

HOPF BIFURCATION IN KOLMOGOROV SYSTEMS OF DEGREE 4 IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

JAUME LLIBRE¹ AND Y. PAULINA MANCILLA-MARTÍNEZ²

ABSTRACT. We consider the Kolmogorov system of degree 4 in $\mathbb{R}^2$ having an equilibrium point in the positive quadrant and the Kolmogorov system of degree 4 in $\mathbb{R}^3$ having an equilibrium point in the positive octant. Note that from a biological point of view the two equilibria considered must be in the positive quadrant and octant. We give the conditions under which the equilibrium points of the planar and spatial systems will be a Hopf point and a zero-Hopf point, respectively. The explicit conditions for the existence of limit cycles bifurcating from these two equilibria are obtained using the averaging theory. Moreover we characterize the stability or instability of these limit cycles.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Kolmogorov systems are an extension given in 1936 by Andrey Kolmogorov [10] of the Lotka–Volterra systems which have the form $dx_i/dt = x_i(a_{1i}x_1 + a_{2i}x_2 + a_{0i})$ with $i = 1, 2$ in $\mathbb{R}^2$, and $dx_i/dt = x_i(a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + a_{0i})$ with $i = 1, 2, 3$ in $\mathbb{R}^3$, and were proposed, independently developed by Alfred J. Lotka in 1925 [14] and Vito Volterra in 1926 [23], as a model for the research of interactions between two species. Kolmogorov extended the Lotka–Volterra systems of degree two to arbitrary degree.

Many applications of the Lotka–Volterra and the Kolmogorov systems can be mentioned, the main areas of application are in biology where have been applied to model diverse natural phenomena such as the evolution between species in competition (see for instance May and Leonard [16]), the time evolution of conflicting species (see for more details May [17] and [9, 24] as examples), also in economics (where [5], [7] and [22] are some examples), and in other areas as chemical reactions [8], hydrodynamics [4], the evolution of electrons, ions and neutral species in plasma physics [15], to name a few.

An isolated periodic orbit in the set of all periodic orbits of a differential system is called a *limit cycle*. A limit cycle can be *stable*, *unstable*, or *semi-stable* and this depend on the characteristic exponent of its associated Poincaré map, if at least one characteristic exponent has positive real part then the limit cycle is *unstable*, and if all the characteristic exponents have negative real part the limit cycle is *stable*.

The existence of a limit cycle is an important phenomenon in the dynamics of the differential systems, its analysis is significant in the study of the nonlinear differential systems (see for instance [18]). In the study of the limit cycles an important role is played by the Hopf bifurcations (see for instance [20]).

A *Hopf equilibrium point* π of a differential system in $\mathbb{R}^2$ is an equilibrium point such that the jacobian matrix of the differential system at π has eigenvalues

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$\pm\omega i$, $\omega > 0$. A *Hopf bifurcation* is a local bifurcation in which a Hopf equilibrium point of a differential system switches its stability and some small limit cycles can bifurcate from the equilibrium point. An equilibrium point changes its stability when a pair of complex conjugate eigenvalues of the linearization around the equilibrium point crosses in the complex plane the imaginary axis.

It is known that the Lotka–Volterra systems in $\mathbb{R}^2$ cannot have limit cycles (see [2]), so such systems can not exhibit a Hopf bifurcation. Kolmogorov systems of degree 3 in $\mathbb{R}^2$ were studied in [11] finding that one limit cycle can bifurcate from a Hopf equilibrium point, using the averaging theory of order two.

The first part of our work is focus on Kolmogorov systems of degree 4 in $\mathbb{R}^2$ having the finite equilibria (x_0, y_0) located in the region of biological interest, i.e. in the positive quadrant. Without loss of generality we can assume that the equilibrium is $(1, 1)$, because after the rescaling $(x, y) \rightarrow (x/x_0, y/y_0)$ we get the equilibrium that we want.

According with the previous description a Kolmogorov system of degree 4 in the plane having the equilibrium point $(1, 1)$ can be written as

$$(1) \quad \begin{aligned} \dot{x} &= x(a_1(x-1) + a_2(y-1) + a_3(x-1)^2 + a_4(x-1)(y-1) + a_5(y-1)^2 \\ &\quad + a_6(x-1)^3 + a_7(x-1)^2(y-1) + a_8(x-1)(y-1)^2 + a_9(y-1)^3), \\ \dot{y} &= y(b_1(x-1) + b_2(y-1) + b_3(x-1)^2 + b_4(x-1)(y-1) + b_5(y-1)^2 \\ &\quad + b_6(x-1)^3 + b_7(x-1)^2(y-1) + b_8(x-1)(y-1)^2 + b_9(y-1)^3), \end{aligned}$$

where the derivative with respect to the time t is denoted with a dot.

Here we shall study the existence of limit cycles for the Kolmogorov systems (1) emerging from a Hopf equilibrium point. For this purpose, we first characterize the conditions in order that the point $(1, 1)$ be a Hopf equilibrium point of system (1). Second, we will use the averaging theory of first and second order to provide sufficient conditions for the existence of limit cycles emerging from this Hopf equilibrium point.

In order to state the principal result for the Hopf bifurcation in the plane we introduce the following notation:

$$\begin{aligned} A &= \frac{\alpha_{12}}{2\omega}, \\ B &= \frac{1}{8b_1\omega^3((b_1-b_2)^2 + \omega^2)}\omega^2(-2a_3^2b_1b_2 + a_3a_4b_1^2 + a_3b_2(b_1^2 + b_1(3b_2 + b_4) - \\ &\quad 4b_2b_3) - a_4b_1^2b_2 + a_4b_1b_2b_3 + 3a_6b_1b_2^2 - 2a_7b_1^2b_2 + a_8b_1^3 + 2b_1^3b_5 + 3b_1^3b_9 - \\ &\quad b_1^2b_2b_4 + b_1^2b_2b_5 - 2b_1^2b_2b_8 - b_1^2b_4b_5 + b_1b_2^2b_3 - b_1b_2^2b_4 + b_1b_2^2b_7 + 2b_1b_2b_3b_5 + \\ &\quad b_1b_2b_4^2 + 2b_2^3b_3 - 2b_2^2b_3b_4) - (-2a_3b_2 + a_4b_1 + b_2(b_1 + b_2 - b_4) + 2b_1b_5)(-b_2 \cdot \\ &\quad (-a_4b_1^2 + b_1^2b_5b_2(a_3b_1 - b_1b_4 + b_2b_3)) - a_5b_1^3) + \omega^4(2a_3(b_1 - b_3) + 3a_6b_1 \\ &\quad + b_1b_7 + b_2b_3 - b_3b_4), \end{aligned}$$

where α_{12} is defined in the relation $a_1 = -b_2 + \varepsilon\alpha_{11} + \varepsilon\alpha_{12}$.

Thus our main result concerning the Kolmogorov systems (1) is the following.

Theorem 1. *Consider the Kolmogorov system (1) with $a_1 = -b_2 + \varepsilon\alpha_{11} + \varepsilon\alpha_{12}$, $a_2 = -(b_2^2 + \omega^2)/b_1 + \varepsilon\alpha_{21} + \varepsilon\alpha_{22}$, $b_1 \neq 0$ and $AB < 0$. Then one small limit cycle bifurcates from the Hopf equilibrium point $(1, 1)$ of system (1) when $\varepsilon = 0$. If $A > 0$ (resp. $A < 0$) the limit cycle is stable (resp. unstable).*

Theorem 1 is proved in Section 2, where we provide the explicit initial condition of the limit cycle bifurcating from the Hopf equilibrium point $(1, 1)$ when $\varepsilon = 0$.

Now we apply Theorem 1 for finding the limit cycle bifurcating from the Hopf equilibrium $(1, 1)$ of the differential system

$$(2) \quad \begin{aligned} \dot{x} &= x \left(x^3 - 3x^2 + xy - x/10000 + y^3 - 2y^2 - 5y + 70001/10000 \right), \\ \dot{y} &= y \left(x^3 + x^2y - 4x^2 - 2xy + 6x + y^3 - 3y^2 + 6y - 6 \right). \end{aligned}$$

According with Theorem 1 and its proof the initial condition for the periodic orbit $(x(t, 1/100), y(t, 1/100))$ bifurcating from the Hopf equilibrium $(1, 1)$ is

$$\left(x \left(0, \frac{1}{100} \right), y \left(0, \frac{1}{100} \right) \right) = \left(1 + \frac{3}{100\sqrt{7}}, 1 - \frac{1}{100\sqrt{7}} \right).$$

Drawing numerically the periodic orbit through this initial condition we obtain the periodic orbit of Figure 1.

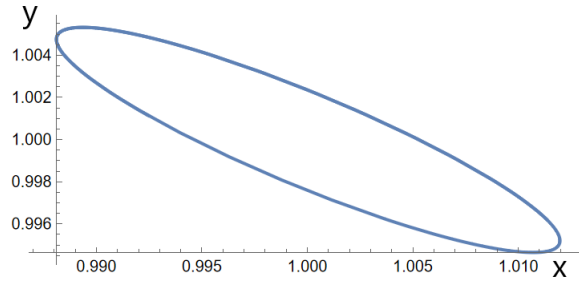


FIGURE 1. The limit cycle of system (2) bifurcating from the Hopf equilibrium $(1, 1)$ when $\varepsilon = 1/100$.

An isolated equilibrium point of a differential system in $\mathbb{R}^3$ whose linear part has eigenvalues $\pm\omega i$, $\omega > 0$ and 0 is called a *zero-Hopf equilibrium point*. As the Hopf bifurcation in $\mathbb{R}^2$ also in $\mathbb{R}^3$ small limit cycles can emerge from a zero-Hopf equilibrium point.

Particular Lotka-Volterra systems of degree 2 in $\mathbb{R}^3$ has been studied and the existence of limit cycles has been proved (see [6], [19]). Kolmogorov systems of degree 3 in $\mathbb{R}^3$ were studied in [11] showing that two limit cycles can bifurcate from a zero-Hopf equilibrium point, using the averaging theory of order one.

We also study the Kolmogorov systems of degree 4 in the space having the finite equilibria (x_0, y_0, z_0) in the positive octant, i.e. where these systems have biological sense. As before we can assume, using the rescaling $(x, y, z) \rightarrow (x/x_0, y/y_0, z/z_0)$, that this finite equilibrium point is located at $(1, 1, 1)$.

In short, a general Kolmogorov system of degree 4 in $\mathbb{R}^3$ with an equilibrium point at $(1, 1, 1)$ can be written as

$$\begin{aligned}
 \dot{x} &= x (a_1(x-1) + a_2(y-1) + a_3(z-1) + a_4(x-1)^2 + a_5(y-1)^2 + \\
 &\quad a_6(z-1)^2 + a_7(x-1)(y-1) + a_8(x-1)(z-1) + a_9(y-1)(z-1) + \\
 &\quad a_{10}(x-1)^3 + a_{11}(y-1)^3 + a_{12}(z-1)^3 + a_{13}(x-1)^2(y-1) + \\
 &\quad a_{14}(x-1)^2(z-1) + a_{15}(x-1)(y-1)^2 + a_{16}(y-1)^2(z-1) + \\
 &\quad a_{17}(z-1)^2(x-1) + a_{18}(z-1)^2(y-1)), \\
 \dot{y} &= y (b_1(x-1) + b_2(y-1) + b_3(z-1) + b_4(x-1)^2 + b_5(y-1)^2 + \\
 &\quad b_6(z-1)^2 + b_7(x-1)(y-1) + b_8(x-1)(z-1) + b_9(y-1)(z-1) + \\
 (3) \quad &\quad b_{10}(x-1)^3 + b_{11}(y-1)^3 + b_{12}(z-1)^3 + b_{13}(x-1)^2(y-1) + \\
 &\quad b_{14}(x-1)^2(z-1) + b_{15}(x-1)(y-1)^2 + b_{16}(y-1)^2(z-1) + \\
 &\quad b_{17}(z-1)^2(x-1) + b_{18}(z-1)^2(y-1)), \\
 \dot{z} &= z (c_1(x-1) + c_2(y-1) + c_3(z-1) + c_4(x-1)^2 + c_5(y-1)^2 + \\
 &\quad c_6(z-1)^2 + c_7(x-1)(y-1) + c_8(x-1)(z-1) + c_9(y-1)(z-1) + \\
 &\quad c_{10}(x-1)^3 + c_{11}(y-1)^3 + c_{12}(z-1)^3 + c_{13}(x-1)^2(y-1) + \\
 &\quad c_{14}(x-1)^2(z-1) + c_{15}(x-1)(y-1)^2 + c_{16}(y-1)^2(z-1) + \\
 &\quad c_{17}(z-1)^2(x-1) + c_{18}(z-1)^2(y-1)).
 \end{aligned}$$

For systems (3) we first characterize the conditions in order that the point $(1, 1, 1)$ be a zero-Hopf equilibrium point, and then we will study the existence of limit cycles bifurcating from this equilibrium point using the averaging theory of first order.

To enunciate the next theorem we need to introduce some notation. Let

$$\begin{aligned}
 \alpha &= b_2^3 c_1 - b_1 b_2^2 c_2 + 2b_2 b_3 c_1 c_2 - b_1 b_3 c_2^2 - b_1 b_2 c_2 c_3 + b_3 c_1 c_2 c_3 - b_1 c_2 c_3^2 + \\
 &\quad b_2 c_1 \omega^2 - b_1 c_2 \omega^2 \\
 \beta &= b_2^2 b_3 c_1 - b_1 b_2 b_3 c_2 + b_2^2 c_1 c_2 + b_2 b_3 c_1 c_3 - 2b_1 b_3 c_2 c_3 + b_3 c_1 c_3^2 - b_1 c_3^3 + \\
 &\quad b_3 c_1 \omega^2 - b_1 c_3 \omega^2, \\
 (4) \quad \gamma &= b_1 b_2 c_1 + b_3 c_1^2 - b_1^2 c_2 - b_1 c_1 c_3, \\
 s_1 &= b_3 c_2 \beta_{11} - b_2 c_3 \beta_{11} - b_3 c_1 \beta_{12} + b_1 c_3 \beta_{12} + b_2 c_1 \beta_{13} - b_1 c_2 \beta_{13}, \\
 s_2 &= j l s_1 + k l m \omega^3, \quad s_3 = k l \omega^3, \quad s_4 = 2 k m \omega + j s_1, \\
 \Delta &= (s_2 + m s_3)^2 + 8 j s_2 s_3, \quad \lambda_{\pm} = -\frac{s_2 + m s_3 \pm \sqrt{\Delta}}{2 j l \omega}.
 \end{aligned}$$

The coefficients j, k, l, m that appear in the definitions of s_2, s_3, s_4, Δ and $\lambda_{\pm}$ and also in the statement of the next theorem are defined later on in the averaged function of order one of systems (3).

Theorem 2. *Consider the Kolmogorov system (3) with $a_1 = -b_2 - c_3 + \varepsilon \beta_{11}$, $a_2 = -\alpha/\gamma + \varepsilon \beta_{12}$, $a_3 = -\beta/\gamma + \varepsilon \beta_{13}$, $s_1 s_4 \neq 0$, $k s_2 < 0$, then two small limit cycles bifurcate from the zero-Hopf equilibrium point $(1, 1, 1)$ of systems (3) when $\varepsilon = 0$. The kind of stability or inestability of these limit cycles is described in what follows.*

- (a) *If $s_1 > 0$, $s_4 m > 0$ and either $\Delta > 0$, $\lambda_- > 0, \lambda_+ > 0$; or $\Delta \leq 0$, $\text{Re}(\lambda_+) > 0$, $\text{Re}(\lambda_-) > 0$; then both limit cycles are locally unstable with unstable manifolds of dimension 3.*
- (b) *If $s_1 < 0$, $s_4 m < 0$ and either $\Delta > 0$, $\lambda_- < 0$, $\lambda_+ < 0$; or $\Delta \leq 0$, $\text{Re}(\lambda_-) < 0$, $\text{Re}(\lambda_+) < 0$; then both limit cycles are locally stable with a stable manifold of dimension 3.*
- (c) *If $s_1 s_4 m < 0$, $\Delta > 0$, $\lambda_- \lambda_+ < 0$, the two limit cycles are unstable, both have two local stable manifolds of dimension 2 and two local unstable manifolds of dimension 2, these manifolds of dimension 2 near the limit cycle are topological cylinders.*
- (d) *If either $s_1 > 0$, $s_4 m > 0, \Delta > 0$, $\lambda_- \lambda_+ < 0$; or $s_1 s_4 m < 0$, $\Delta > 0$, $\lambda_- > 0$, $\lambda_+ > 0$; or $s_1 s_4 m < 0$, $\Delta \leq 0$, $\text{Re}(\lambda_-) > 0$, $\text{Re}(\lambda_+) > 0$; then the two*

limit cycles are unstable. One of them has locally an unstable manifold of dimension 3 and the other has two local stable manifolds of dimension 2 and two local unstable manifolds of dimension 2, these manifolds of dimension 2 near the limit cycle are topological cylinders.

- (e) If either $s_1 > 0, s_4 m > 0, \Delta > 0, \lambda_- < 0, \lambda_+ < 0$; or $s_1 < 0, s_4 m < 0, \Delta > 0, \lambda_- > 0, \lambda_+ > 0$; or $s_1 > 0, s_4 m > 0, \Delta \leq 0, \operatorname{Re}(\lambda_-) < 0, \operatorname{Re}(\lambda_+) < 0$; or $s_1 < 0, s_4 m < 0, \Delta \leq 0, \operatorname{Re}(\lambda_-) > 0, \operatorname{Re}(\lambda_+) > 0$; then one limit cycle is locally stable with a stable manifold of dimension 3, and the other limit cycle is unstable with an unstable manifold of dimension 3.
- (f) If either $s_1 < 0, s_4 m < 0, \Delta > 0, \lambda_- \lambda_+ < 0$; or $s_1 < 0, s_4 m > 0, \Delta > 0, \lambda_- < 0, \lambda_+ < 0$; or $s_1 < 0, s_4 m > 0, \Delta \leq 0, \operatorname{Re}(\lambda_-) < 0, \operatorname{Re}(\lambda_+) < 0$; then one limit cycle is locally stable with a stable manifold of dimension 3 and the other limit cycle is locally unstable with two stable manifolds of dimension 2 and two local unstable manifolds of dimension 2, these manifolds of dimension 2 near the limit cycle are topological cylinders.

Theorem 2 is proved in Section 3, where the initial conditions for the two limit cycles of Theorem 2 are given.

Now we apply Theorem 2 for finding the two limit cycles bifurcating from the zero-Hopf equilibrium $(1, 1, 1)$ of the differential system

$$(5) \quad \begin{aligned} \dot{x} &= x \left(x^3 - 2x^2 + xz - \frac{299x}{100} + y^3 - 2y^2 - \frac{1097y}{100} + z^3 - 2z^2 + 2z + \frac{349}{25} \right), \\ \dot{y} &= y (x + y^3 - 3y^2 + yz + 4y + z^3 - 2z^2 - 3), \\ \dot{z} &= z (x^3 - 2x^2 + xz^2 - 2xz + 4x + y^3 - 2y^2 + yz + y + z^3 - 3z^2 + 3z - 4). \end{aligned}$$

According with Theorem 2 and its proof the two initial conditions for the two periodic orbits $(x_j(t, 1/100), y_j(t, 1/100), z_j(1/100))$ ($j = 1, 2$) bifurcating from the zero-Hopf equilibrium $(1, 1, 1)$ are

$$\left(\frac{29001}{29000}, \frac{57999}{58000}, \frac{57997}{58000} \right)$$

and

$$\left(\frac{34001}{34000} + \frac{3}{3400\sqrt{3485}}, \frac{67999}{68000} - \frac{1}{6800\sqrt{3485}}, \frac{47393909 - \sqrt{3485}}{47396000} \right).$$

Drawing numerically these two periodic orbits using these two initial conditions we obtain the two periodic orbits of Figure 2.

The averaging theory that we use for proving Theorems 1 and 2 is a classical tool that allows to study the dynamics of nonlinear periodic differential systems. We present in the Appendix A the theorems associated to the averaging theory of first and second order that we need for proving our Theorems 1 and 2.

2. PROOF OF THEOREM 1

The equilibrium point $(1, 1)$ of the Kolmogorov system (1) will be a Hopf equilibrium if the eigenvalues associated to its Jacobian matrix M are $\pm\omega i$, $\omega > 0$. To this purpose we need to impose that $a_1 = -b_2$, and $a_2 = -(b_2^2 + \omega^2)/b_1$, with $b_1 \neq 0$ because then the characteristic polynomial of the matrix M is $\lambda^2 + \omega^2$. Then the Jacobian matrix of system (1) at the equilibrium point $(1, 1)$ is

$$M = \begin{pmatrix} -b_2 & -\frac{b_2^2 + \omega^2}{b_1} \\ b_1 & b_2 \end{pmatrix},$$

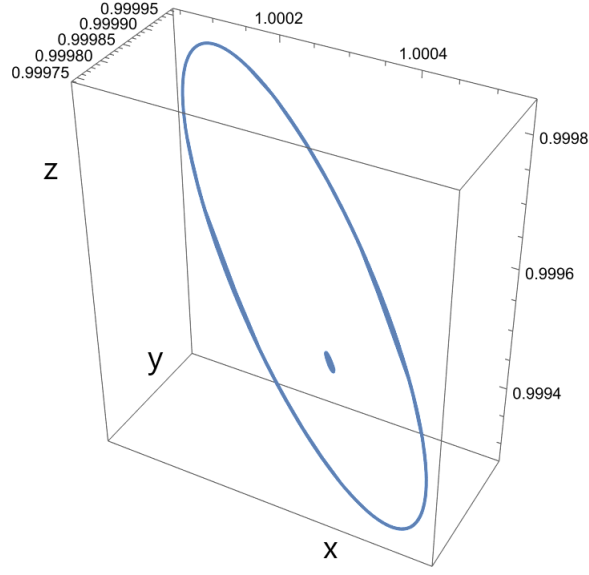


FIGURE 2. The two limit cycles of system (5) bifurcating from the zero-Hopf equilibrium $(1, 1, 1)$ when $\varepsilon = 1/100$.

with eigenvalues $\pm\omega i$.

Note that if $b_1 = 0$ never the eigenvalues of the equilibrium $(1, 1)$ are purely imaginary and we cannot have a Hopf bifurcation. In summary the condition on a_2 is well defined and covers all the cases where a Hopf equilibrium can exist.

Now we introduce a perturbation to the parameters a_1 and a_2 in order to analyze the existence of limit cycles that can bifurcate from the Hopf equilibrium point $(1, 1)$. Let ε be a sufficiently small parameter, and set

$$(6) \quad \begin{aligned} a_1 &= -b_2 + \varepsilon\alpha_{11} + \varepsilon^2\alpha_{12}, \\ a_2 &= -\frac{b_2^2 + \omega^2}{b_1} + \varepsilon\alpha_{21} + \varepsilon^2\alpha_{22}. \end{aligned}$$

To proceed with the analysis we need to translate the Hopf equilibrium point $(1, 1)$ to the origin of coordinates, for this we use the translation $(x, y) = (X + 1, Y + 1)$. After we express the linear part at the origin of the differential system $(\dot{X}, \dot{Y})$ into its real Jordan normal form J using the change of variables $(u, v) = P(X, Y)$, where the matrix P is such that $J = PMP^{-1}$, here

$$J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 \\ \frac{b_2 - b_1}{\omega} & \frac{-b_1b_2 + b_2^2 + \omega^2}{b_1\omega} \end{pmatrix}.$$

Now the differential system in the variables (u, v) becomes

$$(7) \quad \begin{aligned} \dot{u} &= A_1(u, v) + \varepsilon A_2(u, v) + \varepsilon^2 A_3(u, v) + O(\varepsilon^3), \\ \dot{v} &= B_1(u, v) + \varepsilon B_2(u, v) + \varepsilon^2 B_3(u, v) + O(\varepsilon^3), \end{aligned}$$

where

$$\begin{aligned}
A_2 &= \frac{1}{((b_1 - b_2)^2 + \omega^2)^2} (b_1^2 - b_1(b_2(u+2) + \omega v) + (u+1)(b_2^2 + \omega^2)) \\
&\quad (u(b_1 b_2)(\alpha_{21} b_1 - \alpha_{11} b_2) + b_1 \omega v(\alpha_{21} - \alpha_{11}) + \alpha_{11} \omega^2 u), \\
A_3 &= \frac{1}{((b_1 - b_2)^2 + \omega^2)^2} (b_1^2 - b_1(b_2(u+2) + \omega v) + (u+1)(b_2^2 + \omega^2)) \\
&\quad (u(b_1 - b_2)(\alpha_{22} b_1 - \alpha_{12} b_2) + b_1 \omega v(\alpha_{22} - \alpha_{12}) + \alpha_{12} \omega^2 u), \\
B_2 &= \frac{-(b_1 - b_2)}{\omega((b_1 - b_2)^2 + \omega^2)^2} (b_1^2 - b_1(b_2(u+2) + v\omega) + (u+1)(b_2^2 + \omega^2)) \\
&\quad (u(b_1 - b_2)(\alpha_{21} b_1 - \alpha_{11} b_2) + b_1 v\omega(\alpha_{21} - \alpha_{11}) + \alpha_{11} u\omega^2), \\
B_3 &= \frac{-(b_1 - b_2)}{\omega((b_1 - b_2)^2 + \omega^2)^2} (b_1^2 - b_1(b_2(u+2) + v\omega) + (u+1)(b_2^2 + \omega^2)) \\
&\quad (u(b_1 - b_2)(\alpha_{22} b_1 - \alpha_{12} b_2) + b_1 v\omega(\alpha_{22} - \alpha_{12}) + \alpha_{12} u\omega^2).
\end{aligned}$$

We do not provide the explicit expression of A_1 and B_1 because we will need several pages for writing them. But they can be obtained easily using an algebraic manipulator as mathematica. In fact we have provided the mathematica file where all the computations of this paper can be found. In particular the explicit expression of A_1 and B_1 .

After putting the differential system with the Hopf equilibrium point at the origin in its real Jordan form in the system (7), we proceed to write system (7) in the polar coordinates (r, θ) defined by $(u, v) = (r \cos \theta, r \sin \theta)$. Thus system (7) becomes

$$\begin{aligned}
(8) \quad \dot{r} &= C_1(r, \theta) + \varepsilon C_2(r, \theta) + \varepsilon^2 C_3(r, \theta), \\
\dot{\theta} &= D_1(r, \theta) + \varepsilon D_2(r, \theta) + \varepsilon^2 D_3(r, \theta),
\end{aligned}$$

where

$$\begin{aligned}
C_2 &= \frac{1}{\omega((b_1 - b_2)^2 + \omega^2)^2} [r((b_2 - b_1) \sin \theta + \omega \cos \theta)(r \cos \theta(-b_1 b_2 + b_2^2 + \omega^2) + (b_1 - b_2)^2 - b_1 \omega r \sin \theta + \omega^2)(\cos \theta((b_1 - b_2)(\alpha_{21} b_1 - \alpha_{11} b_2) + \alpha_{11} \omega^2) + b_1 \omega(\alpha_{21} - \alpha_{11}) \sin \theta)], \\
C_3 &= \frac{1}{\omega((b_1 - b_2)^2 + \omega^2)^2} [r((b_2 - b_1) \sin \theta + \omega \cos \theta)(r \cos \theta(-b_1 b_2 + b_2^2 + \omega^2) + (b_1 - b_2)^2 - b_1 \omega r \sin \theta + \omega^2)(\cos \theta((b_1 - b_2)(\alpha_{22} b_1 - \alpha_{12} b_2) + \alpha_{12} \omega^2) + b_1 \omega(\alpha_{22} - \alpha_{12}) \sin \theta)], \\
D_2 &= -\frac{1}{[\omega((b_1 - b_2)^2 + \omega^2)^2]} ((b_1 - b_2) \cos \theta + \omega \sin \theta)(r \cos \theta(-b_1 b_2 + b_2^2 + \omega^2) + (b_1 - b_2)^2 - b_1 \omega r \sin \theta + \omega^2)(\cos \theta((b_1 - b_2)(\alpha_{21} b_1 - \alpha_{11} b_2) + \alpha_{11} \omega^2) + b_1 \omega(\alpha_{21} - \alpha_{11}) \sin \theta)], \\
D_3 &= -\frac{1}{\omega((b_1 - b_2)^2 + \omega^2)^2} [((b_1 - b_2) \cos \theta + \omega \sin \theta)(r \cos \theta(-b_1 b_2 + b_2^2 + \omega^2) + (b_1 - b_2)^2 - b_1 \omega r \sin \theta + \omega^2)(\cos \theta((b_1 - b_2)(\alpha_{22} b_1 - \alpha_{12} b_2) + \alpha_{12} \omega^2) + b_1 \omega(\alpha_{22} - \alpha_{12}) \sin \theta)],
\end{aligned}$$

and we do not provide the explicit expression of C_1 and D_1 because we will need several pages for writing them, but their are in the mathematica file.

We take $R = r/\varepsilon$ as the new variable. And after that we write the differential system $(\dot{R}, \dot{\theta})$ as the differential equation $r' = \dot{r}/\dot{\theta}$, choosing the angular variable θ as the new independent variable or time. Now the prime denotes the derivative with respect to the new time θ .

We calculate R' and expand it in power series up to second order in the variable ε as follows

$$(9) \quad R' = \varepsilon F_1(R, \theta) + \varepsilon^2 F_2(R, \theta) + O(\varepsilon^3),$$

where F_1 and F_2 are huge polynomials in the variables R with coefficients functions depending on $\sin \theta$ and $\cos \theta$. Again we prefer not exhibit the functions F_1 and F_2 having degrees 2 and 3 in the variable R , respectively, due to their huge size.

The previous differential system (9) is written in the normal form (12) of the Appendix A, where $T = 2\pi$, $x = R$, $t = \theta$ and $\varepsilon^3 \rho(\theta, x, \varepsilon) = O(\varepsilon^3)$, then we can apply the averaging theory to system (9). The function F_1 is 2π -periodic in θ and is C^2 in the variable R in an open interval $I = \{R : 0 < R < R_*\}$, $R_* > 0$.

Now we can obtain the averaged function of first order applying Theorem 3 of Appendix A, and we get

$$f_1(R) = \frac{\alpha_{11}}{2\omega} R.$$

It is obvious than the averaged function of order one $f_1(R)$ has no solution in the interval I . Thus we did not find any small limit cycle bifurcating from the origin using the averaging theory of first order, and we will use the averaging theory of second order. Doing $\alpha_{11} = 0$ we obtain $f_1(R) \equiv 0$, and we can apply the averaging theory of order two.

In order to obtain the averaged function of second order, following (13) of the Appendix A, we need to consider $F_2(R, \theta)$ and $y_1 = R y_{11} + R^2 y_{12}$, where y_{11} and y_{12} are given in the Appendix B. Doing some computations we obtain the averaged function of order two

$$f_2(R) = R(A + B R^2),$$

where the expression of A and B are defined before the statement of Theorem 1.

We see that $R_0 = \sqrt{-A/B}$ is one positive real root in I of $f_2(R)$ under the assumption $AB < 0$ of Theorem 1. Furthermore under this hypothesis we have that $(df_2/dR)(R_0) = -2A \neq 0$. Then for $|\varepsilon|$ small enough, Theorem 3 implies that system (9) has one 2π -periodic solution namely $R(\theta, \varepsilon)$ that satisfies $R(0, \varepsilon) \rightarrow \sqrt{-A/B}$ when $\varepsilon \searrow 0$. About the stability of this limit cycle we have that this periodic solution is stable (resp. unstable) if $A > 0$ (resp. $A < 0$).

In order to recover the original variables, we go back through the changes of variables and get for the original system (1) a periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that $(x(0, \varepsilon), y(0, \varepsilon))$ is

$$\left(1 + \varepsilon \frac{\sqrt{-A/B}(-b_1 b_2 + b_2^2 + \omega^2)}{b_1^2 - 2b_1 b_2 + b_2^2 + \omega^2}, 1 + \varepsilon \frac{b_1 \sqrt{-A/B}(b_1 - b_2)}{b_1^2 - 2b_1 b_2 + b_2^2 + \omega^2} \right) + O(\varepsilon^2).$$

Note that when $\varepsilon \rightarrow 0$ the initial condition $(x(0, \varepsilon), y(0, \varepsilon))$ tends to the Hopf equilibrium point $(1, 1)$. Consequently, this family of periodic orbits parametrized by ε emerges from the equilibrium $(1, 1)$, so we have a Hopf bifurcation.

The previous analysis proves Theorem 1.

3. PROOF OF THEOREM 2

The proof of Theorem 2 essentially follows the same steps than the proof of Theorem 1.

In order that the equilibrium $(1, 1, 1)$ of the differential system (3) be a zero-Hopf equilibrium point, i.e. the eigenvalues of the Jacobian matrix M of system (3)

evaluated at $(1, 1, 1)$ be $0, \pm\omega i$, with $\omega > 0$, we need the conditions:

$$a_1 = -b_2 - c_3, \quad a_2 = -\alpha/\gamma, \quad a_3 = -\beta/\gamma,$$

where the expressions of α, β and γ were defined in (4). Note that under these conditions the characteristic polynomial of the matrix M is $\lambda(\lambda^2 + \omega^2)$.

To analyze the number of limit cycles that can bifurcate from the zero-Hopf equilibrium point $(1, 1, 1)$ we consider a parameter ε sufficiently small and introduce the perturbation

$$a_1 = -b_2 - c_3 + \varepsilon\beta_{11}, \quad a_2 = -\alpha/\gamma + \varepsilon\beta_{12}, \quad a_3 = -\beta/\gamma + \varepsilon\beta_{13}.$$

We transform system (3) in the way that the zero-Hopf equilibrium point be at the origin of coordinates using the translation $(x, y, z) = (X + 1, Y + 1, Z + 1)$, and after we write the linear part of system $(\dot{X}, \dot{Y}, \dot{Z})$ at $(0, 0, 0)$ in its real Jordan normal form J_3 using the change of variables $(u, v, w) = P_3(X, Y, Z)$, where

$$J_3 = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & -\frac{d_1}{d_2} \\ -\frac{\beta_1}{d_2\omega} & \frac{\alpha d_2 + c_2 d_1 \gamma}{d_2 \gamma \omega} & \sigma \\ 1 & \frac{c_1 \omega^2 - \beta_1}{\gamma} & -\frac{\beta_2}{\gamma} \end{pmatrix}.$$

Here we have rewritten β as $\beta = -b_3\beta_1 - c_3\beta_2 + b_3c_1\omega^2$, and where $\sigma = (-b_1d_2\alpha + c_1c_3d_1\gamma + d_2\gamma((b-2+c_3)^2 + d_1d_2))$, $d_1 = b_3c_2 - b_2c_3$ and $d_2 = b_2c_1 - b_1c_2$. Now the differential system $(\dot{u}, \dot{v}, \dot{w})$ has its linear part at the origin in its real Jordan normal form. All details are given in mathematica file.

In $\mathbb{R}^3$ we introduce the cylindrical coordinates (r, θ, w) defined by $(u, v, w) = (r \sin \theta, r \cos \theta, w)$. The differential system $(\dot{u}, \dot{v}, \dot{w})$ in the new coordinates (r, θ, w) becomes the differential system $(\dot{r}(r, \theta, w), \dot{\theta}(r, \theta, w), \dot{w}(r, \theta, w))$, that we do not write due to its huge size. See the attached mathematica file.

The following step is to introduce the change of variables $(r, w) = (R\varepsilon, W\varepsilon)$. After that we arrive to a new differential system that has the form

$$\begin{aligned} \dot{R} &= \dot{r}/\varepsilon = P_{11}(R, \theta, W)\varepsilon + O(\varepsilon^2), \\ \dot{\theta} &= \dot{\theta} = \omega + O(\varepsilon), \\ \dot{W} &= \dot{w}/\varepsilon = P_{12}(R, \theta, W)\varepsilon + O(\varepsilon^2), \end{aligned}$$

where P_{ij} are C^2 functions, 2π -periodic in θ . Finally considering the variable θ as the new time we get the new differential system:

$$(10) \quad R' = \frac{\dot{R}}{\dot{\theta}} = \varepsilon F_{11}(R, W, \theta) + O(\varepsilon^2), \quad W' = \frac{\dot{W}}{\dot{\theta}} = \varepsilon F_{12}(R, W, \theta) + O(\varepsilon^2),$$

the prime denotes derivative with respect to the variable θ . See the expressions of F_{11} and F_{12} in the attached mathematical file.

Now we can apply the averaging theory to system (10) because it is in the normal form (12) of Theorem 3 with $T = 2\pi$, $x = (R, W)$, $t = \theta$ and $\varepsilon^2 \rho(\theta, x, \varepsilon) = O(\varepsilon^2)$. In this case the functions of system (10) are C^2 in the variables (R, W, θ) and 2π -periodic in θ .

The next step is to calculate the averaged function f_1 of order one using Theorem 3. Since we are in $\mathbb{R}^3$ the averaged function of order one is

$$\begin{aligned} f_1(R, W) &= (f_{11}(R, W), f_{12}(R, W)) \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{F_{11}(R, W, \theta)}{\omega} d\theta, \frac{1}{2\pi} \int_0^{2\pi} \frac{F_{12}(R, W, \theta)}{\omega} d\theta \right) \\ &= \left(R(k + jW), lR^2 - \frac{s_1}{\omega^3} W + mW^2 \right), \end{aligned}$$

the coefficients j, k, l, m are too long to explicit them here, see the Mathematica file. We can choose the parameters of system (3) in a such way that $j k l m s_1 \neq 0$.

Looking for solutions of $f_1(R, W) = (0, 0)$ we get that for the first component function, the equation $f_{11}(R, W) = 0$ has solution for $R_1 = 0$, or for $W_2 = -k/j$, if $kj \neq 0$. Considering s_1 and s_2 as in (4) we get for $R_1 = 0$ that the second component function $f_{12}(R, W)$ is zero for $W_1 = s_1/(m\omega^3)$ if $s_1 m \neq 0$. For the case $W = W_2$ the second component function $f_{12}(R, W) = 0$ has the solution $R_2 = (\sqrt{-ks_2})/(jl\omega^{3/2})$, which is positive if $ks_2 < 0$ and $jl > 0$. Thus we have two solutions $(R_1, W_1) = (0, s_1/(m\omega^3))$ and $(R_2, W_2) = (\sqrt{-ks_2}/(jl\omega^{3/2}), -k/j)$ for system $f_1(R, W) = (0, 0)$ with $R_{1,2} \leq 0$.

To prove the existence of associated limit cycles we need to show that the Jacobian of the function $f_1(R, W)$ exists and is not zero at these two solutions. Since the Jacobian matrix of $f_1(R, W)$ is

$$(11) \quad C = \begin{pmatrix} jW + k & jR \\ 2lR & 2mW - \frac{s_1}{\omega^3} \end{pmatrix},$$

we get that the determinant of the Jacobian matrix at the solution (R_1, W_1) is equal to $s_1 s_4/(m\omega^6)$, and at the solution (R_2, W_2) it is equal to $2ks_2/(j\omega^3)$, where s_4 was given in (4). Then we get for the solution (R_1, W_1) that its determinant is not zero because from the assumptions of Theorem 2 $s_1 s_4 \neq 0$, and their associated eigenvalues are equal to s_1/ω^3 and $s_4/(m\omega^3)$. Thus for $|\varepsilon|$ sufficiently small, system (10) has one 2π -periodic solutions $(R_1(\theta, \varepsilon), W_1(\theta, \varepsilon))$ such that $(R_1(0, \varepsilon), W_1(0, \varepsilon)) \rightarrow (R_1, W_1)$ when $\varepsilon \rightarrow 0$.

We can analyze the stability of the periodic solution $(R_1(\theta, \varepsilon), W_1(\theta, \varepsilon))$ computing the eigenvalues of the Jacobian matrix C at (R_1, W_1) , that correspond to the characteristic exponent of the Poincaré map defined in a neighborhood of the periodic solution. Since $\omega > 0$ we have that if $s_1 > 0$ and $s_4 m > 0$, then the periodic solution is unstable having an unstable manifold of dimension three. If $s_1 < 0$ and $s_4 m > 0$, or $s_1 > 0$ and $s_4 m < 0$, then the periodic solution is unstable having an unstable and a stable manifold, both of dimension two. And the periodic solution is stable with a stable manifold of dimension three if $s_1 < 0$ and $s_4 m < 0$.

On the other hand, the determinant of matrix C (see (11)) evaluated at (R_2, W_2) is not zero for $s_4 \neq 0$, then for $|\varepsilon|$ sufficiently small, system (10) has a second 2π -periodic solution $(R_2(\theta, \varepsilon), W_2(\theta, \varepsilon))$ such that $(R_2(0, \varepsilon), W_2(0, \varepsilon)) \rightarrow (R_2, W_2)$ when $\varepsilon \rightarrow 0$. The eigenvalues associated to the matrix C evaluated at (R_2, W_2) are equal to $\lambda_{\pm} = -\frac{s_2 + ms_3 \pm \sqrt{\Delta}}{2jl\omega}$, where $\Delta = (s_2 + ms_3)^2 + 8js_2s_3$ (s_3 is given in (4)).

We have that if $\Delta > 0$ and $\lambda_- \lambda_+ < 0$, then the periodic solution is unstable with an unstable and a stable manifold of dimension two. The periodic solution is stable if $\Delta > 0$ and $\lambda_{\pm} > 0$, or $\Delta \leq 0$ and $Re(\lambda_{\pm}) = -(s_2 + ms_3)/(2jl\omega) > 0$, having a stable manifold of dimension three. Finally, the periodic solution is unstable with

an unstable manifold of dimension three if $\Delta > 0$ and $\lambda_{\pm} < 0$, or $\Delta \leq 0$ and $\text{Re}(\lambda_{\pm}) < 0$.

Using the previous information we get that if $s_1 s_4 \neq 0$, then system (10) has two 2π -periodic solutions $(R_1(\theta, \varepsilon), W_1(\theta, \varepsilon))$ and $(R_2(\theta, \varepsilon), W_2(\theta, \varepsilon))$ for $|\varepsilon|$ small enough such that $(R_j(0, \varepsilon), W_j(0, \varepsilon)) \rightarrow (R_j, W_j)$ for $j = 1, 2$ when $\varepsilon \searrow 0$.

From the analysis of the eigenvalues of the matrix C evaluated at both solutions (R_1, W_1) and (R_2, W_2) , and considering that if $\lambda_- < 0$ and $\lambda_+ < 0$, then $s_1 \omega^3 < 0$, we have proved in Theorem 2 statements (a)–(f).

Finally, going back through the changes of variables we obtain two periodic solutions $(x_j(t, \varepsilon), y_j(t, \varepsilon), z_j(t, \varepsilon))$ ($j = 1, 2$) bifurcating from the zero-Hopf equilibrium $(1, 1, 1)$ at $\varepsilon = 0$ such that

$$x_j(0, \varepsilon) = 1 + \varepsilon \frac{X_j}{D} + O(\varepsilon^2), \quad y_j(0, \varepsilon) = 1 - \varepsilon \frac{Y_j}{D} + O(\varepsilon^2), \quad z_j(0, \varepsilon) = 1 - \varepsilon \frac{Z_j}{D} + O(\varepsilon^2),$$

where

$$\begin{aligned} X_j &= \gamma c_1 d_2^2 R_j \sigma \omega^3 + (\gamma c_2 d_1 + \alpha d_2)(\beta_2 d_2 R_j - \gamma d_1 W_j) - \beta_1 \gamma d_2^2 R_j \sigma \omega, \\ Y_j &= \gamma (\beta_1 \gamma d_1 W_j + \gamma d_2^2 \sigma \omega (R_j - W_j) - \beta_1 \beta_2 d_2 R_j), \\ Z_j &= \gamma d_2 (R_j (\beta_1^2 - \beta_1 c_1 \omega^2 - \gamma c_2 d_1) + \gamma c_2 d_1 W_j + \alpha d_2 (W_j - R_j)), \\ D &= \gamma d_1 (\beta_1^2 - \beta_1 c_1 \omega^2 - \gamma c_2 d_1) + d_2^2 (\alpha \beta_2 + \gamma \sigma \omega (c_1 \omega^2 - \beta_1)) + \gamma d_1 d_2 (\beta_2 c_2 - \alpha), \end{aligned}$$

for $j = 1, 2$.

This complete the proof of Theorem 2.

APPENDIX A. AVERAGING THEORY

We give a summary of the averaging theory of first and second order, this theory provides sufficient conditions for the existence of periodic orbits for a periodic differential system depending on a small parameter being a good tool for studying the existence of limit cycles. The averaging theory has a long history started with the classical works of Lagrange and Laplace. See [3, 12, 21] for additional details and for the proof of the next theorem.

Theorem 3. *Let*

$$(12) \quad \dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 \rho(t, x, \varepsilon),$$

be a differential system in $\mathbb{R}^n$, where the functions $F_1, F_2: \mathbb{R} \times D \rightarrow \mathbb{R}^n$, $\rho: \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ are continuous and T -periodic in the first variable, having D an open subset of $\mathbb{R}^n$. Suppose that the following assumptions hold:

- (a) $F_1(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R and $D_x F_1$ are locally Lipschitz with respect to x , and ρ is differentiable with respect to ε . We define the averaged functions of first and second order $f_1, f_2: D \rightarrow \mathbb{R}^n$ as

$$(13) \quad \begin{aligned} f_1(z) &= \int_0^T F_1(s, z) ds, \\ f_2(z) &= \int_0^T [D_z F_1(s, z) y_1(s, z) + F_2(s, z)] ds \end{aligned}$$

where

$$y_1(s, z) = \int_0^s F_1(t, z) dt.$$

- (b) Let $V \subset D$ be an open and bounded set and for $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$ assume that $f_1(z) \not\equiv 0$ and there is an $a \in V$ such that $f_1(a) = 0$ and $d_B(f, a) \neq 0$, where $d_B(f, a)$ is the Brouwer degree of the function $f_1: V \rightarrow \mathbb{R}^n$ at the

point a . If $f_1(z) \equiv 0$ and $f_2(z) \not\equiv 0$, and there is an $a \in V$ such that $f_2(a) = 0$ with $d_B(f_2, a) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small, there exists an isolated T -periodic solution $x(t, \varepsilon)$ of the system such that $x(0, \varepsilon) \rightarrow a$ when $\varepsilon \rightarrow 0$. The kind of stability or instability of the limit cycle $x(t, \varepsilon)$ is given by the eigenvalues of the Jacobian matrix $D_2(f_1(z) + \varepsilon f_2(z))|_{z=a}$ that provide the stability or instability of the Poincaré map associated to the periodic orbit $x(t, \varepsilon)$.

Note that a sufficient condition for showing that the Brouwer degree of a function f at a point a is nonzero, is that the Jacobian of the function f at a be nonzero when it is defined, see [13].

When the first part of assumption (b) holds (i.e. $f_1(z) \not\equiv 0$) Theorem 3 provides the averaging theory of first order, and it provides the averaging theory of second order when the second part of assumption (b) holds (i.e. $f_1(z) \equiv 0$ and $f_2(z) \not\equiv 0$).

APPENDIX B. NOTATIONS

In this section we introduce the notation of y_{11} and y_{12} introduced in the statement and proof of Theorem 2.

$$y_{11} = -\frac{a_{21}b_1 \sin \theta (\sin \theta (b_1^2 - 2b_1b_2 + b_2^2 - \omega^2) - 2\omega(b_1 - b_2) \cos \theta)}{2\omega^2(b_1^2 - 2b_1b_2 + b_2^2 + \omega^2)}$$

$$y_{12} = \frac{1}{12b_1\omega^2(b_1^2 - 2b_1b_2 + b_2^2 + \omega^2)^2} [4((b_3 - b_1)\omega^6 + (a_4b_1^2 + 4b_2b_1^2 + 2b_5b_1^2 + a_3(b_2 - 3b_1)b_1 - b_2(2b_2 + 3b_3 + b_4)b_1 + 3b_2^2b_3)\omega^4 - (b_1 - b_2)^2(b_1^3 + (-2b_2 + b_4 - 3b_5)b_1^2 + 2a_3(b_1 - b_2)b_1 + b_2(b_2 + 2b_4)b_1 - 3b_2^2b_3)\omega^2 - a_5b_1^3(b_1 - b_2)^3 + (b_1 - b_2)^3b_2(a_4b_1^2 - b_5b_1^2 - b_2(a_3b_1 - b_4b_1 + b_2b_3))) + 3((b_1 - b_2)^2 + \omega^2)((b_1 - b_3)\omega^4 + (b_1^3 - a_4b_1^2 - 4b_2b_1^2 + b_4b_1^2 - 3b_5b_1^2 + b_2^2b_1 + a_3(3b_1 - b_2)b_1 + b_2b_3b_1 + b_2b_4b_1 - 2b_2^2b_3)\omega^2 + (b_1 - b_2)(a_5b_1^3 + b_2(-a_4b_1^2 + b_5b_1^2 + b_2(a_3b_1 - b_4b_1 + b_2b_3)))) \cos \theta + ((b_1 - b_3)\omega^6 + (-6b_1^3 - a_4b_1^2 + 2b_2b_1^2 + 3b_3b_1^2 - 3b_4b_1^2 + b_5b_1^2 + 2b_2^2b_1 + a_3(3b_1 - b_2)b_1 + 3b_2b_3b_1 + b_2b_4b_1 - 3b_2^2b_3)\omega^4 + (b_1 - b_2)(b_1^4 + 3a_4b_1^3 - 3a_5b_1^3 + 3b_2b_1^3 + b_4b_1^3 - 3b_5b_1^3 - 3b_2^2b_1^2 - 3b_2b_3b_1^2 + 4b_2b_4b_1^2 - b_2^3b_1 - a_3(b_1^2 + 4b_2b_1 - 2b_2^2)b_1 - 3b_2^2b_3b_1 - 2b_2^2b_4b_1 + 3b_2^2b_3)\omega^2 + (b_1 - b_2)^3(a_5b_1^3 + b_2(-a_4b_1^2 + b_5b_1^2 + b_2(a_3b_1 - b_4b_1 + b_2b_3)))) \cos(3\theta) + 3b_1\omega((b_1 - b_2)^2 + \omega^2)(-b_2^3 + 3a_3b_2^2 + b_3b_2^2 + b_4b_2^2 + b_1^2b_2 - 2a_3b_1b_2 - 2b_1b_4b_2 - 2b_1b_5b_2 + a_5b_1^2 + (3a_3 - b_2 + b_3 + b_4)\omega^2 + a_4b_1(b_1 - 2b_2) + 3b_1^2b_5) \sin \theta + b_1\omega((a_3 - 4b_1 + b_2 + 3b_3 - b_4)\omega^4 - (-4b_1^3 + a_5b_1^2 + b_3b_1^2 - 3b_4b_1^2 + 6b_2^2b_1 + a_4(2b_2 - 3b_1)b_1 + 6b_2b_3b_1 + (3b_1 - 2b_2)b_5b_1 - 2b_2^2 + a_3(3b_1^2 - 2b_2^2) - 6b_2^2b_3 + 2b_2^2b_4)\omega^2 - (b_1 - b_2)^2(-3a_5b_1^2 + a_4(b_1 + 2b_2)b_1 - (b_1 + 2b_2)b_5b_1 + b_2(b_1^2 + 2b_4b_1 - a_3(2b_1 + b_2) - b_2(b_2 + 3b_3) + b_2b_4))) \sin(3\theta)]$$

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¹ DEPARTAMENT DE MATEMÀTIQUES, FACULTAT DE CIÈNCIES UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

Email address: jllibre@mat.uab.cat

² DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS, UNIVERSIDAD DEL Bío-Bío, CASILLA 5-C, CONCEPCIÓN, CHILE

Email address: ymartinez@ubiobio.cl