*

$$\begin{aligned}\dot{x} &= a_2y + a_3z, \\ \dot{y} &= b_3z - a_2x, \\ \dot{z} &= -c_4 + a_3x + b_3y + c_4(x^2 + y^2 - z^2),\end{aligned}\tag{6}$$

with $c_4 \neq 0$. On the hyperboloid of one sheet, the differential systems described in equation (6) exhibit two saddle equilibrium points. However, when $a_2 = 0$, no equilibrium points are present. These phase portraits are topologically equivalent to those depicted in Figures 1(a) and 4, respectively.

Note that Theorem 1 of [16] does not cover the cases of linear and quadratic polynomial differential systems (5) and (6) with invariant hyperboloids. For instance, consider the hyperboloid of one sheet given by $Q = x^2 + y^2 - z^2 - 1$. The theorem states that all polynomial differential systems preserving this hyperboloid as an invariant can be expressed as

$$\dot{x} = QA - 2yD - 2zE, \quad \dot{y} = QB + 2xD + 2zF, \quad \dot{z} = QC - 2xE + 2yF.$$

where A, B, C, D, E and F are arbitrary polynomials. However, the theorem does not specify how to select these six polynomials A, B, C, D, E and F such that the resulting system becomes either linear or quadratic.

Theorem 3. *The quadratic polynomial differential systems (4) in $\mathbb{R}^3$ for which the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$ is invariant are*

$$\begin{aligned}\dot{x} &= a_2y + a_3z, \\ \dot{y} &= b_3z - a_2x, \\ \dot{z} &= c_4 + a_3x + b_3y + c_4(x^2 + y^2 - z^2),\end{aligned}\tag{7}$$

with $c_4 \neq 0$. On the hyperboloid of two sheets, the differential systems described in equation (7) exhibit two center equilibrium points. However, when $a_2 = 0$, no equilibrium points are present. These phase portraits are topologically equivalent to those depicted in Figures 1(a) and Figure 2(a), respectively.

In the next theorem we do not consider the case that in the linear differential system (5) we have $a_2 = a_3 = b_3 = 0$.

Theorem 4. *The phase portrait on the sphere of the infinity of the linear differential system (5), having the invariant hyperboloids of one or two sheets invariant, are given in*

- (a) Figure 5(a) if either $b_3 = 0$, $a_3^2 - a_2^2 < 0$ and $a_3 \neq 0$; or $b_3 = 0$, $a_3^2 - a_2^2 < 0$ and $a_3 \neq 0$, or $b_3(a_3^2 - a_2^2) \neq 0$ and $a_3^2 + b_3^2 - a_2^2 < 0$.
- (b) Figure 5(b) if either $a_2 = a_3 = 0$ and $b_3 \neq 0$; or $a_2 = b_3 = 0$ and $a_3 \neq 0$; or $a_3 = b_3 = 0$ and $a_2 \neq 0$; or $b_3 \neq 0$, $a_3^2 - a_2^2 = 0$ and $a_2 \neq 0$; or $b_3(a_3^2 - a_2^2) \neq 0$ and $a_3^2 + b_3^2 - a_2^2 > 0$; or $b_3 = 0$ and $a_3^2 - a_2^2 > 0$.
- (c) Figure 5(c) if either $b_3 = 0$, $a_3^2 - a_2^2 = 0$ and $a_2 = 0$; or $b_3(a_3^2 - a_2^2) \neq 0$ and $a_3^2 + b_3^2 - a_2^2 = 0$.

Theorem 5. *The phase portrait on the sphere of the infinity of the quadratic differential systems (6) and (7), having the invariant hyperboloids of one or two sheets invariant respectively, is given in Figure 6.*

We note that the invariant hyperboloids of one sheet reached the infinity at the endpoint of the positive z -axis, while the hyperboloids of two sheets reached the infinity at the two endpoints of the z -axis.

Other paper studying the easiest systems having invariant spheres or cylinders can be found in [15; 20; 21].

Theorem 1 is proved in section 3, Theorems 2 and 3 are proved in section 4, and Theorems 4 and 5 are proved in section 5. In section 2 we recall some basic results that we need for proving our theorems.

2. Preliminary result

2.1. The classification of the conics

We present in the next proposition the well known classification of conics. Consider the following general expression of a conic

$$ax^2 + by^2 + 2dxy + 2gx + 2ey + c = 0. \quad (8)$$

Proposition 1. *For*

$$D = \begin{vmatrix} a & d & g \\ d & b & e \\ g & e & c \end{vmatrix}, \quad D' = \begin{vmatrix} a & d \\ d & b \end{vmatrix},$$

the real conic defined by equation (8) exhibits the following form

$$D \neq 0 \begin{cases} D' > 0 \text{ Ellipse,} \\ D' < 0 \text{ Hiperbola,} \\ D' = 0 \text{ Parabola,} \end{cases}$$

and

$$D = 0 \begin{cases} D' > 0 \text{ Point,} \\ D' < 0 \text{ Two intersecting lines,} \\ D' = 0 \text{ Two parallel or coincident lines.} \end{cases}.$$

For a proof of Proposition 1 see for instance [11].

2.2. Poincaré compactification in $\mathbb{R}^3$

Let $X = (P, Q, R)$ be a polynomial vector field in $\mathbb{R}^3$ of degree n , where n is the maximum of the degrees of the polynomials P , Q and R . The Poincaré compactified system, denoted by $p(X)$, is an analytic vector field on the 3-dimensional Poincaré sphere as $\mathbb{S}^3 = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^3; y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$.

We identify the hyperplane $\mathbb{R}^3$ with the hyperplane $y_4 = 1$ of $\mathbb{R}^4$ tangent to the Poincaré sphere $\mathbb{S}^3$ at the north pole $(0, 0, 0, 1)$, the northern hemisphere, the southern hemisphere and the equator of the sphere $\mathbb{S}^3$ are $H_+ = \{y \in \mathbb{S}^3 : y_4 > 0\}$, $H_- = \{y \in \mathbb{S}^3 : y_4 < 0\}$ and $\mathbb{S}^2 = \{y \in \mathbb{S}^3; y_4 = 0\}$, respectively.

The projections of the hyperplane $y_4 = 1$ on the sphere $\mathbb{S}^3$ are defined by the maps $f_{\pm} : \mathbb{R}^3 \longrightarrow H_{\pm}$, where $f_{\pm}(y_1, y_2, y_3, 1) = (y_1^2 + y_2^2 + y_3^2 + 1)^{-1/2}(y_1, y_2, y_3, 1)$. These diffeomorphisms create two copies of the vector field X : one X_+ in H_+ , and the other X_- in H_- . Together they define the vector field $X' = X_+ \cup X_-$ on $\mathbb{S}^3 \setminus \mathbb{S}^2$. Furthermore the infinity of the hyperplane $y_4 = 1$ (i.e. $\mathbb{R}^3$) is identified with the equator $\mathbb{S}^2$ of the Poincaré sphere $\mathbb{S}^3$.

The *Poincaré compactified vector field*, $p(X)$ is the analytic extension of X' from $H_+ \cup H_-$ to $\mathbb{S}^3$, defined as $p(X) = y_4^{n-1} X'$. The *Poincaré Ball* $\mathbb{B}$ is obtained by projecting the closed northern hemisphere onto the hyperplane $y_4 = 0$ under the map $(y_1, y_2, y_3, y_4) \mapsto (y_1, y_2, y_3, 0)$. The dynamics of the vector field $p(X)$ near $\mathbb{S}^2$ correspond to the behavior of the vector field X near infinity.

We define the local charts $\phi_i : U_i \rightarrow \mathbb{R}^3$ and $\psi_i : V_i \rightarrow \mathbb{R}^3$ by $\phi_i(y_1, y_2, y_3, y_4) = \psi_i(y_1, y_2, y_3, y_4) = (y_a/y_i, y_b/y_i, y_c/y_i) = (z_1, z_2, z_3)$, where $a \neq i$, $b \neq i$, $c \neq i$ and $a < b < c$. In the local chart U_1 the expression of the vector field $p(X)$ is

$$\begin{aligned} \dot{z}_1 &= z_3^n \left(Q \left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3} \right) - z_1 P \left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3} \right) \right), \\ \dot{z}_2 &= z_3^n \left(R \left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3} \right) - z_2 P \left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3} \right) \right), \\ \dot{z}_3 &= -z_3^{n+1} P \left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3} \right). \end{aligned}$$

Similarly in the local chart U_2 the expression of the vector field $p(X)$ is

$$\begin{aligned} \dot{z}_1 &= z_3^n \left(P \left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3} \right) - z_1 Q \left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3} \right) \right), \\ \dot{z}_2 &= z_3^n \left(R \left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3} \right) - z_2 Q \left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3} \right) \right), \\ \dot{z}_3 &= -z_3^{n+1} Q \left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3} \right). \end{aligned}$$

Finally, in the local chart U_3 the expression of the vector field $p(X)$ is

$$\begin{aligned} \dot{z}_1 &= z_3^n \left(P \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3} \right) - z_1 R \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3} \right) \right), \\ \dot{z}_2 &= z_3^n \left(Q \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3} \right) - z_2 R \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3} \right) \right), \\ \dot{z}_3 &= -z_3^{n+1} R \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3} \right). \end{aligned}$$

The expressions of $p(X)$ on the charts V_1 , V_2 and V_3 are the same as those on the charts U_1 , U_2 and U_3 , respectively, but multiplied by $(-1)^{n-1}$. In the charts U_i and V_i for $i = 1, 2, 3$, the points at infinity are of the form $(z_1, z_2, 0)$. The infinity $\mathbb{S}^2$ is invariant under the flow of $p(X)$. For more details on Poincaré compactification, see [5].

The equilibrium points of the vector field X are referred to as finite equilibrium points of X or of $p(X)$, while the equilibrium points of the vector field $p(X)$ located at the infinity $\mathbb{S}^2$ are referred to as infinite equilibrium points of X or of $p(X)$.

For studying the infinite equilibrium points on $\mathbb{S}^2$ it is sufficient to study the infinite equilibrium points $(z_1, z_2, 0)$ in the local chart U_1 , the infinite equilibrium points $(0, z_2, 0)$ in the local chart U_2 , and see if the origin $(0, 0, 0)$ of the local chart U_3 is or not an infinite equilibrium point.

2.3. Normally hyperbolic manifolds

Let φ_t be a smooth flow on a manifold M and suppose C is a submanifold of M consisting entirely of equilibrium points for the flow. C is said to be normally hyperbolic if the tangent bundle to M over C splits into three subbundles TC , E^s , and E^u invariant under φ_t and satisfying

- (i) φ_t contracts E^s exponentially,
- (ii) φ_t expands E^u exponentially,
- (iii) TC is the tangent bundle of C .

For normally hyperbolic submanifolds one has the usual existence of smooth stable and unstable manifolds together with the persistence of these invariant manifolds under small perturbations. More precisely, we have the following result.

Theorem 6. *Let C be a normally hyperbolic submanifold of equilibrium points for φ_t . Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$, respectively. Furthermore, both C and the stable and unstable manifolds are permanent under small perturbations of the flow.*

For a proof of this theorem see [10].

3. Proof of Theorem 1

From (2) we force the hyperboloid $x^2 + y^2 - z^2 = 1$ to be an invariant surface of the differential system (3), and we obtain

$$a_0 = a_1 = b_0 = b_2 = c_0 = c_3 = 0, \quad b_1 = -a_2, \quad c_1 = a_3, \quad c_2 = b_3.$$

By substituting these conditions into the differential system (3), we get the differential system (5). Furthermore, we can check that $H = H(x, y, z) = x^2 + y^2 - z^2$ is a first integral of system (5). So the first part of Theorem 1 is proved.

Note that the differential system (5) is invariant under the symmetry $(x, y, z) \rightarrow (-x, -y, -z)$. Moreover, this differential system has the straight line $\lambda(1, -a_3/b_3, a_2/b_3)$, varying $\lambda \in \mathbb{R}$, filled with equilibrium points. These equilibrium points on the invariant hyperboloid $H = h$ are

$$\left(\pm \frac{b_3 \sqrt{h}}{\sqrt{b_3^2 - a_2^2 + a_3^2}}, \mp \frac{a_3 \sqrt{h}}{\sqrt{b_3^2 - a_2^2 + a_3^2}}, \pm \frac{a_2 \sqrt{h}}{\sqrt{b_3^2 - a_2^2 + a_3^2}} \right).$$

So there are three cases:

(i) If $h(b_3^2 - a_2^2 + a_3^2) > 0$ there are two equilibrium points on the hyperboloid $H = h$. The eigenvalues of the Jacobian matrix of the differential system (5) at these equilibrium points are 0 and $\pm \sqrt{b_3^2 - a_2^2 + a_3^2}$. In the following, we will show that $b_3^2 - a_2^2 + a_3^2 > 0$, the eigenvalues are real with opposite signs, indicating that the two equilibrium points on the hyperboloid $H = h$ are saddles. On the other hand, if $b_3^2 - a_2^2 + a_3^2 < 0$, the eigenvalues are purely imaginary (complex with a zero real part), classifying these equilibrium points as centers.

On the invariant hyperboloid $H = h$ with $z = \pm \sqrt{x^2 + y^2 - h}$, the differential system reduces to

$$\dot{x} = a_2 y \pm a_3 \sqrt{x^2 + y^2 - h}, \quad \dot{y} = -a_2 x \pm b_3 \sqrt{x^2 + y^2 - h}.$$

Now the two equilibrium points of this differential system are

$$\left(\pm \frac{b_3 \sqrt{h}}{\sqrt{b_3^2 - a_2^2 + a_3^2}}, \mp \frac{a_3 \sqrt{h}}{\sqrt{b_3^2 - a_2^2 + a_3^2}} \right).$$

It is easy to check that

$$F(x, y) = a_3 y + a_2 z - b_3 x,$$

is a first integral of the differential system (5). The existence of this first integral prevents the existence of foci, so when $b_3^2 - a_2^2 + a_3^2 < 0$ the two equilibrium points are centers.

(ii) If $h = 0$ a single equilibrium point exists at the origin of the cone $H = 0$.

(iii) If $h(b_3^2 - a_2^2 + a_3^2) < 0$ or $b_3^2 - a_2^2 + a_3^2 = 0$ and $h \neq 0$, there are no equilibrium points on the hyperboloid $H = h$.

To determine the orbits on the hyperboloids $H = h$, we examine the intersections of the two first integrals, F and H . Since $F = f$ represents planes and $H = h$ represents hyperboloids, these intersections yield conic orbits. We analyze two cases: when $b_3 \neq 0$ and when $b_3 = 0$.

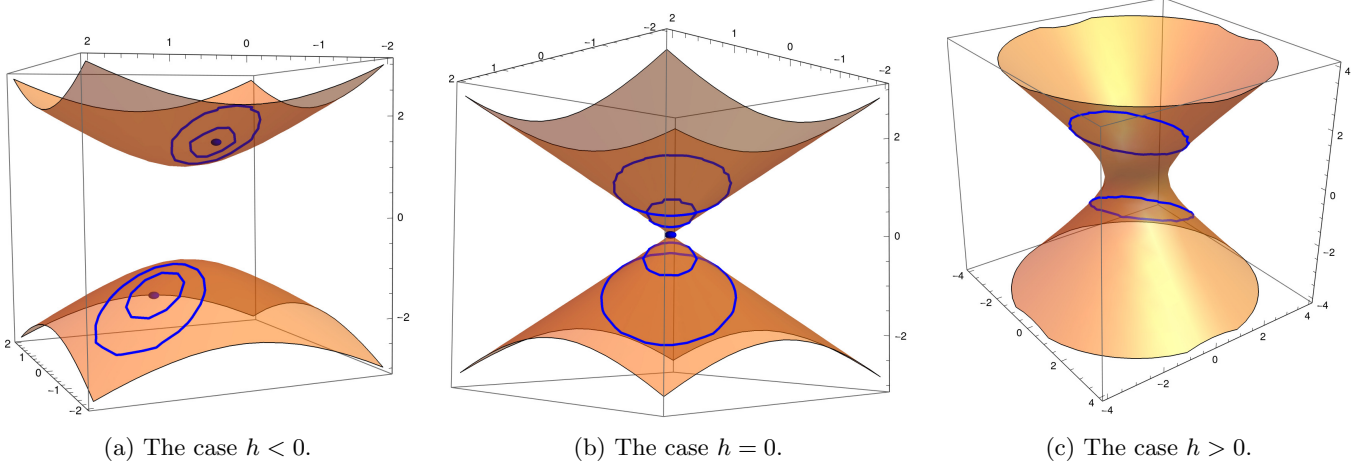


Fig. 1: The phase portrait of system (5) when $D \neq 0$ and $D' > 0$.

3.1. The case $b_3 \neq 0$

From the invariant plane $F = f$ with $f \in \mathbb{R}$, we obtain

$$x = \frac{a_3 y + a_2 z - f}{b_3},$$

Substituting x into $H = h$, we obtain the conic

$$H - h = \left(\frac{a_3^2}{b_3^2} + 1 \right) y^2 + \left(\frac{a_2^2}{b_3^2} - 1 \right) z^2 + 2 \frac{a_2 a_3}{b_3^2} yz - 2 \frac{a_3 f}{b_3^2} y - 2 \frac{a_2 f}{b_3^2} z + \frac{f^2}{b_3^2} - h = 0,$$

then comparing this conic with the conic (8) we get that

$$a = \frac{a_3^2}{b_3^2} + 1, \quad b = \frac{a_2^2}{b_3^2} - 1, \quad c = \frac{f^2}{b_3^2} - h, \quad d = \frac{a_2 a_3}{b_3^2}, \quad e = -\frac{a_2 f}{b_3^2}, \quad g = -\frac{a_3 f}{b_3^2}.$$

Applying Proposition 1 to this conic we obtain that

$$D = \frac{(b_3^2 - a_2^2 + a_3^2)h - f^2}{b_3^2} \text{ and } D' = \frac{a_2^2 - a_3^2 - b_3^2}{b_3^2}.$$

3.1.1. The case $D' > 0$

Then the orbits $\{F = f\} \cap \{H = h\}$ are ellipses. These ellipses must surround some center. The corresponding phase portrait of the differential system (5) on the hyperboloid of two sheets $H = h$ with $h < 0$ is illustrated in Figure 1(a).

In the case of $h = 0$ the quadratic $H = 0$ is a cone and the differential system (5) exhibits a single equilibrium point at the origin, that is a center. See the corresponding phase portrait in Figure 1(b).

When $h > 0$ the differential system (5) has no equilibrium points, and its phase portrait on the hyperboloid of one sheet is shown in Figure 1(c).

3.1.2. The case $D \neq 0$ and $D' < 0$

Then the orbits $\{F = f\} \cap \{H = h\}$ are hyperbolas. If $h < 0$ the differential system (5) does not possess any equilibrium points. Then its phase portrait on the hyperboloid of two sheets $H = h$ is depicted in Figure 2(a).

When $h = 0$ the differential system (5) exhibits a single equilibrium point on the cone $H = 0$, the origin, with eigenvalues $\pm \sqrt{b_3^2 - a_2^2 + a_3^2}$. Therefore this equilibrium is a saddle. Its phase portrait is drawn in Figure 2(b).

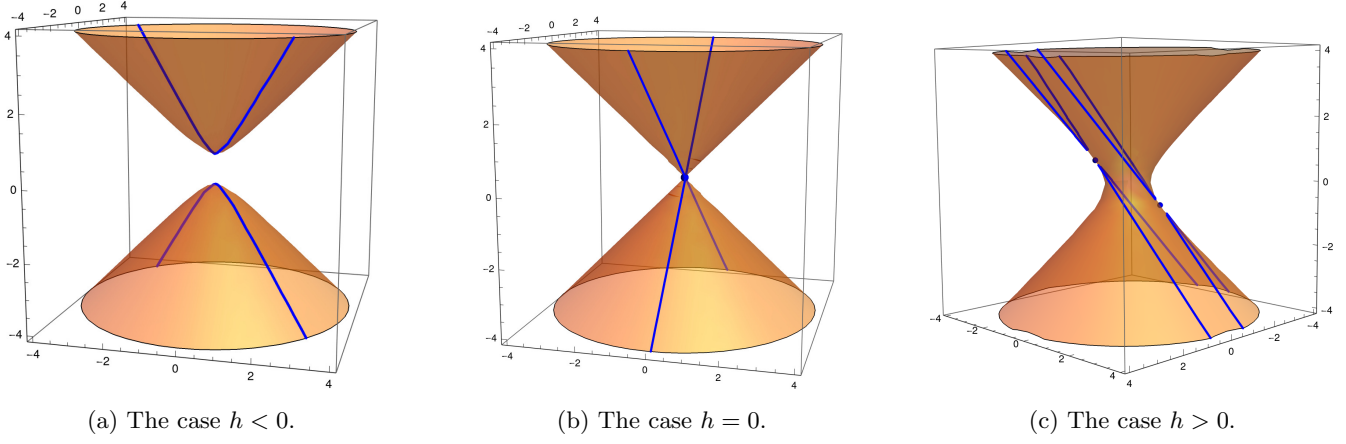


Fig. 2: The phase portrait of system (5) when $D \neq 0$ and $D' < 0$.

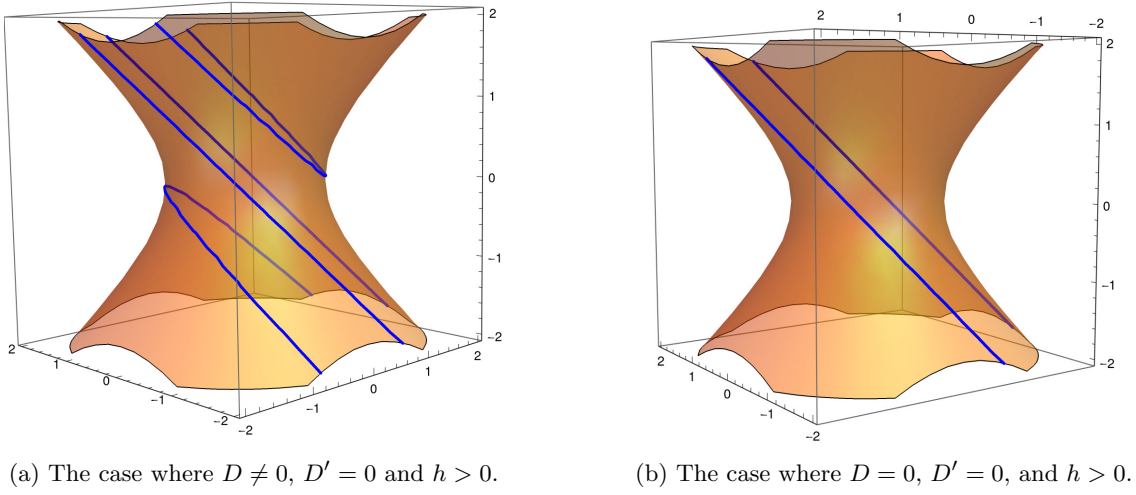


Fig. 3: The phase portrait of system (5).

If $h > 0$ the differential system (5) possesses two equilibrium points on the hyperboloid of one sheet $H = h$, with eigenvalues $\pm\sqrt{b_3^2 - a_2^2 + a_3^2}$. Hence the equilibria are saddles. The phase portrait of the system (5) on $H = h$ is depicted in Figure 2(c).

3.1.3. The case $D \neq 0$ and $D' = 0$

Then the orbits $\{F = f\} \cap \{H = h\}$ are parabolas. In this scenario the differential system (5) has no equilibrium points if $h \neq 0$.

If $h < 0$, the differential system (5) on the hyperboloid of two sheets has a phase portrait equivalent to that shown in Figure 2(a).

If $h = 0$ the differential system (5) has one equilibrium point on the vertex of the cone $H = 0$. Refer to the equivalent phase portrait shown in Figure 2(b).

If $h > 0$ the differential system (5) on the hyperboloid of one sheet $H = h$ has the phase portrait drawn in Figure 3(a).

3.1.4. The case $D = 0$ and $D' < 0$

Under the conditions $D = 0$ and $D' < 0$, we have $h > 0$ and $b_3^2 - a_2^2 + a_3^2 > 0$. Therefore, the differential system (5) possesses two equilibrium points on the hyperboloid of one sheet $H = h$, with eigenvalues $\pm\sqrt{b_3^2 - a_2^2 + a_3^2}$. Hence, these equilibria are saddles. By Proposition 1, the invariant set

$\{F = \pm\sqrt{(b_3^2 - a_2^2 + a_3^2)h}\} \cap \{H = h\}$ on the hyperboloid of one sheet $H = h$ is formed by two straight lines intersecting at a point which are separatrices of these saddles. Figure 2(c) portrays the phase portrait of system (5) on $H = h$.

3.1.5. The case $D = 0$ and $D' = 0$

When $f = 0$, Proposition 1 implies that the invariant set $\{F = 0\} \cap \{H = h\}$ consists of two parallel or coincident lines. Observe the topologically equivalent phase portrait with two coincident lines shown in Figure 2(b) on the cone $H = 0$. Refer to Figure 3(b), which illustrates the phase portrait with two parallel lines on the hyperboloid of one sheet $H = h$ when $h > 0$. Note that there are no orbits of system (5) on the hyperboloid of two sheets $H = h$ when $h < 0$.

3.2. The case $b_3 = 0$ and $a_3 \neq 0$.

From the invariant plane $F = f$ with $f \in \mathbb{R}$, we obtain

$$y = \frac{f - a_2 z}{a_3}$$

Substituting y into $H = h$, we obtain the conic

$$H - h = x^2 + \left(\frac{a_2^2}{a_3^2} - 1\right) z^2 - \frac{2a_2 f}{a_3^2} z + \frac{f^2}{a_3^2} - h = 0,$$

then comparing this conic with the conic (8) we get that

$$a = 1, \quad b = \left(\frac{a_2^2}{a_3^2} - 1\right), \quad c = \frac{f^2}{a_3^2} - h, \quad d = 0, \quad e = -\frac{a_2 f}{a_3^2}, \quad g = 0.$$

Applying Proposition 1 to this conic we obtain that

$$D = \frac{-f^2 - a_2^2 h + a_3^2 h}{a_3^2}, \quad \text{and} \quad D' = \frac{a_2^2}{a_3^2} - 1.$$

3.2.1. The case $D \neq 0$ and $D' > 0$

The orbits $\{F = f\} \cap \{H = h\}$ are ellipses. These ellipses are centered around a specific equilibrium point. Figure 1(a) depicts the corresponding phase portrait of the differential system (5) on the hyperboloid of two sheets $H = h$ where $h < 0$.

For the scenario where $h = 0$, the quadratic $H = 0$ forms a cone, and the differential system (5) displays a unique equilibrium point located at the origin, this equilibrium is center. Refer to Figure 1(b) for the associated phase portrait.

For $h > 0$, the differential system (5) lost equilibrium points, and Figure 1(c) illustrates its phase portrait on the hyperboloid of one sheet.

3.2.2. The case $D \neq 0$ and $D' < 0$

The intersections of orbits $\{F = f\} \cap \{H = h\}$ are hyperbolas. In cases where $h < 0$ the differential system (5) does not exhibit any equilibrium points. Figure 2(a) illustrates the corresponding phase portrait on the hyperboloid of two sheets $H = h$.

For the case where $h = 0$, the differential system (5) displays a unique equilibrium point at the origin on the cone $H = 0$. The eigenvalues associated with this equilibrium are $\pm\sqrt{a_3^2 - a_2^2}$, indicating that this equilibrium point is a saddle. The corresponding phase portrait is illustrated in Figure 2(b).

When $h > 0$ the differential system (5) has two equilibrium points situated on the hyperboloid of one sheet $H = h$, each with eigenvalues $\pm\sqrt{a_3^2 - a_2^2}$. Consequently, these equilibria are identified as saddles. Figure 2(c) portrays the phase portrait of system (5) on $H = h$.

3.2.3. The case $D \neq 0$ and $D' = 0$

In this case, the trajectories defined by the intersection of $\{F = f\} \cap \{H = h\}$ are parabolas. If $h \neq 0$ the differential system (5) does not possess any equilibrium points in this particular scenario.

For $h < 0$, the phase portrait of the differential system (5) on the hyperboloid of two sheets is illustrated in Figure 3(a).

When $h = 0$, the differential system (5) possesses a single equilibrium point situated at the vertex of the cone $H = 0$. Refer to Figure 2(b) for visualization.

In the case $h > 0$ the phase portrait of the differential system (5) on the hyperboloid of one sheet $H = h$ is depicted in Figure 3(c).

3.2.4. The case $D = 0$ and $D' > 0$

Then $h < 0$ from Proposition 1, the orbit $\{F = f\} \cap \{H = h\}$ are points. Figure 1(a) illustrates the phase portrait of the differential system (5) on the hyperboloid of two sheets $H = h$.

3.2.5. The case $D = 0$ and $D' < 0$

In the case $h > 0$, from Proposition 1, the invariant set $\{F = f\} \cap \{H = h\}$ on the hyperboloid of one sheet $H = h$ consists of two straight lines intersecting at a point. Consequently, this invariant set is formed by four trajectories and one equilibrium point. Refer to Figure 2(c) for visualization.

3.2.6. The case $D = 0$ and $D' = 0$

When $f = 0$, and according to Proposition 1, the invariant set $\{F = 0\} \cap \{H = h\}$ consists of two parallel or coincident lines. On the cone $H = 0$, this is illustrated in Figure 2(b). For the hyperboloid of one sheet ($H = h$ with $h > 0$), it is shown in Figure 3(c). However, there are no orbits of system (5) on the hyperboloid of two sheets when $h < 0$.

3.3. The case $b_3 = 0$, $a_3 = 0$ and $a_2 \neq 0$

From the invariant plane $F = f$ with $f \in \mathbb{R}$, we obtain

$$z = \frac{f}{a_2}.$$

Substituting into $H = h$ we obtain the conic

$$H - h = x^2 + y^2 - \frac{f^2}{a_2^2} - h = 0,$$

then comparing this conic with the conic (8) we get that

$$a = 1, \quad b = 1, \quad c = -\frac{f^2}{a_2^2} - h, \quad d = 0, \quad e = 0, \quad g = 0.$$

Applying Proposition 1 to this conic we obtain that

$$D = -\frac{a_2^2 h + f^2}{a_2^2} \text{ and } D' = 1.$$

Thus, $D' > 0$.

3.3.1. The case $D \neq 0$

To ensure that D is not equal to zero, it is necessary to consider the following conditions: either h is greater than zero, or in the case where h equals zero, f must not be equal to zero. Additionally, if h is less than zero, it should not be equal to $-f^2/a_2^2$.

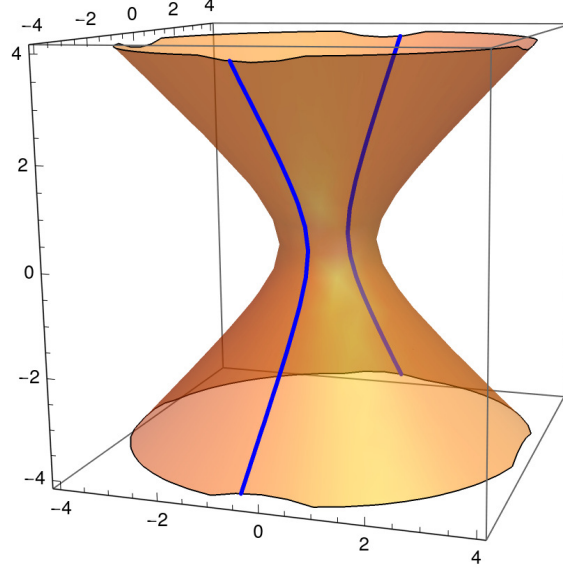


Fig. 4: The phase portrait of system (7) when $a_2 = 0$.

In the case $h < 0$ system (5) has two equilibrium points which are $(0, 0, \pm\sqrt{-h})$. The orbits $\{F = f\} \cap \{H = h\}$ are ellipses. These ellipses are centered around of the equilibrium points, that are centers. Figure 1(a) depicts the corresponding phase portrait of this differential system on the hyperboloid.

In the case $h = 0$, system (5) has the equilibrium $(0, 0, 0)$. The orbits $\{F = f\} \cap \{H = h\}$ are ellipses. These ellipses are centered around of the equilibrium point. Figure 1(b) depicts the corresponding phase portrait of this differential system on the hyperboloid.

In the case $h > 0$ system (5) has no equilibrium points. The intersections of the orbits $\{F = f\} \cap \{H = h\}$ result in elliptical shapes. Figure 1(c) depicts the corresponding phase portrait of this differential system on the hyperboloid.

3.3.2. The case $D = 0$

Then $h = -f^2/a_2^2 < 0$. Under these conditions, the system represented by equation (5) possesses two equilibrium points $(0, 0, \pm\sqrt{-h})$, situated on the surface $H = h$. The orbits $\{F = f\} \cap \{H = h\}$ are points. The phase portrait of this case is topologically equivalent to the one represented in Figure 1(a).

3.4. The case $b_3 = a_3 = a_2 = 0$.

Then the differential system (5) becomes $\dot{x} = \dot{y} = \dot{z} = 0$. So the whole space $\mathbb{R}^3$ is filled with equilibria.

4. Proof of Theorems 2 and 3

4.1. The case of one sheet

Using (2), we force the hyperboloid $x^2 + y^2 - z^2 = 1$ to be an invariant surface of the differential system (4), and we get

$$\begin{aligned} a_0 = a_1 = b_0 = b_2 = c_3 = c_5 = c_6 = c_8 = k_0 = k_1 = k_2 = 0, \quad b_1 = -a_2, \quad c_0 = -c_4, \\ c_1 = a_3, \quad c_2 = b_3, \quad c_7 = c_4, \quad c_9 = -c_4, \quad k_3 = -2c_4, \end{aligned}$$

where these coefficients belong to system (4) and the cofactor $k_0 + k_1x + k_2y + k_3z$ corresponds to the invariant surface $x^2 + y^2 - z^2 = 1$. Substituting these conditions into the differential system (4) yields the differential system (6). Furthermore, we can confirm that $x^2 + y^2 - z^2 = 1$ is an invariant of the system (6). This proves the first part of the theorem 2.

Now let us find the equilibrium point on the hyperboloid of the one sheet $x^2 + y^2 - z^2 - 1 = 0$. For that, we use $z = \pm\sqrt{x^2 + y^2 - 1}$ to reduce the study in $\mathbb{R}^2$.

By substituting z , the system (6) becomes

$$\dot{x} = a_2 y \pm a_3 \sqrt{x^2 + y^2 - 1}, \quad \dot{y} = -a_2 x \pm b_3 \sqrt{x^2 + y^2 - 1}, \quad (9)$$

In what follows without loss of generality we assume that $a_2 \geq 0$, because if $a_2 < 0$ changing the sign of the time a_2 becomes positive.

4.1.1. The case $a_2 > 0$

To find the phase portrait of the system (9), we need a first integral. We change the variables to polar coordinates and solve the differential equation, we get the following first integral

$$H = \pm a_2 \sqrt{x^2 + y^2 - 1} + a_3 y - b_3 x.$$

If $-a_2^2 + a_3^2 + b_3^2 > 0$ the differential system (9) has two equilibrium points

$$\left(\pm \frac{b_3}{\sqrt{-a_2^2 + a_3^2 + b_3^2}}, \mp \frac{a_3}{\sqrt{-a_2^2 + a_3^2 + b_3^2}} \right),$$

and its eigenvalues are $\pm\sqrt{-a_2^2 + a_3^2 + b_3^2}$ and $\mp\sqrt{-a_2^2 + a_3^2 + b_3^2}$, respectively, so they are saddles. Then from the nature of the equilibrium points and the first integral $H = \pm a_2 \sqrt{x^2 + y^2 - 1} + a_3 y - b_3 x$ the phase portrait of system (6) in this case is topologically equivalent to one given in Figure 2(c).

4.1.2. The case $a_2 = 0$

System (9) becomes

$$\dot{x} = \pm a_3 \sqrt{x^2 + y^2 - 1}, \quad \dot{y} = \pm b_3 \sqrt{x^2 + y^2 - 1}.$$

This differential system with a rescaling of the time becomes

$$\dot{x} = a_3, \quad \dot{y} = b_3$$

the orbits of this system are all the straight lines with director vector (a_3, b_3) . Consequently removing to this phase portrait the open disc $\{(x, y) : x^2 + y^2 < 1\}$, we have the phase portrait of the differential system which is topologically equivalent to that given in Figure 4. Here, the proof of Theorem 2 has been finalized.

4.2. The case of two sheets

Using (2), we force the hyperboloid $x^2 + y^2 - z^2 = -1$ to be an invariant surface of the differential system (4), and we get

$$a_0 = a_1 = b_0 = b_2 = c_3 = c_5 = c_6 = c_8 = k_0 = k_1 = k_2 = 0, \quad b_1 = -a_2, \quad c_0 = c_4, \\ c_1 = a_3, \quad c_2 = b_3, \quad c_7 = c_4, \quad c_9 = -c_4, \quad k_3 = -2c_4.$$

Substituting these conditions into the differential system (4) yields the differential system (7). Furthermore, we can confirm that $x^2 + y^2 - z^2 = -1$ is an invariant of the system (7). This proves the first part of the theorem 3.

Now let us find the equilibrium point on the hyperboloid of the one sheet $x^2 + y^2 - z^2 + 1 = 0$. For that, we use $z = \pm\sqrt{x^2 + y^2 + 1}$ to reduce the study in $\mathbb{R}^2$.

By substituting z , the system (7) becomes

$$\dot{x} = a_2 y \pm a_3 \sqrt{x^2 + y^2 + 1}, \quad \dot{y} = -a_2 x \pm b_3 \sqrt{x^2 + y^2 + 1}, \quad (10)$$

In what follows without loss of generality we assume that $a_2 \geq 0$, because if $a_2 < 0$ changing the sign of the time a_2 becomes positive.

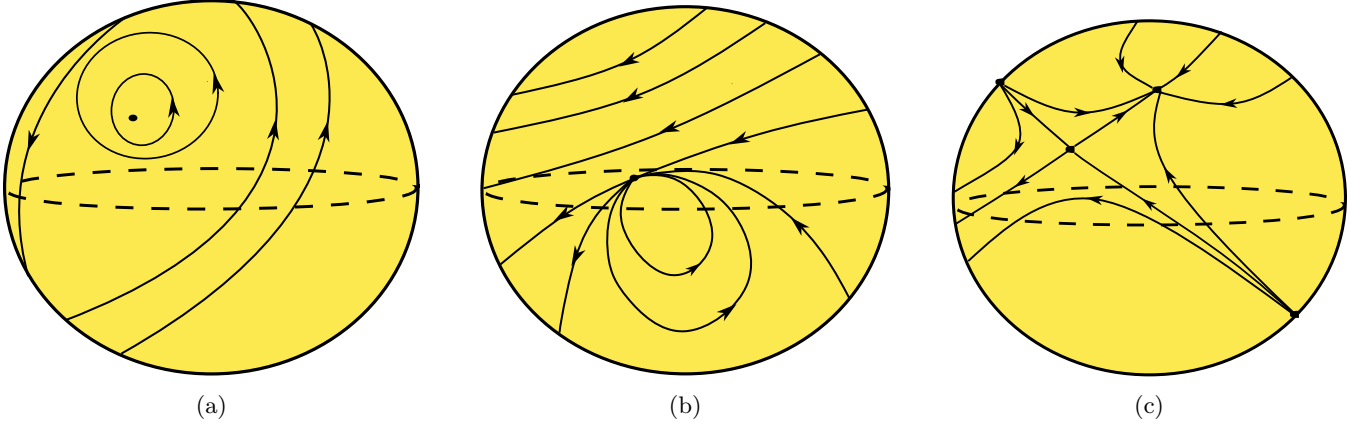


Fig. 5: The phase portrait on the infinite sphere $\mathbb{S}^2$ of the linear differential system (5).

4.2.1. The case $a_2 > 0$

To find the phase portrait of the system (10), we need a first integral. We change the variables to polar coordinates and solve the differential equation, we get the following first integral

$$H = \pm a_2 \sqrt{x^2 + y^2 + 1} + a_3 y - b_3 x.$$

If $a_2^2 - a_3^2 - b_3^2 > 0$, the differential system described in equation (10) possesses two equilibrium points given by

$$\left(\pm \frac{b_3}{\sqrt{a_2^2 - a_3^2 - b_3^2}}, \mp \frac{a_3}{\sqrt{a_2^2 - a_3^2 - b_3^2}} \right).$$

Additionally, its eigenvalues are $\mp i\sqrt{a_2^2 - a_3^2 - b_3^2}$ and $\pm i\sqrt{a_2^2 + a_3^2 + b_3^2}$, respectively, indicating that these equilibrium points are centers. Given the characteristics of the equilibrium points and the first integral $H = \pm a_2 \sqrt{x^2 + y^2 + 1} + a_3 y - b_3 x$, the phase portrait of system (7) in this scenario is topologically equivalent to that depicted in Figure 1(a).

4.2.2. The case $a_2 = 0$

System (10) becomes

$$\dot{x} = \pm a_3 \sqrt{x^2 + y^2 + 1}, \quad \dot{y} = \pm b_3 \sqrt{x^2 + y^2 + 1},$$

this differential system with a rescaling of the time becomes

$$\dot{x} = a_3, \quad \dot{y} = b_3$$

the orbits of this system are all the straight lines with director vector (a_3, b_3) . Consequently removing to this phase portrait the open disc $\{(x, y) : x^2 + y^2 < 1\}$, we have the phase portrait of the differential system which is topologically equivalent to that given in Figure 4. With that, the proof of Theorem 3 is concluded.

5. Proofs of Theorem 4 and 5

Proof. [Proof of Theorem 4] From subsection 2.2, it follows that the linear differential system (5) in the local chart U_1 can be written as

$$\begin{aligned} \dot{z}_1 &= -a_2 - a_2 z_1^2 + b_3 z_2 - a_3 z_1 z_2, \\ \dot{z}_2 &= a_3 + b_3 z_1 - a_2 z_1 z_2 - a_3 z_2^2, \\ \dot{z}_3 &= -z_3(a_2 z_1 + a_3 z_2). \end{aligned} \tag{11}$$

In the local chart U_2 , the linear differential system (5) takes the form

$$\begin{aligned}\dot{z}_1 &= a_2 + a_2 z_1^2 + a_3 z_2 - b_3 z_1 z_2, \\ \dot{z}_2 &= b_3 + a_3 z_1 + a_2 z_1 z_2 - b_3 z_2^2, \\ \dot{z}_3 &= z_3(a_2 z_1 - b_3 z_2).\end{aligned}\tag{12}$$

In the local chart U_3 , the linear differential system (5) takes the form

$$\begin{aligned}\dot{z}_1 &= a_3 - a_3 z_1^2 + a_2 z_2 - b_3 z_1 z_2, \\ \dot{z}_2 &= b_3 - a_2 z_1 - a_3 z_1 z_2 - b_3 z_2^2, \\ \dot{z}_3 &= -z_3(a_3 z_1 + b_3 z_2).\end{aligned}\tag{13}$$

We divide the proof of this theorem into six cases.

Case 1: $b_3 = 0$, $a_3^2 - a_2^2 < 0$, and $a_3 \neq 0$. From (11), there are no infinite equilibria in the local chart U_1 . From (12), there is an infinite equilibrium point at $(0, -a_2/a_3, 0)$ in the local chart U_2 , and according to [14], this equilibrium on the infinite sphere $\mathbb{S}^2$ is a center. From (13), the origin of the local chart U_3 is not an infinite equilibrium. Therefore, the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(a).

Case 2: $b_3 = 0$, $a_3^2 - a_2^2 < 0$, and $a_3 = 0$. From (11) and (12), there are no infinite equilibria in the local charts U_1 and U_2 . The differential system (13) restricted to $z_3 = 0$ becomes $\dot{z}_1 = z_2$ and $\dot{z}_2 = -z_1$. Therefore, the origin $(0, 0, 0)$ of the local chart U_3 is a center on the infinite sphere $\mathbb{S}^2$. Hence, the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(a).

Case 3: $b_3(a_3^2 - a_2^2) \neq 0$ and $a_3^2 + b_3^2 - a_2^2 < 0$. From (11), there is an infinite equilibrium point at $(-a_2/b_3, a_2/b_3, 0)$ in the local chart U_1 , and according to [14], this equilibrium on the infinite sphere $\mathbb{S}^2$ is a center. From (12), there are no additional infinite equilibria in the local chart U_2 that are not already contained in U_1 . From (13), the origin of the local chart U_3 is not an infinite equilibrium. Therefore, the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(a).

Case 4: $a_2 = a_3 = 0$ and $b_3 \neq 0$. From (11), there is an infinite equilibrium point at $(0, 0, 0)$ in the local chart U_1 , and it is hyperbolic. Therefore, by the Hartman–Grobman Theorem (see, for instance, [4]), it is a node on the infinite sphere $\mathbb{S}^2$, stable if $b_3 < 0$ and unstable if $b_3 > 0$. From (12), there are two infinite hyperbolic equilibria in the local chart U_2 , namely $(0, \pm 1, 0)$. The eigenvalues of the Jacobian matrix at the equilibrium $(0, 1, 0)$, restricted to $\mathbb{S}^2$, are $-b_3$ and $-2b_3$, while at the equilibrium $(0, -1, 0)$ they are b_3 and $2b_3$. Therefore, by the Hartman–Grobman Theorem, one of these equilibria is a stable node and the other is an unstable node. From (13), the origin of the local chart U_3 is not an infinite equilibrium. Evaluating the first integral $H = x^2 + y^2 - z^2$ on the infinite equilibria, it follows that the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(b).

Case 5: $a_2 = b_3 = 0$ and $a_3 \neq 0$. In this case, the proof that the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(b) is essentially the same as the proof in Case 4.

Case 6: $a_3 = b_3 = 0$ and $a_2 \neq 0$. Again, in this case, the proof that the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(b) is essentially the same as the proof in Case 4.

Case 7: $b_3 \neq 0$, $a_3^2 - a_2^2 = 0$, and $a_2 \neq 0$. From (11), there are two infinite hyperbolic equilibria in the local chart U_1 , namely

$$p_1 = \left(-\frac{a_2}{b_3}, \frac{a_2}{b_3}, 0\right) \quad \text{and} \quad p_2 = \left(-\frac{a_2}{2b_3} + \frac{b_3}{2a_2}, \frac{a_2^2 + b_3^2}{2a_2 b_3}, 0\right) \quad \text{if } a_3 = a_2,$$

and

$$q_1 = \left(\frac{a_2}{b_3}, \frac{a_2}{b_3}, 0\right) \quad \text{and} \quad q_2 = \left(\frac{a_2}{2b_3} - \frac{b_3}{2a_2}, \frac{a_2^2 + b_3^2}{2a_2 b_3}, 0\right) \quad \text{if } a_3 = -a_2.$$

By the Hartman-Grobman Theorem, the equilibria p_1 and q_1 are saddles, the equilibrium p_2 is a stable node, and q_2 is an unstable node. From (12), in the local chart U_2 , there is one infinite hyperbolic equilibrium

$$p_3 = (0, -1, 0) \quad \text{if } a_3 = a_2, \quad \text{and} \quad q_3 = (0, 1, 0) \quad \text{if } a_3 = -a_2.$$

Again, by the Hartman-Grobman Theorem, p_3 is an unstable node and q_3 is a stable node. From (13), the origin of the local chart U_3 is not an infinite equilibrium. Evaluating the first integral $H = x^2 + y^2 - z^2$ on the infinite equilibria, it follows that the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to that of Figure 5(c).

Case 8: $b_3(a_3^2 - a_2^2) \neq 0$ and $a_3^2 + b_3^2 - a_2^2 > 0$. From (11) there are three infinite hyperbolic equilibria in the local chart U_1 , namely

$$p_1 = \left(-\frac{a_3}{b_3}, \frac{a_2}{b_3}, 0 \right), \quad p_{\pm} = \left(\frac{-a_3 b_3 \mp \sqrt{a_2^2(a_3^2 + b_3^2 - a_2^2)}}{a_2^2 - a_3^2}, \frac{a_2^2 b_3 \pm a_3 \sqrt{a_2^2(a_3^2 + b_3^2 - a_2^2)}}{a_3^2 - a_2 a_3^2}, 0 \right).$$

By the Hartman-Grobman Theorem, the equilibrium p_1 is a saddle, and the equilibria $p_{\pm}$ are nodes, one stable and the other unstable. From (12) and (13), there are no more infinite equilibria. Evaluating the first integral $H = x^2 + y^2 - z^2$ on the infinite equilibria, it follows that the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to the one of Figure 5(c).

Case 9: $b_3 = 0$ and $a_3^2 - a_2^2 > 0$. From (11) there are two infinite hyperbolic equilibria in the local chart U_1 , namely

$$p_{\pm} = \left(\frac{\pm a_2}{\sqrt{a_3^2 - a_2^2}}, \frac{\mp a_3}{\sqrt{a_3^2 - a_2^2}}, 0 \right).$$

By the Hartman-Grobman Theorem, these two equilibria are nodes, one stable and the other unstable. From (12) there is the additional infinite hyperbolic equilibrium $(0, -a_2/a_3, 0)$, which, by the Hartman-Grobman Theorem, is a saddle. From (13) the origin of the local chart U_3 is not an equilibrium. Evaluating the first integral $H = x^2 + y^2 - z^2$ on the infinite equilibria, it follows that the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to the one of Figure 5(c).

Case 10: $b_3 = 0$, $a_3^2 - a_2^2 = 0$ and $a_2 \neq 0$. From (11) and (13) there are no infinite equilibria in the local chart U_1 , and the origin of the local chart U_3 is not an infinite equilibrium. However, from (12), the equilibrium $(0, -a_2/a_3, 0)$ is non-hyperbolic, and by Theorem 3.5 of [8], it is formed by an elliptic and a hyperbolic sector, separated by two parabolic sectors. Then, using the first integral $H = x^2 + y^2 - z^2$, the two separatrices of the hyperbolic sector connect, and hence the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to the one in Figure 5(c).

Case 11: $b_3(a_3^2 - a_2^2) \neq 0$ and $a_3^2 + b_3^2 - a_2^2 = 0$. From (12) and (13), there are no infinite equilibria in the local chart U_2 that do not appear in the local chart U_1 , and the origin of the local chart U_3 is not an infinite equilibrium. From (11) in the local chart U_1 , there is the equilibrium

$$\left(-\frac{a_3}{b_3}, \frac{\sqrt{a_3^2 + b_3^2}}{b_3}, 0 \right)$$

if $a_2 = \sqrt{a_3^2 + b_3^2}$, or the equilibrium

$$\left(-\frac{a_3}{b_3}, -\frac{\sqrt{a_3^2 + b_3^2}}{b_3}, 0 \right)$$

if $a_2 = -\sqrt{a_3^2 + b_3^2}$. This equilibrium is non-hyperbolic and, by Theorem 3.5 of [8], it is formed by an elliptic and a hyperbolic sector, separated by two parabolic sectors. Then, using the arguments from Case 10, the phase portrait on the infinite sphere $\mathbb{S}^2$ is topologically equivalent to the one in Figure 5(c).

In short, Theorem 4 is proved. $\blacksquare$

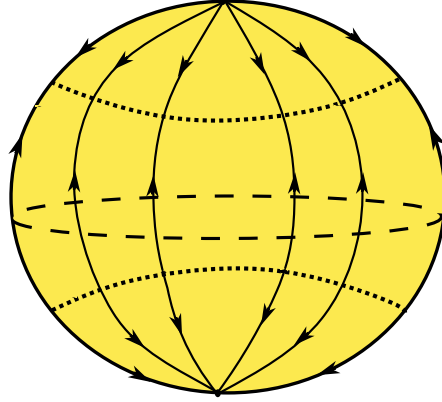


Fig. 6: The phase portrait on the infinite sphere $\mathbb{S}^2$ of the quadratic differential system (6) and (7).

Proof. [Proof of Theorem 5] We present the proof only for the quadratic differential system (6). The proof for the system (7) is similar.

According to Subsection 2.2, the quadratic differential system (6) takes the following form in the local chart U_1

$$\begin{aligned} \dot{z}_1 &= -z_3(a_2 - b_3z_2 + a_2z_1^2 + a_3z_1z_2), \\ \dot{z}_2 &= c_4 + a_3z_3 + c_4z_1^2 + b_3z_1z_3 - c_4z_2^2 - c_4z_3^2 - a_2z_1z_2z_3 - a_3z_2^2z_3, \\ \dot{z}_3 &= -z_3^2(a_2z_1 + a_3z_2). \end{aligned} \quad (14)$$

Similarly, in the local chart U_2 , the quadratic differential system (6) is written as

$$\begin{aligned} \dot{z}_1 &= z_3(a_2 + a_3z_2 + a_2z_1^2 - b_3z_1z_2), \\ \dot{z}_2 &= c_4 + b_3z_3 + c_4z_1^2 + a_3z_1z_3 - c_4z_2^2 - c_4z_3^2 + a_2z_1z_2z_3 - b_3z_2^2z_3, \\ \dot{z}_3 &= z_3^2(a_2z_1 - b_3z_2). \end{aligned} \quad (15)$$

Finally, in the local chart U_3 , the quadratic differential system (6) takes the form

$$\begin{aligned} \dot{z}_1 &= c_4z_1 + a_3z_3 + a_2z_2z_3 - c_4z_1^3 - a_3z_1^2z_3 - c_4z_1z_2^2 - b_3z_1z_2z_3 + c_4z_1z_3^2, \\ \dot{z}_2 &= c_4z_2 + b_3z_3 - a_2z_1z_3 - c_4z_2^3 - c_4z_1^2z_2 - a_3z_1z_2z_3 - b_3z_2^2z_3 + c_4z_2z_3^2, \\ \dot{z}_3 &= z_3(c_4 - c_4z_1^2 - a_3z_1z_3 - c_4z_2^2 - b_3z_2z_3 + c_4z_3^2). \end{aligned} \quad (16)$$

From system (14) in the local chart U_1 , there are two straight lines of equilibrium points given by $(z_1, \pm\sqrt{1+z_1^2}, 0)$, where $z_1 \in \mathbb{R}$. The eigenvalues of the Jacobian matrix of the differential system (6), restricted to $\mathbb{S}^2$ at these equilibria, are 0 and $\mp 2c_4\sqrt{1+z_1^2}$. Therefore, these two lines of equilibrium points form normally hyperbolic manifolds. According to Theorem 6, two trajectories arrive at or depart from each equilibrium point on these lines.

The system (15) in the local chart U_2 admits two additional equilibria at $(0, \pm 1, 0)$, which do not appear in the local chart U_1 . These two equilibria, together with those found in the local charts U_1 , V_1 , and V_2 , form two circles of equilibria on the sphere at infinity $\mathbb{S}^2$. At each of these equilibria, two orbits either arrive or depart.

It follows from (16) that the origin $(0, 0, 0)$ in the local chart U_3 is a hyperbolic node, having eigenvalue c_4 with multiplicity two. By the Hartman–Grobman theorem, this node is stable if $c_4 < 0$ and unstable if $c_4 > 0$.

Considering all the information on the equilibria of the differential system (6) on the sphere at infinity $\mathbb{S}^2$, it follows that the phase portrait on $\mathbb{S}^2$ is topologically equivalent to that shown in Figure 6. This completes the proof of Theorem 5. ■

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