

CENTERS OF CUBIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. An equilibrium point p of a differential system in the plane $\mathbb{R}^2$ is a center if there exists a neighborhood U of p such that $U \setminus \{p\}$ is filled with periodic orbits. A difficult classical problem in the qualitative theory of differential systems in the plane is the problem of distinguishing between a focus and a center.

In this paper we characterize when the origin of coordinates is a center of the following cubic polynomial differential systems

$$\dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_2xy + a_3y^2 + A,$$

where A is an arbitrary nonzero monomial of degree 3. Moreover we provide all topologically different phase portraits when $A = a_4x^3$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The idea of a center first emerged in Huygens work 1656 on the pendulum clock (see [14, 19]). However, it was not until 1881, with the research of Poincaré (see [20]), and later on 1908, with the work of Dulac (see [8]), that the concept of a center was clearly and rigorously defined.

The problem of the center can be formulated as follows: Determine within the class of real planar polynomial vector fields all the ones possessing a center.

A polynomial differential system in the plane $\mathbb{R}^2$ of degree n is a differential system of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

being P and Q polynomials in the variables x and y . The degree n of the polynomial differential system is the maximum of the degrees of the polynomials P and Q . As usual the dot denotes derivative with respect to the time t .

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Poincaré inquired about determining the class of planar polynomial differential systems that have non-degenerate equilibria (i.e., with non-zero eigenvalues) which are centers. To achieve this, it is sufficient to consider those systems with equilibria having purely imaginary eigenvalues. Such an equilibrium can only be a focus or a center. Poincaré's theorem states that a necessary and sufficient condition in order that a polynomial differential system has a center at an equilibrium point with purely imaginary eigenvalues is the existence of a local analytic first integral in the neighborhood of the equilibrium point. Lyapunov extended this result to analytic vector fields. The search for an analytic first integral led Poincaré/Lyapunov to develop an algorithm for computing quantities (Lyapunov constants) $V_i, i = 1, 2, 3, \dots$ of the equilibrium point. A necessary and sufficient condition for having a center is then the vanishing of all these constants.

The classification of centers has been done just for polynomial differential system of degree two by the authors Kapteyn [15] and Bautin [4]. For higher degrees only exist partial results. For a polynomial differential system of degree two to annihilate all constants V_i it suffices to have $V_1 = V_3 = V_5 = 0$. For cubic differential systems, the problem remains open and highly challenging, but there are many partial results for the cubic centers, see for instance without trying to be exhaustive some papers from 2019 up to now [1, 5, 6, 7, 11, 13, 16, 17, 18, 22, 23].

While there exists an algorithm to compute the Lyapunov constants for equilibria with purely imaginary eigenvalues, determining the conditions under which the annihilation of some of these constants implies the annihilation of all remains unresolved for cubic or higher degree polynomial differential systems. This annihilation gives rise to an algebraic set known as the center variety in the space of all coefficients of the polynomial vector field. In this paper we consider four cubic families of polynomial vector fields and for these we classify their centers. For more details on the Lyapunov constants see [21]. Sometimes the computations of the Lyapunov constants are easier using complex notations, see for instance [12].

Our principal result in this paper is given in the following theorem:

Theorem 1. *Consider the differential system*

$$(1) \quad \dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_2xy + a_3y^2 + A,$$

where A is a nonzero monomial of degree 3. Then the following statements holds.

- (a) *If $A = a_4x^3$, then system (1) has a center at the origin of coordinates if, and only if either $a_2 = 0$, or $a_1 = a_3 = 0$.*
- (b) *If $A = a_4x^2y$, then system (1) has a center at the origin of coordinates if, and only if $a_4 = a_1a_2$ and $a_3 = 0$.*
- (c) *If $A = a_4xy^2$, then system (1) has a center at the origin of coordinates if, and only if either $a_2 = 0$ or $a_1 = a_3 = 0$.*
- (d) *If $A = a_4y^3$, then system (1) never has a center at the origin.*

Theorem 1 is proved in Sections 3 to 6.

We provide the global phase portraits in the Poincaré disc of the differential systems (1) with $A = a_4x^3$.

Theorem 2. *The topological phase portraits of the differential systems (1) with $A = a_4x^3$ having a center at the origin of coordinates are given in Figures 1 and 2.*

Theorem 2 is proved in Section 7.

In Section 2 we recall some basic results.

2. PRELIMINARY RESULTS

2.1. Equilibrium points. A point $(x_0, y_0) \in \mathbb{R}^2$ is an equilibrium point of the differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, if, and only if $P(x_0, y_0) = Q(x_0, y_0) = 0$.

The equilibrium point (x_0, y_0) is *hyperbolic* if, and only if all eigenvalues of the Jacobian matrix of the differential system evaluated at (x_0, y_0) have real part different from zero.

The equilibrium point (x_0, y_0) is *semi-hyperbolic* if, and only if the Jacobian matrix of the differential system evaluated at (x_0, y_0) presents only one eigenvalue equal to zero.

The equilibrium (x_0, y_0) is *nilpotent* if, and only if both eigenvalues of the Jacobian matrix of the differential system evaluated at (x_0, y_0) are zero, but the Jacobian matrix is not identically zero.

The equilibrium (x_0, y_0) is *linearly zero* if, and only if its Jacobian matrix is identically zero.

The classification of the local phase portraits of the hyperbolic, semi-hyperbolic and nilpotent equilibrium points can be found in Theorems 2.15, 2.19 and 3.5 (of [3]) of [10], respectively. These points are called non-degenerate.

When an equilibrium point is a linearly zero equilibrium point its local phase portrait can be studied using vertical blow ups. For more details on the blow up method, see for instance [2].

2.2. Vertical blow up. Consider a real planar polynomial differential system given by

$$(2) \quad \dot{x} = P(x, y) = P_n(x, y) + \dots, \quad \dot{y} = Q(x, y) = Q_n(x, y) + \dots,$$

with P and Q being coprime polynomials, P_n and Q_n being homogeneous polynomials of degree n and the dots represent higher order terms in the variables x and y . Since $n > 0$ the origin is an equilibrium point of system (2). Moreover suppose that the origin is a linearly zero equilibrium point, i.e., $n \geq 2$. Then we shall study its local phase portrait using vertical blow ups.

When we want to do a vertical blow up, we do the change of variables $(x, y) = (u, uv)$. This change of variables send the origin to the straight line $u = 0$, so studying the equilibria of the new system in the variables (u, v) on $u = 0$ we shall get the local phase

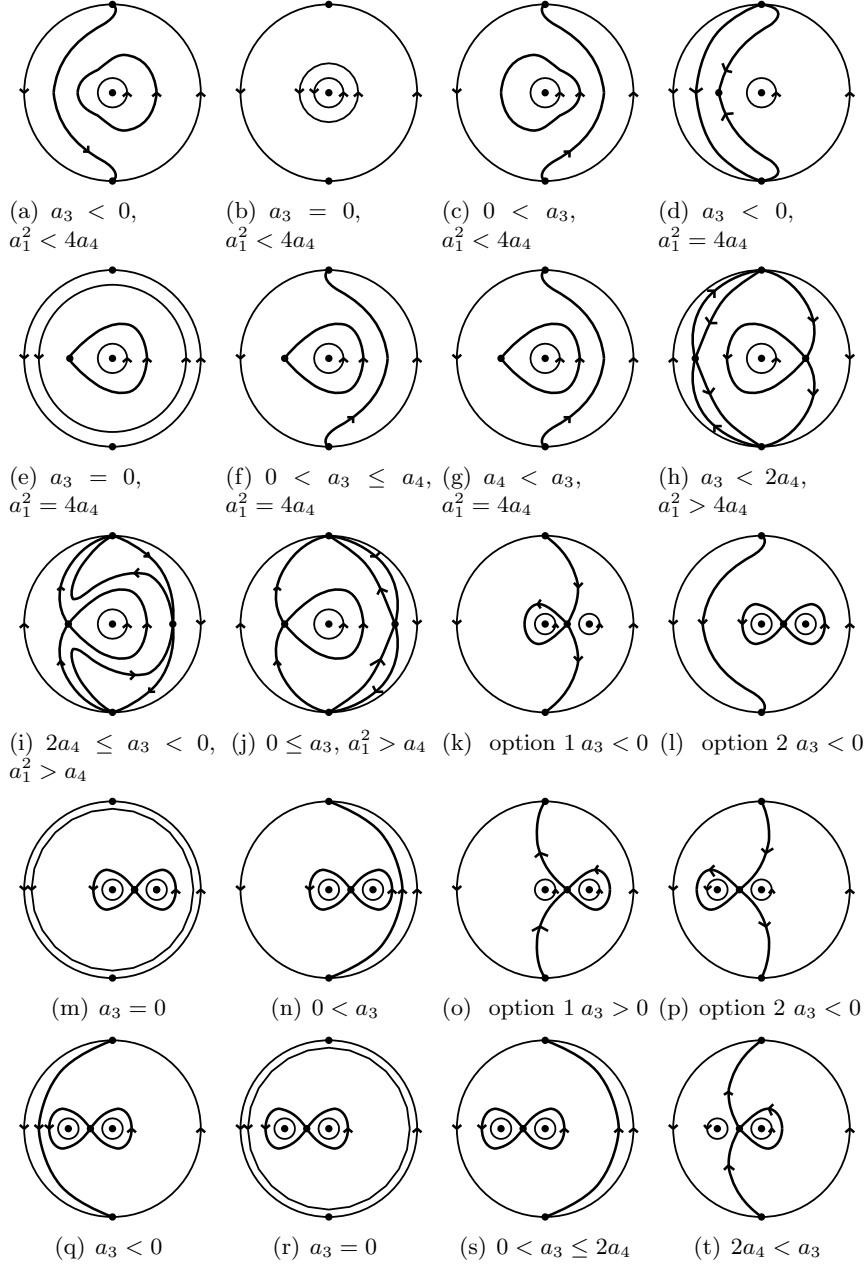
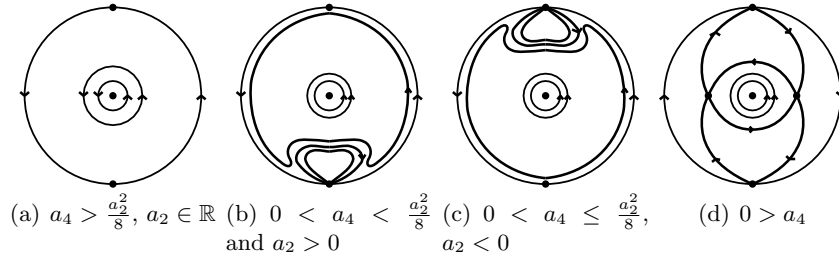


FIGURE 1. $a_2 = 0$. In figures (k)–(o) we have $a_1 < 0$ and $0 < a_4 < \frac{a_1^2}{4}$. In figures (p)–(t) we have $a_1 > 0$ and $0 < a_4 < \frac{a_1^2}{4}$.

portrait of origin in the variables (x, y) , always that in the new system the equilibrium points on $u = 0$ are all non linearly zero. If exists a linearly zero equilibrium point on

FIGURE 2. $a_1 = a_3 = 0$.

$u = 0$ we need to use again another vertical blow up, after translate the linearly zero point to the origin, in case that the equilibrium point is not at the origin.

The change of coordinates $(x, y) = (u, uv)$ is good if $x = 0$ is not a characteristic direction, i.e., if x is not a factor of the homogeneous polynomial

$$P_n(x, y)y - Q_n(x, y)x.$$

If x is factor of the last homogeneous polynomial, we do the change of variables $(x, y) = (u + \alpha v, v)$ with $\alpha \neq 0$. Note that this change of coordinates just do a rotation (twist) of the axis $x = 0$, which goes to the straight line $u = -\alpha v$. When the new system has not $u = 0$ as characteristic direction we can do a vertical blow up $(u, v) = (u_1, u_1 v_1)$, and study the local phase portraits at the equilibrium points on the axis $u_1 = 0$ to get the local phase portraits at the origin in (x, y) , always that all equilibrium points on the axis $u_1 = 0$ are not linearly zero, otherwise we need to do additional vertical blow ups, after moving the linearly zero points to the origin of coordinates. It is known that after a finite number of vertical blow ups (see [9]) we get a system with all points are non linearly zero, so we can found the local phase portrait of the origin in the variables (x, y) .

2.3. The Poincaré compactification. A powerful tool to study the global phase portrait of a polynomial differential system in $\mathbb{R}^2$ is the Poincaré compactification. this technique involves to extend the planar system to the unit sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$.

Consider the plane $\mathbb{R}^2$ tangent to the point $(0, 0, 1)$ of the sphere $\mathbb{S}^2$, we project each point of this plane onto the sphere using the straight line connecting the point and the origin of coordinates. The intersection of this line with the sphere establishes a relationship between points on the sphere and points on the plane, as illustrated in Figure 3. This process yields to two equivalent differential systems to the original one on the plane; one on the northern hemisphere $H_N = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_3 > 0\}$, and another on the southern hemisphere $H_S = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_3 < 0\}$.

This method was introduced by Poincaré in [20]. We formalize the concept in general terms in the following paragraphs. For more details, see Chapter 5 of [10].

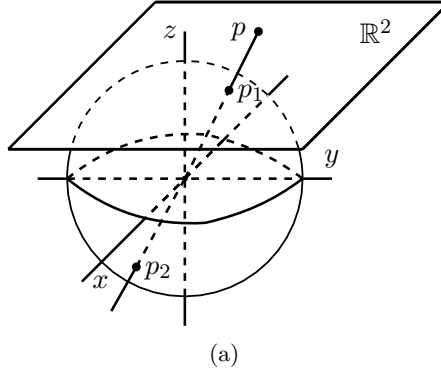


FIGURE 3. The Poincaré sphere, the point p in the tangent plane to the point $(0,0,1)$ of the sphere has two projections on the sphere p_1 and p_2 .

Consider the polynomial differential system

$$(3) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where P and Q are polynomial being d the maximum of the degrees of the polynomials P and Q .

We consider the plane $\{(x_1, x_2, 1); x_1, x_2 \in \mathbb{R}\}$ of $\mathbb{R}^3$ identified with the plane $\mathbb{R}^2$, where we have the differential system (3). This plane is tangent at the nord pole $(0,0,1)$ of the 2-dimensional sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$. We define the hemispheres H_N , H_S and the equator $\mathbb{S}^1 \equiv \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_3 = 0\}$ of the sphere $\mathbb{S}^2$.

In order to study a vector field over $\mathbb{S}^2$ we consider six local charts in $\mathbb{S}^2$, $\varphi_i : U_i \rightarrow \mathbb{R}^2$ and $\psi_i : V_i \rightarrow \mathbb{R}^2$, $i = 1, 2, 3$, given by

$$\varphi_i(x_1, x_2, x_3) = \psi_i(x_1, x_2, x_3) = \left(\frac{x_j}{x_i}, \frac{x_k}{x_i} \right),$$

with $j, k \neq i$ and $j < k$. The sets (U_i, φ_i) and (V_i, ψ_i) are the *local charts* over $\mathbb{S}^2$, where

$$U_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_i > 0\} \text{ and } V_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_i < 0\}.$$

Let $P^{N,S} : \mathbb{R}^2 \rightarrow H_{N,S}$ be the central projections from the tangent plane $\mathbb{R}^2 = (x_1, x_2, 1)$ to $\mathbb{S}^2$ given by

$$P^{N,S}(x_1, x_2) = \pm \left(\frac{x_1}{\Delta}, \frac{x_2}{\Delta}, \frac{1}{\Delta} \right),$$

where $\Delta = \sqrt{x_1^2 + x_2^2 + 1}$. The maps $P^{N,S}$ induces over $H_{N,S}$ vector fields analytically conjugate with the vector field of the differential system (3). Indeed, P^N induces on $H_N = U_3$ the vector field $X_1(P^N(x_1, x_2, 1)) = DP^N(x_1, x_2, 1)X(x_1, x_2, 1)$, and P^S induces on $H_S = V_3$ the vector field $X_2(P^S(x_1, x_2, 1)) = DP^S(x_1, x_2, 1)X(x_1, x_2, 1)$. Thus we obtain a vector field on $\mathbb{S}^2 \setminus \mathbb{S}^1$ that admits an analytic extension $p(X)$ to the

whole $\mathbb{S}^2$. The vector field $p(X)$ on $\mathbb{S}^2$ is called the *Poincaré compactification* of the vector field $X = (P, Q)$.

Denote $(u, v) = \varphi_i(x_1, x_2, x_3) = \psi_i(x_1, x_2, x_3)$. Then the expression of the differential system associated to the vector field $p(X)$ in the chart U_1 is

$$u' = v^d \left[Q \left(\frac{1}{v}, \frac{u}{v} \right) - uP \left(\frac{1}{v}, \frac{u}{v} \right) \right], \quad v' = -v^{d+1} P \left(\frac{1}{v}, \frac{u}{v} \right).$$

The expression of $p(X)$ in U_2 is

$$u' = v^d \left[P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right) \right], \quad v' = -v^{d+1} Q \left(\frac{u}{v}, \frac{1}{v} \right).$$

The expression of $p(X)$ in U_3 is

$$u' = P(u, v), \quad v' = Q(u, v).$$

For $i = 1, 2, 3$ the expression of $p(X)$ in the chart V_i differs of the expression in U_i only by the multiplicative constant $(-1)^{d-1}$.

Note that we can identify the infinity of $\mathbb{R}^2$ with the equator $\mathbb{S}^1$. Two points for each direction in $\mathbb{R}^2$ provide two antipodal points in $\mathbb{S}^1$. An equilibrium point of $p(X)$ in $\mathbb{S}^1$ is called *an infinite equilibrium point* and an equilibrium point in $\mathbb{S}^2 \setminus \mathbb{S}^1$ is called a *finite equilibrium point*. Observe that the coordinates of the infinite equilibrium points are of the form $(u, 0)$ in the charts U_1, V_1, U_2 and V_2 . Thus, if $(x_1, x_2, 0) \in \mathbb{S}^1$ is an infinite equilibrium point, then its antipode $(-x_1, -x_2, 0)$ is also an infinite equilibrium point.

The image of the closed northern hemisphere of $\mathbb{S}^2$ under the projection $(x_1, x_2, x_3) \rightarrow (x_1, x_2, 0)$ is the *Poincaré disc*, denoted by $\mathbb{D}^2$.

3. PROOF OF STATEMENT (a) OF THEOREM 1

In polar coordinates (r, θ) where $x = r \cos \theta$ and $y = r \sin \theta$ the differential system (1) writes

$$\begin{aligned} \dot{r} &= r^2 \sin \theta (a_1 \cos^2 \theta + a_2 \sin \theta \cos \theta + a_3 \sin^2 \theta) + a_4 r^3 \sin \theta \cos^3 \theta, \\ \dot{\theta} &= 1 + r \cos \theta (a_1 \cos^2 \theta + a_2 \sin \theta \cos \theta + a_3 \sin^2 \theta) + a_4 r^2 \cos^4 \theta. \end{aligned}$$

Taking the variable θ as the new independent variable we obtain the equivalent differential equation

$$(4) \quad r' = \sum_{i=1}^{\infty} R_i(\theta) r^i,$$

where $R_i(\theta)$ are analytic functions which depends of θ through $\cos \theta$ and $\sin \theta$. Here the prime denotes derivative with respect to the variable θ , i.e., $r' = dr/d\theta$. Define the solution

$$(5) \quad r(\theta) = \sum_{i=1}^{\infty} v_i(\theta) x^i.$$

Replacing (5) in (4) and equating the coefficients in the exponents of variable x , we obtain linear differential equations. Solving recursively each linear differential equation with the initial conditions $v_1(0) = 1$ and $v_i(0) = 0$, $i = 2, 3, \dots$, we obtain $V_i = v_i(2\pi)$, $i = 2, 3, \dots$, which are the Lyapunov constants. We recall that systems (1) has a center if, and only if all the Lyapunov constants are zero. The first two Lyapunov constants are $V_2 = v_2(2\pi) = 0$ and $V_3 = v_3(2\pi) = -\frac{1}{4}\pi a_2(a_1 + a_3)$. So we have two cases $a_2 = 0$, or $a_2 \neq 0$ and $a_1 = -a_3$.

Case 1: $a_2 = 0$. So we get the system

$$\dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_3y^2 + a_4x^3.$$

Since this differential system is invariant under the symmetry $(x, y, t) \rightarrow (x, -y, -t)$, and the origin of coordinates is either a weak focus or a center, it is a center.

Case 2: $a_2 \neq 0$ and $a_1 = -a_3$. In this $V_2 = V_3 = 0$, $V_4 = v_4(2\pi) = 0$ and $V_5 = v_5(2\pi) = \frac{1}{24}a_1a_2a_4\pi$. Since $a_2 \neq 0$ and $a_4 \neq 0$, if $v_5(2\pi) = 0$, then $a_1 = 0$. So we get the system

$$\dot{x} = -y, \quad \dot{y} = x + a_2xy + a_4x^3.$$

Since this differential system is invariant under the symmetry $(x, y, t) \rightarrow (-x, y, -t)$, and the origin of coordinates is either a weak focus or a center, it is a center.

4. THE PROOF OF STATEMENT (b) OF THEOREM 1

In similar way to Section 3 we obtain the Lyapunov constants $V_2 = v_2(2\pi) = 0$, $V_3 = v_3(2\pi) = -\frac{1}{4}\pi[a_2(a_1 + a_3) - a_4]$ and $V_5 = v_5(2\pi) = \frac{1}{24}\pi a_2 a_3 (a_1 + a_3)(5a_1 + 3a_3)$. From $v_3(2\pi)$ we get $a_4 = a_2(a_1 + a_3)$, since $a_4 \neq 0$ we get $a_2 \neq 0$ and $a_1 + a_3 \neq 0$. So, in $v_5(2\pi)$ we have the cases $a_3 = 0$ or $5a_1 + 3a_3 = 0$. The case $5a_1 + 3a_3 = 0$ is impossible because if we set $a_1 = -3a_3/5$, implies $v_7(2\pi) = 3\pi a_2 a_3^5/1000$. Hence $a_3 = 0$ and from $a_1 = -3a_3/5$ we get $a_1 = 0$ in contradiction with $a_1 + a_3 \neq 0$. So we get the unique case $a_3 = 0$. Thus we have the system

$$\dot{x} = -y, \quad \dot{y} = x + a_2xy + a_1x^2 + a_1a_2x^2y.$$

This system has a first integral

$$H(x, y) = \frac{1}{6}x^2(2a_1x + 3) - \frac{\log(e^{-a_2y}(a_2y + 1))}{a_2^2}.$$

Note that $H(0, 0) = (0, 0)$. Therefore H is defined at the origin of coordinates. Since the origin is either a center or a weak focus, we conclude that it is a center.

5. PROOF OF STATEMENT (c) THEOREM 1

The proof of this statement is identically to the proof of statement (a) of Theorem 1.

6. THE PROOF OF STATEMENT (d) OF THEOREM 1

Proof. In similar way to Section 3 we obtain the Lyapunov constants $V_2 = v_2(2\pi) = 0$ and $V_3 = v_3(2\pi) = -\frac{1}{4}\pi[a_2(a_1 + a_3) - 3a_4]$. Doing $a_4 = (a_1 + a_3)a_2/3$ we get $V_4 = 0$ and

$$V_5 = v_5(2\pi) = -\frac{1}{72}\pi a_2(a_1 + a_3)(15a_1^2 + 9a_1a_3 + 2a_2^2).$$

Since $a_4 \neq 0$ we get that $v_5(2\pi) = 0$ if, and only if $15a_1^2 + 9a_1a_3 + 2a_2^2 = 0$. If $a_1 \neq 0$, then $a_3 = (-15a_1^2 - 2a_2^2)/9a_1$, so $V_6 = 0$ and

$$V_7 = v_7(2\pi) = -\frac{5\pi a_2(3a_1^2 + a_2^2)^3}{5832a_1},$$

which never is zero, because $a_4 \neq 0$. Now if $a_1 = 0$, then $15a_1^2 + 9a_1a_3 + 2a_2^2 = 0$ implies $a_2 = 0$, which is a contradiction because $a_4 \neq 0$. In summary, we can not vanish all Lyapunov constants. Hence the differential system can not have a center.

7. GLOBAL PHASE PORTRAITS OF SYSTEM (1) WHEN $A = a_4x^3$

First we analyze the equilibrium points (both finite and infinite) of system (1) with $A = a_4x^3$ having a center at the origin of coordinates, and after we provide its the phase portraits in the Poincaré disc. The other phase portraits of the systems corresponding to statements (b), (c) and (d), could be studied in similar way.

7.1. Finite equilibrium points of system (1) when $A = a_4x^3$ and $a_2 = 0$. In Theorem 1 statement (a) systems (1) presents two options. Here we consider the case where $a_2 = 0$. Thus we have the differential systems

$$(6) \quad \dot{x} = -y, \quad \dot{y} = x + a_1x^2 + a_3y^2 + a_4x^3,$$

that have the following possible equilibrium points $(0, 0)$, $((-a_1 \pm \sqrt{a_1^2 - 4a_4})/(2a_4), 0)$. So we get the following cases:

1. $(0, 0)$ is the unique equilibrium point if, and only if $a_1^2 < 4a_4$;
2. $(0, 0)$ and $(-a_1/(2a_4), 0)$ are equilibria if, and only if $a_1^2 = 4a_4$;
3. $(0, 0)$ and $P_{\pm} = ((-a_1 \pm \sqrt{a_1^2 - 4a_4})/(2a_4), 0)$ are equilibria if, and only if $a_1^2 > 4a_4$.

Recall that the $(0, 0)$ is a local center. Thus, we need to provide the behavior of the other equilibrium points. In the point $(-a_1/(2a_4), 0)$ systems (6) with $a_1^2 = 4a_4$, has the Jacobian matrix

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

So, by Theorem 3.5 of [10] this point is a nilpotent cusp.

In the points $P_{\pm}$ the system has the Jacobian matrix $M_{\pm}$ given by

$$M_+ = \begin{pmatrix} 0 & -1 \\ \frac{a_1(\sqrt{a_1^2 - 4a_4} + a_1)}{2a_4} - 2 & 0 \end{pmatrix} \quad \text{and} \quad M_- = \begin{pmatrix} 0 & -1 \\ -\frac{4\sqrt{a_1^2 - 4a_4}}{a_1 + 2\sqrt{a_1^2 - 4a_4}} & 0 \end{pmatrix},$$

respectively. Using the trace and the determinant of these matrices we get the following conditions:

- i. If $a_1 < 0$ and $0 < a_4 < a_1^2/4$, then P_- is a saddle. Using the Lyapunov constants we have that the point P_+ is a center;
- ii. If $a_1 > 0$ and $0 < a_4 < a_1^2/4$, then P_+ is a saddle. Using the Lyapunov constants we have that the point P_- is a center;
- iii. If $a_4 < 0$, then $P_{\pm}$ are saddles.

Thus we have completed the classification of the finite equilibrium points.

7.2. Infinite equilibrium points. From subsection 7.1 we distinguish three cases.

7.2.1. Cases 1 and 3. Since these two cases have the same expressions at infinity when we do the Poincaré compactification we study them together. The Poincaré compactification of system (1) in the local chart (U_1, φ_1) is

$$(7) \quad \dot{u} = v(a_1 + u^2(a_3 + v) + v) + a_4, \quad \dot{v} = uv^3.$$

This system has no infinite equilibrium points, because when $v = 0$ we get $a_4 = 0$, which is impossible. recall that the points of the infinity are of the form $(u, 0)$. The Poincaré compactification of system (1) in the local chart (U_2, φ_2) is

$$(8) \quad \begin{aligned} \dot{u} &= -a_1u^3v - a_3uv - a_4u^4 - u^2v^2 - v^2, \\ \dot{v} &= -a_1u^2v^2 - a_3v^2 - a_4u^3v - uv^3. \end{aligned}$$

When $v = 0$ the unique equilibrium point is $(0, 0)$. At this point the Jacobian matrix is zero, so we shall need to use the blow up technique to determine the local phase portrait of this point.

7.2.2. Case 2. The Poincaré compactification of system (1) in the local chart (U_1, φ_1) is

$$(9) \quad \dot{u} = \frac{a_1^2}{4} + a_1v + v(u^2(a_3 + v) + v), \quad \dot{v} = uv^3.$$

This system has not infinite equilibrium points, because when $v = 0$ we get $a_4 = a_1^2/4 \neq 0$. The Poincaré compactification of system (1) in local chart (U_2, φ_2) is

$$(10) \quad \dot{u} = -\frac{1}{4}a_1^2u^4 - a_1u^3v - v(u(a_3 + uv) + v), \quad \dot{v} = -\frac{1}{4}v(u(a_1u + 2v)^2 + 4a_3v).$$

When $v = 0$ the unique equilibrium point is $(0, 0)$. At this point the Jacobian matrix is zero, so again we shall need to use the blow up technique to determine the behavior of this point.

7.3. Blow up for system (1). From subsection 6.1 we have three cases, we only do **Case 1** in order to illustrate the procedure. The other cases are similar. If we want to do a vertical blow up in the direction u , we first compute if u is a characteristic direction. So, from (8) we get that the characteristic directions are the linear factor of $-v^3$. Since u is not a factor of above homogeneous polynomial we do a vertical blow up, that is, we do the change of variables $(u, v) = (u_1, u_1 v_1)$, and we get

$$(11) \quad \dot{u}_1 = -u_1^2 (a_1 u_1^2 v_1 + a_3 v_1 + a_4 u_1^2 + u_1^2 v_1^2 + v_1^2), \quad \dot{v}_1 = u_1 v_1^3.$$

Eliminating the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ doing a rescaling of the time we obtain

$$(12) \quad \dot{u}_1 = -u_1 (a_1 u_1^2 v_1 + a_3 v_1 + a_4 u_1^2 + u_1^2 v_1^2 + v_1^2), \quad \dot{v}_1 = v_1^3.$$

On the straight line $u_1 = 0$ the unique equilibrium point is the $(0, 0)$, in it the Jacobian matrix of the differential system is zero. Consequently we need to perform another blow up. First, we verify if u_1 is a characteristic direction of system (12), and it is because the characteristic direction are the linear factors of $-u_1 v_1 (a_3 v_1)$. Since u_1 is a linear factor in the above expression we do the twist $(u_1, v_1) = (u_2 - v_2, v_2)$, and we obtain the following system

$$\dot{u}_2 = a_3 v_2^2 - a_4 u_2^3 + 3a_4 u_2^2 v_2 - 3a_4 u_2 v_2^2 - u_2 v_2^2 + \mathcal{O}_4(u_2, v_2), \quad \dot{v}_2 = v_2^2.$$

By performing the vertical blow up $(u_2, v_2) = (u_3, u_3 v_3)$ and eliminating the common factor u_3 , through a rescaling of time we obtain the following system

$$\begin{aligned} \dot{u}_3 &= u_3 (-a_4 u_3 - a_3 v_3 + 3a_4 u_3 v_3 + a_3 v_3^2) + \mathcal{O}(u_3, v_3), \\ \dot{v}_3 &= -(-1 + v_3) v_3 (a_4 u_3 + a_3 v_3) + \mathcal{O}_2(u_3, v_3). \end{aligned}$$

On the straight line $u_3 = 0$ we obtain two equilibrium points, namely, $(0, 0)$ and $(0, 1)$, whose Jacobian matrices are

$$(13) \quad M(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad M(0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & -a_3 \end{pmatrix},$$

respectively. Here we have two cases, namely, $a_3 = 0$ and $a_3 \neq 0$.

If $a_3 = 0$ we repeat the entire procedure with the condition $a_3 = 0$. As result, we obtain

$$\begin{aligned} \dot{u}_3 &= u_3 (-a_4 + 3a_4 v_3 - a_1 u_3 v_3 - v_3^2 - 3a_4 v_3^2) + \mathcal{O}(u_3, v_3), \\ \dot{v}_3 &= -((-1 + v_3) v_3 (a_4 - 2a_4 v_3 + a_1 u_3 v_3 + 2v_3^2 + a_4 v_3^2) + \mathcal{O}_3(u_3, v_3), \end{aligned}$$

which has two equilibrium points, namely, $(0, 0)$ and $(0, 1)$ that are hyperbolic saddles. So in this case the blow up ends here.

If $a_3 \neq 0$ we translate the point $(0, 1)$ to origin and, using Theorem 2.19 of [10], we determine that this point is a semi-hyperbolic saddle-node. For the point $(0, 0)$ we need to perform a vertical blow up because its Jacobian matrix is zero. Therefore, we first need to verify if u_3 is a characteristic direction, and it is because the characteristic

direction are the linear factors of $-2u_3v_3(a_4u_3+a_3v_3)$. Then we do the twist $(u_3, v_3) = (u_4 - v_4, v_4)$, and we get

$$\begin{aligned} \dot{u}_4 &= a_3u_4v_4^2 - a_3u_4v_4 + 2a_3v_4^2 + 3a_4u_4^2v_4 - a_4u_4^2 - 9a_4u_4v_4 + 3a_4u_4v_4 - 2a_4v_4^2 + \mathcal{O}(u_4, v_4), \\ \dot{v}_4 &= (-1 + v_4)v_4(-a_4u_4 - a_3v_4 + a_4v_4 + 2a_4u_4v_4 - 2a_4v_4^2) + \mathcal{O}_3(u_4, v_4). \end{aligned}$$

Doing the blow up $(u_4, v_4) = (u_5, u_5v_5)$ and eliminating the common factor u_5 through a rescaling of the time we get

$$(14) \quad \begin{aligned} \dot{u}_5 &= u_5(-a_4 - a_3v_5 + 3a_4v_5 + 3a_4u_5v_5 + 2a_3v_5^2 - 2a_4v_5^2) + \mathcal{O}(u_5, v_5), \\ \dot{v}_5 &= -(-1 + v_5)v_5(2a_4 + 2a_3v_5 - 2a_4v_5 - 6a_4u_5v_5) + \mathcal{O}_3(u_5, v_5), \end{aligned}$$

which has on $u_5 = 0$ the equilibrium points $(0, 0)$, $(0, 1)$ and $(0, a_4/(a_4 - a_3))$. The first two points are hyperbolic saddles, because the Jacobian matrices are

$$M(0, 0) = \begin{pmatrix} -a_4 & 0 \\ 0 & 2a_4 \end{pmatrix}, \quad M(0, 1) = \begin{pmatrix} a_3 & 0 \\ 0 & -2a_3 \end{pmatrix},$$

respectively. To address the other point, we again consider two cases: $a_3 = a_4$, in which case this point does not exist. In the case where $a_3 \neq a_4$ we translate the point to the origin, $(u_5, v_5) = (u_6, v_6 + a_4/(a_4 - a_3))$, and write the Jacobian matrix at the origin in its real Jordan form. Thus, we obtain the differential system

$$\dot{u}_6 = 0, \quad \dot{v}_6 = -\frac{a_3a_4}{a_3 - a_4}v_6,$$

that satisfies the assumptions of Theorem 2.19 from [10] (page 74) with

$$\lambda = -\frac{a_3a_4}{a_3 - a_4}, \quad \bar{a}_3 = \frac{a_3a_4^2}{2(a_3 - a_4)^3},$$

where the term $\bar{a}_3$ represents a_m in the cited theorem. Thus we have different cases according to Theorem 2.19. As $a_4 \neq 0$, we consider the case $a_4 > 0$. The case $a_4 < 0$ is similar. Therefore we obtain:

- $\frac{a_4}{a_4 - a_3} < 0$. In this case $a_4 - a_3 < 0$. Thus, we have $\lambda < 0$ and $\bar{a}_3 > 0$, so the point $(0, \frac{a_4}{a_4 - a_3})$ is a saddle;
- If $0 < \frac{a_4}{a_4 - a_3} < 1$, from $0 < \frac{a_4}{a_4 - a_3}$ we get $a_4 - a_3 > 0$, and from $\frac{a_4}{a_4 - a_3} < 1$ and $a_4 - a_3 > 0$, we get $a_3 < 0$, so $\lambda < 0$ and $\bar{a}_3 > 0$, so $(0, \frac{a_4}{a_4 - a_3})$ is an unstable node;
- $\frac{a_4}{a_4 - a_3} > 1$. We obtain that if $a_4 > 0$, then $a_3 > 0$, and from $\frac{a_4}{a_4 - a_3} > 1$, we get $a_4 - a_3 > 0$. Thus $\lambda > 0$ and $\bar{a}_3 < 0$, so $(0, \frac{a_4}{a_4 - a_3})$ is a saddle.

We have concluded the blow up for the case where $(0, 0)$ is the unique equilibrium point. We illustrate the consecutive steps of the blown up in the case $a_3 = a_4 > 0$ in Figure 4, using equations (11) to (14). By following a similar process for the other cases in the subsection 7.1, we obtain the phase portraits, which are summarized in Figure 1.

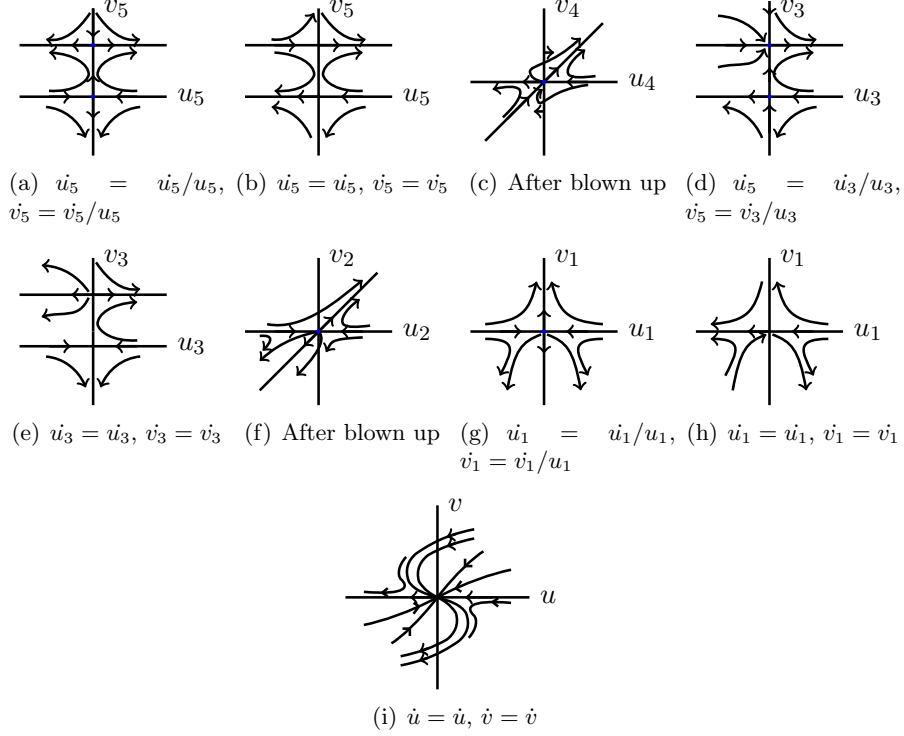


FIGURE 4. Blow up at $(0,0)$ when $a_1^2 < 4a_4$. Case $a_3 = a_4 > 0$.

7.4. The phase protrait. In this section we explain how to get the phase portrait for the cases (a), (b), (e), (m), (q) from Figure 1; the others are get in similar way.

We shall study the phase portrait of Figure 1(a). In Figure 5(a) the unique finite equilibrium is the origin, that is a center, and two equilibria at infinity, denoted by P and Q . This phase portraits has the two separatrices S_1 and S_2 . The periodic orbits of the center rotate in a counterclockwise sense. Taking into account the behavior of the differential system on the axes, we obtain the phase portrait of Figure 5(b).

Suppose the separatrix S_1 has its ω -limit at the point Q without connecting with the separatrix S_2 , as it is shown in Figure 5(c). Note that in this case the separatrix S_2 has its α -limit in the last periodic orbit c surrounding the center at the origin, but this is a contradiction. Indeed, if we do a transversal section L to the pereiodic orbit c and consider on L the Poincaré map, we get that in the region bounded by c the Poincaré map is the identity, but outside to c is not the identity, this is a contradiction because the system is a polynomial and consequently the Poincaré map is an analytic function, and consequently the Poincaré map in all its domain of definition is the identity map.

If we suppose that the separatrix S_2 has its ω -limit in P , we arrive to a similar contradiction to the previous case. Thus, the only feasible scenario is that the separatrix S_1 and the separatrix S_2 connect, as shown in Figure 1(a).

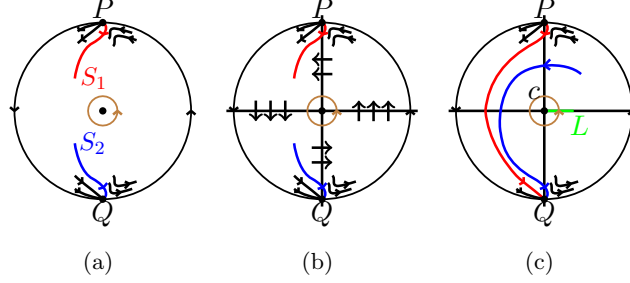


FIGURE 5. Possible phase portrait of system (6) when the origin is the unique finite equilibrium and $a_3 = a_4 > 0$.

Now we shall study the phase portrait of Figure 1(b). The unique finite equilibrium is the origin, which is a center, and two equilibria at infinity P and Q , which have the local phase portraits of Figure 6(a). Suppose that the orbit S_1 is not closed. Since the Poincaré disc is compact, its α -limit and ω -limit are in the disc. From Poincaré-Bendixson Theorem and the obtained from blow up, we get that either its α -limit or its ω -limit is the last periodic orbit c surrounding the origin (see Figure 6(b)), but again we have a contradiction, using the analytic Poincaré map in a transversal section to the periodic orbit c .

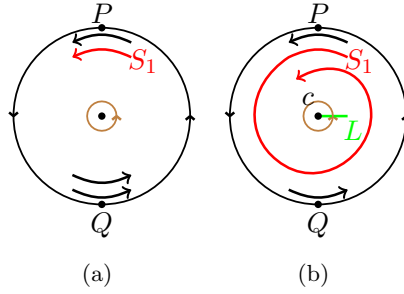


FIGURE 6. Possible phase portrait of system (6) when the origin is the unique finite equilibrium and $a_3 = 0$ and $a_1^2 < 4a_4$.

We study now the phase portrait of Figure 1(e). From the Poncaré compactification and the analysis of finite equilibrium points we have the local phase portraits of Figure 7(a). We want to prove that the separatrices S_2 and S_3 connect as in Figure 1 (e). For this, suppose that S_2 and S_3 do not connect, as in Figure 7(a). Then again we obtain a contradiction using the last periodic orbit surrounding the center. In fact this system is Hamiltonian with Hamiltonian $H = (x^2 + y^2)/2 + a_1 x^3/3 + a_1^2 x^4/16$. Using the first integral H we also can see that the separatrices S_2 and S_3 connect, and that all the other finite orbits with the exception of the origin are periodic orbits.

We analyze now the phase portrait of Figure 1(m). From the analysis of finite and infinite equilibrium points we obtain Figure 8(a), again the α - and ω -limits of

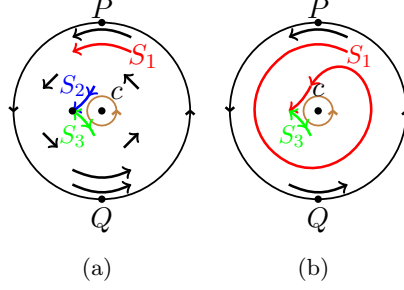


FIGURE 7. Possible phase portrait of system (6) when $a_3 = 0$, $a_1^2 = 4a_4$.

the separatrixes S_i , $i = 2, 3, 4, 5$ cannot be the last periodic orbit surrounding one of the two centers of the system. These separatrixes only can connect as it is indicated in Figure 1(m). This also follows from the fact that again the differential system is Hamiltonian with Hamiltonian $H = (x^2 + y^2)/2 + a_1x^3/3 + a_4x^4/4$.

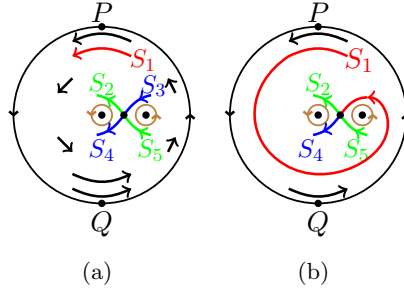


FIGURE 8. Possible phase portrait of system (6) when $a_3 = 0$.

Now we study the phase portrait of Figure 1(q). From the analysis of infinity and finite equilibrium points we get Figure 9(a). Note that any separatrix S_i can be as α - or ω -limit the last periodic orbit surrounding one of the two centers. The differential system has the first integral

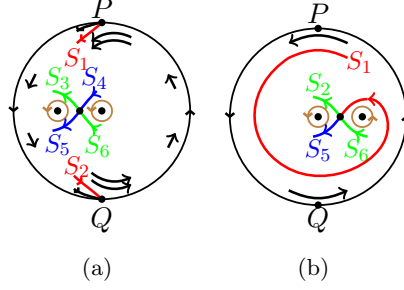
$$H = e^{2a_3x} (2a_1a_3 - 2a_3^2 - 3a_4 - 2a_3(2a_1a_3 - 2a_3^2 - 3a_4)x + 2a_3^2(2a_1a_3 - 3a_4)x^2 + 4a_3^3(a_3y^2 + a_4x^3)).$$

Using this first integral the phase portrait of Figure 1(q) follows.

The remaining phase portraits of Figure 1 can be analyzed in a similar way. This completes the proof of Theorem 1.

7.5. Finite equilibrium points of system (1) when $A = a_4x^3$ and $a_1 = a_3 = 0$. Now we consider the differential system

$$(15) \quad \dot{x} = -y, \quad \dot{y} = x + a_2xy + a_4x^3,$$

FIGURE 9. Possible phase portrait of system (6) when $a_3 < 0$.

which have the following possible finite equilibrium points $(0, 0)$, $(\pm 1/\sqrt{-a_4}, 0)$. So we get the following cases:

1. $(0, 0)$ is the unique equilibrium point if, and only if $0 < a_4$;
2. $(0, 0)$ and $(\pm 1/\sqrt{-a_4}, 0)$ are equilibria if, and only if $a_4 < 0$.

Recall that the $(0, 0)$ is a local center. Thus, we need provide the local phase portrait of the other equilibria. In the equilibria $(\pm 1/\sqrt{-a_4}, 0)$, system (15) with $a_4 < 0$ has the Jacobian matrix

$$M = \begin{pmatrix} 0 & -1 \\ -2 & \frac{a_2 a_4}{(-a_4)^{3/2}} \end{pmatrix}.$$

The matrix M has determinant -2 , and consequently these points are saddles.

Now doing the same process like above to find the infinite equilibrium points we found that for both cases the Poincaré compactification of system (15) in the local chart (U_1, φ_1) is

$$\dot{u} = v(u(a_2 + uv) + v) + a_4, \quad \dot{v} = uv^3.$$

This system has no infinite equilibrium points, because when $v = 0$ we get $a_4 \neq 0$. The Poincaré compactification of system (15) in local chart (U_2, φ_2) is

$$\dot{u} = -v(u^2(a_2 + v) + v) - a_4 u^4, \quad \dot{v} = -v(uv(a_2 + v) + a_4 u^3).$$

When $v = 0$ the unique equilibrium is $(0, 0)$. We have two cases, we only do the **case**

1. Doing the same process of subsection 7.3 we get

$$\begin{aligned} \dot{u}_3 &= u_3 \left(-a_2 v_3^3 + 2a_2 v_3^2 - a_2 v_3 + a_4 v_3^3 - 3a_4 v_3^2 + 3a_4 v_3 - a_4 + u_3^2 v_3^5 - 3u_3^2 v_3^4 + 3u_3^2 v_3^3 \right. \\ &\quad \left. - u_3^2 v_3^2 + 2v_3^3 - v_3^2 \right), \\ \dot{v}_3 &= -((v_3 - 1)v_3 (-a_2 v_3^2 + a_2 v_3 + a_4 v_3^2 - 2a_4 v_3 + a_4 + u_3^2 v_3^4 - 2u_3^2 v_3^3 + u_3^2 v_3^2 + 2v_3^2)), \end{aligned}$$

which has in $u_3 = 0$ the equilibrium points $(0, 0)$, $(0, 1)$, $(0, (\sqrt{a_2^2 - 8a_4 + a_2 - 2a_4})/(2(a_2 - a_4 - 2)))$, and $(0, (\sqrt{a_2^2 - 8a_4 - a_2 + 2a_4})/(2(a_4 - a_2 + 2)))$. The first two points are hyperbolic saddles, because the Jacobian matrices are

$$M(0, 0) = \begin{pmatrix} -a_4 & 0 \\ 0 & a_4 \end{pmatrix}, \quad M(0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

respectively. To address the other two points, we consider the cases: $a_2 = 2 + a_4$ and $a_2^2 < 8a_4$ in these cases these points do not exist, or we consider the case $a_2 \neq 2 + a_4$ and $a_2^2 \geq 8a_4$. These two cases are treated in similar way as before. We illustrate the consecutive steps of the blown up in the case $a_2^2 < 8a_4$ in Figure 10. By following a similar process for the other cases, we obtain the phase portraits which are summarized in Figure 2.

This completes the proof of Theorem 2.

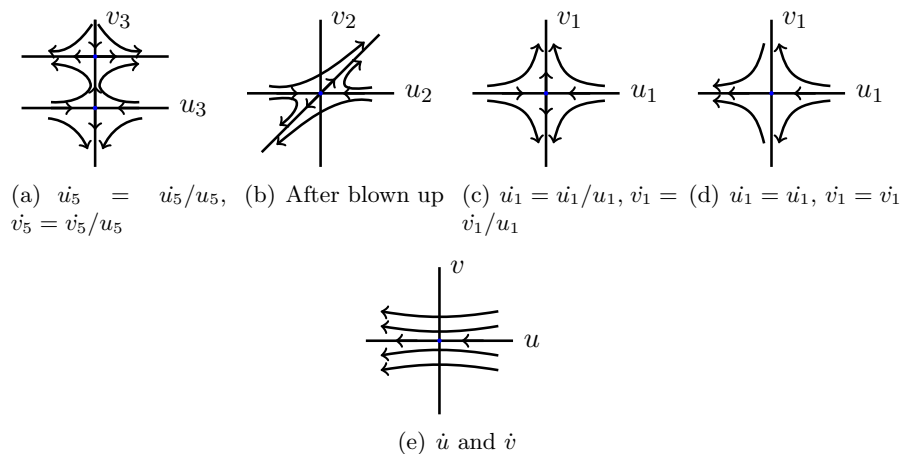


FIGURE 10. Blow up at $(0,0)$ when $a_2^2 < 8a_4$.

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