On self-dual completely regular codes with covering radius $\rho \leq 3$

J. Borges[a], V. Zinoviev[b]

[a]*Department of Information and Communications Engineering, Universitat Autònoma de Barcelona,*
[b]*A.A. Kharkevich Institute for Problems of Information Transmission, Russian Academy of Sciences,*

Abstract

We give a complete classification of self-dual completely regular codes with covering radius $\rho \leq 3$. For $\rho = 1$ the results are almost trivial. For $\rho = 2$, by using properties of the more general class of uniformly packed codes in the wide sense, we show that there are two sporadic such codes, of length 8, and an infinite family, of length 4, apart from the direct sum of two self-dual completely regular codes with $\rho = 1$, each one. For $\rho = 3$, in some cases, we use similar techniques to the ones used for $\rho = 2$. However, for some other cases we use different methods, namely, the Pless power moments which allow to us to discard several possibilities. We show that there are only two self-dual completely regular codes with $\rho = 3$ and $d \geq 3$, which are both ternary: the extended ternary Golay code and the direct sum of three ternary Hamming codes of length 4. Therefore, any self-dual completely regular code with $d \geq 3$ and $\rho = 3$ is ternary and has length 12.

We provide the intersection arrays for all such codes.

Keywords: Self-dual codes, completely regular codes, covering radius

2000 MSC: 94B60, 94B25

---

1. Introduction

Denote by $\mathbb{F}_q^n$ the n-dimensional vector space over the finite field of order q, where q is a prime power. The *(Hamming) distance* between two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{F}_q^n$, denoted by $d(\mathbf{v}, \mathbf{u})$, is the number of coordinates in which they differ. The *(Hamming) weight* of a vector $\mathbf{v} \in \mathbb{F}_q^n$, denoted by $\mathrm{wt}(\mathbf{v})$, is the number of nonzero coordinates of $\mathbf{v}$.

A q-ary *code* C of length n is a subset $C \subseteq \mathbb{F}_q^n$. The elements of C are called *codewords*. The *minimum distance* d of C is the minimum distance between any pair of codewords. The *minimum weight* w of C is the minimum weight of any nonzero codeword. A *linear code* with parameters $[n, k, d]_q$ is a q-ary code of length n with minimum distance d, such that it is a k-dimensional subspace of $\mathbb{F}_q^n$. For linear codes, the minimum distance and the minimum weight coincide, $d = w$. A *t-weight* code is a code where the nonzero codewords have t different weights ($t \geq 1$). A linear code of length n is said to be *antipodal* if there is some codeword of weight n.

The *packing radius* of a code C is $e = \lfloor (d-1)/2 \rfloor$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its distance to the code C is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C}\{d(\mathbf{v}, \mathbf{x})\}$ and the *covering radius* of the code C is $\rho = \max_{\mathbf{v} \in \mathbb{F}_q^n}\{d(\mathbf{v}, C)\}$. Note that $e \leq \rho$. If $e = \rho$, then C is a *perfect code*. It is well known that any nontrivial (with more than two codewords) perfect code has $e \leq 3$ [22, 23]. For $e = 1$, linear perfect codes are called Hamming codes which exist for lengths $n = (q^m - 1)/(q-1)$ ($m \geq 2$), dimension $k = n - m$ and minimum distance $d = 3$. For $e = 2$, the only nontrivial perfect code is the ternary Golay code of length 11.

Given two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$, their Euclidean inner product is

$$\mathbf{v}\cdot\mathbf{u}=\sum_{i=1}^{n} v_i u_i \in \mathbb{F}_q.$$

For a linear code C, its (Euclidean) *dual code* is $C^{\perp}=\{\mathbf{x}\in\mathbb{F}_q^n \mid \mathbf{x}\cdot\mathbf{v}=0,\ \forall \mathbf{v}\in C\}$. The code C is *self-dual* if $C=C^{\perp}$. In this case, C and $C^{\perp}$ have the same dimension $n/2$, hence n must be even. For the rest of the paper, the terms inner product and duality refer always to Euclidean inner product and Euclidean duality, unless otherwise stated.

Denote by $\mathbf{0}$ the all-zero vector. The *support* of a vector $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{F}_q^n$ is the set of nonzero coordinate positions of $\mathbf{x}$, $\mathrm{supp}(\mathbf{x})=\{i\in\{1,\dots,n\}\mid x_i\neq 0\}$. Say that a vector $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{F}_q^n$ *covers* a vector $\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{F}_q^n$ if $x_i=y_i$, for all $i=1,\dots,n$ such that $y_i\neq 0$.

For a given code C of length n and covering radius ρ, define

$$C(i) \;=\; \{\mathbf{x}\in\mathbb{F}_q^n:\ d(\mathbf{x},C)=i\},\quad i=0,1,\dots,\rho.$$

The sets $C(0)=C,C(1),\dots,C(\rho)$ are called the *subconstituents* of C.

Say that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *neighbors* if $d(\mathbf{x},\mathbf{y})=1$.

Definition 1 ([17]). *A code C of length n and covering radius ρ is* completely regular *(shortly CR), if for all $l\geq 0$ every vector $\mathbf{x}\in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Define $a_l=(q-1)\cdot n-b_l-c_l$ and set $c_0=b_\rho=0$. The parameters a_l, b_l and c_l $(0\leq l\leq\rho)$ are called* intersection numbers *and the sequence $\{b_0,\dots,b_{\rho-1};c_1,\dots,c_\rho\}$ is called the* intersection array *(shortly* IA*) of C.*

For any $\mathbf{v}\in\mathbb{F}_q^n$ and any $t\in\{0,\dots,n\}$, define $B_{\mathbf{v},t}=|\{\mathbf{x}\in C\mid d(\mathbf{v},\mathbf{x})=t\}|$. Completely regular codes had previously been defined by Delsarte [8, Section 5.2.3]. According to Delsarte's definition, C is CR if $B_{\mathbf{v},t}$ depends only on t and $d(\mathbf{v}, C)$. Later, Neumaier proved that Delsarte's definition is equivalent to Definition 1 as can be seen in [17].

Existence, construction and classification of completely regular codes, in general, are open hard problems (see [6, 7, 13, 17]) of algebraic and combinatorial coding theory.

All linear completely regular codes with covering radius $\rho = 1$ are known [3]. The next case, i.e. completely regular codes with $\rho = 2$, was solved for the special case when the dual codes are antipodal [3, 5]. In the present paper, we classify all self-dual completely regular codes with covering radius $\rho \leq 3$.

In Section 2, we see some definitions and results that we use later. In Section 3, we show that for $\rho = 1$ we only have some trivial codes with length and minimum distance $n = d = 2$, and the ternary Hamming code of length 4. For $\rho = 2$, we prove that the only possible parameters for a self-dual completely regular code are: $[8, 4, 4]_2$, $[8, 4, 3]_3$, and $[4, 2, 3]_q$, apart from the direct sum of two self-dual $[2, 1, 2]_q$ codes. In Section 4, we prove that for $\rho = 3$ the only possibilities are: the direct sum of three self-dual $[2, 1, 2]_q$ codes, a $[12, 6, 6]_3$ code and a $[12, 6, 3]_3$ code. We identify all such codes and show that, indeed, they are self-dual and completely regular. Moreover, all such codes are antipodal except when they are direct sums of other codes. Finally, in Section 5, we summarize the results and briefly discuss about further research on the case $\rho > 3$ and also on Hermitian self-duality and additive codes.

2. Definitions and preliminary results

In this section we see several results we will need in the next section.

2.1. CR and UPWS codes

A q-ary $t-(n,m,\lambda)$-*design* is a collection S of vectors of weight m in $\mathbb{F}_q^n$ with the property that every vector $\mathbf{v}$ of weight t is covered by exactly λ vectors $\mathbf{y}\in S$ $(t\le m\le n)$. As can be seen in [11], any q-ary $t-(n,m,\lambda)$-design is also a q-ary $i-(n,m,\lambda_i)$-design for $0\le i\le t$, where

$$\lambda_i=\lambda\frac{\binom{n-i}{t-i}}{\binom{m-i}{t-i}}(q-1)^{t-i}. \tag{1}$$

Lemma 2 ([11, Thm. 9]). *Let C be a CR code with packing radius e and containing the all-zero vector. Then the codewords of any nonzero weight w form a q-ary e-design and even a q-ary $(e+1)$-design if the minimum distance is $d=2e+2$.*

Now, we see an easy but fundamental property. For a code C of length n, denote by C_w the set of codewords of weight w.

Lemma 3. *If C is a CR code of length n, containing the zero codeword, and with minimum weight d, then $\bigcup_{\mathbf{x}\in C_d}\operatorname{supp}(\mathbf{x})=\{1,\dots,n\}$.*

Proof. Otherwise taking a 1-weight vector $\mathbf{v}$, we would have that $B_{\mathbf{v},d-1}>0$ if the nonzero coordinate is in $\bigcup_{\mathbf{x}\in C_d}\operatorname{supp}(\mathbf{x})$, but $B_{\mathbf{v},d-1}=0$ if not. Hence, C would not be CR. □

Remark 4. *Lemma 3 can be also proven taking into account that the codewords in C_d form a q-ary e-design (see Lemma 2).*

The next property is a construction of CR codes by direct sum. Recall that the direct sum of two codes C_1 and C_2 is defined as

$$C_1 \oplus C_2 = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in C_1, \mathbf{y} \in C_2\}.$$

If C_1 and C_2 are linear codes, then $C_1 \oplus C_2$ is a linear code with generator matrix:

$$G = \left(\begin{array}{c|c} G_1 & \mathbf{0} \\ \hline \mathbf{0} & G_2 \end{array}\right),$$

where G_1 is a generator matrix for C_1 and G_2 is a generator matrix for C_2. For the case of binary perfect codes, the next construction can be found in [21].

Lemma 5. *Let j be a positive integer and let C_i, $i = 1, \ldots, j$ be q-ary CR codes with the same length, dimension, minimum distance, with covering radius $\rho = 1$ and intersection array $\mathrm{IA} = \{b_0, c_1\}$. Then, the direct sum $C = C_1 \oplus \cdots \oplus C_j$ is a CR code with covering radius j and intersection array*

$$\mathrm{IA} = \{b'_0, \ldots, b'_{j-1}; c'_1, \ldots, c'_j\} = \{jb_0, (j-1)b_0, \ldots, b_0; c_1, 2c_1, \ldots, jc_1\}.$$

Proof. Write any vector $\mathbf{x} \in \mathbb{F}_q^{jn}$ as $(x^{(1)}, \ldots, x^{(j)})$, where $x^{(i)} \in \mathbb{F}_q^n$ for all $i = 1, \ldots, j$. Then

$$d(\mathbf{x}, C) = \sum_{i=1}^{j} d(x^{(i)}, C_i) \leq j,$$

since the covering radius of each C_i is one. Taking each $x^{(i)}$ such that $d(x^{(i)}, C_i) = 1$, we have $d(\mathbf{x}, C) = j$. Thus the covering radius of C is j.

Now, we compute the intersection numbers b'_i and c'_i. In turn, this proves that C is CR. Let $\mathbf{z} \in \mathbb{F}_q^{jn}$ such that $\mathbf{z} \in C(i)$, where $0 < i < j$. Since the covering radius of each C_i is one, we can assume without loss of generality that

$$\begin{aligned} d(z^{(1)}, C_1) &= \ldots = d(z^{(i)}, C_i) = 1 \quad \text{and} \\ d(z^{(i+1)}, C_{i+1}) &= \ldots = d(z^{(j)}, C_j) = 0 \end{aligned}$$

On the one hand, any vector $\mathbf{y} \in C(i+1)$ with $d(\mathbf{y}, \mathbf{z}) = 1$ must be of the form

$$\mathbf{y} = (z^{(1)}, \ldots, z^{(i)}, y^{(i+1)}, \ldots, y^{(j)}),$$

where there is a unique $\ell \in \{i+1, \ldots, j\}$ such that $y^{(\ell)} \notin C_\ell$, $d(z^{(\ell)}, y^{(\ell)}) = 1$ and $y^{(k)} = z^{(k)}$, for all $k \in \{i+1, \ldots, j\} \setminus \{\ell\}$. For each ℓ, the number of choices of $y^{(\ell)}$ is b_0, therefore $b'_i = (j-i)b_0$ $(0 < i < j)$ since ℓ has $j-i$ possible values. Note that for $i = 0$ the argument is also valid, hence $b'_0 = jb_0$.

On the other hand, any vector $\mathbf{x} \in C(i-1)$ with $d(\mathbf{x}, \mathbf{z}) = 1$ must be of the form

$$\mathbf{x} = (x^{(1)}, \ldots, x^{(i)}, z^{(i+1)}, \ldots, z^{(j)}),$$

where there is a unique $\ell \in \{1, \ldots, i\}$ such that $x^{(\ell)} \in C_\ell$, $d(z^{(\ell)}, x^{(\ell)}) = 1$ and $x^{(k)} = z^{(k)}$, for all $k \in \{1, \ldots, i\} \setminus \{\ell\}$. For each ℓ, the number of choices of $x^{(\ell)}$ is c_1, therefore $c'_i = ic_i$ $(0 < i < j)$ since ℓ has i possible values. Note that for $i = j$ the argument is also valid, hence $c'_j = jc_1$. □

Definition 6 ([2]). *A code $C \subseteq \mathbb{F}_q^n$ with covering radius ρ is* uniformly packed in the wide sense (UPWS) *if there exist rational numbers $\beta_0, \ldots, \beta_\rho$ such that*

$$\sum_{i=0}^{\rho} \beta_i B_{\mathbf{x},i} = 1, \tag{2}$$

for any $\mathbf{x} \in \mathbb{F}_q^n$. The numbers $\beta_0, \ldots, \beta_\rho$ are called the packing coefficients.

For UPWS codes, there is a generalized version of the celebrate sphere packing condition for perfect codes.

Lemma 7 ([2]). *Let $C \subseteq \mathbb{F}_q^n$ be a UPWS code with covering radius ρ and packing coefficients $\beta_0, \ldots, \beta_\rho$. Then*

$$|C| = \frac{q^n}{\sum_{i=0}^{\rho} \beta_i (q-1)^i \binom{n}{i}}.$$

For a linear code C, denote by s the number of nonzero weights of $C^\perp$. Following to Delsarte [8], we call *external distance* the parameter s.

Lemma 8. *Let C be a linear code with covering radius ρ, packing radius e and external distance s.*

(i) $\rho \leq s$ [8].

(ii) $\rho = s$ if and only if C is UPWS [1].

(iii) If C is CR, then $\rho = s$ [21].

(iv) If C is UPWS and $\rho = e + 1$, then C is CR [11, 20].

Let C be a CR code. Set $p_{i,j} = B_{\mathbf{v},j}$, for any $\mathbf{v}$ such that $d(\mathbf{v}, C) = i$ $(0 \leq i \leq \rho)$. By Lemma 8, any CR code is also a UPWS code. Hence, for any CR code we can apply Lemma 7.

Proposition 9. *Let C be a CR $[n,k,d]_q$ code with covering radius $\rho > 1$. Then, the packing coefficients verify:*

(i) If $d = 2\rho$, then

$$\beta_0 = \ldots = \beta_{\rho-1} = 1; \qquad \beta_\rho = \frac{q^{n-k} - \sum_{i=0}^{\rho-1}(q-1)^i \binom{n}{i}}{(q-1)^\rho \binom{n}{\rho}}. \tag{3}$$

(ii) If $d = 2\rho - 1$, then

$$\begin{aligned} \beta_0 &= \ldots = \beta_{\rho-2} = 1; \;\; \beta_{\rho-1} + \beta_\rho p_{\rho-1,\rho} = 1; \\ \beta_\rho &= \frac{q^{n-k} - \sum_{i=0}^{\rho-1}(q-1)^i \binom{n}{i}}{(q-1)^\rho \binom{n}{\rho} - p_{\rho-1,\rho}(q-1)^{\rho-1}\binom{n}{\rho-1}}. \end{aligned} \tag{4}$$

(iii) If $d = 2\rho - 2$, then

$$\beta_0 = \ldots = \beta_{\rho-3} = 1;\ \beta_{\rho-2} + \beta_\rho p_{\rho-2,\rho} = 1;\ \beta_{\rho-1} p_{\rho-1,\rho-1} + \beta_\rho p_{\rho-1,\rho} = 1;$$

$$\beta_\rho = \frac{q^{n-k} - \sum_{i=0}^{\rho-2}(q-1)^i\binom{n}{i} - p_{\rho-1,\rho-1}^{-1}(q-1)^{\rho-1}\binom{n}{\rho-1}}{(q-1)^\rho\binom{n}{\rho} - p_{\rho-1,\rho}p_{\rho-1,\rho-1}^{-1}(q-1)^{\rho-1}\binom{n}{\rho-1} - p_{\rho-2,\rho}(q-1)^{\rho-2}\binom{n}{\rho-2}}. \tag{5}$$

In all cases β_ρ^{-1} is a natural number.

Proof. (i) Since C is CR, C is also UPWS. For any $i = 0, \ldots, \rho-1$, we have that $p_{i,i} = 1$ because $i < \rho = d/2$. Moreover, for any $j \in \{0, \ldots, \rho\} \setminus \{i\}$, $p_{i,j} = 0$. Indeed, if $d(\mathbf{x}, C) = i$ and $\mathbf{c}, \mathbf{c}' \in C$ are such that $d(\mathbf{c}, \mathbf{x}) = i$ and $d(\mathbf{c}', \mathbf{x}) = j$, then $d(\mathbf{c}, \mathbf{c}') \le i + j < d$ which is a contradiction. Hence, according to Eq. (2) in Definition 6, we have $\beta_i = 1$ for each $i = 0, \ldots, \rho-1$.

Therefore, by Lemma 7, it follows that

$$|C| = q^k = \frac{q^n}{\sum_{i=0}^{\rho-1}(q-1)^i\binom{n}{i} + \beta_\rho(q-1)^\rho\binom{n}{\rho}},$$

from which we obtain Eq. (3).

(ii) Now, for any $i = 0, \ldots, \rho-1$, we have again that $p_{i,i} = 1$ because $i \le \rho - 1 < d/2$. Moreover, for any $j \in \{0, \ldots, \rho-1\} \setminus \{i\}$, $p_{i,j} = 0$. Indeed, if $d(\mathbf{x}, C) = i$ and $\mathbf{c}, \mathbf{c}' \in C$ are such that $d(\mathbf{c}, \mathbf{x}) = i$ and $d(\mathbf{c}', \mathbf{x}) = j$, then $d(\mathbf{c}, \mathbf{c}') \le i + j < d$ which is a contradiction. Hence, according to Eq. (2) in Definition 6, we have $\beta_i = 1$ for each $i = 0, \ldots, \rho-2$. Thus, $p_{\rho-1,\rho-1} = 1$ and $\beta_{\rho-1} + \beta_\rho p_{\rho-1,\rho} = 1$.

Again by Lemma 7 and using $\beta_{\rho-1} = 1 - \beta_\rho p_{\rho-1,\rho}$, Eq. (4) is obtained.

(iii) In this case, and by similar arguments, we have $p_{i,i} = 1$ for any $i = 0, \ldots, \rho-2$. For any $j \in \{0, \ldots, \rho-1\} \setminus \{i\}$, $p_{i,j} = 0$. Indeed, if $d(\mathbf{x}, C) = i$ and $\mathbf{c}, \mathbf{c}' \in C$ are such that $d(\mathbf{c}, \mathbf{x}) = i$ and $d(\mathbf{c}', \mathbf{x}) = j$, then

$d(\mathbf{c}, \mathbf{c}') \leq i + j < d$ which is a contradiction. Hence, we have $\beta_i = 1$ for all $i = 0, \ldots, \rho - 3$. Thus, $p_{\rho-2,\rho-2} = 1$ and $\beta_{\rho-2} + \beta_\rho p_{\rho-2,\rho} = 1$, since $p_{\rho-2,\rho-1} = 0$. On the other hand, $\beta_{\rho-1} p_{\rho-1,\rho-1} + \beta_\rho p_{\rho-1,\rho} = 1$.

Again by Lemma 7, using $\beta_{\rho-2} = 1 - \beta_\rho p_{\rho-2,\rho}$ and $\beta_{\rho-1} = 1 - \frac{\beta_\rho p_{\rho-1,\rho}}{p_{\rho-1,\rho-1}}$, Eq. (5) is obtained.

In every case (i), (ii) and (iii), it is clear that $p_{\rho,i} = 0$, for all $i = 0, \ldots, \rho-1$ and thus $\beta_\rho p_{\rho,\rho} = 1$ and β_ρ^{-1} is a natural number. □

We are interested in the case when C is self-dual and CR with covering radius $\rho = 2$ or $\rho = 3$.

Corollary 10. *Let C be a self-dual CR $[2k, k, 4]_q$ code with covering radius $\rho = 2$. Then, the packing coefficient β_2 is:*

$$\beta_2 = \frac{q^k - 1 - 2k(q-1)}{(q-1)^2 k(2k-1)},$$

and β_2^{-1} is a natural number.

Proof. Straightforward substituting $\rho = 2$ and $n = 2k$ in Eq. (3). □

Corollary 11. *Let C be a self-dual CR $[2k, k, d]_q$ code with covering radius $\rho = 3$. Then, the packing coefficient β_3 is:*

(i) If $d = 6$, then

$$\beta_3 = 3\frac{q^k - 1 - 2k(q-1) - k(2k-1)(q-1)^2}{(q-1)^3 k(2k-1)(2k-2)}.$$

(ii) If $d = 5$, then

$$\beta_3 = 3\frac{q^k - 1 - 2k(q-1) - k(2k-1)(q-1)^2}{k(2k-1)(q-1)^2[(2k-2)(q-1) - 3p_{2,3}]},$$

where $0 \leq p_{2,3} \leq \frac{2(q-1)(k-1)}{3}$.

(iii) If $d = 4$, then

$$\beta_3 = 3\frac{(\lambda+1)(q^k-1) - k(q-1)[2(\lambda+1)+(2k-1)(q-1)]}{k(2k-1)(q-1)^2[(\lambda+1)(2k-2)(q-1) - 2\lambda(\lambda+1) - 6\lambda(q-2) - 3\lambda']},$$

where $\lambda = p_{2,2} - 1$ and $\lambda' = p_{2,3} - 2\lambda(q-2)$. Moreover, $1 \leq \lambda \leq k-1$.

Proof. (i) Put $\rho = 3$ and $n = 2k$ in Eq. (3).

(ii) Again put $\rho = 3$ and $n = 2k$ in Eq. (4).

A CR $[n, k, d]_q$ code with $\rho = e + 1$ is a quasi-perfect uniformly packed code [11]. In this case, as can be seen in [2], the packing parameters verify:

$$\beta_0 = \ldots = \beta_{e-1} = 1; \quad \beta_e = 1 - s/m; \quad \beta_{e+1} = 1/m; \tag{6}$$

where m and s are integer values and:

$$0 \leq s \leq \frac{(q-1)(n-e)}{e+1}.$$

Since $d = 5$ and $\rho = 3$, we are in the case of a quasi-perfect uniformly packed code. By Proposition 9(ii), $\beta_2 + \beta_3 p_{2,3} = 1$. Combining with the expressions (6), we obtain that $s = p_{2,3}$ and it follows the bound for $p_{2,3}$.

(iii) By Lemma 2, the codewords in C_4 form a q-ary $2-(2k, 4, \lambda)$-design. Consider a 2-weight vector $\mathbf{v}$. Such vector is covered by λ codewords in C_4 and it is also at distance 2 from the zero codeword. Thus, $p_{2,2} = \lambda + 1$. The codewords at distance 3 from $\mathbf{v}$ are:

(a) The μ codewords in C_4 containing the support of $\mathbf{v}$ and covering just one of the nonzero coordinates of $\mathbf{v}$.

(b) The λ' codewords in C_5 (if C_5 is not empty) covering $\mathbf{v}$.

For (a), let $X = \{\mathbf{x} \in C_4 \mid \mathrm{supp}(\mathbf{v}) \subset \mathrm{supp}(\mathbf{x})\}$. There are $(q-1)^2$ vectors with the same support that $\mathbf{v}$. Each one of these vectors is covered by λ vectors in C_4. Thus, $|X| = \lambda(q-1)^2$. Let $Y \subset C_4$ be the set of codewords of weight 4 which are multiples of some codeword in C_4 covering $\mathbf{v}$. Clearly, $|Y| = \lambda(q-1)$. Hence, for any codeword in $X \setminus Y$, we have two multiples that cover exactly one nonzero coordinate of $\mathbf{v}$. This means that

$$\mu = 2\frac{|X \setminus Y|}{q-1} = 2[\lambda(q-1) - \lambda] = 2\lambda(q-2).$$

For (b), simply consider that C_5 form a $2-(2k, 5, \lambda')$-design. If $C_5 = \emptyset$, then we set $\lambda' = 0$. As a consequence, we have that $p_{2,3} = 2\lambda(q-2) + \lambda'$. Now consider a 1-weight vector $\mathbf{u}$. The codewords at distance 3 from $\mathbf{u}$ are those in C_4 covering $\mathbf{u}$. According to Eq. (1), such number of vectors is $p_{1,3} = \lambda(2k-1)(q-1)/3$.

Substituting $p_{1,3} = \lambda(2k-1)(q-1)/3$, $p_{2,2} = \lambda+1$, and $p_{2,3} = 2\lambda(q-2)+\lambda'$ in Eq. (5), we obtain the expression for β_3.

Finally, note that if $\mathbf{x}, \mathbf{y} \in C_4$ are codewords covering the 2-weight vector $\mathbf{v}$, then $\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{v})$ (otherwise $\mathrm{wt}(\mathbf{x}-\mathbf{y}) < 4 = d$). The union of the supports of the λ vectors covering $\mathbf{v}$ must have cardinality at most $n = 2k$. Therefore, $2 + 2\lambda \leq 2k$, implying $\lambda \leq k-1$. □

2.2. Self-dual two-weight and three-weight codes

We start with three general an easy results on self-dual codes.

Lemma 12. *Let C be a q-ary self-dual code.*

(i) If $q = 2$, then the weight of any codeword is even.

(ii) If $q = 3$, then the weight of any codeword is divisible by 3.

Proof. If C is self-dual, then $\mathbf{x} \cdot \mathbf{x} = 0$, for any codeword $\mathbf{x} \in C$. Therefore (i) is trivial. For (ii), note that for any ternary vector $\mathbf{z} \in \mathbb{F}_3^n$, $\mathbf{z} \cdot \mathbf{z} \equiv \mathrm{wt}(\mathbf{z}) \pmod 3$. □

The next well-known property shows which is the only self-dual perfect code.

Lemma 13. *The only self-dual perfect code is the ternary Hamming* $[4, 2, 3]_3$ *code.*

Proof. For any self-dual $[n, k, d]_q$ code, we have that $n = 2k$. The only perfect codes with minimum distance $d > 3$ are the ternary Golay $[11, 6, 5]_3$ code, the binary Golay $[23, 12, 7]_2$ code and binary repetition $[n, 1, n]_2$ codes of odd length. Since the length of these codes is odd, no one can be self-dual.

For the case of a self-dual perfect code with $d = 3$, hence for a self-dual Hamming $[n, n-m, 3]_q$ code, $n = 2(n-m)$ and thus $n = 2m$ implying

$$\frac{q^m - 1}{q - 1} = 2m.$$

The only solution is $q = 3$ and $m = 2$. Therefore, $n = 4$ and $k = 2$. □

Lemma 14. *If C is a self-dual code, then* $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| \neq 1$, *for any* $\mathbf{x}, \mathbf{y} \in C$.

Proof. Otherwise $\mathbf{x}$ and $\mathbf{y}$ would not be orthogonal vectors. □

Now, we show the nonexistence of a particular self-dual code.

Lemma 15. *There is no self-dual* $[6, 3, 4]_4$ *code.*

Proof. Let C be a $[6, 3, 4]_4$ code and consider a generator matrix for C:

$$G = \left(\begin{array}{c|c} I_3 & P \end{array} \right),$$

where I_3 is the 3×3 identity matrix and P is a 3×3 matrix with nonzero entries, since C has minimum weight 4. If C is self-dual then any row of G must be self-orthogonal, implying that for any row abc of P, we have $1+a^2+b^2+c^2=0$. Thus, $(a+b+c)^2=1$ and $a+b+c=1$. In $\mathbb{F}_4$ and since P has no zero entries, this means that $abc \in \{1xx, x1x, xx1\}$, where $x \neq 0$. If $abc=111$, then it is not orthogonal to any other row $a'b'c' \in \{1xx, x1x, xx1\}$. So, each of the three rows contains exactly one 1. Hence, two rows of P have the same value for x, say α. But such two rows cannot be identical, since the distance must be at least two. Hence, the inner product of these rows is α^2 and they are not orthogonal. Therefore the corresponding rows of G are also non-orthogonal. □

For a code C, let $A_w = |C_w|$. Thus, $\{A_0, A_1, \ldots, A_n\}$ is the weight distribution of C. The Pless power moments [18], as well as the McWilliams identities, relate the weight distribution of C and the weight distribution of $C^\perp$, for a linear code C. The first five Pless power moments can be seen in [12, pp. 259-260]. For a self-dual 3-weight $[2k,k,d]_q$ code with $d \geq 3$ and nonzero weights w_1, w_2 and w_3 the first three equations are:

$$A_{w_1}+A_{w_2}+A_{w_3} = q^k-1 \tag{7}$$

$$w_1A_{w_1}+w_2A_{w_2}+w_3A_{w_3} = q^{k-1}2k(q-1) \tag{8}$$

$$w_1^2A_{w_1}+w_2^2A_{w_2}+w_3^2A_{w_3} = q^{k-2}\left[2k(q-1)(2k(q-1)+1)\right]. \tag{9}$$

As a consequence of these equations, we have the following result.

Lemma 16. *If C is a self-dual 3-weight $[2k,k,d]_q$ code with nonzero weights w_1, w_2, w_3 such that $3 \leq d = w_1 < w_2 < w_3$, then $q(2k-w_3) < 2k$.*

Proof. Combining Eqs. (7) and (8), we get

$$(w_3 - w_1)A_{w_1} + (w_3 - w_2)A_{w_2} = w_3(q^k - 1) - q^{k-1}2k(q-1),$$

which gives

$$(w_3 - w_1)A_{w_1} + (w_3 - w_2)A_{w_2} + w_3 = q^{k-1}[(w_3 - 2k)q + 2k].$$

Obviously, both hand sides must be positive. Thus, we obtain $q(2k - w_3) < 2k$. □

Remark 17. *Lemma 16 can be easily generalized for any self-dual code with $d \geq 3$. With the same argument, one obtains $q(2k - w_r) < 2k$, where w_r is the greatest nonzero weight.*

Note that for any q-ary linear code, A_w is a multiple of $q-1$ (indeed, given any codeword, its $q-1$ multiples are codewords). Hence, we define $B_w = A_w/(q-1)$. Therefore, after dividing each term by $q-1$, Eqs. (7), (8) and (9) become:

$$\begin{aligned} B_{w_1} + B_{w_2} + B_{w_3} &= \frac{q^k - 1}{q-1} && (10) \\ w_1 B_{w_1} + w_2 B_{w_2} + w_3 B_{w_3} &= q^{k-1}2k && (11) \\ w_1^2 B_{w_1} + w_2^2 B_{w_2} + w_3^2 B_{w_3} &= q^{k-2}2k(2k(q-1)+1). && (12) \end{aligned}$$

We shall solve the system of Eqs. (10), (11) and (12) for several different cases. Therefore, we summarize in Table 1 some results we need.

Directly, from Table 1, we can state the nonexistence of certain self-dual 3-weight codes.

Proposition 18. *The following self-dual 3-weight codes do not exist:*

(i) A $[8,4,5]_7$ code.

w_1	w_2	w_3	q	k	$(B_{w_1}, B_{w_2}, B_{w_3})$
5	6	7	7	4	$(168, -280, 512)$
5	6	8	7	4	$(-8/3, 232, 512/3)$
5	7	8	7	4	$(224/3, 232, 280/3)$
3	4	5	q	3	$(q^2 - 5q + 10, 3(-q^2 + 5q - 5), 3(q-1)(q-2))$
4	5	6	q	3	$(15, 6(q-4), q^2 - 5q + 10)$
3	4	6	q	3	$(-2(q-4), 3(2q-3), (q-1)(q-2))$
3	5	6	q	3	$(5, 3(2q-3), q^2 - 5q + 5)$

Table 1: Some results of the system of Eqs. (10), (11) and (12)

(ii) A code with nonzero weights $3, 4, 5$.

Proof. Of course, solving the system of Eqs. (10), (11) and (12) we should obtain positive integer values for B_1, B_2 and B_3.

(i) Let w_1, w_2 and w_3 be the nonzero weights such that $5 = w_1 < w_2 < w_3 \leq 8$, then $(w_1, w_2, w_3) \in \{(5,6,7), (5,6,8), (5,7,8)\}$. These cases correspond to the first three rows in Table 1. In any case we always have negative and/or noninteger values. Consequently, no one of these codes can exist.

(ii) By Lemma 12, $q \geq 4$ and thus, by Lemma 16, $k = 3$. Then, by Eqs. (10), (11) and (12) we have $B_4 = 3(-q^2 + 5q - 5)$ (see the fourth row in Table 1). Hence, $-q^2 + 5q - 5 > 0$ and thus $q < 4$, which is a contradiction by Lemma 12. □

Remark 19. *For the case (i) in Proposition 18, note that a* $[8, 4, 5]_q$ *code meets the singleton bound (*$d \leq n - k + 1$*) and thus it is a maximum distance separable (MDS) code. The weight distribution of such codes is completely determined and, as can be seen in [16, p. 320],*

$$A_d = (q-1)\binom{n}{d}.$$

For $q = 7, n = 8, d = 5$, this gives $A_5 = 336$ and hence $B_5 = 56$, which does not coincide with the results of the system of equations. Therefore, we obtain a contradiction again.

3. Self-dual completely regular codes with $\rho \leq 2$

Let C be a self-dual CR $[n, k, d]_q$ code with covering radius $1 \leq \rho \leq 2$. In this section we give a full classification of such codes. Note that $n = 2k$ (since C is self-dual) and C is a 1-weight code (or equidistant code) or a 2-weight code (because $s = \rho$ by Lemma 8). Since $e \leq \rho$, we have that $1 \leq d \leq 6$. But for $d \geq 5$, $e = \rho$ and C would be a perfect 2-error-correcting code, that is, C would be a ternary Golay $[11, 6, 5]_3$ code which obviously is not self-dual (the extended ternary Golay code is self-dual, but with covering radius 3). Clearly, for $d = 1$ there is no self-dual code. Therefore, C must be a $[2k, k, d]_q$ code with one weight $d = 2$ or with two weights $w_1 = d \in \{2, 3, 4\}$ and w_2, where $d < w_2 \leq n$.

Now we study separately the cases $d = 2$, $d = 3$ and $d = 4$.

3.1. The case $d = 2$

This is a very simple case. If C is a self-dual CR $[n, k, 2]_q$ code, then by Lemmas 3 and 14, C is the direct sum of codes of length 2. If C_i is one such code, then C_i has generator matrix $G_i = (1 \;\; \alpha)$, where $1 + \alpha^2 = 0$. Indeed, such a code is CR with covering radius 1 (and $p_{1,1} = 2$). Therefore we have the following characterization.

Proposition 20. *If C is a self-dual CR $[n, k, 2]_q$ code, then C is a direct sum $C = C_1 \oplus \cdots \oplus C_j$, where C_i is a $[2, 1, 2]_q$ code ($1 \leq i \leq j$) and q is such that -1 is a square in $\mathbb{F}_q$. The covering radius of C is $\rho = j$ and its intersection array is:*

$$\mathrm{IA} = \{2j(q-1), 2(j-1)(q-1), \ldots, 2(q-1); 2, 4, \ldots, 2j\}.$$

Proof. Straightforward from Lemma 5, taking into account that each C_i has covering radius 1 and intersection array $\{2(q-1); 2\}$. $\square$

3.2. The case $d = 3$

Recall that for any code C, the set of codewords of weight w is denoted by C_w.

Lemma 21. *If C is a 2-weight $[n, k, 3]_q$ code such that $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| \in \{0, 3\}$ for all $\mathbf{x}, \mathbf{y} \in C_3$, then C is not CR.*

Proof. Let $\mathbf{x}, \mathbf{y} \in C_3$ such that $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 0$, by the assumption and Lemma 3 such vectors must exist. Then, C has weights 3 and 6. Any other codeword $\mathbf{z} \in C_3$ will have $\mathrm{supp}(\mathbf{z}) = \mathrm{supp}(\mathbf{x})$ or $\mathrm{supp}(\mathbf{z}) = \mathrm{supp}(\mathbf{y})$, otherwise C would have more than two weights.

Without loss of generality, assume that $\mathbf{x} = (x_1, x_2, x_3, 0, 0, 0)$ and $\mathbf{y} = (0, 0, 0, y_1, y_2, y_3)$. Now, the vector $\mathbf{v} = (x_1, v_2, 0, 0, 0, 0)$, where $v_2 \neq x_2$ is clearly at distance 2 to C and, since $d(\mathbf{v}, \mathbf{x}) = d(\mathbf{v}, \mathbf{0}) = 2$, we have $B_{\mathbf{v},2} \geq 2$. Now take $\mathbf{u} = (x_1, 0, 0, y_1, 0, 0)$. Clearly, $d(\mathbf{u}, C) = 2$ but $B_{\mathbf{u},2} = 1$. Therefore, C is not CR. $\square$

Proposition 22. *If C is a self-dual CR $[n, k, 3]_q$ code with covering radius $\rho = 2$, then $n = 4$ or $n = 8$.*

Proof. By Lemmas 3 and 21, there exist codewords $\mathbf{x}, \mathbf{y} \in C_3$, such that $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 2$ and thus $|\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{y})| = 4$. Now, if $\mathbf{z} \in C_3$ has $\mathrm{supp}(\mathbf{z}) \cap (\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{y})) \neq \emptyset$, we claim that $\mathrm{supp}(\mathbf{z}) \subset (\mathrm{supp}(\mathbf{x}) \cup \mathrm{supp}(\mathbf{y}))$. Otherwise, without loss of generality assume that $\mathbf{x} = (1, x_2, x_3, 0, \ldots, 0)$ and $\mathbf{y} = (1, y_2, 0, y_3, 0, \ldots, 0)$. By Lemma 21, we can assume that $\mathbf{z} = (z_1, z_2, 0, 0, z_3, 0, \ldots, 0)$. Now, since $q > 2$ by Lemma

12, we can take a multiple of $\mathbf{z}$, say $\mathbf{z}' = (z_1', z_2', 0, 0, z_3', 0, \ldots, 0)$, such that $z_2' = x_2 - y_2$. Hence, $\mathrm{wt}(\mathbf{x} - \mathbf{y} - \mathbf{z}') = 4$. But we can take another multiple, say $\mathbf{z}''$, such that $z_2'' \neq x_2 - y_2$. In this case, $\mathrm{wt}(\mathbf{x} - \mathbf{y} - \mathbf{z}'') = 5$. So, C has more than two nonzero weights, leading to a contradiction.

As a consequence, we have that C_3 induces a partition of the set of coordinates in 4-subsets, implying that n is a multiple of 4. But for $n > 8$, clearly C would have more than two nonzero weights. Therefore $n = 4$ or $n = 8$. □

For the case $n = 4$, we have the following necessary and sufficient condition.

Proposition 23. *There exists a self-dual CR* $[4, 2, 3]_q$ *code if and only if there exist elements* $\alpha, \beta \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ *such that* $1 + \alpha^2 + \beta^2 = 0$.

Proof. Let C be a self-dual $[4, 2, 3]_q$ code and let G be a generator matrix for C. We can write G as

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Then, by self-duality we have that $1 + a^2 + b^2 = 0$ (also, $1 + c^2 + d^2 = 0$).

Conversely, if $1 + \alpha^2 + \beta^2 = 0$, then consider the code C generated by the matrix

$$\begin{pmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & \beta & -\alpha \end{pmatrix}.$$

Clearly, C is self-dual.

If C is perfect, i.e. the ternary Hamming code (see Lemma 13), then C is CR. If C is not perfect, then $\rho \geq 2$ and, by (i) in Lemma 8, we have $\rho = s = 2$. Combining (ii) and (iv) in Lemma 8, we obtain that C is CR. In fact, C is a quasi-perfect uniformly packed code (see [11, Thm. 3.7]). □

3.3. The case $d = 4$

Proposition 24. *If C is a self-dual CR $[2k, k, 4]_q$ code with covering radius $\rho = 2$, then $k = 4$ and $q = 2$.*

Proof. Define the function

$$f(q, k) = \frac{(q-1)^2 k(2k-1)}{q^k - 1 - 2k(q-1)}.$$

By Corollary 10, $f(q, k)$ equals β_2^{-1} and must be a natural number. For $n = 2k = 4$, C cannot be a 2-weight code with minimum weight 4. Thus $k \geq 3$. It can be checked that the derivatives with respect to q and with respect to k are both negative. Thus, for a fixed k (resp. q), $f(q, k)$ is a decreasing function on q (resp. k). Moreover, q must be less than 16, otherwise $f(q, k) < 1$ ($f(16, 3) = 75/89$). Also, $f(q, k) < 1$ for $k > 6$ ($f(2, 7) = 91/113$). For all the possible values ($2 \leq q < 16$, $3 \leq k \leq 6$), we have computationally checked that the only natural values of $f(q, k)$ are $f(2, 4) = 4$, $f(2, 3) = 15$ and $f(4, 3) = 3$. But $f(2, 3) = 15$ implies $B_{\mathbf{x},2} = 15$ which is not possible for $q = 2$ and $n = 2k = 6$. Indeed, if $\mathbf{x}$ has weight 2, the number of codewords of weight 4 at distance 2 from $\mathbf{x}$ cannot be greater than 2 and, taking into account the zero codeword we have $B_{\mathbf{x},2} \leq 3$.

As a consequence, the only possible values for q and k are $(q, k) \in \{(2, 4), (4, 3)\}$, but by Lemma 15, the case $(q, k) = (4, 3)$ is not possible. □

4. Self-dual completely regular codes with $\rho = 3$

Let C be a self-dual CR $[n, k, d]_q$ code with covering radius $\rho = 3$. In this section we give a full classification of such codes. Note that $n = 2k$ (since C

is self-dual) and C is a 3-weight code (because $s = \rho = 3$ by Lemma 8). Since $e \leq \rho = 3$, we have that $d \leq 8$. But for $d \geq 7$, $e = \rho$ and C would be a perfect 3-error-correcting code, that is, C would be the binary Golay $[23, 12, 7]_2$ code which obviously is not self-dual (the extended binary Golay code is self-dual, but with covering radius 4). Hence, C must be a $[2k, k, d]_q$ code with weights $w_1 = d \in \{2, 3, 4, 5, 6\}$, w_2 and w_3, where $d < w_2 < w_3 \leq n$. But the case $d = 2$ is trivial: the only possibility is the direct sum of three self-dual $[2, 1, 2]_q$ codes. Therefore, we study the cases $d \in \{3, 4, 5, 6\}$.

4.1. The case $d = 6$

Proposition 25. *If C is a self-dual CR $[2k, k, 6]_q$ code with covering radius $\rho = 3$, then $k = 6$ and $q = 3$.*

Proof. Define the function

$$f(q, k) = \frac{(q-1)^3 k(2k-1)(2k-2)}{3[q^k - 1 - 2k(q-1) - k(2k-1)(q-1)^2]}.$$

By Corollary 11(i), $f(q, k)$ equals β_3^{-1} and must be a natural number. For length $2k = n < 8$, C cannot be a 3-weight code with minimum weight 6. Thus $k \geq 4$. It can be checked that the derivatives with respect to q and with respect to k are both negative. Thus, for a fixed k (resp. q), $f(q, k)$ is a decreasing function on q (resp. k). Moreover, for $q > 53$, $f(q, k) < 1$ ($f(59, 4) = 23548/25911$). Also, $f(q, k) < 1$ for $k > 10$ ($f(2, 11) = 770/897$). For all these possible values ($2 \leq q \leq 53$, $4 \leq k \leq 10$), we have computationally checked that the only natural values of $f(q, k)$ are $f(7, 4) = 9$ and $f(3, 6) = 4$. But $f(7, 4) = 9$ implies $\beta_3^{-1} = p_{3,3} = 9$ which is not possible for $n = 2k = 8$. Indeed, if $\mathbf{x}$ has weight 3, the number of codewords of weight 6 at distance 3 from $\mathbf{x}$ cannot be greater than 1 and, taking into account the zero codeword, we have $p_{3,3} \leq 2$.

As a consequence, the only possible values for q and k are $(q, k) = (3, 6)$.

□

4.2. The case $d = 5$

Proposition 26. *If C is a self-dual CR $[2k, k, 5]_q$ code with covering radius $\rho = 3$, then $q = 7$, $k = 4$ and the nonzero weights of C, verify $(w_1, w_2, w_3) \in \{(5, 6, 7), (5, 6, 8), (5, 7, 8)\}$.*

Proof. Assume that C is a self-dual CR $[2k, k, 5]_q$ code. Since $d = 5$, we have $e = 2$ and thus $\rho = e+1$. That is, C is a quasi-perfect uniformly packed code [11].

Define the function

$$g(q, k, s) = \frac{k(2k-1)(q-1)^2[(2k-2)(q-1)-3s]}{3[q^k - 1 - 2k(q-1) - k(2k-1)(q-1)^2]}.$$

By Corollary 11(ii), putting $s = p_{2,3}$, we have that $g(q, k, s)$ equals β_3^{-1} and must be a natural number. For length $2k = n < 8$, C cannot be a 3-weight code with minimum weight 5. Thus $k \geq 4$. Clearly, $g(q, k, s)$ is maximum when $s = 0$ and note that $g(q, k, 0)$ is the same that $f(q, k)$ in the proof of Proposition 25. In this case, as in the proof of Proposition 25, we have that for $q > 53$, $g(q, k, s) < 1$. Also, for $k > 10$, $g(q, k, s) < 1$. Since the minimum weight is 5, we have $q > 3$ by Lemma 12. For all these possible values ($4 \leq q \leq 53$, $4 \leq k \leq 10$, and $0 \leq s \leq \frac{(q-1)(2k-2)}{3}$, according to Corollary 11(ii)), we have computationally checked that the only natural values of $g(q, k, s)$ are

$$g(7, 4, 0) = 9; \;\; g(7, 4, 4) = 8; \;\; g(7, 4, 8) = 7; \;\; g(7, 4, 12) = 6.$$

Hence, in all cases we have $n = 2k = 8$ and $q = 7$.

Now, by Lemma 16 we obtain $w_3 \geq 7$ and since $n = 8$, we conclude $w_3 \in \{7, 8\}$. $\square$

Corollary 27. *There is no self-dual CR* $[2k, k, 5]_q$ *code with covering radius* $\rho = 3$.

Proof. By Proposition 26, such a code would be a $[8, 4, 5]_7$ code with nonzero weights $(w_1, w_2, w_3) = \{(5, 6, 7), (5, 6, 8), (5, 7, 8)\}$. The result then follows from Proposition 18. $\square$

4.3. The case $d = 4$

We start with a very restrictive condition.

Lemma 28. *Let* C *be a self-dual CR* $[2k, k, 4]_q$ *code with covering radius* $\rho = 3$. *If* $q > 2$, *then* $p_{2,2} = 2$.

Proof. Recall that $p_{i,j}$ is the number of codewords at distance j from any vector $\mathbf{v}$ such that $d(\mathbf{v}, C) = i$. Let $\mathbf{v} = (1, \alpha, 0, \ldots, 0)$ be a 2-weight vector. Clearly, $d(\mathbf{v}, C) = 2$ and $\mathbf{0}$ is a codeword at distance 2 from $\mathbf{v}$. Assume that $p_{2,2} > 2$ and let $\mathbf{x}, \mathbf{y} \in C_4$ be codewords such that $d(\mathbf{v}, \mathbf{x}) = d(\mathbf{v}, \mathbf{y}) = 2$. Then, $\mathbf{x}$ and $\mathbf{y}$ cover $\mathbf{v}$. It holds that $\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{v})$ (otherwise $\mathrm{wt}(\mathbf{x} - \mathbf{y}) < 4$). Since $\mathbf{x}$ and $\mathbf{y}$ must be orthogonal vectors, we have that $\alpha^2 = -1$. But this should be true for any nonzero element $\alpha \in \mathbb{F}_q^*$. This only happens in the binary field. Thus, $q = 2$. $\square$

Now we establish the nonexistence of self-dual CR quaternary codes of length $n \geq 6$ and minimum distance $d = 4$.

Proposition 29. *For* $\rho = 3$, *there is no self-dual CR* $[2k, k, 4]_4$ *code.*

Proof. If C is a self-dual CR $[2k, k, 4]_4$ code, then C_4 is a quaternary $2-(2k, 4, \lambda)$-design, by Lemma 2 (where $\lambda = 1$, by Lemma 28 and since $\lambda = p_{2,2} - 1$ by Corollary 11(iii)).

Consider the 2-weight vector $\mathbf{v} = (1, 1, 0, \ldots, 0)$ and let $\mathbf{x} \in C_4$ be a codeword covering $\mathbf{v}$. Without loss of generality, assume $\mathbf{x} = (1, 1, x, x, 0, \ldots, 0)$, where $x \in \mathbb{F}_4^*$ (note that the coordinates of $\mathbf{x}$ not covering $\mathbf{v}$ must be equal because $\mathbf{x} \cdot \mathbf{x} = 0$).

If $x \neq 1$, consider the vector $\mathbf{u} = (1, x, 0, \ldots, 0)$ and let $\mathbf{y} \in C_4$ be a codeword covering $\mathbf{u}$. Note that $\eta = |\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y})| = 3$. Indeed, if $\eta = 2$, then $\mathbf{x} \cdot \mathbf{y} = 1 + x \neq 0$, and if $\eta = 4$, then $\text{wt}(\mathbf{x} - \mathbf{y}) < 4$. Without loss of generality, assume that $\mathbf{y} = (1, x, y, 0, z, 0, \ldots, 0)$, where $y, z \in \mathbb{F}_4^*$. Now, we have that $\mathbf{x} \cdot \mathbf{y} = 1 + x + xy$. By self-duality, $\mathbf{x} \cdot \mathbf{y} = 0$, implying $xy = x^2$ (recall that in $\mathbb{F}_4$, $1 + \alpha + \alpha^2 = 0$ for $\alpha \in \mathbb{F}_4 \setminus \{0, 1\}$), and hence $y = x$. But now, $\mathbf{x} + \mathbf{y} = (0, 1 + x, 0, x, z, 0, \ldots, 0)$ which has weight less than 4 getting a contradiction.

If $x = 1$, then consider the vector $\mathbf{u} = (1, \alpha, 0, \ldots, 0)$ ($\alpha \in \mathbb{F}_4 \setminus \{0, 1\}$) and let $\mathbf{y} \in C_4$ be a codeword covering $\mathbf{u}$. As before, $|\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y})| = 3$ and we can assume $\mathbf{y} = (1, \alpha, y, 0, z, 0, \ldots, 0)$, where $y, z \in \mathbb{F}_4^*$. In this case, we obtain $\mathbf{x} \cdot \mathbf{y} = 1 + \alpha + y$, and since $\mathbf{x} \cdot \mathbf{y} = 0$, $y = \alpha^2$. However, $\mathbf{y} \cdot \mathbf{y} = 0$ implies $1 + \alpha^2 + \alpha + z^2 = 0$, which gives $z = 0$, again getting a contradiction.

□

The following proposition and corollary show the nonexistence of self-dual CR codes for $\rho = 3$ and $d = 4$.

Proposition 30. *For $\rho = 3$, there is no self-dual CR $[2k, k, 4]_7$ code.*

Proof. Consider $\mathbb{F}_7$ as $\mathbb{Z}_7$ and note that $x^2 \in \{1, 2, 4\}$ for any element

$x \in \mathbb{Z}_7^* = \mathbb{Z}_7 \setminus \{0\}$. If C is a self-dual CR $[2k, k, 4]_7$ code, then C_4 is a 7-ary $2-(2k, 4, \lambda)$-design, by Lemma 2 (where $\lambda = 1$, by Lemma 28).

Consider the 2-weight vectors $\mathbf{v} = (1, 1, 0, \ldots, 0)$ and $\mathbf{u} = (1, 2, 0 \ldots, 0)$ and let $\mathbf{x}, \mathbf{y} \in C_4$ be codewords covering $\mathbf{v}$ and $\mathbf{u}$, respectively. Note that $\eta = |\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 3$. Indeed, if $\eta = 2$, then $\mathbf{x} \cdot \mathbf{y} = 3 \neq 0$, and if $\eta = 4$, then $\mathrm{wt}(\mathbf{x} - \mathbf{y}) < 4$. Thus, without loss of generality, assume $\mathbf{x} = (1, 1, a, b, 0, \ldots, 0)$ and $\mathbf{y} = (1, 2, c, 0, d, 0, \ldots, 0)$, where $a, b, c, d \in \mathbb{Z}_7^*$. By self-duality, on the one hand, $\mathbf{x} \cdot \mathbf{x} = 0$, implying $a^2 + b^2 = 5$ and hence $\{a^2, b^2\} = \{1, 4\}$. Then $a \in \{1, 2, 5, 6\}$. On the other hand, $\mathbf{y} \cdot \mathbf{y} = 0$ implies $c^2 + d^2 = 2$. So, $c^2 = d^2 = 1$ and hence $c \in \{1, 6\}$. Therefore, we have $ac \in \{1, 2, 5, 6\}$.

Finally, we obtain a contradiction taking into account that $\mathbf{x} \cdot \mathbf{y} = 0$. Indeed, $\mathbf{x} \cdot \mathbf{y} = 1 + 2 + ac$ implies $ac = 4$. □

Corollary 31. *There is no self-dual CR $[6, 3, 4]_q$ code with covering radius $\rho = 3$ for any prime power q.*

Proof. Assume that C is a self-dual CR $[6, 3, 4]_q$ code. By Lemma 2, the codewords in C_4 form a q-ary $2-(6, 4, \lambda)$-design. Hence, according to Eq. (1), we have:

$$A_4 = \lambda_0 = \lambda \frac{\binom{6}{2}}{\binom{4}{2}} (q-1)^2 = \lambda \frac{5}{2} (q-1)^2. \tag{13}$$

In this case the nonzero weights are $w_1 = 4, w_2 = 5, w_3 = 6$. As can be seen in the fifth row of Table 1, $B_4 = 15$. Hence, $A_4 = B_4(q-1) = 15(q-1)$. Comparing with Eq. (13) and by Lemma 28, we conclude that $q = 7$ and $\lambda = 1$; or $q = 2$ and $\lambda = 6$. But this last binary case is not possible since λ cannot be greater than 2, by Corollary 11(iii).

Now, the result follows from Proposition 30. □

Proposition 32. *There is no self-dual CR* $[2k, k, 4]_q$ *code with covering radius* $\rho = 3$ *and* $q > 2$.

Proof. By Corollary 31 and Lemma 28, we only have to consider the cases where $k \geq 4$ and $\lambda = 1$ (recall that $\lambda = p_{2,2} - 1$ by Corollary 11(iii)). Define the function

$$h(q, k, \lambda') = \frac{1}{3} \cdot \frac{k(2k-1)(q-1)^2[2(2k-2)(q-1) - 4 - 6(q-2) - 3\lambda']}{2(q^k - 1) - k(q-1)[4 + (2k-1)(q-1)]},$$

which is β_3^{-1} for $\lambda = 1$, according to Corollary 11(iii). Therefore, $h(q, k, \lambda')$ must be a positive integer number. Note that $h(q, k, \lambda')$ is maximum when $\lambda' = 0$. For $k > 10$, the value of $h(q, k, \lambda')$ is less than 1. For $4 \leq k \leq 10$, the value of $h(q, k, \lambda')$ is greater than 1 for $q \leq 25$. Hence, we have to consider $h(q, k, \lambda')$ for $4 \leq k \leq 10$ and $4 \leq q \leq 25$ (by Lemma 12, $q \neq 3$). For $q, k \geq 4$, the denominator of $h(q, k, \lambda')$ is positive. Thus, in order to get the numerator positive, we need $\lambda' < [4(k-1)(q-1) - 6q + 8]/3$. Computationally, we have found that the only integer values of $h(q, k, \lambda')$ for these cases are $h(4, 4, 0) = 8$, $h(4, 4, 5) = 2$, $h(4, 6, 2) = 1$ and $h(7, 4, 9) = 1$. These values would correspond to codes with parameters:

$$[8, 4, 4]_4 \text{ with } \lambda' = 0; \quad [8, 4, 4]_4 \text{ with } \lambda' = 5;$$

$$[12, 6, 4]_4 \text{ with } \lambda' = 2; \quad [8, 4, 4]_7 \text{ with } \lambda' = 9.$$

By Proposition 29, the codes with parameters $[8, 4, 4]_4$ and $[12, 6, 4]_4$ cannot be self-dual and CR. Finally, by Proposition 30, a self-dual CR $[8, 4, 4]_7$ does not exist. □

Corollary 33. *There is no self-dual CR* $[2k, k, 4]_q$ *code with covering radius* $\rho = 3$.

Proof. By Proposition 32, we only have to consider the binary case. For $q = 2$, the expression of β_3 in Corollary 11(iii) becomes:

$$\beta_3 = \frac{3}{2} \cdot \frac{(\lambda+1)(2^k-1) - k[2(\lambda+1)+(2k-1)]}{k(2k-1)(\lambda+1)[(k-1)-\lambda]},$$

since $\lambda' = 0$ due to the fact that all weights must be even (see Lemma 12 and thus $C_5 = \emptyset$). Define the function

$$\ell(k,\lambda) = \frac{2}{3} \cdot \frac{k(2k-1)(\lambda+1)[(k-1)-\lambda]}{(\lambda+1)(2^k-1) - k[2(\lambda+1)+(2k-1)]},$$

Clearly, $\ell(k,\lambda)$ equals β_3^{-1} and must be a natural number. For $k > 10$, the value of $\ell(k,\lambda)$ is less than 1. Checking all the values for $3 \leq k \leq 10$ and $0 \leq \lambda \leq k-1$ (according to Corollary 11(iii)), the result is that only $\ell(5,2) = 10$ is a natural number. It corresponds to a $[10,5,4]_2$ code. But such code cannot be self-dual, as can be seen in [12, Example 9.4.2].

The conclusion is that there is no binary self-dual CR code with minimum distance $d = 4$ and covering radius $\rho = 3$. □

4.4. The case $d = 3$

Proposition 34. *If C is a self-dual CR 3-weight $[2k,k,3]_q$ code, then*

$$|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 2, \quad \textit{for some} \;\; \mathbf{x},\mathbf{y} \in C_3.$$

Proof. Otherwise, we would have $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| \in \{0,3\}$ by Lemma 14. Hence, by Lemma 3, the length $2k$ should be divisible by 3 and, in fact, $2k = 6$ (9 is odd and for $2k \geq 12$, C would have more than 3 weights). Therefore, by Proposition 18, the weights of C would be $w_1 = 3$, $w_2 \in \{4,5\}$, $w_3 = 6$. Since $k = 3$, we can apply again Eqs. (10), (11) and (12).

For $w_2 = 4$, as can be seen in row 6 of Table 1, $B_3 = -2(q-4)$ which implies $q < 4$, leading to a contradiction by Lemma 12.

For $w_2 = 5$, as can be seen in the last row of Table 1, $B_3 = 5$. If $|\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y})| = 3$, for $\mathbf{x}, \mathbf{y} \in C_3$, then $\mathbf{x}$ is a multiple of $\mathbf{y}$ (otherwise, taking appropriate multiples, we would get $0 < \text{wt}(\mathbf{x} - \mathbf{y}) < 3$). Hence, $B_3 = 2$ which contradicts the result of the system. □

Corollary 35. *If C is a self-dual CR $[2k, k, 3]_q$ code with covering radius $\rho = 3$, then $k = 6$.*

Proof. By Proposition 34, there exist codewords $\mathbf{x}, \mathbf{y} \in C_3$, such that $|\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y})| = 2$ and thus $|\text{supp}(\mathbf{x}) \cup \text{supp}(\mathbf{y})| = 4$. Now, if $\mathbf{z} \in C_3$ has $\text{supp}(\mathbf{z}) \cap (\text{supp}(\mathbf{x}) \cup \text{supp}(\mathbf{y})) \neq \emptyset$, we claim that $\text{supp}(\mathbf{z}) \subset (\text{supp}(\mathbf{x}) \cup \text{supp}(\mathbf{y}))$. Otherwise, without loss of generality, assume that the vectors $\mathbf{x}$ and $\mathbf{y}$ have values $\mathbf{x} = (1, x_2, x_3, 0, \ldots, 0)$ and $\mathbf{y} = (1, y_2, 0, y_3, 0, \ldots, 0)$. By Lemma 14, we can assume that $\mathbf{z} = (z_1, z_2, 0, 0, z_3, 0, \ldots, 0)$. Now, since $q > 2$, we can take a multiple of $\mathbf{z}$, say $\mathbf{z}' = (z_1', z_2', 0, 0, z_3', 0, \ldots, 0)$, such that $z_2' = x_2 - y_2$. Hence, $\text{wt}(\mathbf{x} - \mathbf{y} - \mathbf{z}') = 4$. But we can take another multiple, say $\mathbf{z}''$, such that $z_2'' \neq x_2 - y_2$. In this case, $\text{wt}(\mathbf{x} - \mathbf{y} - \mathbf{z}'') = 5$. So, C has weights 3, 4 and 5, which is a contradiction, by Proposition 18. Define $C_3(\mathbf{x}) = \{\mathbf{z} \in C_3 \mid \text{supp}(\mathbf{z}) \cap \text{supp}(\mathbf{x}) \neq \emptyset\}$ and $S(\mathbf{x}) = \bigcup_{\mathbf{z} \in C_3(\mathbf{x})} \text{supp}(\mathbf{z})$. We have seen that $|S(\mathbf{x})| = 4$. Note also that any one weight vector with the nonzero coordinate in $S(\mathbf{x})$ is covered by more that one codeword of $C_3(\mathbf{x})$, i.e. $p_{1,2} > 1$.

Now, consider a codeword $\mathbf{x}' \in C_3 \setminus C_3(\mathbf{x})$ and let $\mathbf{v}$ be a 1-weight vector covered by $\mathbf{x}'$. Since $p_{1,2} > 1$, there is some other codeword $\mathbf{y}' \in C_3$ covering $\mathbf{v}$. Hence, $\mathbf{y}' \in C_3(\mathbf{x}')$ and it is not a multiple of $\mathbf{x}'$. Clearly, $|\text{supp}(\mathbf{x}') \cap \text{supp}(\mathbf{y}')| = 2$ (since $d(\mathbf{x}', \mathbf{y}') > 2$ and by Lemma 14). Therefore, as in the case of $S(\mathbf{x})$, we obtain $|S(\mathbf{x}')| = 4$.

Repeating the argument, we have that C_3 induces a partition of the set of coordinates in 4-subsets, say $P_1, \ldots, P_{k/2}$, implying that $2k$ is a multiple of 4. Therefore, k is even.

On the other hand, if we take a 2-weight vector $\mathbf{x}$ with one nonzero coordinate in P_i and the other one in P_j $(i \neq j)$, then $d(\mathbf{x}, C) = 2$ and $B_{\mathbf{x},2} = 1$, since the zero codeword is the only one at distance 2 from $\mathbf{x}$. Now, take any 2-weight vector $\mathbf{y}$ with both nonzero coordinates in P_i. Since $|P_i| = 4$, there exists some codeword $\mathbf{z}$ of weight 3 including the support of $\mathbf{y}$ and (taking the appropriate multiple) such that $d(\mathbf{z}, \mathbf{y}) \leq 2$. If $d(\mathbf{z}, \mathbf{y}) = 2$, then $B_{\mathbf{y},2} > 1$ and the code would not be CR. This means that any 2-weight vector $\mathbf{y}$ with both nonzero coordinates in P_i is at distance 1 from C. In other words, the projection of C in P_i, for any $i = 1, \ldots, k/2$, must be a Hamming code of length $n = (q^m - 1)/(q - 1) = 4$, i.e. a ternary Hamming $[4, 2, 3]_3$ code which is self-dual (see Lemma 13). Since C has covering radius $\rho = 3$, there exists some 3-weight vector $\mathbf{x}$ such that $d(\mathbf{x}, C) = 3$. Thus, $\mathbf{x}$ has the three nonzero coordinates in different P_i's. This implies $k \geq 6$, but for $k > 6$, C would have more than three nonzero weights. As a conclusion $k = 6$. □

4.5. The full classification

Now, from Propositions 20, 22, 23, 24, 25, and Corollaries 27, 33, 35, we obtain the main classification theorem.

Theorem 36. *Let C be a self-dual CR $[n, k, d]_q$ code.*

(i) If $d \leq 2$, then C is the direct sum of j copies ($j = 1, 2, \ldots$) of a $[2, 1, 2]_q$ code with generator matrix $(1 \;\; \alpha)$ such that $\alpha^2 = -1$. Such q-ary code

exists if and only if -1 *is a square in* $\mathbb{F}_q$*. The code* C *has covering radius* $\rho = j$ *and intersection array*

$$\mathrm{IA} = \{2j(q-1), 2(j-1)(q-1), \ldots, 2(q-1); 2, 4, \ldots, 2j\}.$$

(ii) If $d = 3$ *and* $\rho = 1$*, then* C *is the ternary Hamming* $[4,2,3]_3$ *code with intersection array*

$$\mathrm{IA} = \{8; 1\}.$$

509 *(iii) If* $d = 3$ *and* $\rho = 2$*, then* C *is*

(iii.i) the direct sum of two ternary Hamming $[4,2,3]_3$ *codes, that is,* C *is a* $[8,4,3]_3$ *code with weights* $w_1 = 3$ *and* $w_2 = 6$*, and intersection array*

$$\mathrm{IA} = \{16, 8; 1, 2\};$$

510 *or*

511 *(iii.ii) any* $[4,2,3]_q$ *code with generator matrix*

$$G = \begin{pmatrix} 1 & 0 & \alpha & \beta \\ 0 & \xi & \beta & -\alpha \end{pmatrix}, \tag{14}$$

where $\alpha, \beta \in \mathbb{F}_q^*$ *are two elements such that* $1+\alpha^2+\beta^2 = 0$*,* $\xi^2 = 1$ *and* $q > 3$*.* C *has weights* $w_1 = 3$ *and* $w_2 = 4$*, and intersection array*

$$\mathrm{IA} = \{4(q-1), 3(q-3); 1, 12\}.$$

(iv) If $d = 3$ *and* $\rho = 3$*, then* C *is the direct sum of three ternary Hamming* $[4,2,3]_3$ *codes, that is, a* $[12,6,3]_3$ *code with weights* $w_1 = 3$*,* $w_2 = 6$*,* $w_3 = 9$ *and intersection array*

$$\mathrm{IA} = \{24, 16, 8; 1, 2, 3\}.$$

(v) If $d = 4$ *and* $\rho \leq 3$*, then* C *is the extended binary Hamming* $[8,4,4]_2$ *code, with weights* $w_1 = 4$ *and* $w_2 = 8$ *(so, an antipodal code), and with intersection array*

$$\mathrm{IA} = \{8, 7; 1, 4\}.$$

512 *(vi) If* $d = 5$ *and* $\rho \leq 3$*,* C *does not exist.*

(vii) *If* $d = 6$ *and* $\rho \leq 3$*, then* C *is the extended ternary Golay* $[12, 6, 6]_3$ *code, with weights* $w_1 = 6$*,* $w_2 = 9$*,* $w_3 = 12$ *(so, an antipodal code), and with intersection array*

$$\mathrm{IA} = \{24, 22, 20; 1, 2, 12\}.$$

No other self-dual CR codes with $\rho \leq 3$ *exist.*

Proof. (i) Direct from Proposition 20.

(ii) In this case, since $e = \rho = 1$, C is a self-dual perfect single-error-correcting code. Hence, C is a self-dual Hamming code and, by Lemma 13, C is the ternary Hamming $[4, 2, 3]_3$ code. The intersection array is trivial and can be seen, for instance, in family (F.1) of [4].

(iii) By Proposition 22, C has length $n = 4$ or $n = 8$.

(iii.i) If $n = 8$, let C be a self-dual CR $[8, 4, 3]_q$ code with covering radius $\rho = 2$. By the argument in the proof of Proposition 22, the set of coordinates $\{1, \ldots, 8\}$ is partitioned into two 4-subsets, say A and B, such that any codeword of weight 3 has its support contained in A or in B. Since C must be a 2-weight code, these weights are trivially $w_1 = 3$ and $w_2 = 6$. Therefore C is the direct sum $C = C_1 \oplus C_2$ of two 1-weight codes (whose nonzero codewords have weight 3). It is clear that C is self-dual if and only if C_1 and C_2 are self-dual.

On the other hand, if we take a 2-weight vector $\mathbf{x}$ with one nonzero coordinate in A and the other one in B, then $d(\mathbf{x}, C) = 2$ and $B_{\mathbf{x},2} = 1$, since the zero codeword is the only one at distance 2 from $\mathbf{x}$. Now, take any 2-weight vector $\mathbf{y}$ with both nonzero coordinates in A (or in B). Since $|A| = |B| = 4$, there exist some codeword $\mathbf{z}$ of weight 3 including the support of $\mathbf{y}$ and (taking the appropriate multiple) such that $d(\mathbf{z}, \mathbf{y}) \leq 2$ (note that

$q > 2$ by Lemma 12). If $d(\mathbf{z}, \mathbf{y}) = 2$, then $B_{\mathbf{y},2} > 1$ and the code would not be CR. This means that any 2-weight vector $\mathbf{y}$ with both nonzero coordinates in A (or in B) is at distance 1 from C. In other words, C_1 and C_2 must be self-dual Hamming codes. By Lemma 13, C_1 and C_2 are ternary Hamming codes of length 4. Therefore, C is the direct sum of two ternary Hamming $[4, 2, 3]_3$ codes. Indeed, the direct sum of perfect codes is a CR code (see Lemma 5). The intersection array follows from (ii) and Lemma 5.

(iii.ii) If C is a self-dual $[4, 2, 3]_q$ code, then consider a generator matrix for C in the form

$$G = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Multiplying the second row by the appropriate value (in fact bc^{-1}), we can get the matrix

$$G = \begin{pmatrix} 1 & 0 & a & b \\ 0 & \xi & b & \xi' \end{pmatrix}.$$

Since both rows must be orthogonal, we obtain that $\xi' = -a$, and by self-orthogonality of the second row, we have $\xi^2 = 1$. Now, by Proposition 23, such code is CR. By Lemma 12, $q = 2$ is not possible, and for $q = 3$ there are only codewords of weight 3, i.e. the case (ii) with $\rho = 1$. Therefore $q > 3$.

By Proposition 23, the necessary and sufficient condition for the existence of such self-dual CR codes is the existence of elements $\alpha, \beta \in \mathbb{F}_q^*$ such that $1 + \alpha^2 + \beta^2 = 0$. For $q = 2^r$ and $r > 1$, we have $1 + \alpha^2 + \beta^2 = (1 + \alpha + \beta)^2$. Hence, the condition is equivalent to the existence of $\alpha, \beta \in \mathbb{F}_q^*$ such that $1 + \alpha + \beta = 0$. Obviously such values exist always. A generator matrix can then be written as

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & \alpha & \beta \end{pmatrix},$$

since $(1,1,1,1)$ is orthogonal to any codeword and thus it is a codeword. These antipodal codes correspond to the family (F.48) in [4]. The intersection array can also be seen in [4].

However, for odd q, the existence of the values α and β is not guaranteed. For example, it is easy to see that for $q = 5$ there are no such values. Whereas for $q = 7$ we can find these values. For example, considering $\mathbb{F}_7$ as $\mathbb{Z}_7$, a self-dual CR $[4,2,3]_7$ code is generated by the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 5 \end{pmatrix}.$$

For these cases where q is odd, the intersection array must be the same, since all parameters (except q) are the same that for the case of even q.

(iv) In this case, by Corollary 35, C must be a $[12,6,3]_q$ code. By the argument of the proof of Corollary 35 (similar to the case (iii.i)), C is the direct sum of three ternary Hamming $[4,2,3]_3$ codes. The intersection array follows from (ii) and Lemma 5.

(v) By Proposition 24 and Corollary 33, we have that C is a $[8,4,4]_2$ code. This is the well-known binary extended Hamming code of length 8, which is self-dual. Trivially the weights are 4 and $n = 8$. This code falls into the family (F.2) in [4], where the intersection array is also specified.

(vi) Since $d = 5$, we have that $e = 2$. Hence $\rho > 2$, otherwise C would be a perfect doubly-error-correcting code. The only such code is the ternary

Golay $[11, 6, 5]_3$ code, which obviously is not self-dual. For $\rho = 3$ the code C cannot exist by Corollary 27.

(vii) For $d = 6$, again $\rho > 2$, and for $\rho = 3$ we have that C is a $[12, 6, 6]_3$ code, by Proposition 25. As can be seen in [9, 19], any code with these parameters must be the extended ternary Golay code, which is self-dual. The weights of such code are 6, 9 and 12, as can be seen, for example, in [16]. This code corresponds to (S.12) in [4], where the intersection array is also specified. □

5. Concluding remarks and further research

Let q' be a prime power such that -1 is a square in $\mathbb{F}_{q'}$. Then, from Theorem 36, we see that the parameters for self-dual CR codes are

- For $\rho = 1$: $[2, 1, 2]_{q'}$, $[4, 2, 3]_3$.
- For $\rho = 2$: $[4, 2, 2]_{q'}$, $[4, 2, 3]_q$, $[8, 4, 3]_3$, $[8, 4, 4]_2$.
- For $\rho = 3$: $[6, 3, 2]_{q'}$, $[12, 6, 3]_3$, $[12, 6, 6]_3$.

For $\rho = 4$, obviously we have the codes with parameters $[8, 4, 2]_{q'}$ and $[16, 8, 3]_3$, corresponding to the direct sums of four copies of a self-dual $[2, 1, 2]_{q'}$ code and four copies of the ternary Hamming $[4, 2, 3]_3$ code. In addition, we have the binary extended Golay $[24, 12, 8]_2$ code. For $\rho > 4$, apart from the direct sums of copies of a self-dual CR code with $\rho = 1$, it seems that there are no other possibilities. However, the techniques used here become of high complexity for $\rho > 3$.

For $q = p^2$ (p prime) it is often considered Hermitian duality. The Hermitian inner product between two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{F}_q^n$ is defined as

$$< \mathbf{v}, \mathbf{u} >_H = \mathbf{v} \cdot \bar{\mathbf{u}} = \sum_{i=1}^{n} v_i \bar{u}_i \in \mathbb{F}_q,$$

where $\bar{u_i} = u_i^p$ is the conjugation of u_i. Results on Hermitian self-duality can be seen, for example, in [15]. For any code $C \subseteq \mathbb{F}_q^n$, it is easy to verify that the Hermitian dual code $C^{\perp_H} = \{\mathbf{x} \in \mathbb{F}_q^n \mid < \mathbf{x}, \mathbf{v} >_H = 0, \ \forall \mathbf{v} \in C\}$ coincides with $\overline{C^{\perp}} = \{\bar{\mathbf{v}} \mid \mathbf{v} \in C^{\perp}\}$. Hence, the parameters (including the weight distribution) of $C^{\perp_H}$ are the same that those of $C^{\perp}$. Therefore several results of this paper applies also to Hermitian duality. However, there are notorious exceptions as for the case of Lemma 15, where we have proven the nonexistence of a self-dual $[6, 3, 4]_4$ code. But the so-called hexacode (see [12, Sect. 10.3]) is a Hermitian self-dual CR $[6, 3, 4]_4$ code. Thus the classification given by Theorem 36 is not valid for Hermitian self-duality, but it should not be difficult to state a similar classification for Hermitian self-dual CR codes.

Another future work could be the study of self-dual CR additive codes. An additive code is an additive subgroup of $\mathbb{F}_q^n$, hence not necessarily linear. Self-dual additive codes has been studied, for example in [10]. Also, CR additive codes has been studied in several papers, for example in [14].

Acknowledgements

This work has been partially supported by the Spanish Ministerio de Ciencia e Innovación under Grants PID2022-137924NB-I00 (AEI/FEDER UE), RED2022-134306-T, and also by the Catalan AGAUR Grant 2021SGR-00643.

The research of the second author of the paper was carried out at the Institute for Information Transmission Problems of the Russian Academy of Sciences within the program of fundamental research on the topic “Mathematical Foundations of the Theory of Error-Correcting Codes" and was also partly supported by the Higher School of Modern Mathematics MIPT (project No. FSMG-2024-0048), Moscow College of Physics and Technology, 1 Klimentovskiy per., Moscow, Russia.

We thank to the anonymous referees for their remarks and corrections, which have enabled us to greatly improve the paper.

References

[1] L.A. Bassalygo, V.A. Zinoviev, “A note on uniformly packed codes", *Problems Inform. Transmiss.*, vol. 13, no. 3, pp. 22–25, 1977.

[2] L.A. Bassalygo, G.V. Zaitsev, V.A. Zinoviev, “Uniformly packed codes," *Problems Inform. Transmiss.,* vol. 10, no. 1, pp. 9–14, 1974.

[3] J. Borges, J. Rifà, V. A. Zinoviev, “On q-ary linear completely regular codes with $\rho = 2$ and antipodal dual", *Adv. in Maths. of Comms.*, vol. 4, pp. 567–578, 2010.

[4] J. Borges, J. Rifà, V.A. Zinoviev, “On completely regular codes", *Problems Inform. Transmiss.*, vol. 55, no. 1, pp. 1–45, 2019.

[5] J. Borges, V.A. Zinoviev, D.V. Zinoviev, “On the classification of completely regular codes with covering radius two and antipodal duals", *Problems Inform. Transmiss.*, vol. 59, no. 3, pp. 204–216, 2023.

[6] A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer, Berlin, 1989.

[7] E. R. van Dam, J.H. Koolen, H. Tanaka, “Distance-Regular graphs" *The Electronic Journal of Combinatorics.*, DS. no. 22, pp. 1–156.

[8] P. Delsarte, “An Algebraic Approach to the Association Schemes in Coding Theory", *Philips Res. Rep. Suppl.*, 10, 1973.

[9] P. Delsarte, J.M. Goethals, “Unrestricted codes with the Golay parameters are unique," *Discrete Math.*, vol. 12, pp. 211—224, 1975.

[10] S. T. Dougherty, A. Korban, S. Şahinkaya, “Self-dual additive codes," *Appl. Alg. in Eng., Comm. and Comp.*, vol. 33, pp. 569–586, 2022.

[11] J.M. Goethals, H.C.A. Van Tilborg, “Uniformly packed codes", *Philips Res.*, vol. 30, pp. 9–36, 1975.

[12] W.C. Huffman, V. Pless, *Fundamentals of error-correcting codes*, Cambridge Univ. Press, New York, 2003.

[13] J.H. Koolen, D. Krotov, B. Martin, “Completely regular codes: Tables of small parameters for binary and ternary Hamming graphs," Ch. 7 in Shi and Solé, eds. *Completely Regular Codes in Distance Regular Graphs.* CRC Press, Boca Raton FL, 2025. To appear.

[14] J. H. Koolen, W. S. Lee, W. J. Martin, H. Tanaka, “Arithmetic completely regular codes," *Discrete Mathematics and Theoretical Computer Science,* vol. 17, pp. 59–76, 2016.

[15] Lin Sok, Chong Yanga, “Constructions of optimal Hermitian self-dual codes from unitary matrices," *Finite Fields and their Appl.*, vol. 79, 102000, 2022.

[16] F.J. MacWilliams, N.J.A. Sloane, *The theory of error-correcting codes*, North Holland Publishing Company, 1977.

[17] A. Neumaier, “Completely regular codes," *Discrete Maths.*, vol. 106/107, pp. 335—360, 1992.

[18] V. Pless, “Power moment identities on weight distributions in error correcting codes," *Inform. and Control*, vol. 6, pp. 147-–152, 1963.

[19] V. Pless, “On the uniqueness of the Golay codes," *J. Comb. Theory*, vol.5, pp. 215-–228, 1968.

[20] N.V. Semakov, V.A. Zinoviev, G.V. Zaitzev, “Uniformly packed codes", *Problems Inform. Transmiss.*, vol. 7, pp. 38–50, 1971.

[21] P. Solé, “Completely regular codes and completely transitive codes", *Discrete Maths.*, vol.81, no. 2 (1990), pp. 193–201, 1990.

[22] A. Tietäväinen, “On the Nonexistence of Perfect Codes over Finite Fields", *SIAM J. Appl. Math.*, 1973, vol. 24, no. 1, pp. 88-–96, 1993.

[23] V.A. Zinoviev, V.K. Leontiev, “On Non-existence of Perfect Codes over Galois Fields", *Probl. Upravlen. Teor. Inform.*, 1973, vol. 2, no. 2, pp. 123—132 [Probl. Control Inf. Theory (Engl. Transl.), 1973, vol. 2, no. 2, pp. 16-–24].