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THE EFFECT OF A SINGULARITY ON TRANSITION MAPS

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Abstract. Consider a planar autonomous differential equation with a unique degenerated singularity inside a flow box with two transversal sections in such a way that a Poincaré map between them is well defined by the flow. We are interested in understanding what effect this singularity has on the properties of the Poincaré map between the transversal sections. The fiber containing the singularity is usually called singular fiber and, in particular, we want to determine its attractive or repulsive character. We prove that there is a real value whose sign determines the stability of the singular fiber. This value is given by the principal value of a suitable integral constructed from the corresponding differential equation.

1. Introduction and main results.

We are interested in understanding what the effect of having a unique degenerate singularity inside a flow box is on the properties of a Poincaré map, when this map is defined between two transversal sections to the flow, say Σ_α and Σ_ω. See Figure 1. This Poincaré map is also called *transition map*.

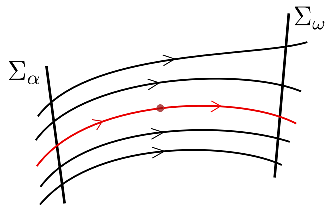

Figure 1. Transition map with a singular fiber containing the unique singularity.

Notice that usual flow boxes have no singularities inside and the transition map Π_α^ω between each two transversal sections is as smooth as the vector field itself. However, our setting involves a degenerate singularity that is the unique singular point on a fiber, which lie between the two transverse sections to the flow. The complement of this singularity consists of its two separatrices. This special fiber is called *singular fiber*. Clearly, this type of singularity allows that the associated return map can be extended continuously to the point where the singular fiber cuts

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with Σ_α, but the regularity of this map Π_α^ω at this cut is not clear at all. Related problems are addressed in the work [8].

In particular, we are interested in recognizing this kind of singularity and in knowing the attractive or repulsive character of its associated singular fibers. These problems often appear when studying the stability of certain planar polycycles and more in particular when people are concerned with so called center-focus problem for degenerated singularities, see for instance [9, 10] and their references. Indeed, the way of going from these two problems to our situation is via the blow-up procedure, that allows to study the flow near the singularities and the polycycles appearing in the corresponding differential equations.

In short, we are involved in the Poincaré map flow study, near a singular fiber, for a family of planar differential systems having a unique degenerate isolated singularity.

The objective of this paper is twofold. Firstly, we seek to characterise the existence of the transition map, defined on open neighbourhoods of isolated degenerate singularities, for orbits close to the singular fiber. In the event of such a transition existing, we designate the singular point as a *monodromic point* by similitude of what happens in the center-focus problem. Secondly, once the existence of the monodromy has been established, the objective is to ascertain the attractive or repulsive character of that singular fiber, for orbits defined near it. A value which sign determines this stability will be given. As we will see this value can be obtained via the computation of the principal value of a suitable integral. Following the previous analogy involving monodromic points, we will name this value as the *first Lyapunov constant* of the singular fiber. As we will see, this constant also depends on the transversal sections.

We note that additional degenerate singularities may exist on the singular fiber. In more complex scenarios—such as the monodromic graphics problem—the attractive or repulsive character of the singular fiber would depend on the *weights* of these singularities. However, since our analysis is restricted to the case of a unique singularity on the singular fiber, this issue does not arise in the present work.

We will use systematically big O and small o Landau notations for real valued functions. For completeness we briefly recall this notation. It is said that $f(x) = \mathrm{O}(g(x))$ at some $x = x_0 \in \mathbb{R}^n$, if

$$\limsup_{x\to x_0} \frac{|f(x)|}{|g(x)|} < \infty,$$

and, unless other value is explicitly given, we will consider $x_0 = 0$. Moreover, when in the above limit $g(x) = |x|^k$, then for shortness we write $f(x) = \mathrm{O}(g(x)) = \mathrm{O}(|x|^k) = \mathrm{O}_k(x)$. Notice that for $n = 2$ and functions expressed in polar coordinates, $f(r, \theta)$, when $g(r, \theta) = r^k$, we simply write $f(r, \theta) = \mathrm{O}(r^k) = \mathrm{O}_k(r)$. Using this notation, the roles of the variables r and θ are interchangeable. We also remark that, in this paper, in most cases the above limit exists. Similarly, the small o notation is used when the above limit is 0.

To establish the differential equations frame of this work, firstly we set $(0, 0)$ as the degenerate isolated singularity on a non-compact singular fiber, and having a well defined transition map between two transversal sections as in Figure 1. Secondly, under these assumptions and under some natural regularity hypotheses, we prove in Lemma 2.1, the rectifiability of its associated singular fiber and that, without loss of generality, generically any smooth enough differential equation in any flow box having a single singularity inside can be written as system

$$\begin{cases} \dot{x} = x^2 + a_{1,1}xy + y^2 + O_3(x,y), \\ \dot{y} = y\big(h_{1,0}x + h_{0,1}y + O_2(x,y)\big), \end{cases} \tag{1}$$

where $a_{1,1}, h_{1,0}, h_{0,1} \in \mathbb{R}$. Consequently, the study of the stability of the singular fiber can be conducted in the strip $y \gtrsim 0$, for the straight line $y = 0$. For notation purposes, throughout this work, we use $y \gtrsim 0$ to point out that y is, at the same time, greater than zero and as close to zero as needed. To fix ideas, given system (1), we restrict our study to its flow defined between two transversal sections, Σ_α and Σ_ω, to its flow through $x = \alpha$ and $x = \omega$ respectively, for $\alpha < 0$, $\omega > 0$ and $y \gtrsim 0$.

For the sake of simplicity, in this work we will restrict our study to a particular case that we believe that already contains the main difficulties of the questions that we address, and that allows to show the difficulties of the problem. More specifically, in this paper we will study the above mentioned questions for next two families of planar differential equations:

$$\dot{x} = f(x) + y^2, \quad \dot{y} = y\, h(x,y), \tag{2}$$

and on its subfamily

$$\dot{x} = f(x) + y^2, \quad \dot{y} = y\, g(x), \tag{3}$$

in the strip $y \gtrsim 0$, where f, g and h are smooth enough functions, $f(0) = g(0) = 0$, $f(x) > 0$, for all $x \in (\alpha,\omega)\backslash\{0\}$, for given $\alpha < 0$, $\omega > 0$, and where $f(x) = x^2 + O_3(x)$ for $x \approx 0$. An analogous study also could be performed in the strip $\{y \lesssim 0\}$. Let

$$\Pi_\alpha^\omega : \Sigma_\alpha \to \Sigma_\omega$$

be the parametrized Poincaré, or transition map, between Σ_α and Σ_ω. See Figure 2.

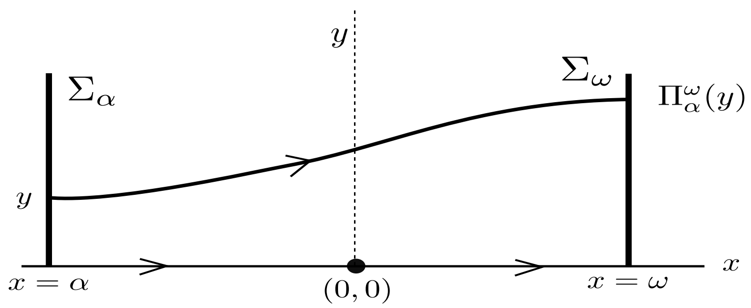

FIGURE 2. Poincaré map $\Pi_\alpha^\omega(y)$ near the singular fiber $y = 0$ of system (2), for $y \gtrsim 0$ and $\alpha < 0$, $\omega > 0$.

Our main goal is to study two questions for system (2):

- Determine when the transition map Π_α^ω is well defined, for $\alpha < 0$ and $\omega > 0$.
- Determine the stability of the singular fiber $y = 0$, for $y \gtrsim 0$, by means of Π_α^ω.

We note that the sign of the displacement map

$$\Delta_\alpha^\omega(y) = \Pi_\alpha^\omega(y) - y, \tag{4}$$

gives the attractive or repulsive character of this fiber in the sense that if $\Delta_\alpha^\omega < 0$ (resp. $\Delta_\alpha^\omega > 0$), then we say that $y = 0$ is a stable (resp. unstable) fiber.

Next Theorem A and its straight consequence, Corollary 1.1, solve the first question for systems (2) and (3), respectively.

Theorem A. *Consider system (2), where $h(x,y) = h_{0,0} + h_{1,0}x + h_{0,1}y + \mathrm{O}_2(x,y)$, is a $\mathcal{C}^2$ differentiable function. Then, the transition map Π_α^ω is well defined for all $\alpha < 0$ and $\omega > 0$, if and only if, $h_{0,0} = 0$ and, either $\delta = (h_{0,1})^2 + 4(h_{1,0} - 1) < 0$, or $h_{0,1} = 0$ and $h_{1,0} = 1$.*

The proof of the above result relies on obtaining the necessary and sufficient conditions for having a local transition map close to the origin $(0,0)$. As we will see, this question is addressed via a suitable blow up of the singularity. See for instance [1, 3] for more information about the blow up technique.

A straightforward consequence of former theorem is next application to system (3).

Corollary 1.1. *Consider system (3), where g is a $\mathcal{C}^2$ differentiable function. Then, the transition map Π_α^ω is well defined for all $\alpha < 0 < \omega$, if and only if, $g(0) = 0$ and $g'(0) \le 1$.*

The answer to our second question is given in next result. It shows when the fiber $y = 0$ is attractive or repulsive for system (3). As usual, whenever it exists, in its statement we use the conventional definition of Cauchy principal value

$$\mathrm{PV}\int_\alpha^\omega p(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\left(\int_\alpha^{-\varepsilon} p(x)\,\mathrm{d}x + \int_\varepsilon^\omega p(x)\,\mathrm{d}x\right),$$

where p is a smooth function in $(\alpha,\omega)\setminus\{0\}$.

Theorem B. *Let us consider system (3) in the case where f and g are $\mathcal{C}^4$ and $\mathcal{C}^3$, respectively, differentiable functions, with $g(0) = 0$ and $g'(0) \le 1$. The transition map for the flow defined between the transversal sections on $x = \alpha < 0$ and $x = \omega > 0$, in $y \gtrsim 0$, is*

$$\Pi_\alpha^\omega(y) = \mathrm{e}^c\, y + \mathrm{o}_1(y), \quad \textit{where} \quad c = \mathrm{PV}\int_\alpha^\omega \frac{g(x)}{f(x)}\,\mathrm{d}x. \tag{5}$$

Note that, for the flow defined between the transversal sections on $x = \alpha < 0$ and $x = \omega > 0$, in $y \gtrsim 0$, the fiber $y = 0$ is attractive (resp. repulsive) if $c < 0$ (resp. $c > 0$). Even though case $c = 0$ is not included in the previous theorem, Proposition 5.1 provides a geometric proof that covers this situation, in the case of a symmetric domain $(-\omega,\omega)$, $\omega > 0$, for the transition map of system (3).

Notice that if in system (3) we would take a positive function f then we would have an actual flow box (that is without singularities) with a well defined smooth transition map. In this case, it is well known that

$$\Pi_\alpha^\omega(y) = \mathrm{e}^c\, y + \mathrm{O}_2(y), \quad \text{where} \quad c = \int_\alpha^\omega \frac{g(x)}{f(x)}\,\mathrm{d}x.$$

For completeness, in Lemma 2.3 we give a proof of this fact. Note the similarities and differences between both formulas.

In Section 2 we prove Theorem A and Corollary 1.1. In Section 3 some technical results are developed, while Section 4 is devoted to the proof of Theorem B. In Section 5 we present results studying the stability of $y = 0$ for some families of type (3), plus a study of the number of fixed points of the transition map, $\Pi_{-\omega}^\omega$ for $\omega > 0$, for a particular family in the frame of system (2). These results can be seen either as some kind of Andronov-Hopf bifurcation from the singular fiber or also as the application of Melnikov integrals in this setting.

2. **On the transition map for system (2).** In this section, first, we prove Lemma 2.1 allowing to reduce the expression of the planar differential equations framed in this paper to the differential systems as given by system (1). After that, we also determine when the transition map is well defined near the singular fiber, both without and with a critical point between the two cross sections. This is Theorem A and its corollary.

The first reduction concerns the rectification of separatrices. Concerning the genericity of the differential system, we assume that its second order terms in Taylor's expansion does not vanish identically.

Lemma 2.1. *Consider a planar differential system of class $\mathcal{C}^3$ having $(0,0)$ as an isolated singularity on a singular fiber and such that, in a neighbourhood of this fiber, has a well-defined transition map between two cross sections through it, as in Figure 1. Then, generically, and after a smooth change of coordinates the differential system can be written as system (1).*

Proof. Using a simple adaptation of Lemma 2.5 of [6], we can prove that the singular fiber between these two transversal sections is rectifiable, in the sense that there exists a diffeomorphism transforming an open segment on this $\mathcal{C}^1$-curve containing the points corresponding to these two cross sections, into a piece of the straight line $y = 0$. Hence, without loss of generality, we can restrict our study to systems written as

$$\dot{x} = f(x, y), \quad \dot{y} = y\, h(x, y), \tag{6}$$

where f, h are $\mathcal{C}^3$ smooth functions on their variables. As a consequence, a transition map can be defined for all $x \in (\alpha, \omega)\backslash\{0\}$ and $y \gtrsim 0$, for a given $\alpha < 0$, $\omega > 0$, except for the origin itself, i.e. having the singular fiber $y = 0$ through the origin.

To fix ideas, we will take $f(x, y) > 0$, for all (x, y) in a neighbourhood of $(0, 0)$, except for the origin itself, and for $y \gtrsim 0$. From the above requirements, and under the generic assumption $a_{2,0}a_{0,2} \neq 0$, we have that

$$f(x, y) = a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + \mathrm{O}_3(x, y),$$
$$h(x, y) = h_{1,0}x + h_{0,1}y + \mathrm{O}_2(x, y),$$

where $a_{2,0}, a_{0,2} > 0$. Finally, by rescaling the (x, y)-variables and renaming the coefficients, the differential system can be written as system (1). □

Concerning transition maps, it is a well-established fact that if these maps are far from equilibrium points, then they are regular maps (as smooth as the differential equations are). Conversely, transition maps close to equilibrium points are not regular in general and, instead, exhibit characteristics of semiregular maps. For further insight, see [12]. Hence, to determine when a transition map is well defined, first let us recall the Poincaré or transition map definition, between two transversal sections to the flow given by a differential system.

Definition 2.2. Consider $a < b$. For each y such that $(a, y) \in \Sigma_a$, let $\Pi_a^b(y)$ be the y-component of the solution of system (2) passing through (a, y) evaluated on Σ_b. If this function is well defined, then we call Π_a^b the transition map between the two transversal sections Σ_a and Σ_b. See Figure 2.

Next technical lemma gives the first term approximation of regular maps development, for system (2).

Lemma 2.3. *Consider a differential system (2) of class $\mathcal{C}^2$ and let $a < b$ such that $0 \notin [a, b]$. Then, the transition map Π_a^b for the flow associated to system (2), between Σ_a and Σ_b, for each $y \gtrsim 0$, is well defined. Furthermore, Π_a^b is a regular transition map that can be written as*

$$\Pi_a^b(y) = C(a, b)y + O_2(y), \tag{7}$$

for $y \gtrsim 0$, where

$$C(a, b) = \exp\left(\int_a^b \frac{h(x, 0)}{f(x)}\,\mathrm{d}x\right).$$

Proof. Since for all $x \in (a, b)$ there exists $k > 0$ such that $k < \dot{x} = f(x) + y^2$, we have that $(y\,h(x, y))/(f(x) + y^2)$, for system (2), is a $\mathcal{C}^1$ function defined on the domain $[a, b] \times [0, y]$, for each $y > 0$. Hence, by using the existence and uniqueness theorem of solutions for the Cauchy problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\,h(x, y)}{f(x) + y^2}, \quad y(a) = \eta,$$

with $0 \lesssim \eta < y$, we have that there exists $y(b)$. See [5, Th. 3.3], for instance. Consequently, the transition map between Σ_a and Σ_b, for each $y \gtrsim 0$, is well defined and differentiable.

Finally, expression (7) follows by using the first order variational equations for the above differential equation. □

Next we prove Theorem A. With respect to its proof, it is remarkable the fact that, as we will see, starting from the local existence of the transition map near the singularity, the existence of the global transition map is proven. To this end, we introduce the local transition map $\Pi_{-\varepsilon}^{\varepsilon}$, $\varepsilon \gtrsim 0$, close to the origin $(0, 0)$, as in Figure 3. Then,

$$\Pi_\alpha^\omega = \Pi_\varepsilon^\omega \circ \Pi_{-\varepsilon}^{\varepsilon} \circ \Pi_\alpha^{-\varepsilon}. \tag{8}$$

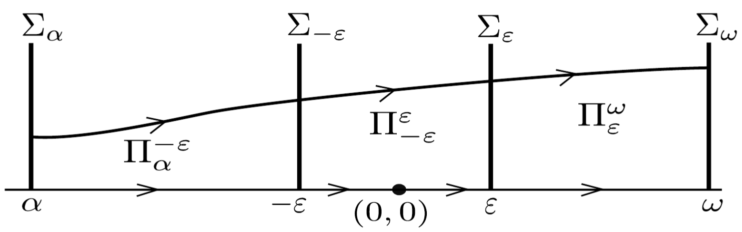

FIGURE 3. Effect of the singular fiber $y = 0$ of system (2), using the intermediate transition map composition, $\Pi_\alpha^\omega = \Pi_\varepsilon^\omega \circ \Pi_{-\varepsilon}^{\varepsilon} \circ \Pi_\alpha^{-\varepsilon}$, $\varepsilon \gtrsim 0$, and $\alpha < 0$, $\omega > 0$.

Proof of Theorem A. Recall that $h(x, y) = h_{0,0} + h_{1,0}x + h_{0,1}y + O_2(x, y)$ and $\delta = (h_{0,1})^2 + 4(h_{1,0} - 1)$. In the case $h_{0,0} \neq 0$, the origin of system (2) is locally a saddle-node equilibrium and, hence, the transition map is not defined close to $(0, 0)$. So, let us assume that $h_{0,0} = 0$.

Since the origin of the system (2) is a degenerate critical point, to study the behaviour of the flow around it, we apply the directional blow up $(x, y) \to (\bar{x}, \bar{y}) = (x, y/x)$. Let $\bar{X}(\bar{x}, \bar{y})$ be the corresponding vector field after the change of variables plus a rescaling of the time variable. We note that $(0,0)$ is always an equilibrium point of $\bar{X}$, and that on $\bar{x}=0$, there also can be other equilibrium points, $(0,\bar{y})$, where $\bar{y}$ is a solution of the equation

$$\bar{y}^2 - h_{0,1}\bar{y} + 1 - h_{1,0} = 0.$$

The study of the blow up at the singularities, in the $(\bar{x},\bar{y})$-plane, gives the differential matrix

$$D\bar{X}(0,\bar{y}) = \begin{pmatrix} 1+\bar{y}^2 & 0 \\ * & k(\bar{y}) \end{pmatrix}$$

where $k(\bar{y}) = -3\bar{y}^2 + 2h_{0,1}\bar{y} + h_{1,0} - 1$.

We note that if one of these singularities is either a node or a saddle-node, then the transition map is not defined close to $(0,0)$. Taking into account this fact, let us review the singularities.

If $h_{1,0} > 1$, the equilibrium point $(0,0)$ is a node and hence this case is ruled out. In the case $h_{1,0} = 1$ and $h_{0,1} \neq 0$, the equilibrium point $(0,0)$ is a saddle node (see [2] for more details) and hence this case is ruled out too.

When $h_{1,0} < 1$, or when $h_{1,0} = 1$ and $h_{0,1} = 0$, the origin is a saddle equilibrium (hyperbolic or elementary degenerate, respectively) whose separatrices coincide with the coordinate axes in the $(\bar{x},\bar{y})$-plane.

Concerning the rest of the singularities, observe that if $\delta < 0$ the only equilibrium point is $(0,0)$ which, as we have seen before, is a well placed saddle. If $\delta = 0$ and $h_{0,1} \neq 0$, then the equilibrium point different from the origin is a saddle node. Hence, this case is also ruled out. If $\delta = 0$ and $h_{0,1} = 0$, then $h_{1,0} = 1$ and the $(0,0)$ is the unique equilibrium point which is a well placed saddle. In the case $\delta > 0$ and $h_{1,0} < 1$, one of the two critical points different from the origin is a node and again this case is ruled out.

From the local study of the equilibrium point at the origin, assuming that $h_{0,0} = 0$ and either

$$\delta = h_{0,1}^2 + 4h_{1,0} - 4 < 0, \quad \text{or} \quad h_{0,1} = 0 \quad \text{and} \quad h_{1,0} = 1,$$

we conclude that the transition map $\Pi_{-\varepsilon}^{\varepsilon}$, for system (2), is well defined for all $\varepsilon \gtrsim 0$. Let us prove that $\Pi_{-\varepsilon}^{\varepsilon}$ extends to Π_{α}^{ω}, for all $\alpha < -\varepsilon \lesssim 0 \lesssim \varepsilon < \omega$. See Figure 3.

For each $0 \lesssim \varepsilon$, let us consider $0 \lesssim \varepsilon < \omega$ and transversal sections Σ_ε and Σ_ω. Using Lemma 2.3 with $a = \varepsilon < \omega = b$, the transition map $\Pi_{\varepsilon}^{\omega}$ for the flow associated to system (2), between Σ_ε and Σ_ω, for each $0 \lesssim y$, is well defined. Analogously, the transition map between Σ_α and $\Sigma_{-\varepsilon}$, $\Pi_{\alpha}^{-\varepsilon}$, for each $0 \lesssim y$ is well defined too.

Hence,

$$\Pi_{\alpha}^{\omega} = \Pi_{\varepsilon}^{\omega} \circ \Pi_{-\varepsilon}^{\varepsilon} \circ \Pi_{\alpha}^{-\varepsilon},$$

is well defined for all $\alpha < 0 \lesssim \varepsilon < \omega$. □

We remark that similar conditions to the ones of Theorem A could be obtained for the more general system (1) but we do not address this problem because we are more interested on system (2). For instance, by using the results of [4] it is readily seen that a necessary condition for the origin to be a singularity of the type we are interested in, which is on the singular fiber $y=0$, is that the homogenous cubic polynomial, constructed from the quadratic jet of the vector field associated to system (1),

$$xy(h_{1,0}x + h_{0,1}y) - y(x^2 + a_{1,1}xy + y^2) = y\big((h_{1,0}-1)x^2 + (h_{0,1}-a_{1,1})xy - y^2\big),$$

has only the real branch $y = 0$, either simple or triple. In other words that either $(h_{0,1} - a_{1,1})^2 + 4(h_{1,0} - 1) < 0$, or that $h_{1,0} = 1$ and $h_{0,1} = a_{1,1}$, respectively. Of course, the results obtained from this point of view, agree with the ones of the Theorem A.

Proof of Corollary 1.1. Notice that system (3) corresponds to system (2) when $h(x, y) \equiv g(x)$. Therefore $h_{0,0} = g(0)$, $h_{1,0} = g'(0)$ and $h_{0,1} = 0$. Hence using former theorem, the necessary and sufficient conditions to have a well-defined transition map Π_α^ω, for system (3), for all $\alpha < 0$ and $\omega > 0$, are $g(0) = 0$ and $g'(0) \leq 1$. □

3. **Technical results for system (3).** Let us assume system (3) in the case where f and g are $\mathcal{C}^4$ and $\mathcal{C}^3$ functions, respectively, on a (α, ω) interval, $\alpha < 0 < \beta$. In this case we write

$$f(x) = x^2 + f_3 x^3 + \mathrm{O}_4(x), \quad g(x) = g_1 x + g_2 x^2 + \mathrm{O}_3(x),$$

where, in accordance with Theorem A for the existence of the transition map Π_α^ω, we assume $g_1 \leq 1$.

To know the stability of the singular fiber $y = 0$ of system (3), we need to know the behaviour of the flow close to the origin for $y \gtrsim 0$. Using the transition map given by the expression (8),

$$\Pi_\alpha^\omega = \Pi_\varepsilon^\omega \circ \Pi_{-\varepsilon}^\varepsilon \circ \Pi_\alpha^{-\varepsilon},$$

and since Lemma 2.3 gives the first term approximation of the regular maps development $\Pi_\alpha^{-\varepsilon}$ and Π_ε^ω we, therefore, only need to study the intermediate transition map $\Pi_{-\varepsilon}^\varepsilon$, for $\varepsilon \gtrsim 0$ and $y \gtrsim 0$.

To get the expression of $\Pi_{-\varepsilon}^\varepsilon$ is a key point for the technical results. To derive it, we use a (θ, r)-polar coordinates blow up to system (3) between the two transversal sections $\Sigma_{-\varepsilon}$ and Σ_ε (see Figure 4.(a)) and, since $\Pi_{-\varepsilon}^\varepsilon$ is only considered for $\varepsilon \gtrsim 0$ and $y \gtrsim 0$, we can restrict polar coordinates to $r \gtrsim 0$.

As a result, after a rescaling of the time variable, we can express system (3) by the equivalent system

$$\left\{ \begin{array}{ll} \dot{\theta} = & \sin\theta \left[-1 + g_1 \cos^2\theta + (g_2 - f_3)\cos^3\theta\, r + \mathrm{O}_2(r)\right], \\ \dot{r} = & r\cos\theta \left[1 + g_1 \sin^2\theta + \big((f_3 - g_2)\cos^3\theta + g_2\cos\theta\big)\, r + \mathrm{O}_2(r)\right], \end{array} \right. \tag{9}$$

for $r \gtrsim 0$ and $0 < \theta < \pi$. See Figure 4.(b). From now on, we can consider the transversal sections $\Sigma_{-\varepsilon}$ and Σ_ε with $\varepsilon \gtrsim 0$, parametrized by the θ-angle in polar coordinates, as

$$\Sigma_{-\varepsilon} = \{(\theta, -\varepsilon/\cos\theta),\, \theta \lesssim \pi\}, \quad \Sigma_\varepsilon = \{(\theta, \varepsilon/\cos\theta),\, \theta \gtrsim 0\}.$$

We note that, although in Cartesian coordinates the transversal sections $\Sigma_{-\varepsilon}$ and Σ_ε are rectilinear segments, in polar coordinates they are not. To make the notation simpler, we will write $\theta \in \Sigma_\varepsilon$ instead of $(\theta, \varepsilon/\cos\theta) \in \Sigma_\varepsilon$. Analogous notation applies to the case $\theta \in \Sigma_{-\varepsilon}$.

Hence, the intermediate transition map $\Pi_{-\varepsilon}^\varepsilon$ can be expressed in terms of the $T_{-\varepsilon}^\varepsilon : \Sigma_{-\varepsilon} \to \Sigma_\varepsilon$ function, named angle transition map, for $\theta \in \Sigma_{-\varepsilon}$, as follows (see Figure 4.(b))

$$\Pi_{-\varepsilon}^\varepsilon(y) = \varepsilon \tan\left(T_{-\varepsilon}^\varepsilon\left(\arctan\left(\frac{-y}{\varepsilon}\right) + \pi\right)\right). \tag{10}$$

A key point in the calculus of $T_{-\varepsilon}^\varepsilon$ is based in next well known symmetry argument with respect $\theta = \pi/2$. The symmetry is given by the change of variables $(\theta, r) \to (\pi - \theta, r)$ applied to system (9) in the (θ, r)-plane, plus reversing the time variable.

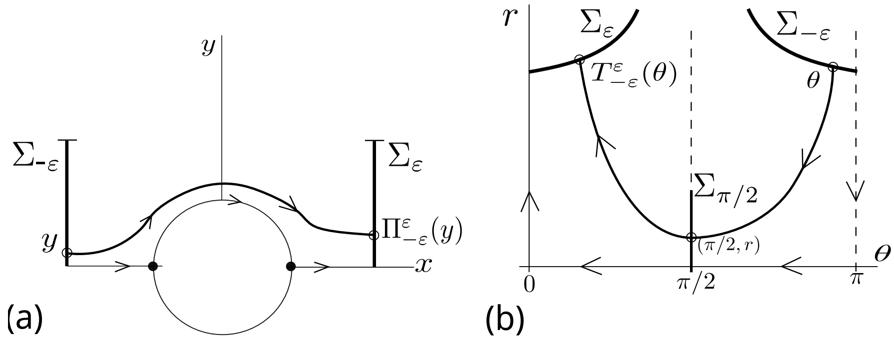

FIGURE 4. System (3) in case $g_1 \le 1$. (a) Behaviour of the intermediate flow close the origin for $y \gtrsim 0$, in Cartesian coordinates, after a polar blow up between the two transversal sections $\Sigma_{-\varepsilon}$ and Σ_ε, with $\varepsilon \gtrsim 0$. (b) Equivalent flow in (θ, r)-polar coordinates given by system (9), $r \gtrsim 0$, through the angle transition map $T^{\varepsilon}_{-\varepsilon}$.

Hence, the flow of system (9) for $\theta \in [\pi/2, \pi)$ corresponds with the flow of that system after applying that change of variables, but for $\theta \in (0, \pi/2]$. See Figure 5.

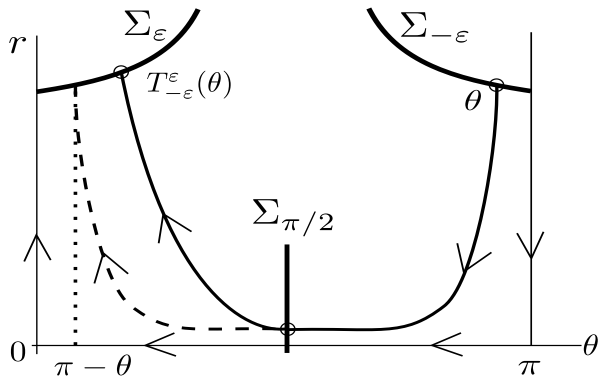

FIGURE 5. The θ-angle transition map $T^{\varepsilon}_{-\varepsilon} : \Sigma_{-\varepsilon} \to \Sigma_\varepsilon$. Solution of system (9) through (θ, r) is the solid line. Dashed line is the symmetry of this solution defined on $[\pi/2, \pi)$, with respect $\theta = \pi/2$; i.e. dashed line is the solution of system (11), with $\lambda = -1$, on $(0, \pi/2]$.

As a consequence, by defining

$$\Sigma_{\pi/2} = \{(\pi/2, r), r \gtrsim 0\},$$

in polar coordinates, we reduce the calculus of the $T^{\varepsilon}_{-\varepsilon}$ angle transition map to the values $\theta \in (0, \pi/2]$. This reduction is proved in next Lemma 3.2 (see Figure 3) for the flow of system

$$\left\{ \begin{array}{l} \dot\theta = P(\theta, r) = \sin\theta \left[-1 + g_1 \cos^2\theta + \lambda\,(g_2 - f_3)\cos^3\theta\, r + \mathrm{O}_2(r)\right], \\ \dot r = Q(\theta, r) = r\cos\theta \left[1 + g_1 \sin^2\theta + \lambda\left((f_3 - g_2)\cos^3\theta + g_2\cos\theta\right) r + \mathrm{O}_2(r)\right], \end{array} \right. \tag{11}$$

when $\theta \in (0, \pi/2]$, $r \gtrsim 0$, with $\lambda = \pm 1$. We note that if $\lambda = 1$, then system (11) represents system (9), while $\lambda = -1$, then it corresponds with system (9) after applying the change of variables $(\theta, r) \to (\pi - \theta, r)$.

Finally, from expression (10) and Lemma 3.2, and by using Proposition 3.3 (case $g_1 < 1$) and Proposition 3.5 (case $g_1 = 1$), the expression of the intermediate transition map $\Pi^{\varepsilon}_{-\varepsilon}$ is derived. Next, we prove all these results. We want to point out that the technique in the proofs of both previous propositions are quite similar, so we will emphasize the details in the proof of the former proposition and less in the latter.

Definition 3.1. Let us denote by σ_λ, $\lambda = \pm 1$, the transition map from $\Sigma_{\pi/2}$ to Σ_ε given by the flow of system (11). See Figure 3.

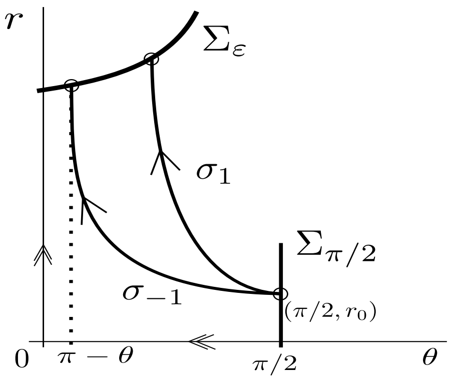

FIGURE 6. The transition map σ_1 (resp. σ_{-1}) is given by the flow of system (11) with $\lambda = 1$ (resp. $\lambda = -1$), with $r_0 = \sigma_{-1}^{-1}(\pi - \theta)$.

Lemma 3.2. *Let us consider system* (11)*. If $\theta \in \Sigma_{-\varepsilon}$ and $\varepsilon \gtrsim 0$, then*

$$T^{\varepsilon}_{-\varepsilon}(\theta) = \sigma_1(\sigma_{-1}^{-1}(\pi - \theta)).$$

Proof. Let $T^{\varepsilon}_{-\varepsilon}(\theta) \in \Sigma_\varepsilon$. By the symmetry with respect $\theta = \pi/2$, the solution of system (11), with $\lambda = 1$, and initial condition (θ, r_0), $\theta \in \Sigma_{-\varepsilon}$, moves to the solution of system (11), with $\lambda = -1$, and final condition $(\pi - \theta, r_0)$, $\pi - \theta \in \Sigma_\varepsilon$. This solution intersects to $\Sigma_{\pi/2}$ at the point $(\pi/2, \sigma_{-1}^{-1}(\pi - \theta))$. See Figure 3. Hence,

$$T^{\varepsilon}_{-\varepsilon}(\theta) = \sigma_1\left(\sigma_{-1}^{-1}(\pi - \theta)\right).$$

□

From the proof of Theorem A it follows that, under the hypothesis regarding the existence of the transition map $\Pi^{\varepsilon}_{-\varepsilon}$ for system (3), after a blow-up of the origin, the singular points appearing are either hyperbolic saddles ($g_1 < 1$) or elementary degenerate saddles ($g_1 = 1$).

To apply Lemma 3.2, we need the expressions of the transition maps σ_1 and σ_{-1} for system (11). We will prove that the σ_1 and σ_{-1} maps are the composition of semiregular with regular transition maps. See [12], for detailed definitions. To get σ_1 and σ_{-1} we follow the approach given in Lemmas 8 and 9 of [10], based on the assumption $g_1 \le 1$.

In next proposition, we first address the calculation of σ_1 and $\sigma_{\text{-}1}$ when they are associated with a hyperbolic sector of a hyperbolic saddle of system (11), that is when $g_1 < 1$.

Proposition 3.3. *Let us consider system* (11) *where* $g_1 < 1$ *and let* σ_λ *be the transition map from* $\Sigma_{\pi/2} = \{(\pi/2, r), r \gtrsim 0\}$ *to* $\Sigma_\varepsilon = \{(\theta, \varepsilon/\cos\theta), \theta \gtrsim 0\}$, *for* $\varepsilon \gtrsim 0$. *Then,*

$$\sigma_\lambda(r) = \frac{(1-g_1)^{g_1/2}}{\varepsilon^{1-g_1}} \exp(F_\lambda(\varepsilon)) r^{1-g_1} + \mathrm{o}(r^{1-g_1}),$$

where

$$F_\lambda(\varepsilon) = \lambda \int_0^\varepsilon \frac{g_2 - g_1 f_3 + \mathrm{O}_1(t)}{1 + \lambda f_3 t + \mathrm{O}_2(t)} \,\mathrm{d}t. \tag{12}$$

Proof. We consider σ_λ as the composition of the three transition maps: T_1 (from $\Sigma_{\pi/2}$ to $\{\theta = \eta\}$), $\sigma_{\eta,\delta}$ (from $\{\theta = \eta\}$ to $\{r = \delta\}$) and T_2 (from $\{r = \delta\}$ to Σ_ε),

$$\sigma_\lambda(r) = (T_2 \circ \sigma_{\eta,\delta} \circ T_1)(r),$$

where η and δ are small enough positive numbers such that $\delta < \varepsilon$. See Figure 7.

First, let us compute $\sigma_{\eta,\delta}$. Since $\sigma_{\eta,\delta}$ is the transition map of the flow from

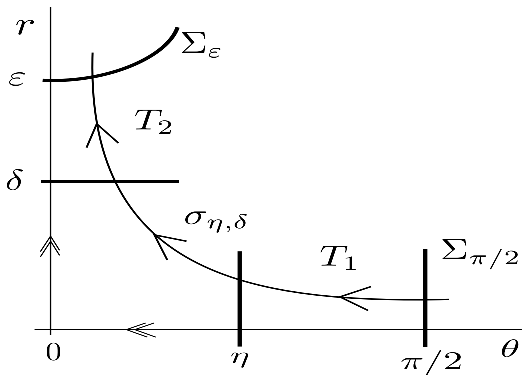

FIGURE 7. Flow of system (11) when $g_1 < 1$, hyperbolic sector of a hyperbolic saddle. We take $\sigma_\lambda(r) = (T_2 \circ \sigma_{\eta,\delta} \circ T_1)(r)$.

$\{\theta = \eta\}$ to $\{r = \delta\}$, it is known that $\sigma_{\eta,\delta}$ is a semiregular map with leading term of order $r^{-a/b}$, where a and b are the eigenvalues of the hyperbolic saddle at the origin. That is,

$$\sigma_{\eta,\delta}(r) = A_{\eta,\delta} r^{-a/b} + \mathrm{o}(r^{-a/b}).$$

See [12, 13, 14, 15], for further details. For $\theta \in (0, \eta)$, since $\eta \gtrsim 0$, by using the trigonometric θ-power series expansions, system (11) writes as

$$\begin{cases} \dot\theta = \widetilde{P}(\theta, r) = -\theta\left(1 - g_1 + \lambda\left(f_3 - g_2\right) r + \mathrm{O}_2(r) + \mathrm{O}_2(\theta)\right), \\ \dot r = \widetilde{Q}(\theta, r) = r\left(1 + \lambda f_3 r + \mathrm{O}_2(r) + \mathrm{O}_2(\theta)\right), \end{cases} \tag{13}$$

and, hence, using an adaptation of [10, Lemma 8], we get that

$$\sigma_{\eta,\delta}(r) = A_{\eta,\delta} r^{1-g_1} + \mathrm{o}(r^{1-g_1}).$$

$$A_{\eta,\delta} = \frac{\eta}{\delta^{1-g_1}} \frac{\exp(F_\lambda(\delta))}{\exp((1-g_1)G(\eta))},$$

for all $\eta, \delta \gtrsim 0$, being

$$F_\lambda(r) = \int_0^r \left[\left(\frac{\widetilde{P}(\theta,t)}{\theta \widetilde{Q}(\theta,t)} \right)\bigg|_{\theta=0} + \frac{1-g_1}{t} \right] \mathrm{d}t, \quad r \gtrsim 0,$$
$$G(\theta) = \int_0^\theta \left[\left(\frac{\widetilde{Q}(t,r)}{r \widetilde{P}(t,r)} \right)\bigg|_{r=0} + \frac{1}{(1-g_1)t} \right] \mathrm{d}t, \quad \theta \gtrsim 0.$$

Note that F_λ and G are well defined functions.

Concerning the transition maps T_1 and T_2 we note that, assuming $\delta < \varepsilon$, they are regular maps for the flow of system (11), between the corresponding cross-sections. Therefore, we can write them as

$$T_1(r) = C_1 r + \mathrm{O}_2(r), \quad \text{and} \quad T_2(\theta) = C_2\theta + \mathrm{O}_2(\theta),$$

where $C_1 = C_1(\eta)$ and $C_2 = C_2(\delta, \varepsilon)$. By using first order variational equations of system (11), we have

$$C_1 = \exp\left(\int_{\pi/2}^{\eta} \frac{\partial}{\partial r} \left(\frac{Q(\theta,r)}{P(\theta,r)} \right)\bigg|_{r=0} \mathrm{d}\theta \right), \; C_2 = \exp\left(\int_\delta^\varepsilon \frac{\partial}{\partial \theta} \left(\frac{P(\theta,r)}{Q(\theta,r)} \right)\bigg|_{\theta=0} \mathrm{d}r \right).$$

Computing C_1 we have

$$C_1 = \left(\frac{(1-g_1\cos^2\eta)^{g_1}}{1-\cos^2\eta} \right)^{\frac{1}{2(1-g_1)}}.$$

Hence, since $\eta \gtrsim 0$, we get

$$C_1 = \eta^{-1/(1-g_1)} (1-g_1)^{\frac{g_1}{2(1-g_1)}} \exp(G(\eta)), \quad C_2 = \left(\frac{\delta}{\varepsilon} \right)^{(1-g_1)} \frac{\exp(F_\lambda(\varepsilon))}{\exp(F_\lambda(\delta))}.$$

Then,

$$\sigma_\lambda(r) = (T_2 \circ \sigma_{\eta,\delta} \circ T_1)(r) = C_2 A_{\eta,\delta} C_1^{1-g_1} r^{1-g_1} + \mathrm{o}(r^{1-g_1})$$
$$= \frac{(1-g_1)^{g_1/2}}{\varepsilon^{1-g_1}} \exp(F_\lambda(\varepsilon)) r^{1-g_1} + \mathrm{o}(r^{1-g_1}).$$

□

In next corollary, when $g_1 < 1$ and $f(x) = x^2$ in system (3), the transition map σ_λ, from $\Sigma_{\pi/2}$ to Σ_ε, for $\varepsilon > 0$, follows from straight computations.

Corollary 3.4. *Let us take $f(x) = x^2$ and $g(x) = g_1 x + g_2 x^2 + \mathrm{O}_3(x)$, with $g_1 < 1$, in system (3). Consider the equivalent system (11) and let σ_λ be its transition map from $\Sigma_{\pi/2}$ to Σ_ε, for $\varepsilon > 0$, $\lambda = \pm 1$. Then, for $\varepsilon > 0$,*

$$\sigma_\lambda(r) = \frac{(1-g_1)^{g_1/2}}{\varepsilon^{1-g_1}} \exp\left(\lambda g_2 \varepsilon + \mathrm{O}_2(\varepsilon)\right) r^{1-g_1} + \mathrm{o}(r^{1-g_1}). \tag{14}$$

Now we address the case in which σ_1 and σ_{-1} are associated with a hyperbolic sector of an elementary degenerate saddle of system (11), that is when $g_1 = 1$.

As was in the previous case, it can be observed that the σ_λ map is again the result of the composition of a flat semiregular map with regular transition maps. See [12], for detailed definitions.

Proposition 3.5. *Let us consider system* (11) *with* $g_1 = 1$*. Let* σ_λ *be the transition map from* $\Sigma_{\pi/2} = \{(\pi/2, r), r \gtrsim 0\}$ *to* $\Sigma_\varepsilon = \{(\theta, \varepsilon/\cos\theta), \theta \gtrsim 0\}$*, for* $\varepsilon > 0$*,* $\lambda = \pm 1$*, and let us take* η *any small enough positive number. Then*

$$\sigma_\lambda(r) = \frac{1}{\sqrt{2}} \exp\left(F_\lambda(\varepsilon)\right) \left(\frac{-1}{\ln|C_1 r + \mathrm{o}(r)|}\right)^{1/2} + \mathrm{o}\left(\left(\frac{-1}{\ln|C_1 r + \mathrm{o}(r)|}\right)^{1/2}\right),$$

with

$$F_\lambda(\varepsilon) = \lambda \int_0^\varepsilon \frac{g_2 - f_3 + \mathrm{O}_1(t)}{1 + \lambda f_3 t + \mathrm{O}_2(t)}\, \mathrm{d}t, \quad C_1 = \exp\left(\int_{\pi/2}^{\eta} \left.\frac{Q(\theta, r)}{rP(\theta, r)}\right|_{r=0} \mathrm{d}\theta\right). \tag{15}$$

Proof. Let σ_λ be the composition of the three transition maps: T_1 (from $\Sigma_{\pi/2}$ to $\{\theta = \eta\}$), $\tau_{\eta,\delta}$ (from $\{\theta = \eta\}$ to $\{r = \delta\}$) and T_2 (from $\{r = \delta\}$ to Σ_ε), where η and δ are small enough positive numbers, that is

$$\sigma_\lambda(r) = (T_2 \circ \tau_{\eta,\delta} \circ T_1)(r). \tag{16}$$

The situation is similar that the one depicted in Figure 7 but changing $\sigma_{\eta,\delta}$ by $\tau_{\eta,\delta}$. Let us obtain each one of the maps of this composition. Since $\tau_{\eta,\delta}$, is a flat semiregular map, we apply Lemma 9 of [10] to system (11), with $\eta, \delta \gtrsim 0$, written as

$$\begin{cases} \dot{r} &= -r(-1 - \lambda f_3 r + O_2(r)), \\ \dot{\theta} &= \theta\left(\lambda\left(g_2 - f_3\right) r - \theta^2 + O_2(r) + O_3(\theta)\right), \end{cases}$$

getting

$$\tau_{\eta,\delta}(r) = \frac{1}{\sqrt{2}} \exp\left(F_\lambda(\delta)\right) \left(\frac{-1}{\ln(r)}\right)^{1/2} + \mathrm{o}\left(\left(\frac{-1}{\ln(r)}\right)^{1/2}\right),$$

expressed in (r, θ)-polar variables, and being function F_λ given as in (15). Furthermore, transition maps T_1 and T_2, assuming $0 \lesssim \delta < \varepsilon$ and $0 \lesssim \eta < \pi/2$, are regular maps between the corresponding cross-sections for the flow of system (11). Therefore, they write as

$$T_1(r) = C_1 r + \mathrm{O}_2(r), \quad \text{and} \quad T_2(\theta) = C_2\theta + \mathrm{O}_2(\theta),$$

where $C_1 = C_1(\eta)$ and $C_2 = C_2(\delta, \varepsilon)$. By using first order variational equations for system (11) we have

$$C_1 = \exp\left(\int_{\pi/2}^{\eta} \left.\frac{P(x, y)}{xQ(x, y)}\right|_{x=0} \mathrm{d}y\right), \quad C_2 = \frac{\exp(F_\lambda(\varepsilon))}{\exp(F_\lambda(\delta))},$$

Hence, using T_1, $\tau_{\eta,\delta}$ and T_2, from expression (16), the proposition follows. □

Next corollary gives the expression of σ_λ when $f(x) = x^2$ in system (3) when $g_1 = 1$.

Corollary 3.6. *Let us take* $f(x) = x^2$ *and* $g(x) = x + g_2 x^2 + \mathrm{O}_3(x)$ *in system* (3)*. Consider the equivalent system* (11)*, and let* σ_λ *be their transition maps,* $\lambda = \pm 1$*. Then, for* $\varepsilon > 0$*,*

$$\sigma_\lambda(r) = A\, D(\eta, r) + \mathrm{o}\left(D(\eta, r)\right), \quad A = \frac{1}{\sqrt{2}} \exp\left(\lambda g_2 \varepsilon + \mathrm{O}_2(\varepsilon)\right),$$

where

$$D(\eta, r) = \left(\frac{-1}{\ln|C_1 r + \mathrm{o}(r)|}\right)^{1/2}, \quad C_1 = \exp\left(\int_{\pi/2}^{\eta} \left.\frac{Q(\theta, r)}{rP(\theta, r)}\right|_{r=0} \mathrm{d}\theta\right),$$

and P *and* Q *are given by system* (11)*.*

4. **Proof of Theorem B.** Since $\alpha < 0 < \omega$, $g(0) = 0$ and $g_1 \leq 1$, using Corollary 1.1, we can assume that the transition map Π_α^ω is well defined for the flow of system (3) in $y \gtrsim 0$.

For every $\varepsilon \gtrsim 0$, using expression (8) we have that the transition map is given as

$$\Pi_\alpha^\omega = \Pi_\varepsilon^\omega \circ \Pi_{-\varepsilon}^\varepsilon \circ \Pi_\alpha^{-\varepsilon},$$

and, from expression (10), the intermediate transition map $\Pi_{-\varepsilon}^\varepsilon$ can be obtained by using the angle transition map, $T_{-\varepsilon}^\varepsilon(\theta)$. Since Lemma 3.2 gives us,

$$T_{-\varepsilon}^\varepsilon(\theta) = \sigma_1(\sigma_{-1}^{-1}(\pi - \theta)),$$

as a consequence, to get Π_α^ω we need to know the expressions of the map σ_λ, $\lambda = \pm 1$.

First, let us prove Theorem B in case $g_1 < 1$. If we consider system (11), Proposition 3.3 gives us

$$\sigma_\lambda(r) = A_\lambda r^{1-g_1} + \mathrm{o}\left(r^{1-g_1}\right), \quad A_\lambda = \frac{(1-g_1)^{g_1/2}}{\varepsilon^{1-g_1}} \exp(F_\lambda(\varepsilon)),$$

where

$$F_\lambda(\varepsilon) = \lambda \int_0^\varepsilon \frac{g_2 - g_1 f_3 + \mathrm{O}_1(t)}{1 + \lambda f_3 t + \mathrm{O}_2(t)}\, \mathrm{d}t.$$

Using Lemma 3.2 and former expression, we have

$$T_{-\varepsilon}^\varepsilon(\theta) = A_1 A_{-1}^{-1}(\pi - \theta) + \mathrm{o}(\pi - \theta).$$

Summing up,

$$T_{-\varepsilon}^\varepsilon(\theta) = \frac{\exp(F_1(\varepsilon))}{\exp(F_{-1}(\varepsilon))}(\pi - \theta) + \mathrm{o}(\pi - \theta). \tag{17}$$

On the other hand, applying Lemma 2.3 with $h(x,y) = g(x)$, we have that the regular transition maps Π_ε^ω and $\Pi_\alpha^{-\varepsilon}$ are given by

$$\Pi_\alpha^{-\varepsilon}(y) = C(\alpha, -\varepsilon) y + \mathrm{O}_2(y), \quad \Pi_\varepsilon^\omega(y) = C(\varepsilon, \omega) y + \mathrm{O}_2(y),$$

for $y \gtrsim 0$, where

$$C(a,b) = \exp\left(\int_a^b \frac{g(x)}{f(x)}\, \mathrm{d}x\right).$$

Since

$$\Pi_\alpha^\omega = \Pi_\varepsilon^\omega \circ \Pi_{-\varepsilon}^\varepsilon \circ \Pi_\alpha^{-\varepsilon},$$

using the expression of $\Pi_{-\varepsilon}^\varepsilon$ given by relation (10), we have that

$$\Pi_\alpha^\omega(y) = \varepsilon\, C_2 \tan\left(\left(\frac{\exp(F_1(\varepsilon))}{\exp(F_{-1}(\varepsilon))}\right) \arctan\left(\frac{C_1 y + \mathrm{o}(y)}{\varepsilon}\right)\right) + \mathrm{o}(y), \tag{18}$$

where

$$C_1 = \exp\left(\int_\alpha^{-\varepsilon} \frac{g(t)}{f(t)}\, \mathrm{d}t\right), \quad C_2 = \exp\left(\int_\varepsilon^\omega \frac{g(t)}{f(t)}\, \mathrm{d}t\right).$$

Thus, since $g_1 < 1$, using a y-power series expansion of the function $\Pi_\alpha^\omega(y)$ as it is given in expression (18) we have

$$\Pi_\alpha^\omega(y) = C_1 C_2 K y + \mathrm{o}(y), \quad K = \frac{\exp(F_1(\varepsilon))}{\exp(F_{-1}(\varepsilon))},$$

for all $\varepsilon \gtrsim 0$. Since $\lim_{\varepsilon \to 0}(F_1(\varepsilon) - F_{-1}(\varepsilon)) = 0$ and

$$\lim_{\varepsilon \to 0} C_1 C_2 = \exp\left(\mathrm{PV} \int_\alpha^\omega \frac{g(t)}{f(t)}, \mathrm{d}t\right),$$

the theorem follows in this case.

Let us prove case $g_1 = 1$. For system (11), Proposition 3.5 gives us that, for $r \gtrsim 0$,

$$\sigma_\lambda(r) = \frac{1}{\sqrt{2}} \exp\left(F_\lambda(\varepsilon)\right) H(r) + \mathrm{o}\left(H(r)\right),$$

with $\lambda = \pm 1$, where

$$F_\lambda(\varepsilon) = \lambda \int_0^\varepsilon \frac{g_2 - f_3 + \mathrm{O}_1(t)}{1 + \lambda f_3 t + \mathrm{O}_2(t)}\,\mathrm{d}t,$$

$$H(r) = \left(\frac{-1}{\ln|C_1 r + \mathrm{O}_2(r)|}\right)^{1/2}, \quad C_1 = \exp\left(\int_{\pi/2}^{\eta} \left.\frac{Q(\theta,r)}{rP(\theta,r)}\right|_{r=0} \mathrm{d}\theta\right),$$

where $P(\theta,r)$ and $Q(\theta,r)$ are given by system (11). Hence,

$$\sigma_{\text{-}1}^{-1}(\theta) = \frac{1}{C_1} \exp\left(-\frac{\exp(2F_{\text{-}1}(\varepsilon))}{2\theta^2}\right) \exp\Big(\mathrm{o}\left(1/\ln|C_1 r + \mathrm{o}_2(r)|\right)\Big).$$

As a consequence, using Lemma 3.2 and former expressions, we get

$$T_{-\varepsilon}^{\varepsilon}(\theta) = \frac{\exp(F_1(\varepsilon))}{\exp(F_{\text{-}1}(\varepsilon))}(\pi - \theta) + \mathrm{o}(\pi - \theta).$$

At this point, we would remark that neither $H(r)$ nor C_1 appear at the first order term of $T_{-\varepsilon}^{\varepsilon}(\theta)$. As a consequence, using relation (10), we get $\Pi_{-\varepsilon}^{\varepsilon}(\theta)$. On the other hand, applying Lemma 2.3 with $h(x,y) = g(x)$, we have that the regular transition maps $\Pi_\alpha^{-\varepsilon}$ and Π_ε^ω are given by

$$\Pi_\alpha^{-\varepsilon}(y) = C(\alpha,-\varepsilon)y + \mathrm{O}_2(y), \quad \Pi_\varepsilon^\omega(y) = C(\varepsilon,\omega)y + \mathrm{O}_2(y),$$

for $y \gtrsim 0$, where

$$C(a,b) = \exp\left(\int_a^b \frac{g(x)}{f(x)}\,\mathrm{d}x\right).$$

Hence, since

$$\Pi_\alpha^\omega = \Pi_\varepsilon^\omega \circ \Pi_{-\varepsilon}^{\varepsilon} \circ \Pi_\alpha^{-\varepsilon},$$

for $\varepsilon \gtrsim 0$, using similar arguments to those the case $g_1 < 1$, the case $g_1 = 1$ follows.

5. **On the transition map for some families.**

In this section we tackle the study of the stability of the singular fiber defined on a symmetric interval, for some families of systems (2) and (3).

More concretely, for system (3) we consider $g(x) = g_{\mathrm{e}}(x) + g_{\mathrm{o}}(x)$, where g_{o} is an odd (resp. g_{e} even) function. In this case we will get Theorem 5.1 and Corollary 5.2.

This section ends with the study of the number of fixed points of the transition map, $\Pi_{-\omega}^\omega$ for $\omega > 0$, for a particular family in the frame of system (2). See Proposition 5.4.

Theorem 5.1. *Consider system (3) with f an even function, positive outside 0, and such that the transition map $\Pi_{-\omega}^\omega$ is well defined for some $\omega > 0$.*

(i) If $g_{\mathrm{e}} \equiv 0$, then $\Pi_{-\omega}^\omega(y) = y$, for all $y > 0$.
(ii) If for every $x \in (-\omega,\omega)\backslash\{0\}$, we have $g_{\mathrm{e}}(x) < 0$ (resp. $g_{\mathrm{e}}(x) > 0$), then $y = 0$ is a stable (resp. unstable) singular fiber on this interval.

Proof of Theorem 5.1. If in system (3) we consider $g_e(x) \equiv 0$, we get the time-reversible system,

$$\dot{x} = f(x) + y^2, \quad \dot{y} = g_o(x)y, \tag{19}$$

where g_o is the odd part of g, which is invariant under the change of variables $(x, y, t) \to (-x, y, -t)$. From this symmetry, the transition map $\Pi_\omega(y) = y$, for all $y > 0$.

On the other hand, let V_1 and V_2 be the vector field associated with the differential system (19) and (3), respectively. We observe that

$$\det(V_1, V_2) = \begin{vmatrix} f(x) + y^2 & g_o(x)y \\ f(x) + y^2 & g(x)y \end{vmatrix} = (f(x) + y^2)g_e(x)y,$$

and then the sign of $\det(V_1, V_2)$ is given by the sign of g_e. Moreover when this determinant does not change sign, this means that the orbits of vector field V_1 are without contact with the orbits of V_2. Hence, the displacement map evaluated on each solution of system (3) has the sign of the $\det(V_1, V_2)$, and the result follows. □

Corollary 5.2. *Consider system (3) where g is an analytic function written as $g(x) = \sum_{i\geq 1} g_i x^i$ with $g_1 \leq 1$.*

(i) If $g_{2k} = 0$ for all positive integer k, then $\Pi^\omega_{-\omega}(y) = y$, for all $y \gtrsim 0$, $\omega > 0$.
(ii) Assume that there exists a positive integer k such that g_{2k} is the first non-zero coefficient of $g_e(x)$. Then, there exists $\omega' \gtrsim 0$ such that if $g_{2k} < 0$ (resp. $g_{2k} > 0$), $y = 0$ is a stable (resp. unstable) singular fiber on $(-\omega', \omega')$.

Proof of Corollary 5.2. As it is proved in Theorem A, if $g(0) \neq 0$, the origin of system (3) is locally a saddle-node equilibrium and, hence, the transition map is not defined close to $(0, 0)$. This fact supports that $g(0) = 0$.

In case $g_1 \leq 1$, using Corollary 1.1, it follows that the transition map $\Pi^\omega_{-\omega}$ is well defined between the two transversal sections $\Sigma_{-\omega}$ and Σ_ω to the flow of system (3), for all $\omega > 0$. Hence, the displacement map, $\Delta^\omega_{-\omega}$, is also well defined between $\Sigma_{-\omega}$ and Σ_ω.

By one hand, if $g_{2i} = 0$ for all $i \geq 1$, then g is an odd function and, using Theorem 5.1.(i), $\Pi^\omega_{-\omega}(y) = y$, for all $y \gtrsim 0$. As a consequence, the statement (i) of this corollary follows.

To prove statement (ii), let us assume that there exists a positive integer k such that g_{2k} is the first non-zero coefficient of $g_e(x)$. Then, there exists $\omega' \gtrsim 0$ such that, if $g_{2k} < 0$ (resp. $g_{2k} > 0$), we have $g_e(x) < 0$ (resp. $g_e(x) > 0$), for all $x \in (-\omega', \omega') \backslash \{0\}$. By using Theorem 5.1.(ii), we have that $y = 0$ is a stable (resp. unstable) singular fiber on $(-\omega', \omega')$. □

Notice that the change of stability of a singular fiber can be used to bifurcate from it a zero for the displacement map. This can be understood as a kind of Andronov-Hopf bifurcation on the cylinder, specially when the differential equation is 2ω-periodic in the x-variable.

In this section, as complementary result to the study of the stability of the singular fiber $y = 0$, we also present a result on the number of zeroes of the displacement map when it is a perturbation of the identity. These zeros are controlled by a kind of Melnikov function associated to a differential equation. This equation is a perturbation of another one with displacement map identically zero. More concretely, let us consider

$$\dot{x} = A(x) + y^2, \quad \dot{y} = -A'(x)y + \varepsilon B(x)y^2, \tag{20}$$

where A and B are smooth enough functions defined on the interval $[-1,1]$, A is non-negative, even, such that $A(0)=0$ and vanishing only at $x=0$, and ε is a small enough real number. Next results concern the number of fixed points of the transition map Π_{-1}^{1} defined between Σ_{-1} and Σ_1 for system (20) in $y \gtrsim 0$.

We note that for $\varepsilon=0$, system (20) is a Hamiltonian system with Hamiltonian function $H(x,y)=A(x)y+y^3/3$ and that, from Theorem 5.1.(i), $\Pi_{-1}^{1}(y)=y$ for all $y>0$.

Proposition 5.3. *Let $y(x)=y(x;\rho,\varepsilon)$ be a solution of the initial value problem (20), such that $y(-1;\rho,0)=\rho$, for $\rho>0$. Let us write*

$$y(x;\rho,\varepsilon)=y_0(x;\rho)+\varepsilon y_1(x;\rho)+\mathrm{O}_2(\varepsilon). \tag{21}$$

Let Π_{-1}^{1} be the transition map defined between the two transversal sections to the flow of system (20), through $x=-1$ and $x=1$. Then,

(i)

$$y_0^3(x;\rho)+3A(x)y_0(x;\rho)-3A(1)\rho-\rho^3=0. \tag{22}$$

(ii)

$$y_1(x;\rho)=\frac{\int_{-1}^{x}B(t)y_0^2(t;\rho)\,\mathrm{d}t}{A(x)+y_0^2(x;\rho)}. \tag{23}$$

(iii) Define $M_1(\rho)=(A(1)+\rho^2)y_1(1;\rho)$. Then

$$M_1(\rho)=\int_{-1}^{1}B(t)\frac{\left(\left(S(\rho)+\sqrt{S^2(\rho)+A^3(t)}\,\right)^{2/3}-A(t)\right)^2}{\left(S(\rho)+\sqrt{S^2(\rho)+A^3(t)}\,\right)^{2/3}}\,\mathrm{d}t,$$

where $S(\rho)=\frac{1}{2}(\rho^3+3A(1)\rho)$. Moreover, if for $\varepsilon=0$, $\rho=\rho^*$ is such that $M_1(\rho^*)=0$ and $M_1'(\rho^*)\neq 0$; then, for ε small enough, there exists $\rho_0(\varepsilon)$ such that $\Pi_{-1}^{1}(\rho_0(\varepsilon))=\rho_0(\varepsilon)$ and $\lim_{\varepsilon\to 0}\rho_0(\varepsilon)=\rho^*$.

Proof. When $\varepsilon=0$, system (20) is a Hamiltonian system with Hamiltonian function $H(x,y)=A(x)y+y^3/3$. Hence, since $y_0(x;\rho)$ is a solution of this system, we have that

$$A(x)y_0(x;\rho)+\frac{1}{3}y_0^3(x;\rho)=A(1)\rho+\frac{1}{3}\rho^3,$$

which proves part (i) of this proposition.

To prove part (ii), by imposing that function (21) is solution of system (20), we have that $y_1=y_1(x;\rho)$ satisfies the linear differential equation

$$(A(x)+y_0^2)y_1'=-(A'(x)+2y_0y_0')y_1+B(x)y_0^2,$$

where the derivatives stand for derivatives with respect the variable x. Since $y_1(-1;\rho)=0$, after some computations the expression of $y_1(x;\rho)$ follows.

To prove part (iii), by imposing that ρ is such that $y(1;\rho,\varepsilon)=\rho$, we get that $y_1(1;\rho)=0$. Then, from expression (23),

$$M_1(\rho)=\int_{-1}^{1}B(t)y_0^2(t;\rho)\,\mathrm{d}t=0.$$

Since, $y_0(x;\rho)$ is a solution of the cubic equation (22), by Cardano's formula,

$$\int_{-1}^{1}B(t)\left(G(S(\rho),t)-\frac{A(t)}{G(S(\rho),t)}\right)^2\,\mathrm{d}t=0,$$

where

$$G(S(\rho),x)=\sqrt[3]{S(\rho)+\sqrt{S(\rho)^2+A^3(x)}},$$

expression that coincides with the one in the statement. The proof finishes by applying the implicit function theorem to

$$T(\rho,\varepsilon)=\frac{y(1;\rho,\varepsilon)-\rho}{\varepsilon}=\frac{M_1(\rho)}{A(1)+\rho^2}+\mathrm{O}(\varepsilon)=0,$$

at the point $(\rho,\varepsilon)=(\rho^*,0)$. This can be done, precisely because $M_1'(\rho^*)\neq 0$. □

Proposition 5.4. *Consider the system of differential equation* (20) *defined on the cylinder* $(x,y)\in[-1,1]\times\mathbb{R}^+$, *with*

$$A(x)=\sin^2\left(\frac{\pi x}{2}\right),\quad B(x)=\cos\left(\frac{\pi x}{2}\right)\sum_{k=0}^{3}b_k\sin^{2k}\left(\frac{\pi x}{2}\right),$$

and $b_k\in\mathbb{R}$. *Then, for* ε *small enough, there are values of these parameters such that it has at least 3 limit cycles bifurcating from the continuum of periodic solutions defined on* $y>0$ *and existing for* $\varepsilon=0$.

Proof. By using Proposition 5.3.(iii), and the change of variables $u=\sin\left(\frac{\pi x}{2}\right)$, we have

$$M_1(\rho)=\frac{2}{\pi}\sum_{k=0}^{3}b_kI_k(S^3),$$

where $S^3=\frac{1}{2}(\rho^3+3\rho)$ and

$$I_k(S^3)=\int_{-1}^{1}\frac{\left(\left(S^3+\sqrt{S^6+u^6}\right)^{\frac{2}{3}}-u^2\right)^2u^{2k}}{\left(S^3+\sqrt{S^6+u^6}\right)^{\frac{2}{3}}}\mathrm{d}u,$$

for $k=0,\dots,3$. For each k, after some work it can be seen that

$$\begin{aligned}I_k(S^3)=&\frac{2^{\frac{5}{3}}S^2}{2k+1}\,{}_3F_2\left(-\frac{1}{3},\frac{1}{6},\frac{k}{3}+\frac{1}{6};\frac{1}{3},\frac{7}{6}+\frac{k}{3};-\frac{1}{S^6}\right)-\frac{4}{3+2k}\\&+\frac{2^{\frac{1}{3}}}{S^2\,(5+2k)}\,{}_3F_2\left(\frac{1}{3},\frac{5}{6},\frac{5}{6}+\frac{k}{3};\frac{5}{3},\frac{11}{6}+\frac{k}{3};-\frac{1}{S^6}\right),\end{aligned}$$

where ${}_3F_2$ is the generalized $(3,2)$-hypergeometric function and, hence, it is an analytical function on an interval $(0,S_0)$, for some $S_0\gtrsim 0$. Furthermore, since the Wronskian $W(I_0,I_1,I_2,I_3)(S)\neq 0$ in a small enough open interval $(0,S_1), S_1\leq S_0$, the four functions are linearly independent on that interval. Additionally, if S_1 is small enough, all these functions have constant sign on the interval $(0,S_1)$. As a consequence, by applying Lemma 4.5(ii) of [7], for b_i, $i=0,\dots,3$ suitable constants, it is possible to choose b_k in such a way that $M_1(\rho)$ has at least three simple zeroes in $(0,S_1)$. By Proposition 5.3, when ε is small enough, each one of these three zeroes gives rise to a limit cycle for the given system of differential equations. □

We can use the same proof technique from the previous proposition when $A(x)=x^2$ and $B(x)=\sum_{k\geq 0}^{n}b_kx^{2k}$, obtaining similar results. Notice that then the zeroes of Π_{-1}^1 are no more limit cycles.

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