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On fractional parabolic BMO and Lip_α caloric capacities

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Abstract
In the present paper we characterize the removable sets for solutions of the fractional heat equation satisfying some parabolic BMO or Lip_α normalization conditions. We do this by introducing associated fractional caloric capacities, that we show to be comparable to a certain parabolic Hausdorff content.

Keywords Fractional heat equation · Singular integrals · Removable singularities · Capacities

Mathematics Subject Classification Primary 42B20; Secondary 28A12

1 Introduction

In this paper we characterize removable sets for solutions of the fractional heat equation under certain parabolic BMO or Lip_α normalization conditions. Our main motivation stems from the results obained in [14] and [13]. The study conducted by Mateu, Prat and Tolsa in [14] explores removable singularities for regular $(1, 1/2)-$Lipschitz solutions of the classical heat equation, associated with the operator

$$\Theta := (-\Delta_x) + \partial_t, \quad \text{where} \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Here $(-\Delta_x)$ is the usual Laplacian, computed with respect to the spatial variables. In [13], the authors extend the study to the fractional heat equation, defined via the s-heat operator

$$\Theta^s := (-\Delta_x)^s + \partial_t, \;\; s \in (0, 1].$$

For $s = 1$, we recover the classical heat equation, while for $s < 1$, the operator $(-\Delta_x)^s$, commonly referred to as s-fractional Laplacian or s-Laplacian, requires an alternative definition.

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It is typically introduced through its Fourier transform:

$$\widehat{(-\Delta_x)^s} f(\xi, t) = |\xi|^{2s} \widehat{f}(\xi, t),$$

or via the singular integral representation

$$\begin{aligned}(-\Delta_x)^s f(x,t) &= c_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{f(x,t) - f(y,t)}{|x-y|^{n+2s}} \mathrm{d}y \\ &= c'_{n,s} \int_{\mathbb{R}^n} \frac{f(x+y,t) - 2f(x,t) + f(x-y,t)}{|y|^{n+2s}} \mathrm{d}y.\end{aligned}$$

These representations are equivalent and highlight that $(-\Delta_x)^s$ is no longer a local operator and that as $s \to 1$, one recovers the expression of $(-\Delta_x)$. The reader may consult [4, §3] or [19] for details on the properties of $(-\Delta_x)^s$.

To study removable sets in this context, we introduce the *s-parabolic distance* between two points $\overline{x} := (x, t)$, $\overline{y} := (y, \tau)$ in $\mathbb{R}^{n+1}$, defined as

$$|\overline{x} - \overline{y}|_{p_s} = \text{dist}_{p_s}(\overline{x}, \overline{y}) := \max\left\{|x-y|, |t-\tau|^{\frac{1}{2s}}\right\}, \qquad \text{for } 0 < s \leq 1.$$

This leads naturally to the notions of s-parabolic cubes and s-parabolic balls. We convey that $B(\overline{x}, r)$ will be the *s-parabolic ball* centered at $\overline{x}$ with radius r, where the spatial coordinates are contained in a Euclidean ball B_1 of radius r, while the temporal coordinate lies in a real interval I of length $(2r)^{2s}$. On the other hand, an *s-parabolic cube* Q of side length ℓ is a set of the form

$$I_1 \times \cdots \times I_n \times I_{n+1},$$

where $I_1, \ldots, I_n$ are intervals of length ℓ, while I_{n+1} is another interval of length ℓ^{2s}. We write $\ell(Q) = \ell$.

Let us recall that a function f is said to be (1, 1/2)-Lipschitz regular if, as precised in [14], it is such that

$$\|\nabla_x f\|_{L^\infty(\mathbb{R}^{n+1})} < \infty, \qquad \|\partial_t^{1/2} f\|_{*,p_1} < \infty. \tag{1}$$

Here $\|\cdot\|_{*,p_1}$ stands for the usual BMO($\mathbb{R}^{n+1}$) norm but computed with respect to 1-*parabolic* cubes. As shown by Hofmann and Lewis [10, Lemma 1], [11, Thm. 7.4], such functions satisfy

$$\|f\|_{\text{Lip}_{1/2},t} := \sup_{\substack{x \in \mathbb{R}^n \\ t,u \in \mathbb{R}, t \neq u}} \frac{|f(x,t) - f(x,u)|}{|t-u|^{1/2}} \lesssim \|\nabla_x f\|_{L^\infty(\mathbb{R}^{n+1})} + \|\partial_t^{1/2} f\|_{*,p_1}.$$

Thus a (1, 1/2)−Lipschitz function is Lipschitz in the spatial variables and 1/2-Lipschitz in time. This explains the term (1, 1/2)−Lipschitz caloric capacity introduced in [14], defined for a compact set $E \subset \mathbb{R}^{n+1}$ as

$$\Gamma_\Theta(E) = \sup\{|\langle \Theta f, 1 \rangle|\},$$

the supremum taken over all (1, 1/2)−Lipschitz regular functions f satisfying the heat equation on $\mathbb{R}^{n+1} \setminus E$ and with the norms in (1) smaller or equal than one.

A key result in [14] establishes the equivalence between the removability of the compact set E for (1, 1/2)−Lipschitz solutions of the heat equation and the fact that $\Gamma_\Theta(E)$ vanishes.

In this paper, we aim at characterizing different variants of the previous Lipschitz caloric capacity, replacing the previous estimates with parabolic BMO or Lip_α conditions for $\nabla_x f$ and $\partial_t^{1/2} f$. More generally, we analyze removable sets for solutions of the s-fractional heat equation with s-parabolic gradient $(\nabla_x, \partial_t^{\frac{1}{2s}})$ satisfying either an s-parabolic BMO or Lip_α

condition. The reader who is not familiar with the notion of removability may conceive removable sets as those which "do not matter" when solving the Θ^s-equation, $0 < s \leq 1$. This has to be understood in the sense that any solution defined on their complement that satisfies the above $(1, \frac{1}{2s})$-gradient estimates, can be extended to verify the Θ^s-equation throughout the entire domain, including the set itself.

Our main result characterizes removability in terms of two different capacities: one requiring the $(1, \frac{1}{2s})$-gradient of solutions of the Θ^s-equation satisfy s-parabolic BMO estimates, and another one requiring s-parabolic Lip_α bounds. These capacities, denoted by $\Gamma_{\Theta^s,*}$ and $\Gamma_{\Theta^s,\alpha}$ respectively, are related to certain s-parabolic Hausdorff contents $\mathcal{H}^m_{\infty,p_s}$, which are defined as in the Euclidean case (see [15], for instance), just replacing the Euclidean distance by the parabolic distance introduced above. Our main result reads as follows:

Theorem *Let $s \in (1/2, 1]$, $\alpha \in (0, 1)$ and $E \subset \mathbb{R}^{n+1}$ compact set. Then,*

$$\Gamma_{\Theta^s,*}(E) \approx_{n,s} \mathcal{H}^{n+1}_{\infty,p_s}(E),$$
$$\text{if } \alpha < 2s - 1, \quad \Gamma_{\Theta^s,\alpha}(E) \approx_{n,s,\alpha} \mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E).$$

Moreover, the nullity of these capacities is equivalent to the removability of the corresponding compact set for solutions satisfying $(1, \frac{1}{2s})$-gradient estimates in either s-parabolic BMO or Lip_α, assuming $\alpha < 2s - 1$ in the latter case.

We further study the same type of question for a generalization of the capacities presented by Mateu and Prat in [13, §4 & §7]. That is, we will ask for a characterization of removable sets for solutions of the Θ^s-equation satisfying conditions of the form

$$\|(-\Delta)^\sigma f\| < \infty, \qquad \|\partial_t^{\sigma/s} f\| < \infty, \qquad s \in (0, 1] \text{ and } \sigma \in [0, s).$$

Symbols $\|\cdot\|$ can refer both to s-parabolic BMO norms or both to s-parabolic Lip_α seminorms, giving rise to the capacities $\gamma^\sigma_{\Theta^s,*}$ and $\gamma^\sigma_{\Theta^s,\alpha}$ respectively. We prove the following:

Theorem *For any $s \in (0, 1]$, $\sigma \in [0, s)$, $\alpha \in (0, 1)$ and $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\gamma^\sigma_{\Theta^s,*}(E) \approx_{n,s,\sigma} \mathcal{H}^{n+2\sigma}_{\infty,p_s}(E),$$
$$\text{if } \alpha < 2s - 2\sigma, \quad \gamma^\sigma_{\Theta^s,\alpha}(E) \approx_{n,s,\sigma,\alpha} \mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E).$$

The nullity of these capacities is equivalent to the removability of the corresponding compact set for solutions satisfying $(\sigma, \sigma/s)$-Laplacian estimates in either s-parabolic BMO or Lip_α, assuming $\alpha < 2s - 2\sigma$ in the latter case.

The previous study has been motivated by the one carried out for the BMO variant of analytic capacity by Kaufman [12] and Verdera [22] (for a brief overview the reader may consult [2, §13.5.1]); and that for the Lip_α variant of the same capacity in the direction presented by Mel'nikov [16] or O'Farrell [17]. We remark that the results presented here also generalize those of [9, §5 & §6].

A brief overview of the paper is as follows. Sects. 2 and 3 are devoted to kernel estimates and growth estimates for the so-called *admissible* functions. Since some of the proofs in Sect. 2 are rather long and intricate, we defer them to a separate section at the end of the paper, namely Sect. 6. In Sect. 4, we establish several important properties of potentials defined with respect to positive Borel measures satisfying certain growth conditions. Finally, Sect. 5 introduces the various capacities under consideration and characterizes them in terms of appropriate s-parabolic Hausdorff contents.

About the notation: Constants appearing in the sequel may depend on the dimension of the ambient space and the parameter s, and their value may change at different occurrences. They will frequently be denoted by the letters c or C. The notation $A \lesssim B$ means that there exists C, so that $A \leq CB$. Moreover, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$, while $A \simeq B$ will mean $A = CB$. If the reader finds expressions of the form $\lesssim_\beta$ or $\approx_\beta$, for example, this indicates that the implicit constants depend on n, s and β.

Since Laplacian operators (fractional or not) will frequently appear in our discussion and will be always taken with respect to spatial variables, we will adopt the notation:

$$(-\Delta)^s := (-\Delta_x)^s, \qquad s \in (0,1], \quad \text{and we convey } (-\Delta)^0 := \text{Id}.$$

We will also write $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^{n+1})}$. Finally, we stress that an important parameter which will play a fundamental role in Sect. 2 is

$$2\zeta := \min\{1, 2s\}.$$

2 Basic notation and kernel estimates

We begin by noticing that the *s-parabolic distance* between $\overline{x} := (x,t)$, $\overline{y} := (y,\tau)$ in $\mathbb{R}^{n+1}$, defined in the introduction as

$$|\overline{x}-\overline{y}|_{p_s} = \text{dist}_{p_s}(\overline{x},\overline{y}) := \max\left\{|x-y|, |t-\tau|^{\frac{1}{2s}}\right\}, \qquad \text{for } 0 < s \leq 1,$$

is, in fact, equivalent to

$$\text{dist}_{p_s}(\overline{x},\overline{y}) \approx \left(|x-y|^2 + |t-\tau|^{1/s}\right)^{1/2}.$$

The *s-parabolic dilation* of factor $\lambda > 0$, written δ_λ, is given by

$$\delta_\lambda(x,t) = \left(\lambda x, \lambda^{2s} t\right).$$

To ease notation, since we will always work with s-parabolic distances, we will write λQ to denote $\delta_\lambda(Q)$, the s-parabolic cube concentric with Q of side length $\lambda \ell(Q)$.

As the reader may suspect, the notion of *s-parabolic BMO space*, BMO_{p_s}, refers to the space of usual BMO functions (strictly, equivalence classes of functions where constants are identified as 0) obtained by replacing Euclidean cubes by s-parabolic ones. Similarly, a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is said to be *s-parabolic* Lip_α for some $0 < \alpha < 1$, shortly Lip_{α,p_s}, if

$$\|f\|_{\text{Lip}_{\alpha,p_s}} := \sup_{\overline{x},\overline{y}\in\mathbb{R}^{n+1}} \frac{|f(\overline{x}) - f(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^\alpha} \lesssim 1.$$

For each $s \in (0,1]$, the fundamental solution $P_s(x,t)$ to the Θ^s-equation, i.e. that associated with the operator

$$\Theta^s := (-\Delta)^s + \partial_t,$$

is the inverse spatial Fourier transform of $e^{-4\pi^2 t|\xi|^{2s}}$ for $t > 0$, and it equals 0 if $t \leq 0$. For the special case $s = 1$, we retrieve the classical *heat kernel*, given by:

$$W(\overline{x}) := P_1(\overline{x}) = c_n t^{-\frac{n}{2}} \phi_{n,1}(|x|t^{-\frac{1}{2}}), \qquad \text{if } t > 0,$$

where $\phi_{n,1}(\rho) := e^{-\rho^2/4}$, independent of n. Although the expression of P_s is not explicit in general, Blumenthal and Getoor [3, Theorem 2.1] established that for $s < 1$,

$$P_s(\overline{x}) = c_{n,s}\, t^{-\frac{n}{2s}} \phi_{n,s}\left(|x|t^{-\frac{1}{2s}}\right) \chi_{t>0}, \tag{2}$$

Here, $\phi_{n,s}$ is a smooth function, radially decreasing and satisfying, for $0 < s < 1$,

$$\phi_{n,s}(\rho) \approx \big(1+\rho^2\big)^{-(n+2s)/2}, \tag{3}$$

being an exact equality if $s = 1/2$ [21]. Therefore,

$$P_s(\overline{x}) \approx \frac{t}{|\overline{x}|_{p_s}^{n+2s}} \chi_{t>0}.$$

The function $\phi_{n,s}$ is tightly related to the Fourier transform of $e^{-4\pi^2|\xi|^{2s}}$. Indeed, taking the spatial Fourier transform in both sides of identity (2), we get

$$e^{-4\pi^2 t|\xi|^{2s}} = c_{n,s}\, t^{-\frac{n}{2s}} \Big[\phi_{n,s}\big(|\cdot|\, t^{-\frac{1}{2s}}\big)\Big]^{\wedge}(\xi).$$

Recall that for $\lambda > 0$, the dilation $f_\lambda := f(\lambda x)$ satisfies $\widehat{f_\lambda}(\xi) = \lambda^{-n}\widehat{f}(\lambda^{-1}\xi)$. Then,

$$e^{-4\pi^2 t|\xi|^{2s}} = c_{n,s}\, \widehat{\phi_{n,s}\big(|\cdot|\big)}\big(\xi t^{\frac{1}{2s}}\big), \qquad \text{that implies} \qquad e^{-4\pi^2|\xi|^{2s}} \simeq \widehat{\phi_{n,s}\big(|\cdot|\big)}(\xi).$$

The above relations will allow us to obtain explicit bounds for the derivatives of $\phi_{n,s}$. Let us present our first lemma. Although it can be deduced straightforwardly from [7, Theorem 1.1], we shall give a detailed proof for the sake of clarity and completeness.

Lemma 2.1 *Let* $s \in (0, 1]$ *and* $\beta \in (0, 1)$. *We define the following function in* $\mathbb{R}^n$*:*

$$\psi_{n,s}^{(\beta)}(x) := (-\Delta)^\beta \phi_{n,s}\big(|x|\big).$$

Then,

1. $\phi_{n,s}'(\rho) \simeq -\rho\, \phi_{n+2,s}(\rho)$.
2. $|\psi_{n,s}^{(\beta)}(x)| \lesssim_\beta \big(1+|x|^2\big)^{-(n+2\beta)/2}$.
3. $\nabla\psi_{n,s}^{(\beta)}(x) \simeq -x\, \psi_{n+2,s}^{(\beta)}(x)$.

Proof We begin by proving *1* for $s < 1$ (the case $s = 1$ is trivial). To do so, we will use the explicit integral representation for the inverse Fourier transform of a radial function in [6, §B.5] or [20, §IV.I]. Applying it to the Fourier transform $e^{-4\pi^2|\xi|^{2s}}$ we get

$$\phi_{n,s}(|z|) = 2\pi|z|^{1-n/2}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r|z|)\mathrm{d}r, \quad \text{for any } z \in \mathbb{R}^n \setminus \{0\},$$

where J_k is the classical Bessel function of order k [1, §9]. Since we are interested in the derivatives of $\phi_{n,s}$ as a radial real variable function, let us rewrite the previous expression in terms of $\rho \in (0,\infty)$ so that it reads as

$$\phi_{n,s}(\rho) = 2\pi\rho^{1-n/2}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\mathrm{d}r. \tag{4}$$

Therefore, to estimate the derivatives of $\phi_{n,s}$ we need to determine first if we can differentiate under the integral sign. To that end, we use the following recurrence relation for classical Bessel functions [1, §9.1.27],

$$J_k'(x) = \frac{k}{x}J_k(x) - J_{k+1}(x).$$

This recurrence formula together with (4) remain valid for the case $k = -1/2$, conveying that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$. In our case these imply

$$\partial_\rho J_{n/2-1}(2\pi r\rho) = \left(\frac{n}{2}-1\right)\rho^{-1} J_{n/2-1}(2\pi r\rho) - 2\pi r\, J_{n/2}(2\pi r\rho).$$

If we differentiated under the integral sign in (4), we would get integrands of the form

$$e^{-r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho), \qquad e^{-r^{2s}} r^{n/2+1} J_{n/2}(2\pi r\rho).$$

Notice that both are bounded by integrable functions in the domain of integration, locally for each $\rho > 0$ (by the boundedness of the functions J_k for $n > 1$, and by that of $\cos x$ if $n = 1$). Hence, we can indeed differentiate under the integral sign to compute $\phi'_{n,s}$, obtaining the desired result:

$$\begin{aligned}
\phi'_{n,s}(\rho) &= 2\pi\Bigg[\left(1-\frac{n}{2}\right)\rho^{-n/2}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\mathrm{d}r \\
&\qquad + \rho^{1-n/2}\partial_\rho\left(\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\mathrm{d}r\right)\Bigg] \\
&= 2\pi\Bigg[\left(1-\frac{n}{2}\right)\rho^{-n/2}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(r\rho)\mathrm{d}r \\
&\qquad \rho^{1-n/2}\left(\frac{n}{2}-1\right)\rho^{-1}\left(\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2} J_{n/2-1}(2\pi r\rho)\mathrm{d}r\right) \\
&\qquad - 2\pi\rho\,\rho^{1-(n+2)/2}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{(n+2)/2} J_{(n+2)/2-1}(r\rho)\mathrm{d}r\Bigg] \\
&= -2\pi\rho\,\phi_{n+2,s}(\rho).
\end{aligned}$$

Next we prove statement 2. Observe that for $s \in (0,1]$ and $\beta \in (0,1)$, we have $\widehat{\psi^{(\beta)}_{n,s}}(\xi) = |\xi|^{2\beta} e^{-4\pi^2|\xi|^{2s}}$, which is an integrable function, and thus $\psi^{(\beta)}_{n,s}$ is bounded (in fact, since the product of $\widehat{\psi^{(\beta)}_{n,s}}$ by any polynomial is also integrable, we infer that $\psi^{(\beta)}_{n,s}$ is smooth). By the integral representation formula for inverse Fourier transforms of radial functions,

$$\psi^{(\beta)}_{n,s}(x) = 2\pi|x|^{1-n/2}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|)\mathrm{d}r, \qquad x \in \mathbb{R}^n \setminus \{0\}. \tag{5}$$

Now, we apply [18, Lemma 1] to deduce the desired decaying property $|\psi^{(\beta)}_{n,s}(x)| = O\big(|x|^{-n-2\beta}\big)$, for $|x|$ large. Hence, since $\psi^{(\beta)}_{n,s}$ is bounded, we deduce the desired bound $|\psi^{(\beta)}_{n,s}(x)| \lesssim_\beta \big(1+|x|^2\big)^{-(n+2\beta)/2}$.

We are left to control the norm of $\nabla\psi^{(\beta)}_{n,s}$, provided the latter is well-defined. We claim that this is the case, since we can differentiate under the integral sign in (5). Indeed, by the recurrence relation satisfied by the derivatives of J_k we get

$$|\nabla_x J_{n/2-1}(r|x|)| = \left|\left(\frac{n}{2}-1\right)\frac{1}{|x|} J_{n/2-1}(2\pi r|x|) - 2\pi r J_{n/2}(2\pi r|x|)\right|.$$

So the resulting integrands to study are terms of the form

$$e^{-4\pi^2 r^{2s}} r^{n/2+2\beta}|J_{n/2-1}(2\pi r|x|)|, \qquad e^{-4\pi^2 r^{2s}} r^{n/2+2\beta+1}|J_{n/2}(2\pi r|x|)|,$$

both bounded by the integrable functions $C_1 e^{-r^{2s}} r^{n/2+2\beta}$ and $C_2 e^{-r^{2s}} r^{n/2+2\beta+1}$ for some constants C_1, C_2 depending on n, s and β, and locally for each $x \in \mathbb{R}^n$ with $|x| > 0$. Hence, we can differentiate under the integral sign in (5) and obtain

$$\begin{aligned}\nabla\psi_{n,s}^{(\beta)}(x) = 2\pi\Bigg[\Big(1-\frac{n}{2}\Big)\frac{x}{|x|^{n/2+1}}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|)\mathrm{d}r \\ + \Big(\frac{n}{2}-1\Big)\frac{x}{|x|^{n/2+1}}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|)\mathrm{d}r \\ - 2\pi\frac{x}{|x|^{n/2}}\int_0^\infty e^{-4\pi^2 r^{2s}} r^{(n+2)/2+2\beta} J_{(n+2)/2-1}(2\pi r|x|)\mathrm{d}r\Bigg] \\ = -2\pi x\,\psi_{n+2,s}^{(\beta)}(x).\end{aligned}$$

□

Using the above lemma together with (3) we can estimate the derivatives of $\phi_{n,s}$ and $\psi_{n,s}^{(\beta)}$. In particular, the following relations hold:

$$\text{If } s<1,\quad \phi'_{n,s}(\rho)\approx\frac{-\rho}{(1+\rho^2)^{(n+2s+2)/2}},\qquad \phi''_{n,s}(\rho)\approx\frac{-1+(2\pi-1)\rho^2}{(1+\rho^2)^{(n+2s+4)/2}},\tag{6}$$

$$|\nabla\psi_{n,s}^{(\beta)}(x)|\lesssim_\beta\frac{|x|}{(1+|x|^2)^{(n+2\beta+2)/2}}.\tag{7}$$

2.1 Estimates for $\nabla_x P_s$ and $\Delta^\beta P_s$

We shall now present some growth estimates for the kernels P_s. Our first result provides bounds for $\nabla_x P_s$, $s\in(0,1)$. These estimates are analogous to those of [14, Lemma 5.4] which cover the case $s=1$. In the forthcoming results, the parameter $2\zeta:=\min\{1,2s\}$ will play an important role.

Theorem 2.2 *The following estimates hold for any $\overline{x}\neq 0$ and $s\in(0,1)$:*

$$|\nabla_x P_s(\overline{x})|\lesssim\frac{|xt|}{|\overline{x}|_{p_s}^{n+2s+2}},\qquad |\Delta P_s(\overline{x})|\lesssim\frac{|t|}{|\overline{x}|_{p_s}^{n+2s+2}},\qquad |\partial_t\nabla_x P_s(\overline{x})|\lesssim\frac{|x|}{|\overline{x}|_{p_s}^{n+2s+2}}.$$

The last bound is only valid for points with $t\neq 0$. Also, if $\overline{x}'$ is such that $|\overline{x}-\overline{x}'|_{p_s}\le|\overline{x}|_{p_s}/2$,

$$|\nabla_x P_s(\overline{x})-\nabla_x P_s(\overline{x}')|\lesssim\frac{|\overline{x}-\overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}.$$

Proof To simplify the arguments below, we specify the dependence of P_s with respect to n. Let us write $P_{s,n+1}$ to refer to the fundamental solution of the Θ^s-equation in $\mathbb{R}^{n+1}$ and use the following abuse of notation: given $\overline{x}=(x_1,\dots,x_n,t)\in\mathbb{R}^{n+1}$, write

$$\begin{aligned}P_{s,n+3}(\overline{x}) &:= P_{s,n+3}(x_1,\dots,x_n,0,0,t),\\ P_{s,n+5}(\overline{x}) &:= P_{s,n+5}(x_1,\dots,x_n,0,0,0,0,t).\end{aligned}$$

This way, we directly apply relations (2) and (6) to obtain for each $t>0$,

$$|\nabla_x P_s(\overline{x})|\simeq t^{-\frac{n+1}{2s}}|\phi'_{n,s}(|x|t^{-\frac{1}{2s}})|\simeq|xP_{s,n+3}(\overline{x})|\approx\frac{|xt|}{|\overline{x}|_{p_s}^{n+2s+2}}.$$

The bounds for $\Delta P_{s,n+1}$ and $\partial_t\nabla_x P_{s,n+1}$ can be obtained from the previous result and (2). Indeed,

$$|\Delta P_{s,n+1}(\overline{x})| \simeq P_{s,n+3}(\overline{x}) + |x|^2 P_{s,n+5}(\overline{x}) \lesssim \frac{|t|}{|\overline{x}|_{p_s}^{n+2s+2}},$$
$$|\partial_t \nabla_x P_{s,n+1}(\overline{x})| \lesssim \frac{|x|}{t}\Big(P_{s,n+3}(\overline{x}) + |x|^2 P_{s,n+5}(\overline{x})\Big) \lesssim \frac{|x|}{|\overline{x}|_{p_s}^{n+2s+2}}.$$

For the final estimate, we recover the notation $P_s := P_{s,n+1}$. Let $\overline{x}' = (x', t') \in \mathbb{R}^{n+1}$ with $|\overline{x} - \overline{x}'|_{p_s} \leq |\overline{x}|_{p_s}/2$ and use the definition of $\operatorname{dist}_{p_s}$ to obtain

$$|\overline{x}|_{p_s} \leq 2|\overline{x}'|_{p_s} \quad \text{and} \quad |x'| \geq |x| - \frac{|\overline{x}|_{p_s}}{2}. \tag{8}$$

Put $\widehat{x} = (x', t)$ and write

$$|\nabla_x P_s(\overline{x}) - \nabla_x P_s(\overline{x}')| \leq |\nabla_x P_s(\overline{x}) - \nabla_x P_s(\widehat{x})| + |\nabla_x P_s(\widehat{x}) - \nabla_x P_s(\overline{x}')|.$$

We observe that the first term in the above inequality satisfies the desired bound,

$$|x - x'| \sup_{\xi \in [x,x']} |\Delta P_s(\xi, t)| \lesssim \frac{|x - x'|}{|\overline{x}|_{p_s}^{n+2}} \leq \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}} \left(\frac{|\overline{x} - \overline{x}'|_{p_s}}{|\overline{x}|_{p_s}}\right)^{1-2\zeta} \leq \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}.$$

Regarding the second term, assume without loss of generality $t > t'$. If $t' > 0$, use $|\overline{x}'|_{p_s} \geq |\overline{x}|_{p_s}/2$ so that we also have

$$|t - t'| \sup_{\tau \in [t,t']} |\partial_t \nabla_x P_s(x', \tau)| \lesssim \frac{|t - t'|}{|\overline{x}|_{p_s}^{n+2s+1}} \leq \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}} \left(\frac{|\overline{x} - \overline{x}'|_{p_s}}{|\overline{x}|_{p_s}}\right)^{2s-2\zeta} \lesssim \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}},$$

If $t < 0$ then $|\nabla_x P_s(\widehat{x}) - \nabla_x P_s(\overline{x}')| = 0$, and the estimate becomes trivial. Then, we are left to study the case $t > 0$ and $t' < 0$. These two conditions imply that the p_s-ball

$$B(\overline{x}) := \left\{ \overline{y} \in \mathbb{R}^{n+1} \, : \, |\overline{x} - \overline{y}|_{p_s} \leq \frac{|\overline{x}|_{p_s}}{2} \right\} \ni \overline{x}'$$

intersects the hyperplane $\{t = 0\}$. Since the radius of $B(\overline{x})$ also depends on $\overline{x}$, the previous property imposes the following condition over $\overline{x}$,

$$t^{1/s} \leq \frac{x_1^2 + \cdots + x_n^2}{3}, \quad \text{that is} \quad t^{\frac{1}{2s}} \leq \frac{|x|}{\sqrt{3}},$$

which is attained if the point $(x, 0)$ belongs to $\partial B(\overline{x})$. Therefore $|\overline{x}|_{p_s} := \max\left\{|x|, t^{\frac{1}{2s}}\right\} = |x|$, so by (8) we get $|x'| \geq |x|/2$, and this in turn implies

$$\frac{|\overline{x}|_{p_s}}{2} \leq |\overline{x}'|_{p_s} \leq |\overline{x} - \overline{x}'|_{p_s} + |\overline{x}|_{p_s} \leq \frac{3|x|}{2} \leq 3|x'|. \tag{9}$$

Using this last inequality we can finally conclude:

$$|\nabla_x P_s(\widehat{x}) - \nabla_x P_s(\overline{x}')| = |\nabla_x P_s(x', t) - \nabla_x P_s(x', 0)| \lesssim |t| \sup_{\tau \in (0,t]} |\partial_t \nabla_x P_s(x', \tau)|$$
$$\lesssim \frac{|t|}{|x'|^{n+2s+1}} \lesssim \frac{|t|}{|\overline{x}|_{p_s}^{n+2s+1}} \leq \frac{|t - t'|}{|\overline{x}|_{p_s}^{n+2s+1}} \lesssim \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+1+2\zeta}}.$$

□

Theorem 2.3 *Let $s \in (0,1]$ and $\beta, \gamma \in [0,1)$. Then, for any $\overline{x} \neq 0$ we have,*

1. $|(-\Delta)^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|\overline{x}|_{p_s}^{n+2\beta}}$,
2. $|(-\Delta)^\gamma (-\Delta)^\beta P_s(\overline{x})| \lesssim_{\beta,\gamma} \dfrac{1}{|\overline{x}|_{p_s}^{n+2\beta+2\gamma}}$, 3. $|\nabla_x (-\Delta)^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{|x|}{|\overline{x}|_{p_s}^{n+2\beta+2}}$.

Moreover, for any $\overline{x} \neq (x,0)$,

$$4.\ |\partial_t (-\Delta)^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|\overline{x}|_{p_s}^{n+2\beta+2s}}.$$

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ is such that $|\overline{x} - \overline{x}'|_{p_s} \leq |\overline{x}|_{p_s}/2$,

$$5.\ |(-\Delta)^\beta P_s(\overline{x}) - (-\Delta)^\beta P_s(\overline{x}')| \lesssim_\beta \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|\overline{x}|_{p_s}^{n+2\beta+2\zeta}}.$$

Proof We shall also assume $\beta > 0$, since the case $\beta = 0$ is already covered in [13, Lemma 2.2]. For the sake of notation, in this proof we will write $\phi := \phi_{n,s}$ and $\psi := \psi_{n,s}^{(\beta)}$, and we also set $K_\beta := (-\Delta)^\beta P_s$. Let us begin by applying the integral representation of K_β together with relation (2) to obtain for $t > 0$,

$$\begin{aligned} K_\beta(x,t) := (-\Delta)^\beta P_s(x,t) &\simeq_\beta \text{p.v.} \int_{\mathbb{R}^n} \frac{P_s(x,t) - P_s(y,t)}{|x-y|^{n+2\beta}} \mathrm{d}y \\ &= t^{-\frac{n}{2s}} \text{p.v.} \int_{\mathbb{R}^n} \frac{\phi\big(|x|t^{-\frac{1}{2s}}\big) - \phi\big(|y|t^{-\frac{1}{2s}}\big)}{|x-y|^{n+2\beta}} \mathrm{d}y \\ &= t^{-\frac{n+2\beta}{2s}} \text{p.v.} \int_{\mathbb{R}^n} \frac{\phi\big(|x|t^{-\frac{1}{2s}}\big) - \phi(|z|)}{|xt^{-\frac{1}{2s}} - z|^{n+2\beta}} \mathrm{d}z = t^{-\frac{n+2\beta}{2s}} \psi\big(xt^{-\frac{1}{2s}}\big). \end{aligned}$$

Using the estimate proved in Lemma 2.1 for ψ we deduce the desired bound:

$$\big|K_\beta(x,t)\big| \lesssim_\beta \frac{t^{-\frac{n+2\beta}{2s}}}{\big(1 + |x|^2 t^{-1/s}\big)^{(n+2\beta)/2}} = \frac{1}{\big(t^{1/s} + |x|^2\big)^{(n+2\beta)/2}} \approx \frac{1}{|\overline{x}|_{p_s}^{n+2\beta}}.$$

We shall continue by studying estimate *2* in a similar way. Indeed,

$$\begin{aligned} (-\Delta)^\gamma K_\beta(x,t) &\simeq_\gamma \text{p.v.} \int_{\mathbb{R}^n} \frac{K_\beta(x,t) - K_\beta(y,t)}{|x-y|^{n+2\gamma}} \mathrm{d}y \\ &\simeq_\beta t^{-\frac{n+2\beta}{2s}} \text{p.v.} \int_{\mathbb{R}^n} \frac{\psi\big(xt^{-\frac{1}{2s}}\big) - \psi\big(yt^{-\frac{1}{2s}}\big)}{|x-y|^{n+2\gamma}} \mathrm{d}y \\ &= t^{-\frac{n+2\beta+2\gamma}{2s}} \text{p.v.} \int_{\mathbb{R}^n} \frac{\psi\big(xt^{-\frac{1}{2s}}\big) - \psi(z)}{|xt^{-\frac{1}{2s}} - z|^{n+2\gamma}} \mathrm{d}z = t^{-\frac{n+2\beta+2\gamma}{2s}} (-\Delta)^\gamma \psi\big(xt^{-\frac{1}{2s}}\big). \end{aligned}$$

Set $\Psi := (-\Delta)^\gamma \psi(\,\cdot\,)$ and notice that

$$\widehat{\Psi}(\xi) = |\xi|^{2\gamma} |\xi|^{2\beta} e^{-4\pi^2 |\xi|^{2s}} = |\xi|^{2\beta+2\gamma} e^{-4\pi^2 |\xi|^{2s}}.$$

Thus, since $\widehat{\Psi}$ is integrable, Ψ is the radial bounded function in $\mathbb{R}^n$ given by

$$\Psi(z) = 2\pi |z|^{1-n/2} \int_0^\infty e^{-4\pi^2 r^{2s}} r^{n/2+2\beta+2\gamma} J_{n/2-1}(2\pi r|z|) \mathrm{d}r,$$

By [18, Lemma 1] Ψ decays as

$$|\Psi(z)| = O\big(|z|^{-n-2\beta-2\gamma}\big), \qquad \text{for } \ |z| \ \text{ large.}$$

Therefore

$$|\Psi(z)| \lesssim_{\beta,\gamma} \big(1+|z|^2\big)^{-(n+2\beta+2\gamma)/2}.$$

So analogously to the proof of *1*, we deduce the desired result:

$$\big|(-\Delta)^{\gamma} K_{\beta}(x,t)\big| \lesssim_{\beta,\gamma} \frac{t^{-\frac{n+2\beta+2\gamma}{2s}}}{\big(1+|x|^2 t^{-1/s}\big)^{(n+2\beta+2\gamma)/2}} \approx \frac{1}{|\overline{x}|_{p_s}^{n+2\beta+2\gamma}}.$$

Regarding estimate *3*, notice that

$$|\nabla_x K_{\beta}(x,t)| \simeq_{\beta} \Big|\nabla_x\Big(t^{-\frac{n+2\beta}{2s}}\psi\big(xt^{-\frac{1}{2s}}\big)\Big)\Big| = t^{-\frac{n+2\beta+1}{2s}}\big|\nabla\psi\big(xt^{-\frac{1}{2s}}\big)\big|.$$

Therefore, applying the bound obtained for $\nabla\psi$ in (7) we deduce

$$\big|\nabla_x K_{\beta}(x,t)\big| \lesssim_{\beta} t^{-\frac{n+2\beta+1}{2s}} \frac{|x|t^{-\frac{1}{2s}}}{\big(1+|x|^2 t^{-1/s}\big)^{(n+2\beta+1)/2}} \approx \frac{|x|}{|\overline{x}|_{p_s}^{n+2\beta+2}}.$$

We move on to estimate *4*, that is, the one concerning $\partial_t K_{\beta}(\overline{x})$ at points of the form $\overline{x} \neq (x,0)$. Observe that the previous derivative is well defined if $t>0$, since the expression of K_{β} can be written as

$$\begin{aligned} K_{\beta}(x,t) &\simeq t^{-\frac{n+2\beta}{2s}}(-\Delta)^{\beta}\phi\big(|x|t^{-\frac{1}{2s}}\big) \\ &\simeq_{\beta} |x|^{1-n/2}\bigg(\frac{1}{t^{\frac{n+4\beta+2}{4s}}}\int_0^{\infty} e^{-4\pi^2 r^{2s}} r^{n/2+2\beta} J_{n/2-1}(2\pi r|x|t^{-\frac{1}{2s}})\mathrm{d}r\bigg), \end{aligned}$$

so differentiating under the integral sign, it is clear that temporal derivatives of any order exist in $\mathbb{R}^{n+1}\setminus\{t=0\}$. We claim now that the operators ∂_t and $(-\Delta)^{\beta}$ commute when applied to P_s. To prove this, let us first observe that for each $t_0>0$ fixed we have

$$\big[(-\Delta)^{\beta}\big(\partial_t P_s\big)\big]^{\wedge}(\xi,t_0) = |\xi|^{2\beta}\widehat{\partial_t P_s}(\xi,t_0) = |\xi|^{2\beta}\int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle}\partial_t P_s(x,t_0)\mathrm{d}x.$$

If we can bound $\partial_t P_s$ by an integrable function on $\mathbb{R}^n$ in a neighborhood of t_0, we will be able to locally differentiate outside the integral sign for each t_0. If $0<s<1$, this is a consequence of [21, Equation 2.6] and (3). Indeed,

$$|\partial_t P_s(x,t_0)| \lesssim \frac{1}{t_0}|P_s(x,t_0)| \lesssim \frac{1}{t_0^{\frac{n+2s}{2s}}}\bigg[\frac{1}{\big(1+|x|^2 t_0^{-1/s}\big)^{(n+2s)/2}}\bigg].$$

On the other hand, if $s=1$ by definition we have

$$|\partial_t W(x,t_0)| \lesssim \bigg(1+\frac{|x|^2}{t_0}\bigg)\frac{1}{t_0^{n/2+1}}e^{-|x|^2/(4t_0)}.$$

In both cases we obtain a bounded function of x that decreases like $|x|^{-n-2}$ at infinity (for the case $s=1$, see [14, Lemma 2.1]) and thus it is integrable on $\mathbb{R}^n$. Therefore, differentiating outside the integral sign we have

$$\big[(-\Delta)^{\beta}\big(\partial_t P_s\big)\big]^{\wedge}(\xi,t_0) = \partial_t\big[(-\Delta)^{\beta}P_s\big]^{\wedge}(\xi,t_0), \qquad \forall t_0>0.$$

So we are left to check whether we can enter ∂_t inside the previous Fourier transform, that is, whether the following holds

$$\partial_t\big[(-\Delta)^\beta P_s\big]^\wedge(\xi,t_0)=\big[\partial_t(-\Delta)^\beta P_s\big]^\wedge(\xi,t_0).$$

Again, the latter is just a matter of being able to bound $|\partial_t(-\Delta)^\beta P_s|=|\partial_t K_\beta|$ locally for each $t_0>0$ by an integrable function, so that we can differentiate under the integral defining the Fourier transform. We know that

$$|\partial_t K_\beta(x,t_0)|=\left|\partial_t\Big[t^{-\frac{n+2\beta}{2s}}\psi\big(xt^{-\frac{1}{2s}}\big)\Big]_{t=t_0}\right|\lesssim_\beta C_1(t_0)\big|\psi\big(xt_0^{-\frac{1}{2s}}\big)\big|+C_2(t_0)|x|\big|\nabla\psi\big(xt_0^{-\frac{1}{2s}}\big)\big|.$$

For the first summand, using that $|\psi|$ is bounded and decays as $|x|^{-n-2\beta}$, we deduce the desired integrability condition. For the second summand we can argue exactly in the same manner, using that $|\nabla\psi|$ is bounded and decays as $|x|^{-n-2\beta-1}$. Hence, we conclude that ∂_t and $(-\Delta)^\beta$ commute.

The previous commutativity relation and [13, Eq. 2.5] yield the following for $t>0$,

$$\begin{aligned}\partial_t K_\beta(x,t)&=\partial_t\big[(-\Delta)^\beta P_s\big](x,t)=(-\Delta)^\beta\big(\partial_t P_s\big)(x,t)\\&=(-\Delta)^\beta\Big[-(-\Delta)^s P_s\Big](x,t)=-(-\Delta)^s K_\beta(x,t),\end{aligned}$$

where we have commuted the operators $(-\Delta)^s$ and $(-\Delta)^\beta$, that can be easily checked via their Fourier transform. Then, applying *2* with $\gamma=s$ we are done.

Finally, regarding estimate *5*, we can follow the same proof to that presented for the last estimate in Theorem 2.2, using estimates *3* and *4* from above. □

2.2 Estimates for $\partial_t^\beta P_s$

In this subsection we obtain similar estimates now for the kernel $\partial_t^\beta P_s$, with $\beta\in(0,1)$. Since the proofs of such estimates are rather long and intricate, we shall present them in a separate section, namely Sect.6. Recall that the *β-temporal derivative* of $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is defined, provided it exists, as

$$\partial_t^\beta f(x,t):=\int_{\mathbb{R}}\frac{f(x,\tau)-f(x,t)}{|\tau-t|^{1+\beta}}\mathrm{d}\tau.$$

The study below considers the cases $s<1$ and $s=1$ separately. In the following theorems, which generalize results of [13, Lemma 2.2] and [14, Lemma 2.1], we get dimensional restrictions that in the end will not matter for our purposes.

Theorem 2.4 *For any $\beta,s\in(0,1)$ and $\overline{x}=(x,t)\neq(0,t)$, the following hold:*

1. If $n>1,\quad |\partial_t^\beta P_s(\overline{x})|\lesssim_\beta\dfrac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}$,

2. If $n=1$ *and* $\beta>1-\dfrac{1}{2s},\quad |\partial_t^\beta P_s(\overline{x})|\lesssim_{\beta,\alpha}\dfrac{1}{|x|^{1-2s+\alpha}|\overline{x}|_{p_s}^{2s(1+\beta)-\alpha}},\quad\forall\alpha\in(2s-1,4s).$

Moreover, for every n,

3. $|\nabla_x\partial_t^\beta P_s(\overline{x})|\lesssim_\beta\dfrac{1}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}}$, *4.* $|\partial_t\partial_t^\beta P_s(\overline{x})|\lesssim_\beta\dfrac{1}{|x|^n|\overline{x}|_{p_s}^{2s(1+\beta)}}$, *for* $t\neq0$.

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ *is such that* $|\overline{x} - \overline{x}'|_{p_s} \leq |x|/2$,

$$5.\ |\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\overline{x}')| \lesssim_\beta \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

We carry out the same study for the case $s = 1$, obtaining the following estimates:

Theorem 2.5 *For any* $\beta \in (0, 1)$ *and* $\overline{x} = (x, t) \neq (0, t)$, *the following hold:*

$$1.\ \text{For}\quad n > 2, \qquad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-2}|\overline{x}|_{p_1}^{2+2\beta}},$$

$$2.\ \text{For}\quad n = 2, \qquad |\partial_t^\beta W(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{\alpha}|\overline{x}|_{p_1}^{2+2\beta-\alpha}}, \quad \forall \alpha \in (0, 2+2\beta],$$

$$3.\ \text{For}\quad n = 1, \qquad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|\overline{x}|_{p_1}^{1+2\beta}}.$$

Moreover, for every n,

$$4.\ |\nabla_x \partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}, \qquad 5.\ |\partial_t \partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n}|\overline{x}|_{p_1}^{2+2\beta}}.$$

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ *is such that* $|\overline{x} - \overline{x}'|_{p_1} \leq |x|/2$, *then*

$$6.\ |\partial_t^\beta W(\overline{x}) - \partial_t^\beta W(\overline{x}')| \lesssim_\beta \frac{|\overline{x} - \overline{x}'|_{p_1}}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}.$$

3 Growth estimates for admissible functions

We will say that a positive Borel measure μ in $\mathbb{R}^{n+1}$ has upper s-parabolic *growth of degree* ρ (with constant C) or simply s*-parabolic* ρ*-growth* if there is some constant $C(n, s) > 0$ such that for any s-parabolic ball $B(\overline{x}, r)$,

$$\mu\big(B(\overline{x}, r)\big) \leq C r^\rho.$$

It is clear that this property is invariant if formulated using cubes instead of balls. We will be interested in a generalized version of such growth that can be defined not only for measures, but also for general distributions. To introduce such notion we present the concept of admissible function:

Definition 3.1 Let $s \in (0, 1)$. Given $\phi \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, we will say that it is an *admissible* function for an s-parabolic cube Q if $\text{supp}(\phi) \subset Q$ and

$$\|\phi\|_\infty \leq 1, \qquad \|\nabla_x \phi\|_\infty \leq \ell(Q)^{-1}, \qquad \|\partial_t \phi\|_\infty \leq \ell(Q)^{-2s}, \qquad \|\Delta \phi\|_\infty \leq \ell(Q)^{-2}.$$

Remark 3.1 If ϕ is a $\mathcal{C}^2$ function supported on Q s-parabolic cube with $\|\phi\|_\infty \leq 1$, $\|\nabla_x \phi\|_\infty \leq \ell(Q)^{-1}$ and $\|\Delta \phi\|_\infty \leq \ell(Q)^{-2}$, then it also satisfies

$$\|(-\Delta)^s \phi\|_\infty \lesssim \ell(Q)^{-2s}.$$

Indeed, begin by observing that translations in $\mathbb{R}^n$ commute with ∇_x and $(-\Delta)^s$. From it, it is clear that we may assume Q to be centered at the origin. Assuming this, let us fix $t \in \mathbb{R}$ and compute

$$
\begin{aligned}
(-\Delta)^s\phi(x,t) &:= c_{n,s}\int_{\mathbb{R}^n}\frac{\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)}{|y|^{n+2s}}\mathrm{d}y\\
&= c_{n,s}\int_{2Q}\frac{\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)}{|y|^{n+2s}}\mathrm{d}y\\
&\quad c_{n,s}\int_{\mathbb{R}^n\setminus 2Q}\frac{\phi(x+y,t)-2\phi(x,t)+\phi(x-y,t)}{|y|^{n+2s}}\mathrm{d}y =: I_1+I_2.
\end{aligned}
$$

Regarding I_2, integration in polar coordinates yields

$$
|I_2|\le 4c_{n,s}\int_{\mathbb{R}^n\setminus 2Q}\frac{\mathrm{d}y}{|y|^{n+2s}}\lesssim \ell(Q)^{-2s}.
$$

For I_1, we apply twice the mean value theorem so that

$$
\begin{aligned}
|I_1| &\le c_{n,s}\int_{2Q}\frac{|\langle\nabla_x\phi(x+\eta_1 y,t),y\rangle+\langle\nabla_x\phi(x-\eta_2 y,t),y\rangle|}{|y|^{n+2s-1}}\\
&\lesssim \int_{2Q}\frac{\|\Delta\phi\|_\infty}{|y|^{n+2s-2}}\mathrm{d}y\lesssim \ell(Q)^{-2s}.
\end{aligned}
$$

Definition 3.2 We will say that a distribution T has s-parabolic n-*growth* if there exists some constant $C=C(n,s)>0$ such that, given any s-parabolic cube Q and any function ϕ admissible for Q, we have

$$
|\langle T,\phi\rangle|\le C\ell(Q)^n.
$$

In the end, the results below will help us estimate the growth of distributions of the form φT, for some particular choices of T and a fixed admissible function φ, associated with a fixed s-parabolic cube.

In any case, let us clarify that in the following Theorems 3.1, 3.2, 3.3 and 3.4, we will fix $s\in(0,1]$ and Q and R will be s-parabolic cubes in $\mathbb{R}^{n+1}$ with $Q\cap R\ne\varnothing$. We will write $Q:=Q_1\times I_Q\subset\mathbb{R}^n\times\mathbb{R}$ and analogously for R. Moreover, φ and ϕ will denote $\mathcal{C}^1$ functions with $\mathrm{supp}(\varphi)\subset Q$, $\mathrm{supp}(\phi)\subset R$ and such that $\|\varphi\|_\infty\le 1$ and $\|\phi\|_\infty\le 1$.

Theorem 3.1 *Let $\beta\in(0,1)$, $\alpha\in(0,1)$ and $f:\mathbb{R}^{n+1}\to\mathbb{R}$. Assume $\|\partial_t\varphi\|_\infty\le\ell(Q)^{-2s}$ and $\|\partial_t\phi\|_\infty\le\ell(R)^{-2s}$. Then, if $\ell(R)\le\ell(Q)$,*

1. *If $f\in BMO_{p_s}$,*
$$
|\langle f,\partial_t(\varphi\phi)*_t|t|^{-\beta}\rangle|\lesssim_\beta \|f\|_{*,p_s}\ell(R)^{n+2s(1-\beta)}.
$$
2. *If $f\in Lip_{\alpha,p_s}$ and $\alpha<2s\beta$,*
$$
|\langle f,\partial_t(\varphi\phi)*_t|t|^{-\beta}\rangle|\lesssim_{\beta,\alpha}\|f\|_{Lip_{\alpha,p_s}}\ell(R)^{n+2s(1-\beta)+\alpha}.
$$

Proof Set $g:=\partial_t(\varphi\phi)*_t|t|^{-\beta}$ and begin by proving that g is integrable. Firstly, we observe that

$$
\int_{I_Q\cap I_R}\partial_t(\varphi\phi)(x,u)du=0.
$$

Then, if $c_{Q\cap R}$ denotes the center of $I_Q\cap I_R$, for each $t\notin 2(I_Q\cap I_R)$ we get

$$
\begin{aligned}
|g(x,t)| &= \left|\int_{I_Q\cap I_R}\frac{\partial_t(\varphi\phi)(x,u)}{|t-u|^\beta}\mathrm{d}u\right|\le\int_{I_Q\cap I_R}|\partial_t(\varphi\phi)(x,u)|\left|\frac{1}{|t-u|^\beta}-\frac{1}{|t-c_{Q\cap R}|^\beta}\right|\mathrm{d}u\\
&\lesssim \frac{\ell(I_Q\cap I_R)}{|t-c_{Q\cap R}|^{1+\beta}}\int_{I_Q\cap I_R}|\partial_t(\varphi\phi)(x,u)|\mathrm{d}u
\end{aligned}
$$

$$\lesssim_\beta \frac{\ell(I_Q\cap I_R)}{|t-c_{Q\cap R}|^{1+\beta}}\left(\frac{1}{\ell(Q)^{2s}}+\frac{1}{\ell(R)^{2s}}\right)\ell(I_Q\cap I_R)\lesssim \frac{\ell(I_Q\cap I_R)}{|t-c_{Q\cap R}|^{1+\beta}}. \tag{10}$$

That is, $|g|$ decays as $|t|^{-1-\beta}$ for large values of t. Hence, since $\mathrm{supp}(g)\subset(Q_1\cap R_1)\times\mathbb{R}$, this implies $g\in L^1(\mathbb{R}^{n+1})$. Then, for any constant $c\in\mathbb{R}$ we have

$$|\langle f,g\rangle|=\left|\int(f-c)g\right|\le\int_{2R}|f-c||g|+\int_{\mathbb{R}^{n+1}\setminus 2R}|f-c||g|=:I_1+I_2,$$

where we have used that g has null integral (it can be easily checked taking the Fourier transform, for example). To study I_1, observe that for $t\in 4I_R$ we get

$$|g(x,t)|\le\|\partial_t(\varphi\phi)\|_\infty\int_{-5\ell(I_R)}^{5\ell(I_R)}\frac{\mathrm{d}u}{|u|^\beta}\lesssim_\beta\left(\frac{1}{\ell(R)^{2s}}+\frac{1}{\ell(Q)^{2s}}\right)\ell(I_R)^{1-\beta}\lesssim\ell(R)^{-2s\beta},$$

since $\ell(R)\le\ell(Q)$. Therefore,

$$I_1\lesssim_\beta\frac{1}{\ell(R)^{2s\beta}}\int_{2R}|f-c|.$$

If $f\in\mathrm{BMO}_{p_s}$, pick $c:=f_{2R}$, the average of f over $2R$, so that

$$I_1\lesssim_\beta\ell(R)^{n+2s(1-\beta)}\|f\|_{*,p_s}.$$

If $f\in\mathrm{Lip}_{\alpha,p_s}$, pick $c:=f(\overline{x}_R)$, where $\overline{x}_R$ is the center of $2R$, so that

$$I_1\lesssim_{\beta,\alpha}\ell(R)^{n+2s(1-\beta)+\alpha}\|f\|_{\mathrm{Lip}_{\alpha,p_s}},$$

and we are done with I_1. To study I_2, define the s-parabolic annuli $A_j:=2^jR\setminus 2^{j-1}R$ for $j\ge 2$. Then, since $\mathrm{supp}(g)\subset(Q_1\cap R_1)\times\mathbb{R}$ applying 10 we have

$$I_2=\sum_{j=2}^\infty\int_{A_j\cap\mathrm{supp}(g)}|f(\overline{x})-c||g(\overline{x})|\mathrm{d}\overline{x}\lesssim_\beta\frac{1}{\ell(R)^{2s\beta}}\sum_{j=2}^\infty\frac{1}{2^{2s(1+\beta)j}}\int_{A_j\cap\mathrm{supp}(g)}|f(\overline{x})-c|\mathrm{d}\overline{x}. \tag{11}$$

If $f\in\mathrm{BMO}_{p_s}$, pick again $c:=f_{2R}$ and observe

$$I_2\lesssim_\beta\frac{1}{\ell(R)^{2s\beta}}\sum_{j=2}^\infty\frac{1}{2^{2s(1+\beta)j}}\left(\int_{A_i\cap\mathrm{supp}(g)}|f(\overline{x})-f_{2^jR}|\mathrm{d}\overline{x}+\int_{A_j\cap\mathrm{supp}(g)}|f_{2^jR}-f_{2R}|\mathrm{d}\overline{x}\right),$$

Regarding the first integral, apply Hölder's inequality (with exponent q, to be fixed later) and John-Nirenberg's, so that

$$\begin{aligned}\int_{A_i\cap\mathrm{supp}(g)}|f(\overline{x})-f_{2^jR}|\mathrm{d}\overline{x}&\le\left(\int_{A_j\cap\mathrm{supp}(g)}|f(\overline{x})-f_{2^jR}|^q\mathrm{d}\overline{x}\right)^{\frac{1}{q}}|\mathrm{supp}(g)\cap 2^jR|^{\frac{1}{q'}}\\&\le\|f\|_{*,p_s}(2^j\ell(R))^{\frac{n+2s}{q}}\left[2^{2sj}\ell(R)^{n+2s}\right]^{\frac{1}{q'}}=\|f\|_{*,p_s}2^{j(\frac{n}{q}+2s)}\ell(R)^{n+2s}.\end{aligned}$$

For the second integral we apply [5, Ch. VI, Lemma 1.1] to deduce $|f_{2^jR}-f_{2R}|\lesssim j\|f\|_{*,p_s}\le j$, so

$$\int_{A_j\cap\mathrm{supp}(g)}|f_{2^jR}-f_{2R}|\mathrm{d}\overline{x}\lesssim j\,|\mathrm{supp}(g)\cap 2^jR|=j\|f\|_{*,p_s}2^{2sj}\ell(R)^{n+2s}.$$

Therefore, choosing $q > \frac{n}{2s\beta}$,

$$I_2 \lesssim_\beta \frac{\|f\|_{*,p_s}}{\ell(R)^{2s\beta}} \sum_{j=2}^{\infty} \frac{1}{2^{2s(1+\beta)j}} \big(2^{j(\frac{n}{q}+2s)} + j2^{2sj}\big)\ell(R)^{n+2s} \lesssim \|f\|_{*,p_s}\ell(R)^{n+2s(1-\beta)}.$$

If on the other hand $f \in \mathrm{Lip}_{\alpha,p_s}$, pick $c := f(\overline{x}_R)$ so that Hölder's inequality in (11) yields

$$I_2 \lesssim_\beta \frac{\|f\|_{\mathrm{Lip}_{\alpha,p_s}}}{\ell(R)^{2s\beta}} \sum_{j=2}^{\infty} \frac{\big(2^j\ell(R)\big)^\alpha}{2^{2s(1+\beta)j}} |\mathrm{supp}(g) \cap 2^j R| \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \sum_{j=2}^{\infty} \frac{\ell(R)^{n+2s(1-\beta)+\alpha}}{2^{(2s\beta-\alpha)j}},$$

being this last sum convergent because $\alpha < 2s\beta$, so we are done. □

Theorem 3.2 *Let $\alpha \in (0,1)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume $\|\nabla_x\varphi\|_\infty \le \ell(Q)^{-1}$ and $\|\nabla_x\phi\|_\infty \le \ell(R)^{-1}$. Then, if $\ell(R) \le \ell(Q)$, for each $i = 1, \ldots, n$ we have*

1. *If $f \in BMO_{p_s}$,*
$$|\langle f, \partial_{x_i}(\varphi\phi)\rangle| \lesssim_\beta \|f\|_{*,p_s}\ell(R)^{n+2s-1}.$$
2. *If $f \in Lip_{\alpha,p_s}$,*
$$|\langle f, \partial_{x_i}(\varphi\phi)\rangle| \lesssim_{\beta,\alpha} \|f\|_{Lip_{\alpha,p_s}}\ell(R)^{n+2s-1+\alpha}.$$

Proof First, observe that for any real constant c, we have the identity

$$\langle f, \partial_{x_i}(\varphi\phi)\rangle = \langle f - c, \partial_{x_i}(\varphi\phi)\rangle,$$

Therefore,

$$\begin{aligned}
\langle f, \partial_{x_i}(\varphi\phi)\rangle = \langle f - c, \partial_{x_i}(\varphi\phi)\rangle &\le \int_{Q\cap R} \big|f(\overline{x}) - c\big|\big|\partial_{x_i}(\varphi\phi)(\overline{x})\big|\mathrm{d}\overline{x} \\
&\le \left(\int_R \big|f(\overline{x}) - c\big|^2 \mathrm{d}\overline{x}\right)^{1/2} \left(\int_{Q\cap R} \big|\partial_{x_i}(\varphi\phi)(\overline{x})\big|^2 \mathrm{d}\overline{x}\right)^{1/2} \\
&\lesssim \left(\int_R \big|f(\overline{x}) - c\big|^2 \mathrm{d}\overline{x}\right)^{1/2} \left(\int_{Q\cap R} \Big[\|\nabla_x\varphi\|_\infty^2\|\phi\|_\infty^2 + \|\varphi\|_\infty^2\|\nabla_x\phi\|_\infty^2\Big]\mathrm{d}\overline{x}\right)^{1/2} \\
&\le \left(\int_R \big|f(\overline{x}) - c\big|^2 \mathrm{d}\overline{x}\right)^{1/2} |Q\cap R|^{1/2}\Big(|Q|^{-\frac{1}{n+2s}} + |R|^{-\frac{1}{n+2s}}\Big) \\
&\le \left(\int_R \big|f(\overline{x}) - c\big|^2 \mathrm{d}\overline{x}\right)^{1/2} \left(\frac{|Q\cap R|^{1/2}}{\ell(Q)} + \ell(R)^{\frac{n+2s}{2}-1}\right) \\
&= \left(\int_R \big|f(\overline{x}) - c\big|^2 \mathrm{d}\overline{x}\right)^{1/2} \left(|Q\cap R|^{\frac{n+2s-2}{2(n+2s)}} \frac{|Q\cap R|^{\frac{1}{n+2s}}}{\ell(Q)} + \ell(R)^{\frac{n+2s}{2}-1}\right) \\
&\le \left(\int_R \big|f(\overline{x}) - c\big|^2 \mathrm{d}\overline{x}\right)^{1/2} \ell(R)^{\frac{n+2s}{2}-1}.
\end{aligned}$$

Now, if $f \in \mathrm{BMO}_{p_s}$, choose $c := f_R$ and apply an s-parabolic version of John-Nirenberg's inequality (that admits an analogous proof) to deduce estimate *1*. On the other hand, if $f \in \mathrm{Lip}_{\alpha,p_s}$, choose $c := f(\overline{x}_R)$ to obtain estimate *2*. □

Theorem 3.3 *Let $\beta \in (0,1)$, $\alpha \in (0,1)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume that φ and ϕ are $\mathcal{C}^2$ with $\|\nabla_x\varphi\|_\infty \le \ell(Q)^{-1}$, $\|\Delta\varphi\|_\infty \le \ell(Q)^{-2}$ and $\|\nabla_x\phi\|_\infty \le \ell(R)^{-1}$, $\|\Delta\phi\|_\infty \le \ell(R)^{-2}$. Then, if $\ell(R) \le \ell(Q)$,* *1. If* $f \in BMO_{p_s}$,

$$|\langle f, (-\Delta)^{\beta}(\varphi\phi)\rangle| \lesssim_{\beta} \|f\|_{*,p_s}\ell(R)^{n+2(s-\beta)}.$$

2. If $f \in Lip_{\alpha,p_s}$ *and* $\alpha < 2\beta$,

$$|\langle f, (-\Delta)^{\beta}(\varphi\phi)\rangle| \lesssim_{\beta,\alpha} \|f\|_{Lip_{\alpha,p_s}}\ell(R)^{n+2(s-\beta)+\alpha}.$$

Proof Observe that for any real constant c,

$$\begin{aligned}|\langle f, (-\Delta)^{\beta}(\varphi\phi)\rangle| &= |\langle f - c, (-\Delta)^{\beta}(\varphi\phi)\rangle| \\ &\leq \int_{2R_1\times(I_Q\cap I_R)} |f(\overline{x}) - c|\left|(-\Delta)^{\beta}(\varphi\phi)(\overline{x})\right| \mathrm{d}\overline{x} \\ &\quad + \int_{(\mathbb{R}^n\setminus 2R_1)\times(I_Q\cap I_R)} |f(\overline{x}) - c|\left|(-\Delta)^{\beta}(\varphi\phi)(\overline{x})\right| \mathrm{d}\overline{x} =: I_1 + I_2.\end{aligned}$$

Regarding I_1, observe that for any $\overline{x} \in \mathbb{R}^{n+1}$ by Remark 3.1 we have $|(-\Delta)^{\beta}(\varphi\phi)(\overline{x})| \lesssim_{\beta} \ell(R)^{-2\beta}$. Therefore,

$$I_1 \lesssim_{\beta} \frac{1}{\ell(R)^{2\beta}} \int_{2R_1\times(I_Q\cap I_R)} |f(\overline{x}) - c| \mathrm{d}\overline{x}.$$

Let $\overline{x}_0$ be the center of $2R_1 \times (I_Q \cap I_R)$. Choosing $c := f_{2R}$ or $c := f(\overline{x}_0)$ for $f \in \mathrm{BMO}_{p_s}$ or $f \in \mathrm{Lip}_{\alpha,p_s}$ respectively, we obtain the desired estimates.

Let us turn to I_2. We first notice that, taking the Fourier transform, the operator $(-\Delta)^{\beta}$ can be rewritten as

$$(-\Delta)^{\beta}(\cdot) \simeq_{\beta} \sum_{j=1}^{n} \partial_{x_j}\left(\frac{1}{|x|^{n+2\beta-2}}\right) *_n \partial_{x_j}(\cdot),$$

where the notation $*_n$ is used to stress that the convolution is taken with respect the first n spatial variables. With this, if $x_0 \in \mathbb{R}^n$ denotes the center of $Q_1 \cap R_1$, for any $\overline{x} \in (\mathbb{R}^n \setminus 2R_1) \times (I_Q \cap I_R)$ we get

$$\begin{aligned}\left|(-\Delta)^{\beta}(\varphi\phi)(\overline{x})\right| &\lesssim_{\beta} \sum_{j=1}^{n}\left|\int_{Q_1\cap R_1} \partial_j(\varphi\phi)(z,t)\frac{z_j - x_j}{|z-x|^{n+2\beta}}\mathrm{d}z\right| \\ &= \sum_{j=1}^{n}\left|\int_{Q_1\cap R_1} \partial_j(\varphi\phi)(z,t)\left(\frac{z_j - x_j}{|z-x|^{n+2\beta}} - \frac{x_{0,j} - x_j}{|x_0-x|^{n+2\beta}}\right)\mathrm{d}z\right| \\ &\lesssim_{\beta} \sum_{j=1}^{n}\frac{\ell(R)}{|x_0-x|^{n+2\beta}}\|\nabla_x(\varphi\phi)\|_{\infty}\ell(R)^n \lesssim \frac{\ell(R)^n}{|x_0-x|^{n+2\beta}},\end{aligned} \tag{12}$$

by the mean value theorem. So, defining the cylinders $C_j := 2^j R_1 \times (I_Q \cap I_R)$ for $j \geq 1$, relation (12) implies

$$I_2 \lesssim_{\beta} \frac{1}{\ell(R)^{2\beta}}\sum_{j=1}^{\infty}\frac{1}{2^{j(n+2\beta)}}\int_{C_{j+1}\setminus C_j}\left|f(\overline{x}) - c\right|\mathrm{d}\overline{x},$$

If $f \in \mathrm{BMO}_{p_s}$, we choose $c := f_{2R}$ and proceed as in Theorem 3.1,

$$
\begin{aligned}
I_2 &\lesssim_\beta \frac{1}{\ell(R)^{2\beta}} \sum_{j=1}^{\infty} \frac{1}{2^{j(n+2\beta)}} \left(\int_{C_{j+1}\setminus C_j} \left| f(\overline{x}) - f_{2^j R} \right| \mathrm{d}\overline{x} + \int_{C_{j+1}\setminus C_j} \left| f_{2R} - f_{2^j R} \right| \mathrm{d}\overline{x} \right) \\
&\lesssim \frac{\|f\|_{*,p_s}}{\ell(R)^{2\beta}} \sum_{j=1}^{\infty} \frac{1}{2^{j(n+2\beta)}} \left[\left(2^j \ell(R)\right)^{\frac{n+2s}{q}} |C_{j+1}\setminus C_j|^{\frac{1}{q'}} + j|C_{j+1}\setminus C_j| \right] \\
&\lesssim \frac{\|f\|_{*,p_s}}{\ell(R)^{2\beta}} \sum_{j=1}^{\infty} \frac{1}{2^{j(n+2\beta)}} \left[\ell(R)^{n+2s} 2^{j(n+\frac{2s}{q})+\frac{2s}{q'}} + j\ell(R)^{n+2s}\, 2^{jn+2s} \right] \\
&\lesssim \|f\|_{*,p_s} \ell(R)^{n+2(s-\beta)} \left(1 + \sum_{j=1}^{\infty} \frac{2^{\frac{2}{q'}}}{2^{j(2\beta - \frac{2s}{q})}} \right).
\end{aligned}
$$

Fixing $q > s/\beta$ so that this last sum is convergent, proves the result.

On the other hand, if $f \in \mathrm{Lip}_{\alpha,p_s}$ let $c := f(\overline{x}_0)$ and also proceed as in Theorem 3.1 to deduce

$$
I_2 \lesssim_{\beta,\alpha} \frac{\|f\|_{\mathrm{Lip}_{\alpha,p_s}}}{\ell(R)^{2\beta}} \sum_{j=1}^{\infty} \frac{(2^j \ell(R))^\alpha}{2^{j(n+2\beta)}} |C_{j+1}\setminus C_j| \lesssim \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \ell(R)^{n+2(s-\beta)+\alpha} \sum_{j=1}^{\infty} \frac{1}{2^{(2\beta-\alpha)j}},
$$

that is a convergent sum since $\alpha < 2\beta$ by hypothesis. □

Recall that given $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $\beta \in (0, n)$, we define its n-dimensional β-Riesz transform (whenever it makes sense) as

$$
\mathcal{I}^n_\beta f(\cdot, t) := \frac{1}{|x|^{n-\beta}} * f(\cdot, t),
$$

for each t, where the convolution is thought in a principal value sense. Let us observe that for a test function f, for example, the operators $\mathcal{I}^n_\beta$ and ∂_{x_i} commute.

Theorem 3.4 *Let $\beta \in (0, 1)$, $\alpha \in (0, 1-\beta)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Assume $\|\nabla_x \varphi\|_\infty \le \ell(Q)^{-1}$ and $\|\nabla_x \phi\|_\infty \le \ell(R)^{-1}$. Then, if $\ell(R) \le \ell(Q)$, for each $i = 1, \ldots, n$ we have*

1. *If $f \in BMO_{p_s}$,*
$$
|\langle f, \partial_{x_i} [\mathcal{I}^n_\beta(\varphi\phi)]\rangle| \lesssim_\beta \|f\|_{*,p_s} \ell(R)^{n+2s+\beta-1}.
$$
2. *If $f \in Lip_{\alpha,p_s}$,*
$$
|\langle f, \partial_{x_i} [\mathcal{I}^n_\beta(\varphi\phi)]\rangle| \lesssim_{\beta,\alpha} \|f\|_{Lip_{\alpha,p_s}} \ell(R)^{n+2s+\beta+\alpha-1}.
$$

Proof Notice that for any $c \in \mathbb{R}$,

$$
\begin{aligned}
|\langle f, \partial_{x_i} [\mathcal{I}^n_\beta(\varphi\phi)]\rangle| &= |\langle f - c, \partial_{x_i} [\mathcal{I}^n_\beta(\varphi\phi)]\rangle| \\
&\le \int_{2R_1 \times (I_Q \cap I_R)} |f(\overline{x}) - c| \left| \partial_{x_i} [\mathcal{I}^n_\beta(\varphi\phi)](\overline{x}) \right| \mathrm{d}\overline{x} \\
&\quad + \int_{(\mathbb{R}^n \setminus 2R_1) \times (I_Q \cap I_R)} |f(\overline{x}) - c| \left| \partial_{x_i} [\mathcal{I}^n_\beta(\varphi\phi)](\overline{x}) \right| \mathrm{d}\overline{x} =: I_1 + I_2.
\end{aligned}
$$

Regarding I_1, we have for some conjugate exponents q, q' to be fixed later on,

$$
I_1 \lesssim \left(\int_{2R} |f(\overline{x}) - c|^{q'} \mathrm{d}\overline{x} \right)^{\frac{1}{q'}} \left(\int_{I_Q \cap I_R} \int_{2R_1} |\mathcal{I}^n_\beta[\partial_{x_i}(\varphi\phi)](x, t)|^q \, \mathrm{d}x \mathrm{d}t \right)^{\frac{1}{q}}
$$

$$\lesssim \left(\int_{2R} |f(\overline{x}) - c|^{q'} \mathrm{d}\overline{x} \right)^{\frac{1}{q'}} \left(\int_{I_Q \cap I_R} \|\mathcal{I}_\beta^n [\partial_{x_i}(\varphi\phi)(\cdot, t)]\|_q^q \mathrm{d}t \right)^{\frac{1}{q}}.$$

Choosing $q > \frac{n}{n-\beta}$, we shall apply [6, Theorem 6.1.3] and obtain

$$\begin{aligned} I_1 &\lesssim_\beta \left(\int_{2R} |f(\overline{x}) - c|^{q'} \mathrm{d}\overline{x} \right)^{\frac{1}{q'}} \left(\int_{I_Q \cap I_R} \|\partial_{x_i}(\varphi\phi)(\cdot, t)\|^q_{\frac{qn}{n+q\beta}} \mathrm{d}t \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_{2R} |f(\overline{x}) - c|^{q'} \mathrm{d}\overline{x} \right)^{\frac{1}{q'}} \ell(R)^{\frac{n+q\beta+2s}{q} - 1}. \end{aligned}$$

If we assume $f \in \mathrm{BMO}_{p_s}$, we choose $c := f_{2R}$ and apply a s-parabolic version of John-Nirenberg's inequality to deduce

$$I_1 \lesssim_\beta \|f\|_{*,p_s} \ell(R)^{\frac{n+2s}{q'}} \ell(R)^{\frac{n+q\beta+2s}{q} - 1} = \|f\|_{*,p_s} \ell(R)^{n+2s+\beta-1}.$$

If we assume $f \in \mathrm{Lip}_{\alpha,p_s}$, we choose $c := f(\overline{x}_R)$, being $\overline{x}_R$ the center of R, and obtain

$$I_1 \lesssim_{\beta,\alpha} \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \ell(R)^{\frac{n+2s}{q'} + \alpha} \ell(R)^{\frac{n+q\beta+2s}{q} - 1} = \|f\|_{*,p_s} \ell(R)^{n+2s+\beta+\alpha-1}.$$

To study I_2, we proceed as in Theorem 3.3. For any $\overline{x} \in (\mathbb{R}^n \setminus 2R_1) \times (I_Q \cap I_R)$, if $x_0 \in \mathbb{R}^n$ denotes the center of $Q_1 \cap R_1$, by the mean value theorem we get

$$\begin{aligned} \left|\mathcal{I}_\beta^n[\partial_{x_i}(\varphi\phi)](\overline{x})\right| &= \left| \int_{Q_1 \cap R_1} \partial_{x_i}(\varphi\phi)(z, t) \frac{1}{|z - x|^{n-\beta}} \mathrm{d}z \right| \\ &= \left| \int_{Q_1 \cap R_1} \partial_{x_i}(\varphi\phi)(z, t) \left(\frac{1}{|z - x|^{n-\beta}} - \frac{1}{|x_0 - x|^{n-\beta}} \right) \mathrm{d}z \right| \\ &\lesssim_\beta \sum_{j=1}^n \frac{\ell(R)}{|x_0 - x|^{n-\beta+1}} \|\nabla_x(\varphi\phi)\|_\infty \ell(R)^n \lesssim \frac{\ell(R)^n}{|x_0 - x|^{n-\beta+1}}. \end{aligned}$$

This way, putting $C_j := 2^j R_1 \times (I_Q \cap I_R)$ for $j \geq 1$, as in Theorem 3.3,

$$I_2 \lesssim_\beta \frac{1}{\ell(R)^{-\beta+1}} \sum_{j=1}^\infty \frac{1}{2^{j(n-\beta+1)}} \int_{C_{j+1} \setminus C_j} \left| f(\overline{x}) - c \right| \mathrm{d}\overline{x}.$$

The case $f \in \mathrm{BMO}_{p_s}$ is dealt with analogously as in Theorem 3.3, obtaining

$$I_2 \lesssim_\beta \|f\|_{*,p_s} \ell(R)^{n+2s+\beta-1} \sum_{j=1}^\infty \frac{1}{2^{j(n-\beta+1)}} \left[2^{j(n+\frac{2s}{q})} + j\, 2^{jn} \right],$$

so choosing $q > \frac{2s}{1-\beta}$ we are done. Observe that we also need $\beta < 1$ in order for the above sum to converge. The case $f \in \mathrm{Lip}_{\alpha,p_s}$ can be dealt with as follows

$$I_2 \lesssim_{\beta,\alpha} \frac{\|f\|_{\mathrm{Lip}_{\alpha,p_s}}}{\ell(R)^{-\beta+1}} \sum_{j=1}^\infty \frac{(2^j \ell(R))^\alpha}{2^{j(n-\beta+1)}} |C_{j+1} \setminus C_j| \lesssim \|f\|_{\mathrm{Lip}_{\alpha,p_s}} \sum_{j=1}^\infty \frac{\ell(R)^{n+2s+\beta+\alpha-1}}{2^{(1-\beta-\alpha)j}},$$

and this sum is convergent by the hypothesis $\alpha < 1 - \beta$. □

4 Potentials of positive measures with growth restrictions

The main goal of this section is to deduce some important BMO_{p_s} and $\mathrm{Lip}_{\alpha,p_s}$ estimates of potentials of the form $\partial_t^\beta P_s * \mu$, where μ is a finite positive Borel measure with some upper s-parabolic growth. We begin by proving a generalization of [14, Lemma 4.2] and [13, Lemma 7.2].

Lemma 4.1 *Let $s \in (0,1]$, $\eta \in (0,1)$ and μ be a positive measure in $\mathbb{R}^{n+1}$ which has upper s-parabolic growth of degree $n+2s\eta$. Then*

$$\|P_s * \mu\|_{Lip_{\eta,t}} \lesssim_\eta 1.$$

Proof Let $\overline{x} := (x,t)$, $\widehat{x} := (x,\tau)$ be fixed points in $\mathbb{R}^{n+1}$ with $t \neq \tau$, and set $\overline{x}_0 := (\overline{x}+\widehat{x})/2$. Writing $\overline{y} := (y,u)$ and $B_0 := B(\overline{x}_0, |\overline{x}-\widehat{x}|_{p_s}) = B(\overline{x}_0, |t-\tau|^{\frac{1}{2s}})$, we split

$$\begin{aligned}|P_s * \mu(\overline{x}) - P_s * \mu(\widehat{x})| &\leq \int_{\mathbb{R}^{n+1}\setminus 2B_0} |P_s(x-y,t-u) - P_s(x-y,\tau-u)|\mathrm{d}\mu(\overline{y}) \\ &\quad + \int_{2B_0} |P_s(x-y,t-u) - P_s(x-y,\tau-u)|\mathrm{d}\mu(\overline{y}) =: I_1 + I_2.\end{aligned}$$

Defining the s-parabolic annuli $A_j := 2^{j+1}B_0 \setminus 2^j B_0$ for $j \geq 1$ and arguing as in the last estimate of Theorem 2.2 we get

$$\begin{aligned}I_1 &\lesssim \sum_{j\geq 1}\int_{A_j} \frac{|t-\tau|}{|\overline{x}-\overline{y}|_{p_s}^{n+2s}}\mathrm{d}\mu(\overline{y}) \lesssim |t-\tau| \sum_{j\geq 1} \frac{\mu\big(2^{j+1}B_0\big)}{\big(2^j|t-\tau|^{\frac{1}{2s}}\big)^{n+2s}} \\ &\lesssim |t-\tau|^\eta \sum_{j\geq 1}\frac{1}{2^{2s(1-\eta)}} \simeq_\eta |t-\tau|^\eta,\end{aligned}$$

that is the desired estimate. Regarding I_2, observe that

$$I_2 \leq P_s * (\chi_{2B_0}\mu)(\overline{x}) + P_s * (\chi_{2B_0}\mu)(\widehat{x}).$$

Notice now that

$$P_s * (\chi_{2B_0}\mu)(\overline{x}) \lesssim \int_{2B_0} \frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^n} \leq \int_{|\overline{x}-\overline{y}|_{p_s} \leq 5|t-\tau|^{\frac{1}{2s}}} \frac{\mathrm{d}\mu(\overline{y})}{|\overline{x}-\overline{y}|_{p_s}^n} \lesssim_\eta |t-\tau|^\eta,$$

where we have split the latter domain of integration into (decreasing) s-parabolic annuli. Since this also holds replacing $\overline{x}$ by $\widehat{x}$, we also have $I_2 \lesssim |t-\tau|^\eta$ and we are done. □

The above result allows us to prove that, given a positive measure as in the above statement, we can ensure that the potential $\partial_t^\beta P_s * \mu$ already belongs to BMO_{p_s}.

Lemma 4.2 *Let $s \in (0,1]$, $\beta \in (0,1)$. Let μ be a finite positive Borel measure in $\mathbb{R}^{n+1}$ with upper s-parabolic growth of degree $n+2s\beta$. Then,*

$$\big\|\partial_t^\beta P_s * \mu\big\|_{*,p_s} \lesssim_\beta 1.$$

Proof Fix $\overline{x}_0 \in \mathbb{R}^{n+1}$ and $r > 0$. Consider the s-parabolic ball $B := B(\overline{x}_0, r) = B_0 \times I_0 \subset \mathbb{R}^n \times \mathbb{R}$ and a constant c_B to be determined later. We want to show that c_B can be chosen so that

$$\frac{1}{|B|}\int_B |\partial_t^\beta P_s * \mu(\overline{y}) - c_B|\mathrm{d}\overline{y} \lesssim_\beta 1.$$

To that end, begin by considering the following sets, which define a partition of $\mathbb{R}^{n+1}$:

$$R_1 := 5B, \qquad R_2 := \mathbb{R}^{n+1} \setminus (5B_0 \times \mathbb{R}), \qquad R_3 := (5B_0 \times \mathbb{R}) \setminus 5B,$$

as well as their corresponding characteristic functions χ_1, χ_2 and χ_3. Bearing in mind the estimates proved in Theorems 2.4 and 2.5 for $\partial_t^\beta P_s$ and the fact that μ is finite, it is clear that the quantity $|\partial_t^\beta P_s * (\chi_2\mu)(\overline{x}_0)|$ is also finite. Moreover, notice that $|\partial_t^\beta P_s|$ is bounded by s-parabolically homogeneous functions of degree $-n-2s\beta$ for any dimension. In fact, we deduce the following estimates: given any $\varepsilon, \alpha > 0$, we obtain if $n > 2$,

$$|\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}} \leq \frac{1}{|x|^{n-\varepsilon}|t|^{\frac{\varepsilon+2s\beta}{2s}}}, \qquad \text{if } \varepsilon < 2s(1-\beta).$$

For $n = 2$,

$$\text{if } s < 1, \quad |\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \frac{1}{|x|^{2-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}} \leq \frac{1}{|x|^{2-\varepsilon}|t|^{\frac{\varepsilon+2s\beta}{2s}}}, \qquad \text{if } \varepsilon < 2s(1-\beta),$$

$$\text{if } s = 1, \quad |\partial_t^\beta W(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{\alpha}|\overline{x}|_{p_s}^{2+2\beta-\alpha}} \leq \frac{1}{|x|^{\alpha}|t|^{1+\beta-\frac{\alpha}{2}}}, \qquad \text{if } 2\beta < \alpha < 2.$$

And for $n = 1$,

$$\text{if } s < 1, \quad |\partial_t^\beta P_s(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^{1-2s+\alpha}|\overline{x}|_{p_s}^{2s(1+\beta)-\alpha}} \leq \frac{1}{|x|^{1-2s+\alpha}|t|^{1+\beta-\frac{\alpha}{2s}}}, \quad \text{if } 2s\beta < \alpha < 2s,$$

$$\text{if } s = 1, \quad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|\overline{x}|_{p_s}^{1+2\beta}} \leq \frac{1}{|x|^{\varepsilon}|t|^{\frac{1+2\beta-\varepsilon}{2}}}, \qquad \text{if } 2\beta - 1 < \varepsilon < 1.$$

In light of the above inequalities, and using that $\beta < 1$, it is clear that $\partial_t^\beta P_s$ defines a $\mathcal{L}^{n+1}$-locally integrable function in $\mathbb{R}^{n+1}$ once endowed with the s-parabolic distance. Hence, there exists some $\overline{\xi}_0 \in B$ (that we may think as close as we need to $\overline{x}_0$) such that $|\partial_t^\beta P_s * (\chi_3\mu)(\overline{\xi}_0)|$ is finite. Bearing all these observations in mind, we choose c_B to be

$$c_B := \partial_t^\beta P_s * (\chi_2\mu)(\overline{x}_0) + \partial_t^\beta P_s * (\chi_3\mu)(\overline{\xi}_0).$$

Therefore, we are interested in bounding by a constant the following quantity:

$$\begin{aligned}\frac{1}{|B|}\int_B |\partial_t^\beta P_s * \mu(\overline{y}) - c_B|\mathrm{d}\overline{y} &\leq \frac{1}{|B|}\int_B |\partial_t^\beta P_s * (\chi_1\mu)(\overline{y})|\mathrm{d}\overline{y} \\ &+ \frac{1}{|B|}\int_B |\partial_t^\beta P_s * (\chi_2\mu)(\overline{y}) - \partial_t^\beta P_s * (\chi_2\mu)(\overline{x}_0)|\mathrm{d}\overline{y} \\ &+ \frac{1}{|B|}\int_B |\partial_t^\beta P_s * (\chi_3\mu)(\overline{y}) - \partial_t^\beta P_s * (\chi_3\mu)(\overline{\xi}_0)|\mathrm{d}\overline{y} =: I_1 + I_2 + I_3.\end{aligned}$$

For I_1, simply notice that

$$I_1 \leq \frac{1}{|B|}\int_{5B}\left(\int_B |\partial_t^\beta P_s(\overline{y}-\overline{z})|\mathrm{d}\overline{y}\right)\mathrm{d}\mu(\overline{z}).$$

Using any of the bounds above for $\partial_t^\beta P_s$, depending on n and s, integration in polar coordinates yields

$$I_1 \lesssim_\beta \frac{1}{|B|} r^{2s(1-\beta)} \mu(5B) \lesssim_\beta 1.$$

Regarding I_2, write

$$I_2 \le \frac{1}{|B|} \int_B \left(\int_{R_2} |\partial_t^\beta P_s(\overline{y} - \overline{z}) - \partial_t^\beta P_s(\overline{x}_0 - \overline{z})| \mathrm{d}\mu(\overline{z}) \right) \mathrm{d}\overline{y}.$$

If we name $\overline{x} := \overline{x}_0 - \overline{z}$ and $\overline{x}' := \overline{y} - \overline{z}$, we have in particular

$$|\overline{x} - \overline{x}'|_{p_s} = |\overline{x}_0 - \overline{y}|_{p_s} \le r < \frac{|x_0 - z|}{2} = \frac{|x|}{2},$$

where the second inequality holds because $\overline{z} \in R_2$. Therefore, by the last estimate of Theorems 2.4 and 2.5, writing $2\zeta := \min\{1, 2s\}$ we get

$$\begin{aligned} I_2 &\lesssim_\beta \frac{1}{|B|} \int_B \left(\int_{R_2} \frac{|\overline{y} - \overline{x}_0|_{p_s}^{2\zeta}}{|x_0 - z|^{n+2\zeta-2s} |\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}} \mathrm{d}\mu(\overline{z}) \right) \mathrm{d}\overline{y} \\ &\lesssim r^{2\zeta} \int_{R_2} \frac{\mathrm{d}\mu(\overline{z})}{|x_0 - z|^{n+2\zeta-2s} |\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}}. \end{aligned}$$

Let us split R_2 into proper disjoint pieces. Take the cylinders given by $C_j := 5^j B_0 \times \mathbb{R},\ j \in \mathbb{Z},\ j \ge 1$, as well as the annular cylinders $\widehat{C}_j := C_{j+1} \setminus C_j,\ j \ge 1$. The partition of R_2 we are interested in is given by the disjoint union of all the sets $\widehat{C}_j,\ j \ge 1$, which clearly cover R_2. Therefore

$$I_2 \lesssim_\beta r^{2\zeta} \sum_{j=1}^{\infty} \int_{\widehat{C}_j} \frac{\mathrm{d}\mu(\overline{z})}{|x_0 - z|^{n+2\zeta-2s} |\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}}. \tag{13}$$

At the same time, for each $j \ge 1$, we shall consider a proper partition of $\widehat{C}_j$. Denote $A_k = 5^{k+1}B \setminus 5^k B$ for every positive integer k and define $\widehat{C}_{j,k} := \widehat{C}_j \cap A_k,\ k \ge 1$. Let us make some observations about the sets $\widehat{C}_{j,k}$. First, notice that by definition, for each $j \ge 1$,

$$\widehat{C}_{j,k} = \left[(5^{j+1}B_0 \setminus 5^j B_0) \times \mathbb{R}\right] \cap \left(5^{k+1}B \setminus 5^k B\right).$$

Hence, using that

$$\left[(5^{j+1}B_0 \setminus 5^j B_0) \times \mathbb{R}\right] \cap \left(5^{k+1}B \setminus 5^k B\right) = \varnothing, \qquad \text{for} \qquad k < j,$$

we have that, in fact, $\widehat{C}_j$ can be covered by $\widehat{C}_{j,k}$ for $k \ge j$, that is

$$\widehat{C}_j = \bigcup_{k=1}^{\infty} \widehat{C}_{j,k} = \bigcup_{k=j}^{\infty} \widehat{C}_{j,k}.$$

Secondly, in order to estimate $\mu(\widehat{C}_{j,k})$, observe that for any $k \ge j$, by definition, the set $\widehat{C}_{j,k}$ can be written explicitly as follows:

$$\begin{aligned} \widehat{C}_{j,k} &= \left[\left(5^{j+1}B_0 \setminus 5^j B_0\right) \times \mathbb{R}\right] \cap \left(5^{k+1}B \setminus 5^k B\right) \\ &= \left[\left(5^{j+1}B_0 \setminus 5^j B_0\right) \times \mathbb{R}\right] \\ &\quad \cap \left\{\left[\left(5^{k+1}B_0 \setminus 5^k B_0\right) \times 5^{2s(k+1)} I_0\right] \cup \left[5^k B_0 \times \left(5^{2s(k+1)} I_0 \setminus 5^{2k} I_0\right)\right]\right\}. \end{aligned}$$

Continue by observing that if $k = j$, the intersection with the second element of the union is empty, so

$$\widehat{C}_{j,j} = \big(5^{j+1}B_0 \setminus 5^j B_0\big) \times 5^{2s(j+1)} I_0;$$

while if $k > j$ one has the contrary, that is, the intersection with the first element is empty, and therefore, since $5^{j+1}B_0 \setminus 5^j B_0 \subset 5^k B_0$,

$$\widehat{C}_{j,k} = \big(5^{j+1}B_0 \setminus 5^j B_0\big) \times \big[5^{2s(k+1)} I_0 \setminus 5^{2sk} I_0\big].$$

Observe that $\widehat{C}_{j,j} \subset 5^{j+1}B$, which implies $\mu(\widehat{C}_{j,j}) \leq \mu(5^{j+1}B) \lesssim (5^{j+1}r)^{n+2s\beta}$. On the other hand, for $k > j$, notice that the set $\widehat{C}_{j,k}$ can be covered by disjoint temporal translates of $\widehat{C}_{j,j}$, and the number needed to do it is proportional to the ratio between their respective time lengths, that is

$$\frac{2\big(5^{2s(k+1)} - 5^{2sk}\big)}{5^{2s(j+1)}} \simeq \frac{5^{2sk}}{5^{2sj}}.$$

Therefore, since this last ratio is also valid for the case $k = j$, for every $k \geq j$ we have

$$\mu(\widehat{C}_{j,k}) \simeq \frac{5^{2sk}}{5^{2sj}} \mu(\widehat{C}_{j,j}) \lesssim_\beta \frac{5^{2sk}}{5^{2sj}} \big(5^{j+1}r\big)^{n+2s\beta}.$$

All in all, we finally obtain

$$I_2 \lesssim_\beta r^{2\zeta} \sum_{j=1}^{\infty} \sum_{k \geq j} \int_{\widehat{C}_{j,k}} \frac{\mathrm{d}\mu(\overline{z})}{|x_0 - z|^{n+2\zeta-2s} |\overline{x}_0 - \overline{z}|_{p_s}^{2s(1+\beta)}} \lesssim r^{2\zeta} \sum_{j=1}^{\infty} \sum_{k \geq j} \frac{\mu(\widehat{C}_{j,k})}{(5^j r)^{n+2\zeta-2s} (5^k r)^{2s(1+\beta)}}$$
$$\lesssim_\beta \sum_{j=1}^{\infty} \sum_{k \geq j} \frac{1}{5^{j(2\zeta-2s\beta)} 5^{2s\beta k}} = \sum_{k=1}^{\infty} \frac{1}{5^{2s\beta k}} \sum_{j=1}^{k} \frac{1}{5^{j(2\zeta-2s\beta)}} \lesssim \sum_{k=1}^{\infty} \frac{1}{5^{2s\beta k}} \bigg(1 + \frac{1}{5^{(2\zeta-2s\beta)k}}\bigg) \lesssim_\beta 1.$$

Finally, let us study I_3. Notice that the estimate we want to check is deduced if we prove

$$|\partial_t^\beta P_s * (\chi_3 \mu)(\overline{y}) - \partial_t^\beta P_s * (\chi_3 \mu)(\overline{\xi}_0)| \lesssim_\beta 1,$$

that at the same time, can be obtained if we show that for any $\overline{x}, \overline{y} \in B$ we have

$$|\partial_t^\beta P_s * (\chi_3 \mu)(\overline{x}) - \partial_t^\beta P_s * (\chi_3 \mu)(\overline{y})| \lesssim_\beta 1. \tag{14}$$

It is clear that it suffices to check the latter estimate in two particular cases: when $\overline{x}$ and $\overline{y}$ share their time coordinate, and when they share their spatial coordinate.

Case 1: $\overline{x} = (x,t)$ and $\overline{y} = (y,t)$ points of B. Let us begin by observing that

$$\begin{aligned}
&|\partial_t^\beta P_s * (\chi_3 \mu)(\overline{x}) - \partial_t^\beta P_s * (\chi_3 \mu)(\overline{y})| \\
&= \bigg| \int \frac{P_s * (\chi_3 \mu)(x,\tau) - P_s * (\chi_3 \mu)(x,t)}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \\
&\quad - \int \frac{P_s * (\chi_3 \mu)(y,\tau) - P_s * (\chi_3 \mu)(y,t)}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \bigg| \\
&\leq \int_{|\tau - t| \leq (2r)^{2s}} \frac{|P_s * (\chi_3 \mu)(x,\tau) - P_s * (\chi_3 \mu)(x,t)|}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| \leq (2r)^{2s}} \frac{|P_s * (\chi_3 \mu)(y,\tau) - P_s * (\chi_3 \mu)(y,t)|}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| > (2r)^{2s}} \frac{|P_s * (\chi_3 \mu)(x,\tau) - P_s * (\chi_3 \mu)(x,t) - P_s * (\chi_3 \mu)(y,\tau) + P_s * (\chi_3 \mu)(y,t)|}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

First, we estimate I_1. Argue as in the proof of the last estimate of Theorem 2.2 to obtain

$$|P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)| \le |\tau - t| \int_{R_3} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+2s}} \lesssim_\beta \frac{|t-\tau|}{r^{2s(1-\beta)}},$$

where the last inequality can be obtained by splitting the domain of integration into s-parabolic annuli and using the s-parabolic growth condition of degree $n+2s\beta$ of μ. Thus,

$$I_1 \lesssim_\beta \frac{1}{r^{2s(1-\beta)}} \int_{|\tau-t|\le(2r)^{2s}} \frac{\mathrm{d}\tau}{|\tau-t|^\beta} \lesssim_\beta \frac{(r^{2s})^{(1-\beta)}}{r^{2s(1-\beta)}} = 1.$$

The arguments to obtain $I_2 \lesssim_\beta 1$ are exactly the same (just write y instead of x in the lines above). Concerning the term I_3, we split it as follows

$$\begin{aligned} I_3 \le & \int_{|\tau-t|>(2r)^{2s}} \frac{|P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(y,\tau)|}{|\tau-t|^{1+\beta}}\mathrm{d}\tau \\ & + \int_{|\tau-t|>(2r)^{2s}} \frac{|P_s * (\chi_3\mu)(x,t) - P_s * (\chi_3\mu)(y,t)|}{|\tau-t|^{1+\beta}}\mathrm{d}\tau =: I_{31} + I_{32}. \end{aligned}$$

First, let us deal with integral I_{32}. Since $(x,t), (y,t) \in B$,

$$|P_s * (\chi_3\mu)(x,t) - P_s * (\chi_3\mu)(y,t)| \le |x-y| \, \|\nabla_x P_s * (\chi_3\mu)\|_{\infty,B}.$$

Notice that for any $\overline{z} \in B$, by Theorem 2.2 and the fact that $s\beta < 1$, we have

$$|\nabla_x P_s * (\chi_3\mu)(\overline{z})| \lesssim \int_{R_3} \frac{|z-w|}{|\overline{z}-\overline{w}|_{p_s}^{n+2}}\mathrm{d}\mu(\overline{w}) \lesssim r \int_{\mathbb{R}^{n+1}\setminus 5B} \frac{\mathrm{d}\mu(\overline{w})}{|\overline{z}-\overline{w}|_{p_s}^{n+2}} \lesssim_\beta r^{2s\beta-1}.$$

Therefore, since $|x-y| \le r$,

$$I_{32} \lesssim_\beta r^{2s\beta} \int_{|\tau-t|>(2r)^{2s}} \frac{\mathrm{d}\tau}{|\tau-t|^{1+\beta}} \lesssim_\beta r^{2s\beta} \frac{1}{(r^{2s})^\beta} = 1. \tag{15}$$

Regarding I_{31}, observe that for each τ the points (x,τ) and (y,τ) belong to a temporal translate of B that does not intersect B, since $|\tau-t| > (2r)^{2s}$ and $t \in I_0$. We call it B_τ. Hence, bearing in mind the first estimate of [14, Lemma 2.1] we deduce

$$\begin{aligned} & |P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(y,\tau)| \\ & \le \int_{2B_\tau} |P_s((x,\tau)-\overline{w}) - P_s((y,\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\ & \quad + \int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} |P_s((x,\tau)-\overline{w}) - P_s((y,\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\ & \lesssim \int_{2B_\tau} \frac{\mathrm{d}\mu(\overline{w})}{|(x,\tau)-\overline{w}|_{p_s}^n} + \int_{2B_\tau} \frac{\mathrm{d}\mu(\overline{w})}{|(y,\tau)-\overline{w}|_{p_s}^n} \\ & \quad + |x-y| \int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} |\nabla_x P_s((\widetilde{x},\tau)-\overline{w})|\mathrm{d}\mu(\overline{w}) \\ & \lesssim_\beta r^{2s\beta} + r \int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} \frac{|\widetilde{x}-w|}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}}\mathrm{d}\mu(\overline{w}) \\ & \lesssim r^{2s\beta} + r^2 \int_{[(5B_0\times\mathbb{R})\setminus 5B]\cap(2B_\tau)^c} \frac{\mathrm{d}\mu(\overline{w})}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}} \\ & \le r^{2s\beta} + r^2 \int_{\mathbb{R}^{n+1}\setminus 2B_\tau} \frac{\mathrm{d}\mu(\overline{w})}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}} \lesssim_\beta r^{2s\beta}, \end{aligned} \tag{16}$$

where for both integrals in (16) we have split the domain of integration into (decreasing) s-parabolic annuli; while in the remaining term, $\widetilde{x}$ belongs to the segment joining x and y and we have split the domain of integration into s-parabolic annuli centered at $(x_0, t+s)$. Hence, similarly to (15) we get $I_{31} \lesssim_\beta 1$ and we are done with *Case 1*.

Case 2: $\overline{x} = (x,t)$ *and* $\overline{y} = (x,u)$ *points of* B. Write

$$\begin{aligned}
&|\partial_t^\beta P_s * (\chi_3\mu)(\overline{x}) - \partial_t^\beta P_s * (\chi_3\mu)(\overline{y})| \\
&= \left| \int \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)}{|\tau - t|^{1+\beta}} \mathrm{d}\tau - \int \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,u)}{|\tau - u|^{1+\beta}} \mathrm{d}\tau \right| \\
&\leq \int_{|\tau - t| \leq (2r)^{2s}} \frac{|P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)|}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| \leq (2r)^{2s}} \frac{|P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,u)|}{|\tau - u|^{1+\beta}} \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| > (2r)^{2s}} \left| \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)}{|\tau - t|^{1+\beta}} \right. \\
&\quad \left. - \frac{P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,u)}{|\tau - u|^{1+\beta}} \right| \mathrm{d}\tau =: I_1' + I_2' + I_3'.
\end{aligned}$$

The expressions corresponding to I_1', I_2' can be tackled in the same way as I_1, I_2. Hence, $I_1' \lesssim_\beta 1$ and $I_2' \lesssim_\beta 1$. Finally, for I_3', adding and subtracting $P_s * (\chi_3\mu)(x,t)/|\tau - u|^{1+\beta}$,

$$\begin{aligned}
I_3' \leq &\int_{|\tau - t| > (2r)^{2s}} \left| \frac{1}{|\tau - t|^{1+\beta}} - \frac{1}{|\tau - u|^{1+\beta}} \right| |P_s * (\chi_3\mu)(x,\tau) - P_s * (\chi_3\mu)(x,t)| \mathrm{d}\tau \\
&+ \int_{|\tau - t| > (2r)^{2s}} \frac{1}{|\tau - u|^{1+\beta}} |P_s * (\chi_3\mu)(x,t) - P_s * (\chi_3\mu)(x,u)| \mathrm{d}\tau.
\end{aligned}$$

Since $|\tau - t| > (2r)^{2s}$ we can apply the mean value theorem to deduce

$$\left| \frac{1}{|\tau - t|^{1+\beta}} - \frac{1}{|\tau - u|^{1+\beta}} \right| \lesssim_\beta \frac{|t-u|}{|\tau - t|^{2+\beta}} \lesssim \frac{r^{2s}}{|\tau - t|^{2+\beta}}.$$

In addition, since μ has upper s-parabolic growth of degree $n + 2s\beta$, by Lemma 4.1, with $\eta := \beta$, the time function $P_s * (\chi_3\mu)(x, \cdot)$ is Lip-β. Therefore,

$$I_3' \lesssim_\beta \int_{|\tau - t| > (2r)^{2s}} \frac{r^{2s}}{|\tau - t|^{2+\beta}} |\tau - t|^\beta \mathrm{d}\tau + \int_{|\tau - t| > (2r)^{2s}} \frac{1}{|\tau - u|^{1+\beta}} |t - u|^\beta \mathrm{d}\tau \lesssim_\beta 1.$$

Therefore estimate (14) is satisfied and we are done with I_3 and also with the proof. □

In the same spirit, if we ask the positive measure for an extra α growth, the potential $\partial_t^\beta P_s * \mu$ will satisfy a $\mathrm{Lip}_{\alpha, p_s}$ property. Recall that $2\zeta := \min\{1, 2s\}$.

Lemma 4.3 *Let* $s \in (0,1]$, $\beta \in (0,1)$ *and* $\alpha \in (0, 2\zeta)$ *such that* $2s\beta + \alpha < 2$. *Let* μ *be a positive measure in* $\mathbb{R}^{n+1}$ *which has upper* s*-parabolic growth of degree* $n + 2s\beta + \alpha$. *Then,*

$$\|\partial_t^\beta P_s * \mu\|_{Lip_{\alpha, p_s}} \lesssim_{\beta, \alpha} 1.$$

Proof Fix any $\overline{x}, \overline{y} \in \mathbb{R}^{n+1}$, $\overline{x} \neq \overline{y}$. We have to check if the following holds

$$|\partial_t^\beta P_s * \mu(\overline{x}) - \partial_t^\beta P_s * \mu(\overline{y})| \lesssim_{\beta, \alpha} |\overline{x} - \overline{y}|_{p_s}^\alpha.$$

Begin by choosing the following partition of $\mathbb{R}^{n+1}$

$$R_1 := \big\{\overline{z} \,:\, |\overline{x}-\overline{y}|_{p_s} \le |x-z|/5\big\} \cup \big\{\overline{z} \,:\, |\overline{y}-\overline{x}|_{p_s} \le |y-z|/5\big\},$$
$$R_2 := \mathbb{R}^{n+1} \setminus R_1 = \big\{\overline{z} \,:\, |\overline{x}-\overline{y}|_{p_s} > |x-z|/5\big\} \cap \big\{\overline{z} \,:\, |\overline{y}-\overline{x}|_{p_s} > |y-z|/5\big\},$$

and their corresponding characteristic functions χ_1, χ_2. From the latter we have

$$\begin{aligned}
&\frac{|\partial_t^\beta P_s * \mu(\overline{x}) - \partial_t^\beta P_s * \mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^\alpha} \\
&\quad \le \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha} \int_{|\overline{x}-\overline{y}|_{p_s} \le |x-z|/5} |\partial_t^\beta P_s(\overline{x}-\overline{z}) - \partial_t^\beta P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) \\
&\quad + \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha} \int_{|\overline{y}-\overline{x}|_{p_s} \le |y-z|/5} |\partial_t^\beta P_s(\overline{x}-\overline{z}) - \partial_t^\beta P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) \\
&\quad + \frac{1}{|\overline{x}-\overline{y}|_{p_s}^\alpha} \big|\partial_t^\beta P_s * (\chi_2\mu)(\overline{x}) - \partial_t^\beta P_s * (\chi_2\mu)(\overline{y})\big| =: I_{1,\overline{x}} + I_{1,\overline{y}} + I_2. \qquad (17)
\end{aligned}$$

Regarding $I_{1,\overline{x}}$, name $\overline{\xi} := \overline{x} - \overline{z}$, $\overline{\xi}' := \overline{y} - \overline{z}$ and observe that, in particular, one has

$$|\overline{\xi} - \overline{\xi}'|_{p_s} = |\overline{x}-\overline{y}|_{p_s} < \frac{|x-z|}{2} = \frac{|\xi|}{2},$$

Applying the last estimate either of Theorem 2.4 or Theorem 2.5, we deduce

$$I_{1,\overline{x}} \lesssim_\beta \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-2\zeta}} \int_{|\overline{x}-\overline{y}|_{p_s} \le |x-z|/5} \frac{\mathrm{d}\mu(\overline{z})}{|x-z|^{n+2\zeta-2s} |\overline{x}-\overline{z}|_{p_s}^{2s(1+\beta)}}.$$

Let us split the domain of integration into proper disjoint pieces. For $\overline{x} = (x,t)$, we denote

$$B_{\overline{x}} := B(\overline{x}, |\overline{x}-\overline{y}|_{p_s}) = B_1(x, |\overline{x}-\overline{y}|_{p_s}) \times J_{\overline{x}},$$

where $B_1(x, |\overline{x}-\overline{y}|_{p_s})$ is an Euclidean ball in $\mathbb{R}^n$ and $J_{\overline{x}}$ is a real interval centered at t with length $2|\overline{x}-\overline{y}|_{p_s}^{2s}$. As in Lemma 4.2, take cylinders $C_{j,\overline{x}} := 5^j B_1(\overline{x}, |\overline{x}-\overline{y}|_{p_s}) \times \mathbb{R}$ for $j \ge 1$, as well as the annular cylinders $\widehat{C}_{j,\overline{x}} := C_{j+1,\overline{x}} \setminus C_{j,\overline{x}}$, for $j \ge 1$. We express $\{\overline{z} : |\overline{x}-\overline{y}|_{p_s} \le |x-z|/5\}$ as the disjoint union of the sets $\widehat{C}_{j,\overline{x}}$, so that

$$I_{1,\overline{x}} \lesssim_\beta \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-2\zeta}} \sum_{j=1}^\infty \int_{\widehat{C}_{j,\overline{x}}} \frac{\mathrm{d}\mu(\overline{z})}{|x-z|^{n+2\zeta-2s} |\overline{x}-\overline{z}|_{p_s}^{2s(1+\beta)}}.$$

The above integral can be studied as that appearing in (13), in the study of the term I_2 of Lemma 4.2 (centering now the cylinders in $\overline{x}$ and interchanging the roles of r and $|\overline{x}-\overline{y}|_{p_s}$). Doing so, and taking into account the $n + 2s\beta + \alpha$ growth of μ, one obtains

$$\begin{aligned}
I_{1,\overline{x}} &\lesssim_{\beta,\alpha} \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha-2\zeta}} \sum_{j=1}^\infty \sum_{k\ge j} \frac{(5^{j+1}|\overline{x}-\overline{y}|_{p_s})^{n+2s\beta+\alpha}}{(5^j|\overline{x}-\overline{y}|_{p_s})^{n+2\zeta-2s} (5^k|\overline{x}-\overline{y}|_{p_s})^{2s(1+\beta)}} \frac{5^{2sk}}{5^{2sj}} \\
&= \sum_{j=1}^\infty \sum_{k\ge j} \frac{5^{j(n+2s\beta+\alpha)}}{5^{j(n+2\zeta-2s)} 5^{2s(1+\beta)k}} \frac{5^{2sk}}{5^{2sj}} \\
&\simeq \sum_{j=1}^\infty \sum_{k\ge j} \frac{5^{j(2s\beta+\alpha-2\zeta)}}{5^{2s\beta k}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{5^{2s\beta k}} \sum_{j=1}^{k} \frac{1}{5^{j(2\zeta - 2s\beta - \alpha)}} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{5^{2s\beta k}} \left(1 + \frac{1}{5^{(2\zeta - 2s\beta - \alpha)k}}\right) \lesssim_{\beta,\alpha} 1, \quad \text{if } \alpha < 2\zeta.
\end{aligned}
$$

The study of $I_{1,\overline{y}}$ is analogous, interchanging the roles of $\overline{x}$ and $\overline{y}$. Finally we deal with I_2. We claim that the following estimate holds

$$
\left|\partial_t^{\beta} P_s * (\chi_2\mu)(\overline{x}) - \partial_t^{\beta} P_s * (\chi_2\mu)(\overline{y})\right| \lesssim_{\beta,\alpha} |\overline{x} - \overline{y}|_{p_s}^{\alpha}.
$$

The general case will follows from the following two cases: whether $\overline{x}$ and $\overline{y}$ share their time coordinate, or if they share their spatial coordinate. Indeed, write $\overline{x} = (x, t)$, $\overline{y} = (y, \tau)$ and set $\widehat{x} := (x, \tau)$ so that

$$
\begin{aligned}
&\left|\partial_t^{\beta} P_s * (\chi_2\mu)(\overline{x}) - \partial_t^{\beta} P_s * (\chi_2\mu)(\overline{y})\right| \\
&\quad \leq \left|\partial_t^{\beta} P_s * (\chi_2\mu)(\overline{x}) - \partial_t^{\beta} P_s * (\chi_2\mu)(\widehat{x})\right| + \left|\partial_t^{\beta} P_s * (\chi_2\mu)(\widehat{x}) - \partial_t^{\beta} P_s * (\chi_2\mu)(\overline{y})\right| \\
&\quad \lesssim_{\beta,\alpha} |\overline{x} - \widehat{x}|_{p_s}^{\alpha} + |\widehat{x} - \overline{y}|_{p_s}^{\alpha} = |t - \tau|^{\alpha/2} + |x - y|^{\alpha} \leq 2|\overline{x} - \overline{y}|_{p_s}^{\alpha}, \quad \text{and we are done.}
\end{aligned}
$$

Case 1: $\overline{x} = (x, t)$ and $\overline{y} = (x, u)$. Write $\mu_2 := \chi_2\mu$ and estimate $|\partial_t^{\beta} P_s * \mu_2(\overline{x}) - \partial_t^{\beta} P_s * \mu_2(\overline{y})|$ as follows

$$
\begin{aligned}
&\left| \int \frac{P_s * \mu_2(x, \tau) - P_s * \mu_2(x, t)}{|\tau - t|^{1+\beta}} \mathrm{d}\tau - \int \frac{P_s * \mu_2(x, \tau) - P_s * \mu_2(x, u)}{|\tau - u|^{1+\beta}} \mathrm{d}\tau \right| \\
&\leq \int_{|\tau - t| \leq 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_2(x, \tau) - P_s * \mu_2(x, t)|}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| \leq 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_2(x, \tau) - P_s * \mu_2(x, u)|}{|\tau - u|^{1+\beta}} \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| > 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \left| \frac{P_s * \mu_2(x, \tau) - P_s * \mu_2(x, t)}{|\tau - t|^{1+\beta}} \right. \\
&\quad \left. - \frac{P_s * \mu_2(x, \tau) - P_s * \mu_2(x, u)}{|\tau - u|^{1+\beta}} \right| \mathrm{d}\tau =: I_1 + I_2 + I_3.
\end{aligned}
$$

By a direct application of Lemma 4.1 we are able to obtain, straightforwardly,

$$
\begin{aligned}
I_1 &\lesssim_{\beta,\alpha} \int_{|\tau - t| \leq 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \frac{\mathrm{d}\tau}{|\tau - t|^{1 - \frac{\alpha}{2s}}} \lesssim_{\alpha} |\overline{x} - \overline{y}|_{p_s}^{\alpha} \quad \text{and} \\
I_2 &\lesssim_{\beta,\alpha} \int_{|\tau - t| \leq 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \frac{\mathrm{d}\tau}{|\tau - u|^{1 - \frac{\alpha}{2s}}} \lesssim_{\alpha} |\overline{x} - \overline{y}|_{p_s}^{\alpha}.
\end{aligned}
$$

For I_3, adding and subtracting the term $P_s * \mu_2(x, t)/|\tau - u|^{1+\beta}$ we get

$$
\begin{aligned}
I_3 &\leq \int_{|\tau - t| > 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \left| \frac{1}{|\tau - t|^{1+\beta}} - \frac{1}{|\tau - u|^{1+\beta}} \right| |P_s * \mu_2(x, \tau) - P_s * \mu_2(x, t)| \mathrm{d}\tau \\
&\quad + \int_{|\tau - t| > 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}} \frac{1}{|\tau - u|^{1+\beta}} |P_s * \mu_2(x, t) - P_s * \mu_2(x, u)| \mathrm{d}\tau.
\end{aligned}
$$

Since $|\tau - t| > 2^{2s}|\overline{x} - \overline{y}|_{p_s}^{2s}$ we can apply the mean value theorem to deduce

$$\left|\frac{1}{|\tau - t|^{1+\beta}} - \frac{1}{|\tau - u|^{1+\beta}}\right| \lesssim_\beta \frac{|t-u|}{|\tau - t|^{2+\beta}} \lesssim \frac{|\overline{x} - \overline{y}|_{p_s}^{2s}}{|\tau - t|^{2+\beta}}.$$

Therefore, by Lemma 4.1 with $\eta := \beta + \frac{\alpha}{2s}$, we finally have

$$\begin{aligned} I_3 \lesssim_{\beta,\alpha} & \int_{|\tau - t| > 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|\overline{x} - \overline{y}|_{p_s}^{2s}}{|\tau - t|^{2+\beta}} |\tau - t|^{\beta + \frac{\alpha}{2s}} \mathrm{d}\tau \\ & + \int_{|\tau - t| > 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{1}{|\tau - u|^{1+\beta}} |t-u|^{\beta + \frac{\alpha}{2s}} \mathrm{d}\tau \lesssim_{\beta,\alpha} |\overline{x} - \overline{y}|_{p_s}^{\alpha}. \end{aligned}$$

Therefore $|\partial_t^\beta P_s * \mu_2(\overline{x}) - \partial_t^\beta P_s * \mu_2(\overline{y})| \le I_1 + I_2 + I_3 \lesssim_{\beta,\alpha} |\overline{x} - \overline{y}|_{p_s}^{\alpha}$, and this ends the study of *Case 1*.

Case 2: $\overline{x} = (x,t)$ and $\overline{y} = (y,t)$. To tackle this case, let us first rewrite the set R_2 as

$$R_2 = \Big[5B_1\big(x, |\overline{x} - \overline{y}|_{p_s}\big) \times \mathbb{R}\Big] \cap \Big[5B_1\big(y, |\overline{y} - \overline{x}|_{p_s}\big) \times \mathbb{R}\Big] = \big(5B_{1,x} \times \mathbb{R}\big) \cap \big(5B_{1,y} \times \mathbb{R}\big),$$

Continue rewriting R_2 as follows

$$\begin{aligned} R_2 &= \Big\{5B_{\overline{x}} \cup \big[(5B_{1,x} \times \mathbb{R}) \setminus 5B_{\overline{x}}\big]\Big\} \cap \Big\{5B_{\overline{y}} \cup \big[(5B_{1,y} \times \mathbb{R}) \setminus 5B_{\overline{y}}\big]\Big\} \\ &= \big(5B_{\overline{x}} \cap 5B_{\overline{y}}\big) \cup \Big\{5B_{\overline{x}} \cap \big[(5B_{1,y} \times \mathbb{R}) \setminus 5B_{\overline{y}}\big]\Big\} \\ &\quad \cup \Big\{5B_{\overline{y}} \cap \big[(5B_{1,x} \times \mathbb{R}) \setminus 5B_{\overline{x}}\big]\Big\} \\ &\quad \cup \Big\{\big[(5B_{1,x} \times \mathbb{R}) \setminus 5B_{\overline{x}}\big] \cap \big[(5B_{1,y} \times \mathbb{R}) \setminus 5B_{\overline{y}}\big]\Big\} \\ &=: R_{21} \cup R_{22} \cup R_{23} \cup R_{24}. \end{aligned}$$

Observe that in *Case 2* the real intervals $J_{\overline{x}}$ and $J_{\overline{y}}$ coincide. We name them J. Therefore,

$$\begin{aligned} R_{22} &:= 5B_{\overline{x}} \cap \big[(5B_{1,y} \times \mathbb{R}) \setminus 5B_{\overline{y}}\big] = (5B_{1,x} \times J) \cap \big[5B_{1,y} \times (\mathbb{R} \setminus J)\big] = \varnothing, \\ R_{23} &:= 5B_{\overline{y}} \cap \big[(5B_{1,y} \times \mathbb{R}) \setminus 5B_{\overline{y}}\big] = (5B_{1,y} \times J) \cap \big[5B_{1,x} \times (\mathbb{R} \setminus J)\big] = \varnothing, \end{aligned}$$

meaning that, in fact, $R_2 = R_{21} \cup R_{24}$. Observe also that R_{24} can be rewritten as

$$\begin{aligned} R_{24} :&= \big[(5B_{1,x} \times \mathbb{R}) \setminus 5B_{\overline{x}}\big] \cap \big[(5B_{1,y} \times \mathbb{R}) \setminus 5B_{\overline{y}}\big] \\ &= (5B_{1,x} \cap 5B_{1,y}) \times (\mathbb{R} \setminus J). \end{aligned}$$

Therefore, if χ_{21} and χ_{24} are the characteristic functions of R_{21} and R_{24}, we have, naming $\mu_{21} := \chi_{21}\mu$ and $\mu_{24} := \chi_{24}\mu$,

$$\begin{aligned} I_2 \le & \frac{1}{|\overline{x} - \overline{y}|_{p_s}^{\alpha}} \big|\partial_t^\beta P_s * \mu_{21}(\overline{x}) - \partial_t^\beta P_s * \mu_{21}(\overline{y})\big| \\ & + \frac{1}{|\overline{x} - \overline{y}|_{p_s}^{\alpha}} \big|\partial_t^\beta P_s * \mu_{24}(\overline{x}) - \partial_t^\beta P_s * \mu_{24}(\overline{y})\big| =: I_{21} + I_{24}. \end{aligned}$$

Hence, fixing $j \in \{1, 4\}$, begin by establishing the following estimate

$$\begin{aligned} & |\partial_t^\beta P_s * \mu_{2j}(\overline{x}) - \partial_t^\beta P_s * \mu_{2j}(\overline{y})| \\ & \quad = \left| \int \frac{P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(x,t)}{|\tau - t|^{1+\beta}} \mathrm{d}\tau - \int \frac{P_s * \mu_{2j}(y,\tau) - P_s * \mu_{2j}(y,t)}{|\tau - t|^{1+\beta}} \mathrm{d}\tau \right| \end{aligned}$$

$$\leq \int_{|\tau-t|\leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(x,t)|}{|\tau-t|^{1+\beta}} \mathrm{d}\tau$$
$$+ \int_{|\tau-t|\leq 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(y,\tau) - P_s * \mu_{2j}(y,t)|}{|\tau-t|^{1+\beta}} \mathrm{d}\tau$$
$$+ \int_{|\tau-t|> 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(x,t) - P_s * \mu_{2j}(y,\tau) + P_s * \mu_{2j}(y,t)|}{|\tau-t|^{1+\beta}} \mathrm{d}\tau$$
$$=: C_1 + C_2 + C_3.$$

Lemma 4.1 with $\eta = \beta$ yields $C_1 \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{\alpha}$ and $C_2 \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{\alpha}$, so we focus on C_3. Split it as follows

$$C_3 \leq \int_{|\tau-t|> 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,\tau) - P_s * \mu_{2j}(y,\tau)|}{|\tau-t|^{1+\beta}} \mathrm{d}\tau$$
$$+ \int_{|\tau-t|> 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{|P_s * \mu_{2j}(x,t) - P_s * \mu_{2j}(y,t)|}{|\tau-t|^{1+\beta}} \mathrm{d}\tau =: C_{31} + C_{32}.$$

First, let us deal with integral C_{32}. On the one hand, if $j = 1$, observe that for any $\overline{z} \in 2B_{\overline{x}}$, since $2B_{\overline{x}} \subset R_{21} \subset 5B_{\overline{x}}$, we can contain R_{21} into s-parabolic annuli centered at $\overline{z}$ and (exponentially decreasing) radii proportional to $|\overline{x}-\overline{y}|_{p_s}$. Hence, by [13, Lemma 2.2] and the upper s-parabolic growth of degree $n + 2s\beta + \alpha$ of μ, we deduce

$$|P_s * \mu_{21}(\overline{z})| \lesssim \int_{5B_{\overline{x}} \cap 5B_{\overline{y}}} \frac{\mathrm{d}\mu(\overline{w})}{|\overline{z}-\overline{w}|_{p_s}^{n}} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha}.$$

If $j = 4$, observe that $|P_s * \mu_{24}(x,t) - P_s * \mu_{2j}(y,t)| \leq |x-y| \, \|\nabla_x P_s * \mu_{24}\|_{\infty, 2B_{\overline{x}}}$. So for any $\overline{z} \in 2B_{\overline{x}}$, by Theorem 2.2 we obtain

$$|\nabla_x P_s * \mu_{24}(\overline{z})| \lesssim \int_{(5B_{\overline{x}} \cap 5B_{\overline{y}}) \times (\mathbb{R}\setminus J)} \frac{|z-w|}{|\overline{z}-\overline{w}|_{p_s}^{n+2}} \mathrm{d}\mu(\overline{w})$$
$$\lesssim |\overline{x}-\overline{y}|_{p_s} \int_{\mathbb{R}^{n+1}\setminus(5B_{p,\overline{x}} \cap 5B_{p,\overline{y}})} \frac{\mathrm{d}\mu(\overline{w})}{|\overline{z}-\overline{w}|_{p_s}^{n+2}} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha-1}, \quad \text{since } 2s\beta+\alpha < 2.$$

For the last inequality we can split, for example, the domain of integration into s-parabolic annuli centered at $\overline{z}$ with (exponentially increasing) radii proportional to $2|\overline{x}-\overline{y}|_{p_s}$. Then,

$$C_{32} \lesssim_{\beta,\alpha} |\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha} \int_{|\tau-t|> 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}} \frac{\mathrm{d}\tau}{|\tau-t|^{1+\beta}} \lesssim_{\beta} |\overline{x}-\overline{y}|_{p_s}^{\alpha}. \tag{18}$$

Regarding C_{31}, the points (x,τ) and (y,τ) belong to a temporal translate of $2B_{\overline{x}} \cap 2B_{\overline{y}}$ that does not intersect $2B_{\overline{x}} \cap 2B_{\overline{y}}$, since $|\tau-t| > 2^{2s}|\overline{x}-\overline{y}|_{p_s}^{2s}$. We call it $2B_{\overline{x}}^{\tau} \cap 2B_{\overline{y}}^{\tau}$. For each $j \in \{1,4\}$ and τ (and bearing in mind Theorem 2.2) we deduce

$$|P_s*\mu_{2j}(x,\tau) - P_s * \mu_{2j}(y,\tau)|$$
$$\leq \int_{2B_{\overline{x}}^{\tau} \cap 2B_{\overline{y}}^{\tau}} |P_s((x,\tau)-\overline{w}) - P_s((y,\tau)-\overline{w})| \mathrm{d}\mu(\overline{w})$$
$$+ \int_{R_{2j}\setminus(2B_{\overline{x}}^{\tau} \cap 2B_{\overline{y}}^{\tau})} |P_s((x,\tau)-\overline{w}) - P_s((y,\tau)-\overline{w})| \mathrm{d}\mu(\overline{w})$$
$$\lesssim \int_{2B_{\overline{x}}^{\tau}} \frac{\mathrm{d}\mu(\overline{w})}{|(x,\tau)-\overline{w}|_{p_s}^{n}} + \int_{2B_{\overline{y}}^{\tau}} \frac{\mathrm{d}\mu(\overline{w})}{|(y,\tau)-\overline{w}|_{p_s}^{n}} \tag{19}$$

$$+\,|x-y|\int_{R_{2j}\setminus(2B^\tau_{\overline{x}}\cap 2B^\tau_{\overline{y}})}|\nabla_x P_s((\widetilde{x},\tau)-\overline{w})|\mathrm{d}\mu(\overline{w})$$
$$\lesssim_{\beta,\alpha}|\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha}+|\overline{x}-\overline{y}|_{p_s}^2\int_{\mathbb{R}^{n+1}\setminus(2B^\tau_{\overline{x}}\cap 2B^\tau_{\overline{y}})}\frac{\mathrm{d}\mu(\overline{w})}{|(\widetilde{x},\tau)-\overline{w}|_{p_s}^{n+2}}, \tag{20}$$

where for both integrals of (19) we have split the domain of integration into (exponentially decreasing) s-parabolic annuli; while in the remaining term $\widetilde{x}$ belongs to the segment joining x and y. Observe also that in the last inequality we have used that the spatial distance between any two points of $R_{21}\setminus(2B^\tau_{\overline{x}}\cap 2B^\tau_{\overline{y}})$ and $R_{24}\setminus(2B^\tau_{\overline{x}}\cap 2B^\tau_{\overline{y}})$ is bounded by a multiple of $|x-y|$ and thus of $|\overline{x}-\overline{y}|_{p_s}$. Observe now that, if $\xi:=(x+y)/2$, we have

$$\begin{aligned}2B^\tau_{\overline{x}}\cap 2B^\tau_{\overline{y}}&=B\big((x,t+\tau),2|\overline{x}-\overline{y}|_{p_s}\big)\cap B\big((y,t+\tau),2|\overline{x}-\overline{y}|_{p_s}\big)\\&\supset B\big((\xi,t+\tau),|\overline{x}-\overline{y}|_{p_s}\big)=:\widehat{B}^\tau,\end{aligned}$$

meaning that

$$\mathbb{R}^{n+1}\setminus(2B^\tau_{\overline{x}}\cap 2B^\tau_{\overline{y}})\subset\mathbb{R}^{n+1}\setminus\widehat{B}^\tau.$$

Return to (20) and estimate the remaining integral by another one with the same integrand, but over the enlarged domain $\mathbb{R}^{n+1}\setminus\widehat{B}^\tau$. Afterwards, split the latter into s-parabolic annuli centered at $(\widetilde{x},\tau)$ and (exponentially increasing) radii proportional to $|\overline{x}-\overline{y}|_{p_s}/2$ and use that $2s\beta+\alpha<2$ so that

$$|P_s*\mu_{2j}(x,\tau)-P_s*\mu_{2j}(y,\tau)|\lesssim_{\beta,\alpha}|\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha}+\frac{|\overline{x}-\overline{y}|_{p_s}^2}{|\overline{x}-\overline{y}|_{p_s}^{2-2s\beta-\alpha}}\simeq|\overline{x}-\overline{y}|_{p_s}^{2s\beta+\alpha}.$$

Hence, similarly to (18) we deduce $C_{31}\lesssim_{\beta,\alpha}|\overline{x}-\overline{y}|_{p_s}^\alpha$, which means $I_2\leq I_{21}+I_{24}\lesssim_{\beta,\alpha}1$ and we are done with *Case 2*. This last estimate finally implies

$$\big|\partial_t^\beta P_s*(\chi_2\mu)(\overline{x})-\partial_t^\beta P_s*(\chi_2\mu)(\overline{y})\big|\lesssim_{\beta,\alpha}|\overline{x}-\overline{y}|_{p_s}^\alpha,$$

which means $I_2\lesssim_{\beta,\alpha}1$. So applying it to (17) we conclude that

$$\frac{|\partial_t^\beta P_s*\mu(\overline{x})-\partial_t^\beta P_s*\mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^\alpha}\leq I_{1,\overline{x}}+I_{1,\overline{y}}+I_2\lesssim_{\beta,\alpha}1,$$

and the desired s-parabolic Lip_α condition follows. □

5 The *s*-parabolic BMO and Lip_α caloric capacities

We are finally ready to introduce the s-parabolic BMO and Lip_α variants of the caloric capacities presented in [13, 14]. This section generalizes the concept to include a broader set of variants. The principal result will be that, in any case, such capacities will turn out to be comparable to a certain s-parabolic Hausdorff content. Moreover, we will be able to characterize removable sets for BMO_{p_s} and $\mathrm{Lip}_{\alpha,p_s}$ solutions of the Θ^s-equation in terms of the nullity of the respective capacities. In order to do so, we will need a fundamental lemma that we present before introducing the different capacities. The result below will characterize distributions supported on a compact set with finite d-dimensional Hausdorff measure that satisfy some growth property only for s-parabolic cubes small enough.

Lemma 5.1 *Let $d > 0$ and $E \subset \mathbb{R}^{n+1}$ be a compact set with $\mathcal{H}^d_{p_s}(E) < \infty$. Let T be a distribution supported on E with the property that there exists $0 < \ell_0 \leq \infty$ such that for any $R \subset \mathbb{R}^{n+1}$ s-parabolic cube with $\ell(R) \leq \ell_0$,*

$$|\langle T, \phi\rangle| \lesssim \ell(R)^d, \qquad \forall \phi \;\; \textit{admissible for} \;\; R.$$

Then, T is a signed measure satisfying

$$|\langle T, \psi\rangle| \lesssim \mathcal{H}^d_{p_s}(E)\|\psi\|_\infty, \qquad \forall \psi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1}).$$

Proof We follow the proof of [14, Lemma 6.2]. Let $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$ and $0 < \varepsilon \leq \ell_0/4$. Let Q_i, $i \in I_\varepsilon$ be a collection of s-parabolic cubes with $F \subset \bigcup_{i\in I_\varepsilon} Q_i$ with $\ell(Q_i) \leq \varepsilon$ and

$$\sum_{i\in I_\varepsilon} \ell(Q_i)^d \leq C\mathcal{H}^d_{p_s}(E) + \varepsilon.$$

Now cover each Q_i by a bounded number (depending on the dimension) of dyadic s-parabolic cubes $R_i^1, \dots, R_i^m$ with $\ell(R_i^j) \leq \ell(Q_i)/8$ and apply an s-parabolic version of Harvey-Polking's lemma (that admits an analogous proof, see [8, Lemma 3.1]) to obtain a collection of non-negative functions $\{\varphi_i\}_{i\in I_\varepsilon}$ with $\operatorname{supp}(\varphi_i) \subset 2Q_i$, $c\varphi_i$ admissible for $2Q_i$ and satisfying $\sum_{i\in I_\varepsilon} \varphi_i \equiv 1$ on $\bigcup_{i\in I_\varepsilon} Q_i \supset E$. Now we write

$$|\langle T, \psi\rangle| \leq \sum_{i\in I_\varepsilon} |\langle T, \varphi_i\psi\rangle|.$$

Proceeding as in [14, Lemma 6.2] it can be shown that

$$\eta_i := \frac{\varphi_i\psi}{\|\psi\|_\infty + \ell(Q_i)\|\nabla_x\psi\|_\infty + \ell(Q_i)^{2s}\|\partial_t\psi\|_\infty + \ell(Q_i)^2\|\Delta\psi\|_\infty}$$

is an admissible function for $2Q_i$ (up to a dimensional constant), with $\ell(2Q_i) \leq \ell_0/2$. Therefore, by the growth assumptions on T,

$$\begin{aligned}|\langle T, \psi\rangle| &\lesssim \sum_{i\in I_\varepsilon} \ell(Q_i)^d\big(\|\psi\|_\infty + \ell(Q_i)\|\nabla_x\psi\| + \ell(Q_i)^2\|\partial_t\psi\|_\infty + \ell(Q_i)^2\|\Delta\psi\|_\infty\big)\\ &\lesssim (\mathcal{H}^d_{p_s}(E) + \varepsilon)\big(\|\psi\|_\infty + \varepsilon\|\nabla_x\psi\| + \varepsilon^2\|\partial_t\psi\|_\infty + \varepsilon^2\|\Delta\psi\|_\infty\big),\end{aligned}$$

and making ε tend to 0, we deduce the result. □

5.1 The capacity $\Gamma_{\Theta^s,*}$

The first capacity we introduce is the BMO_{p_s} variant of the caloric capacity first defined in [14] for the usual heat equation.

Definition 5.1 Given $s \in (1/2, 1]$ and $E \subset \mathbb{R}^{n+1}$ compact set, define its BMO_{p_s}*-caloric capacity* as

$$\Gamma_{\Theta^s,*}(E) := \sup|\langle T, 1\rangle|,$$

where the supremum is taken among all distributions T with $\operatorname{supp}(T) \subset E$ and satisfying

$$\|\nabla_x P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1. \tag{21}$$

Such distributions will be called *admissible for* $\Gamma_{\Theta^s,*}(E)$.

Let us also introduce what we will understand as removable sets in this context:

Definition 5.2 A compact set $E \subset \mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with* BMO_{p_s}-$(1, \frac{1}{2s})$*-derivatives* if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|\nabla_x f\|_{*,p_s} < \infty, \qquad \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty,$$

satisfying the Θ^s-equation in $\Omega \setminus E$, also satisfies the previous equation in the whole Ω.

First, we shall prove that if T satisfies (21), then T has upper s-parabolic growth of degree $n+1$. In fact, we shall prove a stronger result:

Theorem 5.2 *Let $s \in (1/2, 1]$ and T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|\nabla_x P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim \ell(R)^{n+1}$.

Proof Let T, Q and φ be as above. Let R be an s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$ (if not, the result is trivial) and ϕ admissible function for R. Since P_s is the fundamental solution of the Θ^s-equation,

$$|\langle \varphi T, \phi\rangle| = |\langle \Theta^s P_s * T, \varphi\phi\rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi\rangle| + |\langle P_s * T, \partial_t(\varphi\phi)\rangle| =: I_1 + I_2.$$

Regarding I_2, observe that defining $\beta := 1 - \frac{1}{2s} \in (0, 1/2]$ we get

$$\partial_t(\varphi\phi) = c\, \partial_t^{1-\beta}\Big(\partial_t(\varphi\phi) *_t |t|^{-\beta}\Big),$$

for some constant c. The latter can be checked via the Fourier transform with respect to the t variable. Therefore, applying Theorem 3.1 we get

$$I_2 \simeq c|\langle \partial_t^{1-\beta} P_s * T, \partial_t(\varphi\phi) *_t |t|^{-\beta}\rangle| \lesssim \ell(R)^{n+2s(1-\beta)} = \ell(R)^{n+1}.$$

To study I_1 we distinguish whether if $s = 1$ or $s < 1$. If $s = 1$, Theorem 3.2 yields

$$I_1 = |\langle \Delta W * T, \varphi\phi\rangle| = |\langle \nabla_x W * T, \nabla_x(\varphi\phi)\rangle| \lesssim \ell(R)^{n+1}.$$

Recall that the operator $(-\Delta)^s$ can be rewritten as

$$(-\Delta)^s(\cdot) \simeq \sum_{i=1}^n \partial_{x_i}\left(\frac{1}{|x|^{n+2s-2}}\right) *_n \partial_{x_i}(\cdot),$$

where $*_n$ indicates that the convolution is taken with respect the first n spatial variables. Therefore, by Theorem 3.4, since $s \in (1/2, 1)$, we have

$$\begin{aligned} I_1 &\lesssim \sum_{i=1}^n \left|\left\langle \partial_{x_i} P_s * T, \partial_{x_i}\left(\frac{1}{|x|^{n+2s-2}}\right) *_n (\varphi\phi)\right\rangle\right| \\ &= \sum_{i=1}^n \left|\left\langle \partial_{x_i} P_s * T, \partial_{x_i}[\mathcal{I}_{2-2s}^n(\varphi\phi)]\right\rangle\right| \lesssim \ell(R)^{n+1}, \end{aligned}$$

and we are done. $\square$

Remark 5.1 Let us observe that in the particular case in which T is compactly supported, we may simply convey that $Q := \mathbb{R}^{n+1}$ and $\varphi \equiv 1$ so that we deduce

$$|\langle T, \phi\rangle| \lesssim \ell(R)^{n+1},$$

for any R s-parabolic cube and ϕ admissible function for R. Therefore, bearing in mind Lemma 5.1, if $E \subset \mathbb{R}^{n+1}$ is a compact set with $\mathcal{H}^{n+1}_{p_s}(E) = 0$ and T is a distribution supported on E and satisfying the BMO_{p_s} estimates of Theorem 5.2, choosing $\ell_0 := \infty$ we get $T \equiv 0$.

Theorem 5.3 *For any $s \in (1/2, 1]$ and $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\Gamma_{\Theta^s,*}(E) \approx \mathcal{H}^{n+1}_{\infty,p_s}(E).$$

Proof Let us first prove

$$\Gamma_{\Theta^s,*}(E) \lesssim \mathcal{H}^{n+1}_{\infty,p_s}(E). \tag{22}$$

Proceed by fixing $\varepsilon > 0$ and $\{A_k\}_k$ a collection of sets in $\mathbb{R}^{n+1}$ that cover E such that

$$\sum_{k=1}^{\infty} \text{diam}_{p_s}(A_k)^{n+1} \leq \mathcal{H}^{n+1}_{\infty,p_s}(E) + \varepsilon.$$

Now, for each k let Q_k an open s-parabolic cube centered at some point $a_k \in A_k$ with side length $\ell(Q_k) = \text{diam}_{p_s}(A_k)$, so that $E \subset \bigcup_k Q_k$. Apply the compactness of E and [8, Lemma 3.1] to consider $\{\varphi_k\}_{k=1}^N$ a collection of smooth functions satisfying, for each k: $0 \leq \varphi_k \leq 1$, $\text{supp}(\varphi_k) \subset 2Q_k$, $\sum_{k=1}^N \varphi_k = 1$ in $\bigcup_{k=1}^N Q_k$ and also $\|\nabla_x\varphi_k\|_\infty \leq \ell(2Q_k)^{-1}$, $\|\partial_t\varphi_k\| \leq \ell(2Q_k)^{-2s}$. Hence, by Theorem 5.2, if T is any distribution admissible for $\Gamma_{\Theta^s,*}(E)$,

$$|\langle T, 1\rangle| = \left|\sum_{k=1}^N \langle T, \varphi_k\rangle\right| \lesssim \sum_{k=1}^N \ell(2Q_k)^{n+1} \simeq \sum_{k=1}^N \text{diam}_{p_s}(A_k)^{n+1} \leq \mathcal{H}^{n+1}_{\infty,p_s}(E) + \varepsilon.$$

Since this holds for any T and $\varepsilon > 0$ can be arbitrarily small, (22) follows.

For the lower bound we will apply (an s-parabolic version of) Frostman's lemma [15, Theorem 8.8], which can be proved using an s-parabolic dyadic lattice, as it is presented in the proof of [14, Lemma 5.1]. Assume then $\mathcal{H}^{n+1}_{\infty,p_s}(E) > 0$ and consider a non trivial positive Borel regular measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+1}_{\infty,p_s}(E)$ and $\mu(B(\overline{x}, r)) \leq r^{n+1}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. If we prove that

$$\|\nabla_x P_s * \mu\|_{*,p_s} \lesssim 1 \qquad \text{and} \qquad \|\partial_t^{\frac{1}{2s}} P_s * \mu\|_{*,p_s} \lesssim 1,$$

we will be done, since this will imply $\Gamma_{\Theta^s,*}(E) \gtrsim \langle \mu, 1\rangle = \mu(E) \gtrsim \mathcal{H}^{n+1}_{\infty,p_s}$. But by Lemma 4.2 we already have $\|\partial_t^{\frac{1}{2s}} P_s * \mu\|_{*,p_s} \lesssim 1$, so we are only left with the BMO_{p_s} norm of $\nabla_x P_s * \mu$. Thus, let us fix an s-parabolic ball $B(\overline{x}_0, r)$ and consider the characteristic function χ_{2B} associated to $2B$. Denote also $\chi_{2B^c} = 1 - \chi_{2B}$. In this setting, we pick

$$c_B := \nabla_x P_s * (\chi_{2B^c}\mu)(\overline{x}_0).$$

Using Theorem 2.2 it easily follows that this last expression is well-defined. Let us now estimate $\|\nabla_x P_s * \mu\|_{*,p_s}$,

$$\begin{aligned}\frac{1}{|B|}\int_B |\nabla_x P_s * \mu(\overline{y}) - c_B|\mathrm{d}\overline{y} \\ &\leq \frac{1}{|B|}\int_B \bigg(\int_{2B} |\nabla_x P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\overline{y} \\ &+ \frac{1}{|B|}\int_B \bigg(\int_{\mathbb{R}^{n+1}\setminus 2B} |\nabla_x P_s(\overline{y}-\overline{z}) - \nabla_x P_s(\overline{x}_0-\overline{z})|\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\overline{y} =: I_1 + I_2.\end{aligned}$$

To deal with I_1 we first notice that by Theorem 2.2 and Tonelli's theorem we have

$$I_1 \lesssim \frac{1}{|B|}\int_{2B}\bigg(\int_B \frac{1}{|\overline{y}-\overline{z}|_{p_s}^{n+1}}\mathrm{d}\overline{y}\bigg)\mathrm{d}\mu(\overline{z}).$$

Writing $B = B_0 \times I_0 \subset \mathbb{R}^n \times \mathbb{R}$, $\overline{y} = (y,t)$, $\overline{z} = (z,u)$ and choosing $0 < \varepsilon < 2s-1$, integration in polar coordinates yields

$$I_1 \lesssim \frac{1}{|B|}\int_{2B}\bigg(\int_{B_0}\frac{\mathrm{d}y}{|y-z|^{n-\varepsilon}}\int_{I_0}\frac{dt}{|t-u|^{\frac{1+\varepsilon}{2s}}}\bigg)\mathrm{d}\mu(\overline{z}) \lesssim \frac{1}{|B|}\big(r^{\varepsilon}(r^{2s})^{1-\frac{1+\varepsilon}{2s}}\big)\mu(2B) \lesssim 1.$$

Regarding I_2, we name $\overline{x} := \overline{x}_0 - \overline{z}$ and $\overline{x}' := \overline{y} - \overline{z}$, and observe that $|\overline{x}-\overline{x}'|_{p_s} \leq |\overline{x}|_{p_s}/2$. Hence, we apply the fourth estimate in Theorem 2.2 with $2\zeta = 1$ since $s > 1/2$, and obtain

$$\begin{aligned} I_2 &\lesssim \frac{1}{|B|}\int_B\bigg(\int_{\mathbb{R}^{n+1}\setminus 2B}\frac{|\overline{y}-\overline{x}_0|_{p_s}}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2}}\mathrm{d}\mu(\overline{z})\bigg)\mathrm{d}\overline{y} \leq r\int_{\mathbb{R}^{n+1}\setminus 2B}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2}} \\ &= r^{2\zeta}\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2}} \lesssim r\sum_{j=1}^{\infty}\frac{(2^{j+1}r)^{n+1}}{(2^jr)^{n+2}} \lesssim \sum_{j=1}^{\infty}\frac{1}{2^j} \lesssim 1,\end{aligned}$$

and we are done. □

Theorem 5.4 *Let $s \in (1/2, 1]$. A compact set $E \subset \mathbb{R}^{n+1}$ is removable for s-caloric functions with BMO_{p_s}-$(1, \frac{1}{2s})$-derivatives if and only if $\Gamma_{\Theta^s,*}(E) = 0$.*

Proof Fix $E \subset \mathbb{R}^{n+1}$ compact set and begin by assuming that is removable. Now pick T admissible for $\Gamma_{\Theta^s,*}(E)$ and observe that defining $f := P_s * T$, we have $\|\nabla_x f\|_{*,p_s} < \infty$, $\|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty$ and $\Theta^s f = 0$ on $\mathbb{R}^{n+1} \setminus E$. So by hypothesis $\Theta^s f = 0$ in $\mathbb{R}^{n+1}$ and therefore $T \equiv 0$. Since T was an arbitrary admissible distribution for $\Gamma_{\Theta^s,*}(E)$, we deduce that $\Gamma_{\Theta^s,*}(E) = 0$.

Let us now assume $\Gamma_{\Theta^s,*}(E) = 0$ and prove the removability of E. Notice that by Theorem 5.3 we get $\mathcal{H}^{n+1}_{\infty,p_s}(E) = 0$ and thus, by [15, Lemma 4.6], we have $\mathcal{H}^{n+1}_{p_s}(E) = 0$. With this in mind, fix Ω any open set and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ any function with $\|\nabla_x f\|_{*,p_s} < \infty$, $\|\partial_t^{\frac{1}{2s}} f\|_{*,p_s} < \infty$ and $\Theta^s f = 0$ on $\Omega \setminus E$. We will assume $\Theta^s f \neq 0$ in Ω and reach a contradiction. The case $\Omega \cap E = \varnothing$ is trivial, so we assume $\Omega \cap E \neq \varnothing$. Define the distribution

$$T := \frac{\Theta^s f}{\|\nabla_x f\|_{*,p_s} + \|\partial_t^{\frac{1}{2s}} f\|_{*,p_s}},$$

which is such that $\|\nabla_x P_s * T\|_{*,p_s} \leq 1$, $\|\partial_t^{\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1$ and $\mathrm{supp}(T) \subset E \cup \Omega^c$. Since $T \neq 0$ in Ω, there exists Q s-parabolic cube with $4Q \subset \Omega$ so that $T \neq 0$ in Q. Observe that $Q \cap E \neq \varnothing$. Then, by definition, there is φ test function supported on Q with $\langle T, \varphi\rangle > 0$. Consider

$$\widetilde{\varphi} := \frac{\varphi}{\|\varphi\|_\infty + \ell(Q)\|\nabla_x\varphi\|_\infty + \ell(Q)^{2s}\|\partial_t\varphi\|_\infty + \ell(Q)^2\|\Delta\varphi\|_\infty},$$

so that $\widetilde{\varphi}$ is admissible for Q. Apply Theorem 5.2 to deduce that $\widetilde{\varphi}T$ has upper s-parabolic growth of degree $n+1$ for cubes R with $\ell(R) \leq \ell(Q)$. Apply Lemma 5.1 to $\widetilde{\varphi}T$ with the compact set $\overline{Q} \cap E$, $\ell_0 := \ell(Q)$ and $d := n+1$. Then,

$$|\langle \widetilde{\varphi}T, \psi \rangle| = 0, \qquad \forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1}),$$

since $\mathcal{H}^{n+1}_{p_s}(\overline{Q} \cap E) = 0$. This would imply $\widetilde{\varphi}T \equiv 0$, which is impossible, since $\langle \varphi, T \rangle > 0$. Therefore $\Theta^s f = 0$ in Ω, and by the arbitrariness of Ω and f we are done. □

5.2 The capacity $\Gamma_{\Theta^s,\alpha}$

We shall now present an s-parabolic Lip_α variant of the caloric capacity presented above.

Definition 5.3 Given $s \in (1/2, 1]$, $\alpha \in (0,1)$ and $E \subset \mathbb{R}^{n+1}$ compact set, define its Lip_{α,p_s}-*caloric capacity* as

$$\Gamma_{\Theta^s,\alpha}(E) := \sup |\langle T, 1 \rangle|,$$

where the supremum is taken among all distributions T with $\text{supp}(T) \subset E$ and satisfying

$$\|\partial_{x_i} P_s * T\|_{\text{Lip}_{\alpha,p_s}} \leq 1, \quad \forall i = 1, \dots, n, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{\text{Lip}_{\alpha,p_s}} \leq 1.$$

Such distributions will be called *admissible for* $\Gamma_{\Theta^s,\alpha}(E)$.

Definition 5.4 A compact set $E \subset \mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with* Lip_{α,p_s}-$(1, \frac{1}{2s})$-*derivatives* if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|\nabla_x f\|_{\text{Lip}_{\alpha,p_s}} < \infty, \qquad \|\partial_t^{\frac{1}{2s}} f\|_{\text{Lip}_{\alpha,p_s}} < \infty,$$

satisfying the Θ^s-equation in $\Omega \setminus E$, also satisfies the previous equation in the whole Ω.

As in the s-parabolic BMO case, if T is a compactly supported distribution satisfying the required normalization conditions, T will present upper s-parabolic growth of degree $n+1+\alpha$. In fact, the following result holds:

Theorem 5.5 *Let $s \in (1/2, 1]$, $\alpha \in (0, 2s-1)$ and T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|\partial_{x_i} P_s * T\|_{Lip_{\alpha,p_s}} \leq 1, \quad \forall i = 1, \dots, n, \qquad \|\partial_t^{\frac{1}{2s}} P_s * T\|_{Lip_{\alpha,p_s}} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ admissible for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi \rangle| \lesssim_\alpha \ell(R)^{n+1+\alpha}$.

Proof Let T, Q and φ be as above. Let us also consider R s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$ and ϕ admissible function for R. We proceed as in the proof of Theorem 5.2 to obtain

$$|\langle \varphi T, \phi \rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi \rangle| + |\langle P_s * T, \partial_t(\varphi\phi) \rangle| =: I_1 + I_2.$$

Regarding I_2, we now define define $\beta := 1 - \frac{1}{2s}$ and observe that $2s\beta = 2s - 1 > \alpha$, so applying Theorem 3.1 we get $I_2 \lesssim_\alpha \ell(R)^{n+1+\alpha}$. The study of I_1 is also analogous to that done in Theorem 5.2. The case $s = 1$ follows in exactly the same way by Theorem 3.2, and if $s \in (1/2, 1)$ we also have

$$I_1 \lesssim \sum_{i=1}^n \left| \left\langle \partial_{x_i} P_s * T, \partial_{x_i} [\mathcal{I}^n_{2-2s}(\varphi\phi)] \right\rangle \right|.$$

So by Theorem 3.4 and condition $\alpha < 2s - 1$ we deduce the desired result. □

Theorem 5.6 *For any* $s \in (1/2, 1]$, $\alpha \in (0, 2s-1)$ *and* $E \subset \mathbb{R}^{n+1}$ *compact set,*

$$\Gamma_{\Theta^s,\alpha}(E) \approx_\alpha \mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E).$$

Proof For the upper bound we proceed analogously as we have done in the proof of Theorem 5.3, using now the growth restriction given by Theorem 5.5. So we focus on the lower bound, which will also rely on Frostman's lemma. Assume then $\mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E) > 0$ and consider a non trivial positive Borel measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+1+\alpha}_{\infty,p_s}(E)$ and $\mu(B(\overline{x}, r)) \leq r^{n+1+\alpha}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. It is enough to check

$$\|\partial_{x_i} P_s * \mu\|_{\text{Lip}_\alpha, p_s} \lesssim_\alpha 1, \quad \forall i = 1, \ldots, n \qquad \text{and} \qquad \|\partial_t^{\frac{1}{2s}} P_s * \mu\|_{\text{Lip}_\alpha, p_s} \lesssim_\alpha 1.$$

Notice that the right inequality follows directly from Lemma 4.3 with $\beta := \frac{1}{2s}$, so we just focus on controlling the s-parabolic Lip_α seminorm of the spatial derivatives of $P_s * \mu$. Fix $i = 1, \ldots, n$ and choose any $\overline{x}, \overline{y} \in \mathbb{R}^{n+1}$ with $\overline{x} \neq \overline{y}$. Consider the following partition

$$\begin{aligned} R_1 &:= \big\{\overline{z} \,:\, |\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2\big\} \cup \big\{z \,:\, |\overline{y}-\overline{x}|_{p_s} \leq |\overline{y}-\overline{z}|_{p_s}/2\big\}, \\ R_2 &:= \mathbb{R}^{n+1} \setminus R_1 = \big\{z \,:\, |\overline{x}-\overline{y}|_{p_s} > |\overline{x}-\overline{z}|_{p_s}/2\big\} \cap \big\{z \,:\, |\overline{y}-\overline{x}|_{p_s} > |\overline{y}-\overline{z}|_{p_s}/2\big\}, \end{aligned}$$

with their corresponding characteristic functions χ_1, χ_2 respectively. This way, we have

$$\begin{aligned} &\frac{|\partial_{x_i} P_s * \mu(\overline{x}) - \partial_{x_i} P_s * \mu(\overline{y})|}{|\overline{x}-\overline{y}|^\alpha_{p_s}} \\ &\quad \leq \frac{1}{|\overline{x}-\overline{y}|^\alpha_{p_s}} \int_{|\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2} |\partial_{x_i} P_s(\overline{x}-\overline{z}) - \partial_{x_i} P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) \\ &\qquad + \frac{1}{|\overline{x}-\overline{y}|^\alpha_{p_s}} \int_{|\overline{y}-\overline{x}|_{p_s} \leq |\overline{y}-\overline{z}|_{p_s}/2} |\partial_{x_i} P_s(\overline{x}-\overline{z}) - \partial_{x_i} P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) \\ &\qquad + \frac{1}{|\overline{x}-\overline{y}|^\alpha_{p_s}} \int_{R_2} |\partial_{x_i} P_s(\overline{x}-\overline{z}) - \partial_{x_i} P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) =: I_1 + I_2 + I_3. \end{aligned}$$

Regarding I_1, apply the fourth estimate of Theorem 2.2 to obtain

$$I_1 \lesssim \frac{1}{|\overline{x}-\overline{y}|^\alpha_{p_s}} \int_{|\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2} \frac{|\overline{x}-\overline{y}|_{p_s}}{|\overline{x}-\overline{z}|^{n+2}_{p_s}} \mathrm{d}\mu(\overline{z}).$$

Split the previous domain of integration into the s-parabolic annuli

$$A_j := 2^{j+1} B\big(\overline{x}, |\overline{x}-\overline{y}|_{p_s}\big) \setminus 2^j B\big(\overline{x}, |\overline{x}-\overline{y}|_{p_s}\big), \qquad \text{for} \qquad j \geq 1,$$

and use that μ has upper parabolic growth of degree $n+1+\alpha$ to deduce

$$\begin{aligned} I_1 &\lesssim \frac{1}{|\overline{x}-\overline{y}|^{\alpha-1}_{p_s}} \sum_{j=1}^\infty \int_{A_j} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|^{n+2}_{p_s}} \\ &\lesssim \frac{1}{|\overline{x}-\overline{y}|^{\alpha-1}_{p_s}} \sum_{j=1}^\infty \frac{(2^{j+1}|\overline{x}-\overline{y}|_{p_s})^{n+1+\alpha}}{(2^j|\overline{x}-\overline{y}|_{p_s})^{n+2}} \\ &\lesssim \sum_{j=1}^\infty \frac{1}{2^{(1-\alpha)j}} \lesssim_\alpha 1. \end{aligned}$$

The study of I_2 is analogous interchanging the roles of $\overline{x}$ and $\overline{y}$. Finally, for I_3, we apply the first estimate of Theorem 2.2 so that

$$\begin{aligned}
I_3 &\lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \int_{R_2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+1}} \\
&+ \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \int_{R_2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|_{p_s}^{n+1}} \\
&\leq \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \int_{|\overline{x}-\overline{y}|_{p_s} > |\overline{x}-\overline{z}|_{p_s}/2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+1}} \\
&+ \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \int_{|\overline{y}-\overline{x}|_{p_s} > |\overline{y}-\overline{z}|_{p_s}/2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|_{p_s}^{n+1}} \\
&=: I_{31} + I_{32}.
\end{aligned}$$

Concerning I_{31}, split the domain of integration into the (decreasing) s-parabolic annuli

$$\widetilde{A}_j := 2^{-j} B\big(\overline{x}, |\overline{x}-\overline{y}|_{p_s}\big) \setminus 2^{-j-1} B\big(\overline{x}, |\overline{x}-\overline{y}|_{p_s}\big), \qquad \text{for} \qquad j \geq -1.$$

Thus, in this case we have

$$\begin{aligned}
I_{31} &\lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \sum_{j=-1}^{\infty} \int_{\widetilde{A}_j} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|_{p_s}^{n+1}} \\
&\lesssim \frac{1}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \sum_{j=-1}^{\infty} \frac{(2^{-j}|\overline{x}-\overline{y}|_{p_s})^{n+1+\alpha}}{(2^{-j-1}|\overline{x}-\overline{y}|_{p_s})^{n+1}} \\
&\lesssim \sum_{j=-1}^{\infty} \frac{1}{2^{\alpha j}} \lesssim_{\alpha} 1.
\end{aligned}$$

On the other hand, for I_{32} we apply the same reasoning but using the partition given by

$$\widetilde{A}'_j := 2^{-j} B_p\big(\overline{y}, |\overline{y}-\overline{x}|_{p_s}\big) \setminus 2^{-j-1} B_p\big(\overline{y}, |\overline{y}-\overline{x}|_{p_s}\big), \qquad \text{for} \qquad j \geq -1,$$

yielding also $I_{32} \lesssim 1$. Combining the estimates obtained for I_1, I_2 and I_3 we deduce

$$\frac{|\partial_{x_i} P_s * \mu(\overline{x}) - \partial_{x_i} P_s * \mu(\overline{y})|}{|\overline{x}-\overline{y}|_{p_s}^{\alpha}} \lesssim_{\alpha} 1,$$

and since the (different) points $\overline{x}$ and $\overline{y}$ were arbitrarily chosen, we deduce the desired s-parabolic Lip_α condition. □

Theorem 5.7 *Let $s \in (1/2, 1]$ and $\alpha \in (0, 2s-1)$. A compact set $E \subset \mathbb{R}^{n+1}$ is removable for s-caloric functions with $\mathrm{Lip}_{\alpha,p_s}$-$(1, \frac{1}{2s})$-derivatives if and only if $\Gamma_{\Theta^s,\alpha}(E) = 0$.*

Proof The proof is completely analogous to that of Theorem 5.4, now using Theorems 5.5 and 5.6, as well as Lemma 5.1 with $d := n + 1 + \alpha$. □

5.3 The capacity $\gamma^{\sigma}_{\Theta^s,*}$

Now, we shall present the BMO_{p_s} variant of the capacities presented in [13, §4 & §7]. To be precise, in the aforementioned reference, Mateu and Prat work with the normalization conditions

$$\|(-\Delta)^{s-\frac{1}{2}} P_s * T\|_{\infty} \leq 1, \qquad \|\partial_t^{1-\frac{1}{2s}} P_s * T\|_{*,p_s} \leq 1,$$

allowing $s \in [1/2, 1)$. In our case we will deal with its s-parabolic BMO variant and we define it more generally as follows:

Definition 5.5 Given $s \in (0, 1]$, $\sigma \in [0, s)$ and $E \subset \mathbb{R}^{n+1}$ compact set, define its Δ^σ-BMO_{p_s}-*caloric capacity* as

$$\gamma^\sigma_{\Theta^s,*}(E) := \sup |\langle T, 1\rangle|,$$

where the supremum is taken among all distributions T with $\text{supp}(T) \subset E$ and satisfying

$$\|(-\Delta)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1.$$

Such distributions will be called *admissible for* $\gamma^\sigma_{\Theta^s,*}(E)$.

Definition 5.6 Let $s \in (0, 1]$ and $\sigma \in [0, s)$. A compact set $E \subset \mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with*BMO_{p_s}-$(\sigma, \sigma/s)$-*Laplacian* if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|(-\Delta)^\sigma f\|_{*,p_s} < \infty, \qquad \|\partial_t^{\sigma/s} f\|_{*,p_s} < \infty,$$

satisfying the Θ^s-equation in $\Omega \setminus E$, also satisfies the previous equation in the whole Ω. If $\sigma = 0$, we will also say that E is *removable for* BMO_{p_s} *s-caloric functions*.

Firstly, we shall prove that if T is a compactly supported distribution satisfying the expected normalization conditions, then T has upper s-parabolic growth of degree $n+2s-2\sigma$. In fact, we prove a stronger result:

Theorem 5.8 *Let $s \in (0, 1]$, $\sigma \in [0, s)$ and T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|(-\Delta)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim_\sigma \ell(R)^{n+2\sigma}$.

Proof Let T, Q and φ be as above, as well as R s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$, and ϕ admissible function for R. We already know, in light of the proof of Theorem 5.2,

$$|\langle \varphi T, \phi\rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi\rangle| + |\langle P_s * T, \partial_t(\varphi\phi)\rangle| =: I_1 + I_2.$$

For I_1, simply apply Theorem 3.3 with $\beta := s - \sigma$ so that

$$I_1 = |\langle (-\Delta)^\sigma P_s * T, (-\Delta)^{s-\sigma}(\varphi\phi)\rangle| \lesssim_\sigma \ell(R)^{n+2\sigma}$$

Regarding I_2, if $\sigma > 0$, observe that defining $\beta := 1 - \sigma/s \in (0, 1)$ we get

$$\partial_t(\varphi\phi) \simeq_\sigma \partial_t^{1-\beta}\Big(\partial_t(\varphi\phi) *_t |t|^{-\beta}\Big),$$

so by Theorem 3.1 we are done. If $\sigma = 0$, we simply have

$$\begin{aligned} I_2 = |\langle P_s * T - (P_s * T)_R, \partial_t(\varphi\phi)\rangle| &\leq \int_{Q\cap R} \big|P_s * T(\overline{x}) - (P * T)_R\big|\big|\partial_t(\varphi\phi)(\overline{x})\big| \mathrm{d}\overline{x} \\ &\leq \ell(R)^{-2s} \int_R \big|P * T(\overline{x}) - (P * T)_R\big| \mathrm{d}\overline{x} \leq \ell(R)^{-2s}\ell(R)^{n+2s}\|P_s * T\|_{*,p_s} \leq \ell(R)^n. \end{aligned}$$

□

Theorem 5.9 *For any $s \in (0,1]$, $\sigma \in [0,s)$ and $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\gamma^{\sigma}_{\Theta^s,*}(E) \approx_\sigma \mathcal{H}^{n+2\sigma}_{\infty,p_s}(E).$$

Proof Again, for the upper bound we proceed analogously as in the proof of Theorem 5.3, using now Theorem 5.8. For the lower bound, we apply Frostman's lemma. Assume then $\mathcal{H}^{n+2\sigma}_{\infty,p_s}(E) > 0$ and consider a non trivial positive Borel measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+2\sigma}_{\infty,p_s}(E)$ and $\mu(B(\overline{x},r)) \leq r^{n+2\sigma}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. We have to prove

$$\|(-\Delta)^\sigma P_s * T\|_{*,p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{*,p_s} \leq 1,$$

If $\sigma > 0$, by Lemma 4.2 with $\beta := \sigma/s$ we already have $\|\partial_t^{\sigma/s} P_s * \mu\|_{*,p_s} \lesssim_\sigma 1$. So we are left to control the BMO_{p_s} norm of $(-\Delta)^\sigma P_s * \mu$ for $\sigma \in [0,s)$. Thus, let us fix an s-parabolic ball $B(\overline{x}_0, r)$ and consider the characteristic function χ_{2B} associated to $2B$. Set also $\chi_{2B^c} = 1 - \chi_{2B}$. In this setting, we pick

$$c_B := (-\Delta)^\sigma P_s * (\chi_{2B^c}\mu)(\overline{x}_0).$$

Using Theorem 2.3 it easily follows that this last expression is well-defined. We estimate $\|(-\Delta)^\sigma P_s * \mu\|_{*,p_s}$ using the previous constant:

$$\begin{aligned}
&\frac{1}{|B|}\int_B |(-\Delta)^\sigma P_s * \mu(\overline{y}) - c_B|\mathrm{d}\overline{y} \\
&\quad \leq \frac{1}{|B|}\int_B \left(\int_{2B} |(-\Delta)^\sigma P_s(\overline{y}-\overline{z})|\mathrm{d}\mu(\overline{z})\right)\mathrm{d}\overline{y} \\
&\qquad + \frac{1}{|B|}\int_B \left(\int_{\mathbb{R}^{n+1}\setminus 2B} |(-\Delta)^\sigma P_s(\overline{y}-\overline{z}) - (-\Delta)^\sigma P_s(\overline{x}_0-\overline{z})|\mathrm{d}\mu(\overline{z})\right)\mathrm{d}\overline{y} =: I_1 + I_2.
\end{aligned}$$

To deal with I_1, notice that by Theorem 2.3, choosing $0 < \varepsilon < 2(s-\sigma)$ and arguing as in Theorem 5.3 we have

$$I_1 \lesssim_\sigma \frac{1}{|B|}\int_{2B}\left(\int_B \frac{\mathrm{d}\overline{y}}{|\overline{y}-\overline{z}|_{p_s}^{n+2\sigma}}\right)\mathrm{d}\mu(\overline{z}) \lesssim \frac{1}{|B|}\big(r^\varepsilon (r^{2s})^{1-\frac{\varepsilon+2\sigma}{2s}}\big)\mu(2B) \lesssim 1$$

by the $n+2\sigma$ growth of μ. Regarding I_2, notice that naming $\overline{x} := \overline{x}_0 - \overline{z}$ and $\overline{x}' := \overline{y} - \overline{z}$, we have $|\overline{x} - \overline{x}'|_{p_s} \leq |\overline{x}|_{p_s}/2$ so we can apply the fifth estimate of Theorem 2.3, that implies

$$I_2 \lesssim_\sigma \frac{1}{|B|}\int_B \left(\int_{\mathbb{R}^{n+1}\setminus 2B} \frac{|\overline{y}-\overline{x}_0|_{p_s}^{2\zeta}}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2\sigma+2\zeta}}\mathrm{d}\mu(\overline{z})\right)\mathrm{d}\overline{y} \leq r^{2\zeta}\int_{\mathbb{R}^{n+1}\setminus 2B} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{z}-\overline{x}_0|_{p_s}^{n+2\sigma+2\zeta}} \lesssim_\sigma 1,$$

again by the by the $n+2\sigma$ growth of μ. □

Theorem 5.10 *Let $s \in (0,1]$ and $\sigma \in [0,s)$. A compact set $E \subset \mathbb{R}^{n+1}$ is removable for s-caloric functions with BMO_{p_s}-$(\sigma, \sigma/s)$-Laplacian if and only if $\gamma^\sigma_{\Theta^s,*}(E) = 0$.*

Proof The proof is analogous to that of Theorem 5.4, applying Theorems 5.8, 5.9 and Lemma 5.1 with $d := n + 2\sigma$. □

5.4 The capacity $\gamma^\sigma_{\Theta^s,\alpha}$

We define now a capacity with an s-parabolic Lip_α normalization condition.

Definition 5.7 Given $\alpha \in (0,1)$, $s \in (0,1]$, $\sigma \in [0,s)$ and $E \subset \mathbb{R}^{n+1}$ compact set, define its Δ^σ-Lip_{α,p_s}-*caloric capacity* as

$$\gamma^{\sigma}_{\Theta^s,\alpha}(E) := \sup |\langle T, 1\rangle|,$$

where the supremum is taken among all distributions T with $\text{supp}(T) \subset E$ and satisfying

$$\|(-\Delta)^\sigma P_s * T\|_{\text{Lip}_{\alpha},p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{\text{Lip}_{\alpha},p_s} \leq 1.$$

Such distributions will be called *admissible for* $\gamma^{\sigma}_{\Theta^s,\alpha}(E)$.

Definition 5.8 Let $\alpha \in (0,1)$, $s \in (0,1]$ and $\sigma \in [0,s)$. A compact set $E \subset \mathbb{R}^{n+1}$ is said to be *removable for s-caloric functions with* Lip_{α,p_s}-$(\sigma, \sigma/s)$-*Laplacian* if for any open subset $\Omega \subset \mathbb{R}^{n+1}$, any function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$\|(-\Delta)^\sigma f\|_{\text{Lip}_{\alpha},p_s} < \infty, \qquad \|\partial_t^{\sigma/s} f\|_{\text{Lip}_{\alpha},p_s} < \infty,$$

satisfying the Θ^s-equation in $\Omega \setminus E$, also satisfies the previous equation in the whole Ω. If $\sigma = 0$, we will also say that E is *removable for* Lip_{α,p_s} *s-caloric functions*.

If T is a compactly supported distribution satisfying the above properties, then T presents upper s-parabolic growth of degree $n + 2\sigma + \alpha$. As in §5.2, the following result will only be valid for a certain range of values of α, dependent on s and σ.

Theorem 5.11 *Let $s \in (0,1]$, $\sigma \in [0,s)$ and $\alpha \in (0,1)$ with $\alpha < 2s - 2\sigma$. Let T be a distribution in $\mathbb{R}^{n+1}$ with*

$$\|(-\Delta)^\sigma P_s * T\|_{Lip_{\alpha},p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{Lip_{\alpha},p_s} \leq 1.$$

Let Q be a fixed s-parabolic cube and φ an admissible function for Q. Then, if R is any s-parabolic cube with $\ell(R) \leq \ell(Q)$ and ϕ is admissible for R, we have $|\langle \varphi T, \phi\rangle| \lesssim \ell(R)^{n+2\sigma+\alpha}$.

Proof Let T, Q and φ be as above, as well as R s-parabolic cube with $\ell(R) \leq \ell(Q)$ and $R \cap Q \neq \varnothing$, and ϕ admissible function for R. Again,

$$|\langle \varphi T, \phi\rangle| \leq |\langle (-\Delta)^s P_s * T, \varphi\phi\rangle| + |\langle P_s * T, \partial_t(\varphi\phi)\rangle| =: I_1 + I_2.$$

For I_1, simply apply Theorem 3.3 with $\beta := s - \sigma$ so that

$$I_1 = |\langle (-\Delta)^\sigma P_s * T, (-\Delta)^{s-\sigma}(\varphi\phi)\rangle| \lesssim_{\sigma,\alpha} \ell(R)^{n+2\sigma+\alpha}.$$

Regarding I_2, if $\sigma > 0$, we define $\beta := 1 - \sigma/s \in (0,1)$ and apply Theorem 3.1. If $\sigma = 0$, let $\overline{x}_R$ be the center of R so that

$$I_2 = |\langle P_s * T - P_s * T(\overline{x}_R), \partial_t(\varphi\phi)\rangle| \leq \ell(R)^{-2s} \int_R |\overline{x} - \overline{x}_R|_{p_s}^{\alpha} \, d\overline{x} \lesssim \ell(R)^{n+\alpha}.$$

□

Theorem 5.12 *Let $s \in (0,1]$, $\sigma \in [0,s)$ and $\alpha \in (0,1)$ with $\alpha < 2s - 2\sigma$. Then, for $E \subset \mathbb{R}^{n+1}$ compact set,*

$$\gamma^{\sigma}_{\Theta^s,\alpha}(E) \approx_{\sigma,\alpha} \mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E).$$

Proof For the upper bound we argue again as in Theorem 5.3, using now Theorem 5.11. For the lower bound, assume $\mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E) > 0$ and apply Frostman's lemma to consider a non trivial positive Borel measure μ supported on E with $\mu(E) \geq c\mathcal{H}^{n+2\sigma+\alpha}_{\infty,p_s}(E)$ and $\mu(B(\overline{x},r)) \leq r^{n+2\sigma+\alpha}$ for all $\overline{x} \in \mathbb{R}^{n+1}$, $r > 0$. It suffices to verify

$$\|(-\Delta)^{\sigma} P_s * T\|_{\mathrm{Lip}_{\alpha},p_s} \leq 1, \qquad \|\partial_t^{\sigma/s} P_s * T\|_{\mathrm{Lip}_{\alpha},p_s} \leq 1.$$

If $\sigma > 0$, by Lemma 4.3 with $\beta := \sigma/s$ we already have $\|\partial_t^{\sigma/s} P_s * \mu\|_{\mathrm{Lip}_{\alpha},p_s} \lesssim_{\sigma,\alpha} 1$. So we are left to estimate $\|(-\Delta)^{\sigma} P_s * \mu\|_{\mathrm{Lip}_{\alpha},p_s}$ for $\sigma \in [0,s)$, and we do it as in Theorem 5.6. Choose any $\overline{x}, \overline{y} \in \mathbb{R}^{n+1}$ with $\overline{x} \neq \overline{y}$ and consider the following partition of $\mathbb{R}^{n+1}$,

$$R_1 := \big\{\overline{z} \,:\, |\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2\big\} \cup \big\{\overline{z} \,:\, |\overline{y}-\overline{x}|_{p_s} \leq |\overline{y}-\overline{z}|_{p_s}/2\big\},$$
$$R_2 := \mathbb{R}^{n+1} \setminus R_1 = \big\{\overline{z} \,:\, |\overline{x}-\overline{y}|_{p_s} > |\overline{x}-\overline{z}|_{p_s}/2\big\} \cap \big\{\overline{z} \,:\, |\overline{y}-\overline{x}|_{p_s} > |\overline{y}-\overline{z}|_{p_s}/2\big\},$$

with their corresponding characteristic functions χ_1, χ_2 respectively. This way, we have

$$\begin{aligned}
&\frac{|(-\Delta)^{\sigma} P_s * \mu(\overline{x}) - (-\Delta)^{\sigma} P_s * \mu(\overline{y})|}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \\
&\leq \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \int_{|\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2} |(-\Delta)^{\sigma} P_s(\overline{x}-\overline{z}) - (-\Delta)^{\sigma} P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) \\
&\quad + \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \int_{|\overline{y}-\overline{x}|_{p_s} \leq |\overline{y}-\overline{z}|_{p_s}/2} |(-\Delta)^{\sigma} P_s(\overline{x}-\overline{z}) - (-\Delta)^{\sigma} P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) \\
&\quad + \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \int_{R_2} |(-\Delta)^{\sigma} P_s(\overline{x}-\overline{z}) - (-\Delta)^{\sigma} P_s(\overline{y}-\overline{z})| \mathrm{d}\mu(\overline{z}) =: I_1 + I_2 + I_3.
\end{aligned}$$

Regarding I_1, the fifth estimate of Lemma 2.3 yields

$$I_1 \lesssim_{\sigma} \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \int_{|\overline{x}-\overline{y}|_{p_s} \leq |\overline{x}-\overline{z}|_{p_s}/2} \frac{|\overline{x}-\overline{y}|^{2\zeta}_{p_s}}{|\overline{x}-\overline{z}|^{n+2\sigma+2\zeta}_{p_s}} \mathrm{d}\mu(\overline{z}).$$

Split the previous domain of integration into s-parabolic annuli centered at $\overline{x}$ with exponentially increasing radii proportional to $|\overline{x}-\overline{y}|_{p_s}$, and deduce as in Theorem 5.6 that $I_1 \lesssim_{\sigma,\alpha} 1$, using now that μ has $n+2\sigma+\alpha$ growth. For I_2, we argue as in I_1 just interchanging the roles of $\overline{x}$ and $\overline{y}$. Finally, for I_3, the first estimate of Lemma 2.3 yields

$$\begin{aligned}
I_3 &\leq \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \int_{R_2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|^{n+2\sigma}_{p_s}} + \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \int_{R_2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|^{n+2\sigma}_{p_s}} \\
&\leq \frac{1}{|\overline{x}-\overline{y}|^{\alpha}_{p_s}} \Bigg(\int_{|\overline{x}-\overline{y}|_{p_s} > |\overline{x}-\overline{z}|_{p_s}/2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{x}-\overline{z}|^{n+2\sigma}_{p_s}} + \int_{|\overline{y}-\overline{x}|_{p_s} > |\overline{y}-\overline{z}|_{p_s}/2} \frac{\mathrm{d}\mu(\overline{z})}{|\overline{y}-\overline{z}|^{n+2\sigma}_{p_s}} \Bigg).
\end{aligned}$$

Both of the above integrals can be dealt with by splitting the domain of integration into exponentially decreasing annuli, centered at $\overline{x}$ and $\overline{y}$ respectively, and using that μ has growth of degree strictly bigger than $n+2\sigma$. Thus, we obtain $I_3 \lesssim_{\sigma,\alpha} 1$ and we are done □

Theorem 5.13 *Let $s \in (0,1]$, $\sigma \in [0,s)$ and $\alpha \in (0,1)$ with $\alpha < 2s - 2\sigma$. A compact set $E \subset \mathbb{R}^{n+1}$ is removable for s-caloric functions with $\mathrm{Lip}_{\alpha,p_s}$-$(\sigma,\sigma/s)$-Laplacian if and only if $\gamma^{\sigma}_{\Theta^s,\alpha}(E) = 0$.*

Proof The proof is analogous to that of Theorem 5.4, applying Theorems 5.11, 5.12 and Lemma 5.1 with $d := n + 2\sigma + \alpha$. □

6 Proofs of Theorems 2.4 and 2.5

Let us recall the statement of Theorem 2.4 and provide its proof:

Theorem 2.4 *For any $\beta, s \in (0,1)$ and $\overline{x} = (x,t) \neq (0,t)$, the following hold:*

1. *If* $n > 1$, $\quad |\partial_t^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}$,
2. *If* $n = 1$ *and* $\beta > 1 - \dfrac{1}{2s}$, $\quad |\partial_t^\beta P_s(\overline{x})| \lesssim_{\beta,\alpha} \dfrac{1}{|x|^{1-2s+\alpha}|\overline{x}|_{p_s}^{2s(1+\beta)-\alpha}}$, $\quad \forall \alpha \in (2s-1, 4s)$.

Moreover, for every n,

3. $|\nabla_x \partial_t^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}}$, 4. $|\partial_t \partial_t^\beta P_s(\overline{x})| \lesssim_\beta \dfrac{1}{|x|^{n}|\overline{x}|_{p_s}^{2s(1+\beta)}}$, *for* $t \neq 0$.

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ is such that $|\overline{x} - \overline{x}'|_{p_s} \leq |x|/2$,

$$5.\ |\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\overline{x}')| \lesssim_\beta \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s}|\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

Proof To prove *1*, we use [13, Equation 2.9] and deduce the existence of a function F_s such that for $t > 0$,

$$P_s(x,t) = \frac{1}{|x|^n} F_s\left(\frac{t}{|x|^{2s}}\right), \tag{23}$$

and such that

$$F_s(u) \approx \frac{u}{\left(1 + u^{1/s}\right)^{(n+2s)/2}}. \tag{24}$$

We extend continuously $F_s(u) := 0$ for $u \leq 0$, so that (23) is verified for any value of t. The existence of F_s is clear, since for $t > 0$ the function P_s can be written as

$$P_s(x,t) = \frac{1}{|x|^n}\left(\frac{t}{|x|^{2s}}\right)^{-\frac{n}{2s}} \phi_{n,s}\left[\left(\frac{t}{|x|^{2s}}\right)^{-\frac{1}{2s}}\right],$$

and defining for $u > 0$, $F_s(u) := u^{-\frac{n}{2s}}\phi_{n,s}\left(u^{-\frac{1}{2s}}\right)$, we are done. Notice that F_s is a bounded continuous function, null for negative values of u, smooth in the domain $u > 0$ and vanishing at ∞. Moreover, using the bounds obtained for ϕ' and ϕ'' we obtain the following estimates for $u > 0$,

$$|F_s'(u)| \lesssim \frac{1}{\left(1 + u^{1/s}\right)^{(n+2s)/2}}, \qquad |F_s''(u)| \lesssim \frac{1}{u\left(1 + u^{1/s}\right)^{(n+2s)/2}}. \tag{25}$$

Let us argue that, in fact, $|F_s''(u)|$ is also a bounded function. Notice that, by definition,

$$\partial_\tau^2 P_s(x,\tau) = \frac{1}{|x|^{n+4s}} F_s''\left(\frac{\tau}{|x|^{2s}}\right) \qquad \Leftrightarrow \qquad \left|F_s''\left(\frac{\tau}{|x|^{2s}}\right)\right| = |x|^{n+4s}\left|\partial_\tau^2 P_s(x,\tau)\right|,$$

and using that P_s is the fundamental solution of the Θ^s-equation and that $\tau > 0$, we have

$$\partial_\tau^2 P_s(x,\tau) = \partial_\tau\left[-(-\Delta)^s P_s(x,\tau)\right].$$

By the commutativity of ∂_τ and $(-\Delta)^s$, we deduce

$$\left|\partial_\tau^2 P_s(x,\tau)\right| = \left|(-\Delta)^s\left[\partial_\tau P_s(x,\tau)\right]\right| = \left|(-\Delta)^s\left[-(-\Delta)^s P_s(x,\tau)\right]\right| \lesssim \frac{1}{|\overline{x}|_{p_s}^{n+4s}}.$$

Therefore,

$$\left|F_s''\left(\frac{\tau}{|x|^{2s}}\right)\right| \lesssim \frac{|x|^{n+4s}}{|\overline{x}|_{p_s}^{n+4s}} = \frac{1}{\max\left\{1, \left(\tau/|x|^{2s}\right)^{1/(2s)}\right\}^{n+4s}} \lesssim \frac{1}{\left[1+\left(\tau/|x|^{2s}\right)^{1/s}\right]^{(n+4s)/2}},$$

that implies the following (improved) bound for F_s'',

$$|F_s''(u)| \lesssim \frac{1}{\left(1+u^{1/s}\right)^{(n+4s)/2}} \le 1, \qquad u > 0. \tag{26}$$

We continue by observing that by a change of variables the following holds,

$$\partial_t^\beta P_s(x,t) = \frac{1}{|x|^n}\left[\partial_t^\beta F_s\left(\frac{\cdot}{|x|^{2s}}\right)\right](t) = \frac{1}{|x|^{n+2s\beta}}\partial^\beta F_s\left(\frac{t}{|x|^{2s}}\right). \tag{27}$$

We shall prove the following inequality,

$$\left|\partial^\beta F_s(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}, \tag{28}$$

where for $u=0$ is just asking for $\left|\partial^\beta F_s(0)\right|$ to be bounded. To verify (28) we distinguish whether if $u=0$, $u<0$ or $u>0$. For $u=0$ observe that by definition and relation (24),

$$\begin{aligned}
\left|\partial^\beta F_s(0)\right| &\le \int_{\mathbb{R}} \frac{|F_s(0)-F_s(w)|}{|0-w|^{1+\beta}}\mathrm{d}w = \int_0^\infty \frac{|F_s(w)|}{w^{1+\beta}}\mathrm{d}w \\
&\lesssim_\beta \int_0^\infty \frac{1}{w^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\mathrm{d}w \\
&= \int_0^1 \frac{\mathrm{d}w}{w^\beta\left(1+w^{1/s}\right)^{(n+2s)/2}} + \int_1^\infty \frac{\mathrm{d}w}{w^\beta\left(1+w^{1/s}\right)^{(n+2s)/2}} \\
&\approx \int_0^1 \frac{\mathrm{d}w}{w^\beta} + \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}+1+\beta}} \lesssim (1-\beta)^{-1} + \left(\frac{n}{2s}+\beta\right)^{-1} \lesssim_\beta 1,
\end{aligned}$$

so case $u=0$ is done. Let us assume $u<0$, so that

$$\left|\partial^\beta F_s(u)\right| \le \int_{\mathbb{R}} \frac{|F_s(w)|}{||u|+w|^{1+\beta}}\mathrm{d}w \lesssim \int_0^\infty \frac{1}{(|u|+w)^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\mathrm{d}w.$$

On the one hand notice that the since $|u|+w>w$, the previous expression is bounded by a constant depending on n, s and β (by the same arguments given for the case $u=0$). On the other hand, observe that

$$\begin{aligned}
\left|\partial^\beta F_s(u)\right| &\lesssim \frac{1}{|u|^{1+\beta}} \int_0^\infty \frac{1}{(w/|u|+1)^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\mathrm{d}w \\
&= \frac{1}{|u|^{1+\beta}}\left[\int_0^1 \frac{1}{(w/|u|+1)^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\mathrm{d}w\right.
\end{aligned}$$

$$+\int_1^\infty \frac{1}{(w/|u|+1)^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \Bigg] =: \frac{1}{|u|^{1+\beta}}(I_1+I_2).$$

Regarding I_1, since the denominators are bigger than 1, we directly have

$$I_1 \lesssim \int_0^1 w \,\mathrm{d}w \leq 1. \tag{29}$$

Turning to I_2, we similarly obtain

$$I_2 \leq \int_1^\infty \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \leq \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}}} = w^{-\frac{n}{2s}+1}\Big|_1^\infty = 1, \tag{30}$$

where notice that $-\frac{n}{2s}+1 < 0$ because $n > 1$ and $s < 1$. Therefore, we also have $\left|\partial^\beta F_s(u)\right| \lesssim |u|^{-1-\beta}$ and we conclude that for $u \leq 0$,

$$\left|\partial^\beta F_s(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}.$$

Let us finally assume $u > 0$. Begin by writing

$$\left|\partial^\beta F_s(u)\right| \leq \int_{|w|\leq u/2} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w + \int_{u/2\leq|w|\leq 2u} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w$$
$$+ \int_{|w|>2u} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w =: I_1+I_2+I_3.$$

We study each of the previous integrals separately. Concerning the first, notice that in its domain of integration $u/2 \leq |w-u| \leq 3u/2$, i.e. $|w-u| \approx u$. We split it as follows

$$I_1 = \int_{-u/2}^0 \frac{|F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w + \int_0^{u/2} \frac{|F_s(w)-F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w =: I_{11}+I_{12}.$$

Observe that I_{11} can be estimated by

$$I_{11} \lesssim \frac{u}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \int_{-u/2}^0 \frac{\mathrm{d}w}{|u|^{1+\beta}} \simeq_\beta \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}}.$$

The expression of the right, viewed as a continuous function of u, tends to zero as $u \to 0$ and decays as $|u|^{-\beta-\frac{n}{2s}}$ as $u \to \infty$. Hence, it is bounded by a constant (depending on n, s and β) and so $I_{11} \lesssim_\beta 1$. On the other hand, to prove that $I_{11} \lesssim_\beta |u|^{-1-\beta}$ it suffices to check that the following expression is bounded by a constant,

$$\frac{u^2}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \approx u F_s(u).$$

Again, it is clear it that tends to zero as $u \to 0$, but observe that it behaves as $|u|^{-\frac{n}{2s}+1}$ as $u \to \infty$, which vanishes only if $n > 2s$, that is, only if $n > 1$, since $s < 1$. But this is satisfied by hypothesis. Therefore we deduce $I_{11} \lesssim_\beta \min\{1, |u|^{-1-\beta}\}$. Regarding I_{12} proceed in a similar manner to obtain

$$I_{12} \lesssim_\beta \frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w + \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}}.$$

The second summand has already been studied in I_{11}. Regarding the first, notice that

$$\int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \leq \int_0^1 \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w + \int_1^\infty \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w$$
$$\leq \int_0^1 w \, \mathrm{d}w + \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}}} \lesssim 1,$$

where we have applied the same arguments as in (29) and (30). On the other hand, by applying the following inequality for $w > 0$,

$$(1 + w^{1/s})^{(n+2s)/2} > w^{1-\beta},$$

that can be checked by a direct computation, we deduce

$$\int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w < \int_0^{u/2} w^\beta \mathrm{d}w \lesssim_\beta u^{1+\beta}.$$

Therefore we conclude

$$I_{12} \lesssim_\beta \frac{1}{u^{1+\beta}} \min\left\{1, u^{1+\beta}\right\} + \min\left\{1, \frac{1}{u^{1+\beta}}\right\} = 2 \min\left\{1, \frac{1}{u^{1+\beta}}\right\},$$

that implies the desired estimate for I_1.

Moving on to I_2, we split it as follows

$$I_2 = \int_{-2u}^{-u/2} \frac{|F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w + \int_{u/2}^{2u} \frac{|F_s(w) - F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w =: I_{21} + I_{22}.$$

The study of I_{21} is exactly the same as the one presented for I_{11}, so we focus on I_{22}. Apply the mean value theorem to obtain

$$I_{22} \leq \sup_{\nu \in [u/2, 2u]} |F_s'(\nu)| \int_{u/2}^{2u} \frac{\mathrm{d}w}{|w-u|^\beta} \lesssim_\beta \sup_{\nu \in [u/2, 2u]} |F_s'(\nu)| \, u^{1-\beta}.$$

Therefore, if we are able to bound $|F_s'|$ by $u^{\beta-1}$ and u^{-2} we will be done. But recalling relation (25), this is equivalent to proving that the following functions are bounded by a constant:

$$\frac{u^{\beta-1}}{\left(1+u^{1/s}\right)^{(n+2s)/2}}, \qquad \frac{u^2}{\left(1+u^{1/s}\right)^{(n+2s)/2}}, \tag{31}$$

that has already been done in I_{11}. Therefore, we are only left to study I_3,

$$I_3 = \int_{-\infty}^{-2u} \frac{|F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w + \int_{2u}^{\infty} \frac{|F_s(w) - F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w =: I_{31} + I_{32}.$$

To deal with I_{31} we first notice that in the domain of integration $|w-u| \approx |w|$, implying

$$I_{31} \approx \frac{u}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \int_{-\infty}^{-2u} \frac{\mathrm{d}w}{|w|^{1+\beta}} \lesssim_\beta \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \lesssim_\beta \min\left\{1, \frac{1}{u^{1+\beta}}\right\}.$$

We study I_{32} by splitting it as

$$I_{32} \leq \int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}} \mathrm{d}w + \int_{2u}^{\infty} \frac{|F_s(u)|}{|w-u|^{1+\beta}} \mathrm{d}w.$$

The second summand is tackled in exactly the same way as I_{31}, so we focus on the first one. Using that $|w-u| \approx |w| \gtrsim u$, we have

$$\begin{aligned}\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}} \mathrm{d}w &\lesssim \frac{1}{u^{1+\beta}} \int_{2u}^{\infty} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \\ &\leq \frac{1}{u^{1+\beta}} \left[\int_0^1 w \,\mathrm{d}w + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n}{2s}}} \right] \lesssim \frac{1}{u^{1+\beta}},\end{aligned}$$

by the same arguments used in (29) and (30). On the other hand, we also have

$$\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}} \mathrm{d}w \lesssim \int_{2u}^{\infty} \frac{1}{w^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \leq \int_0^1 \frac{\mathrm{d}w}{w^{\beta}} + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n}{2s}}}.$$

We already know that the second integral is bounded by a constant for $n > 1$, while the first one is also bounded, since $0 < \beta < 1$. So we conclude that $I_{32} \lesssim_{\beta} \min\{1, |u|^{-1-\beta}\}$ and we obtain the desired bound for I_3 and thus for $|\partial^{\beta} F_s(u)|$ if $u > 0$.

All in all, returning to (27), we finally have

$$\begin{aligned}|\partial_t^{\beta} P_s(x,t)| &= \frac{1}{|x|^{n+2s\beta}} \left| \partial^{\beta} F_s\left(\frac{t}{|x|^{2s}} \right) \right| \lesssim_{\beta} \frac{1}{|x|^{n+2s\beta}} \min\left\{ 1, \frac{|x|^{2s(1+\beta)}}{|t|^{1+\beta}} \right\} \\ &= \frac{1}{|x|^{n-2s}} \min\left\{ \frac{1}{|x|^{2s(1+\beta)}}, \frac{1}{|t|^{\frac{2s(1+\beta)}{2s}}} \right\} = \frac{1}{|x|^{n-2s} |\overline{x}|_{p_s}^{2s(1+\beta)}},\end{aligned}$$

that is estimate *1* in the statement of the lemma.

In order to prove *2*, we follow the same scheme. Indeed, the desired estimate follows once we prove

$$\left| \partial^{\beta} F_s(u) \right| \lesssim_{\beta,\alpha} \min\left\{ 1, \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}} \right\}, \qquad \text{for } 2s-1 < \alpha < 4s.$$

If one followed the same arguments used to prove *1*, in the regime $u < 0$ one already encounters a first bound for which dimension $n = 1$ is troublesome, namely when trying to obtain $|\partial^{\beta} F_s(u)| \lesssim |u|^{-1-\beta+\frac{\alpha}{2s}}$. However, in our current setting we observe that

$$\begin{aligned}&\int_0^{\infty} \frac{1}{(w+|u|)^{1+\beta}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \\ &\lesssim \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}} \int_0^{\infty} \frac{1}{(w/|u|+1)^{1+\beta-\frac{\alpha}{2s}}} \frac{1}{(w+|u|)^{\frac{\alpha}{2s}}} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \\ &\lesssim \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}} \left(\int_0^1 w^{1-\frac{\alpha}{2s}} \mathrm{d}w + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n+\alpha}{2s}}} \right) \lesssim_{\beta,\alpha} \frac{1}{|u|^{1+\beta-\frac{\alpha}{2s}}}, \quad \text{since } 2s-1 < \alpha < 4s,\end{aligned}$$

so the desired bound for $|\partial^{\beta} F_s(u)|$ follows. For the case $u > 0$ we also proceed analogously. Let us comment those steps where the hypotheses on α and β come into play. In I_1, using the same notation as for the case $n > 1$, we obtain the estimates

$$I_{11} \lesssim_{\beta} \frac{u^{1-\beta}}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \qquad \text{and} \qquad I_{12} \lesssim_{\beta} \frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w,$$

expression that we already know to be bounded by a constant. To prove that $I_{11} \lesssim_{\beta,\alpha} |u|^{-1-\beta+\frac{\alpha}{2s}}$ observe that the function

$$\frac{u^{2-\frac{\alpha}{2s}}}{\left(1+u^{1/s}\right)^{(n+2s)/2}} \approx u^{1-\frac{\alpha}{2s}} F_s(u)$$

tends to zero as $u \to 0$, since $\alpha < 4s$. Moreover, it behaves as $|u|^{-\frac{n+\alpha}{2s}+1}$ as $u \to \infty$, which also tends to 0 because $\alpha < 2s-1$. Thus, $I_{11} \lesssim_{\beta,\alpha} \min\{1, |u|^{-1-\beta+\frac{\alpha}{2s}}\}$. On the other hand, since the following holds

$$\left(1+w^{1/s}\right)^{(n+2s)/2} > w^{2-\frac{\alpha}{2s}},$$

we obtain

$$\frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{w}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w < \frac{1}{u^{1+\beta}} \int_0^{u/2} \frac{\mathrm{d}w}{w^{1-\frac{\alpha}{2s}}} \lesssim_{\beta,\alpha} \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}}.$$

Therefore, $I_{12} \lesssim_{\beta,\alpha} \min\{1, |u|^{-1-\beta+\frac{\alpha}{2s}}\}$, hence I_1 satisfies the same estimate. The study of I_2 is completely analogous to that of $n > 1$. Therefore we are only left to study I_3. The arguments can be carried out analogously up to the point of estimating

$$\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}} \mathrm{d}w.$$

Using that $|w-u| \approx |w| \gtrsim u$, we have

$$\begin{aligned}\int_{2u}^{\infty} \frac{|F_s(w)|}{|w-u|^{1+\beta}} \mathrm{d}w &\lesssim \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}} \int_{2u}^{\infty} \frac{w^{1-\frac{\alpha}{2s}}}{\left(1+w^{1/s}\right)^{(n+2s)/2}} \mathrm{d}w \\ &\le \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}} \left[\int_0^1 w^{1-\frac{\alpha}{2s}}\,\mathrm{d}w + \int_1^{\infty} \frac{\mathrm{d}w}{w^{\frac{n+\alpha}{2s}}}\right] \lesssim_{\beta,\alpha} \frac{1}{u^{1+\beta-\frac{\alpha}{2s}}},\end{aligned}$$

since $2s-1 < \alpha < 4s$. Therefore, $I_{32} \lesssim_{\beta,\alpha} \min\{1, |u|^{-1-\beta+\frac{\alpha}{2s}}\}$, and with this we get the desired bound for I_3 and the completion of the proof for the case $n = 1$.

Moving on to estimate *3*, we begin by defining for $u > 0$ the real variable function

$$G_s(u) := u^{-\frac{n+1}{2s}} \phi_n'\left(u^{-\frac{1}{2s}}\right),$$

so that in light of relation (2) we have

$$\nabla_x P_s(x,t) \simeq \frac{x}{|x|^{n+2}} G_s\left(\frac{t}{|x|^{2s}}\right), \qquad \text{for} \quad t > 0, x \neq 0.$$

By (6) it is clear that

$$|G_s(u)| \approx \frac{u}{\left(1+u^{1/s}\right)^{(n+2s+2)/2}}. \tag{32}$$

Hence, as done for F_s, we can extend continuously the definition of G_s by zero for negative values of u. Notice also that the previous estimate implies that G_s is bounded on $\mathbb{R}$.

Our next claim is that the operators ∇_x and ∂_t^{β} commute when applied to P_s. To prove this, it suffices to check that the following integral is locally well-defined for every x and t,

$$\int_{\mathbb{R}} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \mathrm{d}w.$$

Split the domain of integration as

$$\int_{|t-w|<1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \mathrm{d}w + \int_{|t-w|\geq 1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \mathrm{d}w.$$

The second integral is clearly well-defined, since $\nabla_x P_s(x,t) \simeq x/|x|^{n+2} G_s(t/|x|^{2s})$ and we know that G_s is bounded. Thus, directly applying the triangle inequality in the numerator and using that $\beta > 0$, we deduce that, indeed, the second integral is finite. For the first one, we need some more work. We shall distinguish four possibilities:

Case 1: $t \leq -1$. For such values of t the integral becomes null, since $\nabla_x P_s(x,t)$ and $\nabla_x P(x,w)$ are zero.

Case 2: $t \in (-1, 0]$. Observe that in this setting the integral can be rewritten as

$$\begin{aligned}\int_0^{1-|t|} \frac{|\nabla_x P_s(x,w)|}{|w-t|^{1+\beta}} \mathrm{d}w &= \int_0^{1-|t|} \frac{|\nabla_x P_s(x,w) - \nabla_x P_s(x,0)|}{|w-t|^{1+\beta}} \mathrm{d}w \\ &\lesssim \frac{1}{|x|^{n+2s+1}} \int_0^{1-|t|} \frac{|G_s'(\tau/|x|^{2s})|}{|w|^{\beta}} \mathrm{d}w,\end{aligned}$$

for some $\tau \in (0, w)$. By definition, there are constants C_1, C_2 so that for $u > 0$

$$G_s'(u) = C_1\, u^{-(n+2s+1)/(2s)} \phi_{n,s}'\big(u^{-\frac{1}{2s}}\big) + C_2\, u^{-(n+2s+2)/(2s)} \phi_{n,s}''\big(u^{-\frac{1}{2s}}\big),$$

so using the estimates for ϕ_n' and $\phi_{n,s}''$ in (6) we deduce

$$|G_s'(u)| \approx \frac{1}{\big(1+u^{1/s}\big)^{(n+2s+2)/2}}, \tag{33}$$

which is a bounded function. Therefore

$$\frac{1}{|x|^{n+2s+1}} \int_0^{1-|t|} \frac{|G_s'(\tau/|x|^{2s})|}{|w|^{\beta}} \mathrm{d}w \lesssim_\beta \frac{1}{|x|^{n+2s+1}} < \infty,$$

for every $x \neq 0$.

Case 3: $t \in (0, 1]$. The integral we were initially studying can be written as

$$\int_{t-1}^{0} \frac{|\nabla_x P_s(x,t)|}{|t-w|^{1+\beta}} \mathrm{d}w + \int_0^{t+1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \mathrm{d}w.$$

The second integral can be tackled in exactly the same way as the integral in *Case 2*. Regarding the first one, estimate it as follows

$$\begin{aligned}\int_{t-1}^{0} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,0)|}{|t-w|^{1+\beta}} \mathrm{d}w &\leq \frac{1}{|x|^{n+2s+1}} \int_{t-1}^{0} \frac{|G_s'(\tau/|x|^{2s})||t|}{|t-w|^{1+\beta}} \mathrm{d}w \\ &\lesssim \frac{1}{|x|^{n+2s+1}} \int_{t-1}^{0} \frac{|G_s'(\tau/|x|^{2s})|}{|w|^{\beta}} \mathrm{d}w < \infty,\end{aligned}$$

where we have used $|t| \leq |t-w| + |w|$ and also that $|t-w| = (t+|w|) \geq |w|$. The last inequality follows by the same arguments used in *Case 2*.

Case 4: $t > 1$. For this final case, the integral can be estimated as

$$\int_{t-1}^{t+1} \frac{|\nabla_x P_s(x,t) - \nabla_x P_s(x,w)|}{|t-w|^{1+\beta}} \mathrm{d}w \lesssim \frac{1}{|x|^{n+2s+1}} \int_{t-1}^{t+1} \frac{|G_s'(\tau/|x|^{2s})|}{|t-w|^{\beta}} \mathrm{d}w \lesssim_\beta \frac{1}{|x|^{n+2s+1}} < \infty.$$

Thus, we have obtained the desired commutativity between ∂_t^β and ∇_x, which yields

$$\nabla_x \partial_t^\beta P_s(x,t) = \partial_t^\beta \big[\nabla_x P_s\big](x,t) = \frac{x}{|x|^{n+2}}\left[\partial_t^\beta G_s\left(\frac{\cdot}{|x|^{2s}}\right)\right](t) = \frac{x}{|x|^{n+2s\beta+2}}\partial^\beta G_s\left(\frac{t}{|x|^{2s}}\right).$$

Now it is a matter of showing that the following inequality holds

$$\left|\partial^\beta G_s(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}, \tag{34}$$

The proof of (34) is essentially identical to the one given for (28), using the bounds for G_s and G_s' ((32) and (33) respectively) instead of those for F_s and F_s'. The faster decay of G_s and its derivative implies that one does not find any obstacles in (30). In fact, the integral that appears in the current analysis is $\int_1^\infty w^{-\frac{n+1}{2s}} \mathrm{d}w$, which also converges for $n = 1$. So using the previous estimate we deduce, for any $n > 0$,

$$\begin{aligned}
|\nabla_x \partial_t^\beta P_s(x,t)| &= \frac{1}{|x|^{n+2s\beta+1}}\left|\partial_t^\beta G_s\left(\frac{t}{|x|^{2s}}\right)\right| \lesssim_\beta \frac{1}{|x|^{n+2s\beta+1}}\min\left\{1, \frac{|x|^{2s(1+\beta)}}{|t|^{1+\beta}}\right\} \\
&= \frac{1}{|x|^{n-2s+1}}\min\left\{\frac{1}{|x|^{2s(1+\beta)}}, \frac{1}{|t|^{\frac{2s(1+\beta)}{2s}}}\right\} = \frac{1}{|x|^{n-2s+1}|\overline{x}|_{p_s}^{2s(1+\beta)}},
\end{aligned}$$

which proves the statement *3* in our lemma.

We continue by estimating $\partial_t \partial_t^\beta P_s(x,t)$ for $x \neq 0$ and $t \neq 0$. Using (27) we rewrite it as

$$\partial_t \partial_t^\beta P_s(\overline{x}) = \frac{1}{|x|^{n+2s(1+\beta)}}\partial^\beta F_s'\left(\frac{t}{|x|^{2s}}\right),$$

and we claim that the following inequality holds for $u \neq 0$,

$$\left|\partial^\beta F_s'(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}.$$

Let us also recall that we had the following estimates for $u > 0$,

$$|F_s'(u)| \lesssim \frac{1}{\left(1+u^{1/s}\right)^{(n+2s)/2}}, \qquad |F_s''(u)| \lesssim \frac{1}{\left(1+u^{1/s}\right)^{(n+4s)/2}} \leq \frac{1}{u\left(1+u^{1/s}\right)^{(n+2s)/2}}.$$

Observe that, on the one hand,

$$\begin{aligned}
\left|\partial^\beta F_s'(u)\right| &\leq \int_{\mathbb{R}} \frac{|F_s'(u) - F_s'(w)|}{|u-w|^{1+\beta}}\mathrm{d}w \\
&\leq \sup_{\nu\in\mathbb{R}} |F_s''(\nu)| \int_{|u-w|<1} \frac{\mathrm{d}w}{|u-w|^\beta} + 2\sup_{\nu\in\mathbb{R}} |F_s'(\nu)| \int_{|u-w|\geq 1} \frac{\mathrm{d}w}{|u-w|^{1+\beta}} \lesssim_\beta 1,
\end{aligned}$$

by the boundedness of F_s' and F_s'', and the fact that $\beta \in (0,1)$. Therefore we are left to verify $\left|\partial^\beta F_s'(u)\right| \lesssim_\beta |u|^{-1-\beta}$. If $u < 0$, since F_s' is supported on $(0,\infty)$ and $|u-w| > |u|$ for $w \geq 0$, we have

$$\begin{aligned}
\left|\partial^\beta F_s'(u)\right| &\leq \int_0^\infty \frac{|F_s'(w)|}{|u-w|^{1+\beta}}\mathrm{d}w \lesssim \int_0^1 \frac{\mathrm{d}w}{|u-w|^{1+\beta}} + \int_1^\infty \frac{|F_s'(w)|}{|u-w|^{1+\beta}}\mathrm{d}w \\
&\lesssim \frac{1}{|u|^{1+\beta}}\left(1 + \int_1^\infty \frac{\mathrm{d}w}{\left(1+w^{1/s}\right)^{(n+2s)/2}}\right) \leq \frac{1}{|u|^{1+\beta}}\left(1 + \int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}+1}}\right) \lesssim \frac{1}{|u|^{1+\beta}},
\end{aligned}$$

and we are done. If on the other hand $u > 0$, we estimate $|\partial^\beta F_s'|$ in a similar way as $|\partial^\beta F_s|$ in the proof of point *1* of this lemma. Namely, we write

$$\left|\partial^\beta F_s'(u)\right| \le \int_{|w|\le u/2} \frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w + \int_{u/2\le|w|\le 2u} \frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w + \int_{|w|>2u} \frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w =: I_1+I_2+I_3.$$

Regarding I_1, notice that in the domain of integration we have $|w-u|\approx u$, so

$$I_1 \lesssim \int_{-u/2}^{0} \frac{|F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w + \int_0^{u/2} \frac{|F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w + \int_0^{u/2} \frac{|F_s'(w)|}{|w-u|^{1+\beta}}\mathrm{d}w.$$

The first two integrals can be directly bounded by

$$\frac{1}{|u|^{1+\beta}}|F_s'(u)| \int_0^{u/2} \mathrm{d}w \le \frac{1}{|u|^{1+\beta}}\left(\frac{u}{(1+u^{1/s})^{(n+2s)/2}}\right) \le \frac{1}{|u|^{1+\beta}}.$$

For the third,

$$\int_0^{u/2} \frac{|F_s'(w)|}{|w-u|^{1+\beta}}\mathrm{d}w \lesssim \frac{1}{|u|^{1+\beta}}\left(\int_0^1 |F_s'(w)|\mathrm{d}w + \int_1^\infty |F_s'(w)|\mathrm{d}w\right) \lesssim \frac{1}{|u|^{1+\beta}}\left(1+\int_1^\infty \frac{\mathrm{d}w}{w^{\frac{n}{2s}+1}}\right) \lesssim \frac{1}{|u|^{1+\beta}},$$

and we are done with I_1. Moving on to I_2, we split it as follows

$$I_2 = \int_{-2u}^{-u/2} \frac{|F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w + \int_{u/2}^{2u} \frac{|F_s'(w)-F_s'(u)|}{|w-u|^{1+\beta}}\mathrm{d}w =: I_{21}+I_{22}.$$

The study of I_{21} can be carried analogously to that of I_1, since in that domain of integration one has $|w-u|\ge 3|u|/2$, so we focus on I_{22}. Applying the mean value theorem and the bound for $|F_s''|$ of (25) as well as relation (24) we get

$$I_{22} \le \sup_{\nu\in[u/2,2u]} |F_s''(\nu)| \int_{u/2}^{2u} \frac{\mathrm{d}w}{|w-u|^{\beta}} \lesssim_\beta \sup_{\nu\in[u/2,2u]} |F_s''(\nu)|\, u^{1-\beta} \lesssim \frac{u^{1-\beta}}{u(1+u^{1/s})^{(n+2s)/2}} \approx F_s(u)u^{-1-\beta} \le u^{-1-\beta}.$$

So we are left to study I_3. Since in its domain of integration we have $|w-u|\gtrsim w$, we get

$$I_3 \lesssim \int_{-\infty}^{-2u} \frac{|F_s'(u)|}{|w|^{1+\beta}}\mathrm{d}w + \int_{2u}^{\infty} \frac{|F_s'(w)|+|F_s'(u)|}{|w|^{1+\beta}}\mathrm{d}w \lesssim \frac{3}{(1+u^{1/s})^{(n+2s)/2}} \int_{2u}^{\infty} \frac{\mathrm{d}w}{|w|^{1+\beta}} \lesssim_\beta \frac{u^{-\beta}}{(1+u^{1/s})^{(n+2s)/2}} \le u^{-1-\beta},$$

that allows us to finally conclude

$$\left|\partial^\beta F_s'(u)\right| \lesssim_\beta \min\left\{1, \frac{1}{|u|^{1+\beta}}\right\}, \qquad u\neq 0.$$

So using the previous estimate get, for $x\neq 0$ and $t\neq 0$,

$$|\partial_t \partial_t^\beta P_s(x,t)| = \frac{1}{|x|^{n+2s(1+\beta)}}\left|\partial_t^\beta F_s'\left(\frac{t}{|x|^{2s}}\right)\right| \lesssim_\beta \frac{1}{|x|^{n+2s(1+\beta)}} \min\left\{1, \frac{|x|^{2s(1+\beta)}}{|t|^{1+\beta}}\right\}$$

$$= \frac{1}{|x|^n} \min\left\{ \frac{1}{|x|^{2s(1+\beta)}}, \frac{1}{|t|^{\frac{2s(1+\beta)}{2s}}} \right\} = \frac{1}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

Finally, the proof of estimate *5* is analogous to that of *5* in Theorem 2.3. Indeed, let $\overline{x}' = (x', t') \in \mathbb{R}^{n+1}$ such that $|\overline{x} - \overline{x}'|_{p_s} \le |x|/2$, which is a stronger assumption than that of Theorem 2.3. In fact, it can be checked by a direct computation that this already implies $|\overline{x}|_{p_s} \le 2|\overline{x}'|_{p_s}$ and $|x| \le 2|x'|$. Write again $\widehat{x} = (x', t)$ and consider

$$|\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\overline{x}')| \le |\partial_t^\beta P_s(\overline{x}) - \partial_t^\beta P_s(\widehat{x})| + |\partial_t^\beta P_s(\widehat{x}) - \partial_t^\beta P_s(\overline{x}')|.$$

By estimate *2*, the first term in the above inequality now satisfies

$$\begin{aligned} |x - x'| \sup_{\xi \in [x, x']} |\nabla_x \partial_t^\beta P_s(\xi, t)| &\lesssim_\beta \frac{|x - x'|}{|x|^{n-2s+1} |\overline{x}|_{p_s}^{2s(1+\beta)}} \\ &\le \frac{|\overline{x} - \overline{x}'|_{p_s}}{|x|^{n-2s+1} |\overline{x}|_{p_s}^{2s(1+\beta)}} \\ &\lesssim \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s} |\overline{x}|_{p_s}^{2s(1+\beta)}}, \end{aligned}$$

where we have used that $1 - 2\zeta \ge 0$ and that condition $|\overline{x} - \overline{x}'|_{p_s} \le |x|/2$ implies that the line segment joining $\overline{x}$ with $\overline{x}'$ is at a distance of the time axis comparable to $|x|$. Regarding the second term, assume $t > t'$. If t and t' share sign we apply estimate *3* to directly deduce

$$|t - t'| \sup_{\tau \in [t, t']} |\partial_t \partial_t^\beta P_s(x', \tau)| \lesssim_\beta \frac{|t - t'|}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}} \le \frac{|\overline{x} - \overline{x}'|_{p_s}^{2s}}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}} \lesssim \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s} |\overline{x}|_{p_s}^{2s(1+\beta)}}.$$

If on the other hand $t > 0$ and $t' < 0$, we use relation (9), valid also in this case, together with $|x'| \ge |x|/2$ to finally obtain

$$\begin{aligned} &|\partial_t^\beta P_s(\widehat{x}) - \partial_t^\beta P_s(\overline{x}')| \\ &\quad \le |\partial_t^\beta P_s(x', t) - \partial_t^\beta P_s(x', 0)| + |\partial_t^\beta P_s(x', 0) - \partial_t^\beta P_s(x', t')| \\ &\quad \lesssim t \sup_{\tau \in (0, t)} |\partial_t \partial_t^\beta P_s(x', \tau)| + |t'| \sup_{\tau \in (t', 0)} |\partial_t \partial_t^\beta P_s(x', \tau)| \\ &\quad \lesssim_\beta \frac{t + |t'|}{|x'|^n |x'|^{2s(1+\beta)}} \lesssim \frac{|t - t'|}{|x|^n |\overline{x}|_{p_s}^{2s(1+\beta)}} \lesssim \frac{|\overline{x} - \overline{x}'|_{p_s}^{2\zeta}}{|x|^{n+2\zeta-2s} |\overline{x}|_{p_s}^{2s(1+\beta)}}. \end{aligned}$$

□

Now we move on the case $s = 1$. Let us recall the statement of Theorem 2.5:

Theorem 2.5 *For any* $\overline{x} = (x, t) \ne (0, t)$ *and* $\beta \in (0, 1)$*, the following hold:*

$$1.\ \ For\quad n > 2, \qquad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-2} |\overline{x}|_{p_1}^{2+2\beta}},$$

$$2.\ \ For\quad n = 2, \qquad |\partial_t^\beta W(\overline{x})| \lesssim_{\beta,\alpha} \frac{1}{|x|^\alpha |\overline{x}|_{p_1}^{2+2\beta-\alpha}}, \quad \forall \alpha \in (0, 2+2\beta],$$

$$3.\ \ For\quad n = 1, \qquad |\partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|\overline{x}|_{p_1}^{1+2\beta}}.$$

Moreover, for every n,

$$4.\ |\nabla_x \partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}, \qquad 5.\ |\partial_t \partial_t^\beta W(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n}|\overline{x}|_{p_1}^{2+2\beta}}.$$

Finally, if $\overline{x}' \in \mathbb{R}^{n+1}$ *is such that* $|\overline{x}-\overline{x}'|_{p_1} \le |x|/2$, *then*

$$6.\ |\partial_t^\beta W(\overline{x}) - \partial_t^\beta W(\overline{x}')| \lesssim_\beta \frac{|\overline{x}-\overline{x}'|_{p_1}}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}.$$

First, we prove the following auxiliary lemma:

Lemma 6.1 *Let* $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ *be defined as*

$$f_1(t) := \frac{e^{-1/t}}{t^{n/2}} \chi_{t>0}, \qquad f_2(t) := \frac{e^{-1/t}}{t^{n/2+1}} \chi_{t>0}, \qquad f_3(t) := \frac{e^{-1/t}}{t^{n/2+2}} \chi_{t>0}.$$

Then, if $\beta \in (0,1)$, *the following estimates hold*

$$\begin{aligned} &\text{if } \ n > 2, && |\partial_t^\beta f_1(t)| \lesssim_\beta \min\left\{1, |t|^{-1-\beta}\right\}, \\ &\text{if } \ n = 2 \text{ and } \beta > \frac{1}{2}, && |\partial_t^\beta f_1(t)| \lesssim_{\beta,\alpha} \min\left\{1, |t|^{-1-\beta+\alpha/2}\right\}, \quad \forall \alpha \in (0, 2+2\beta], \\ &\text{if } \ n = 1, && |\partial_t^\beta f_1(t)| \lesssim_\beta 1. \end{aligned}$$

In addition, for every n,

$$|\partial_t^\beta f_2(t)| \lesssim_\beta \min\left\{1, |t|^{-1-\beta}\right\}, \qquad |\partial_t^\beta f_3(t)| \lesssim_\beta \min\left\{1, |t|^{-1-\beta}\right\}.$$

For $t = 0$ *the previous estimates have to be understood simply as a bound by a constant depending on n and* β.

Proof We deal first with the estimate concerning $\partial_t^\beta f_1$ for $n > 2$. We distinguish whether if $t = 0$, $t < 0$ or $t > 0$. If $t = 0$ we are done because,

$$|\partial_t^\beta f_1(0)| \le \int_{\mathbb{R}} \frac{|f_1(u) - f_1(0)|}{|u-0|^{1+\beta}} \mathrm{d}u = \int_0^\infty \frac{e^{-1/u}}{u^{(n+2+2\beta)/2}} \mathrm{d}u = \Gamma\left(\frac{n+2\beta}{2}\right) \lesssim_\beta 1,$$

where Γ denotes the usual gamma function.

Let us continue by assuming $t < 0$. By definition,

$$|\partial_t^\beta f_1(t)| \le \int_{\mathbb{R}} \frac{|f_1(u)|}{|u+|t||^{1+\beta}} \mathrm{d}u = \int_0^\infty \frac{e^{-1/u}}{u^{n/2}(u+|t|)^{1+\beta}} \mathrm{d}u.$$

Observe that on the one hand, since $|u+|t|| \ge u$,

$$|\partial_t^\beta f_1(t)| \le \int_0^\infty \frac{e^{-1/u}}{u^{(n+2+2\beta)/2}} \mathrm{d}u \lesssim_\beta 1.$$

On the other hand, since $n > 2$,

$$|\partial_t^\beta f_1(t)| \le \frac{1}{|t|^{1+\beta}} \int_0^\infty \frac{e^{-1/u}}{u^{n/2}(u/|t|+1)^{1+\beta}} \mathrm{d}u \le \frac{1}{|t|^{1+\beta}} \int_0^\infty \frac{e^{-1/u}}{u^{n/2}} \mathrm{d}u \lesssim \frac{1}{|t|^{1+\beta}},$$

Therefore, $|\partial_t^\beta f_1(t)| \lesssim_\beta \min\left\{1, |t|^{-1-\beta}\right\}$ and we are done.

If $t > 0$, we split the integral as follows

$$|\partial_t^\beta f_1(t)| \le \int_{|u|\le t/2} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u + \int_{t/2\le|u|\le 2t} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u + \int_{|u|\ge 2t} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u =: I_1 + I_2 + I_3.$$

In I_1 we have $t/2 \le |u-t| \le 3t/2$. Therefore,

$$I_1 := \int_{-t/2}^{0} \frac{|f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u + \int_0^{t/2} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u \lesssim \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_0^{t/2} \frac{|f_1(u)-f_1(t)|}{t^{1+\beta}}\mathrm{d}u.$$

By the definition of f_1, the last term can be bound by

$$\frac{1}{t^{1+\beta}}\int_0^{t/2}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u + \frac{1}{t^{1+\beta}}\int_0^{t/2}\frac{e^{-1/t}}{t^{n/2}}\mathrm{d}u \simeq \frac{1}{t^{1+\beta}}\int_0^{t/2}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u + \frac{e^{-1/t}}{t^{(n+2\beta)/2}}. \tag{35}$$

We split the remaining integral as follows

$$\int_0^{t/2}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u = \int_0^{1}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u + \int_1^{t/2}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u$$
$$\le e^{-\frac{1}{2t}}\int_0^1 \frac{e^{-\frac{1}{2u}}}{u^{n/2}}\mathrm{d}u + e^{-2/t}\int_1^{t/2}\frac{1}{u^{n/2}}\mathrm{d}u \lesssim e^{-\frac{1}{2t}} + \frac{e^{-\frac{1}{2t}}}{t^{n/2-1}},$$

where in the first inequality we have used $e^{-1/u} \le e^{-\frac{1}{2u}}\, e^{-\frac{1}{2t}}$, which is true for $0 \le u \le t/2$; and in the second the general inequality $e^{-2/t} \le e^{-\frac{1}{2t}}$. In addition, observe that in the last step we have used that $n \ne 2$ in order to compute the corresponding integral. Thus, returning to (35), we obtain

$$I_1 \lesssim \frac{e^{-\frac{1}{2t}}}{t^{1+\beta}} + \frac{e^{-\frac{1}{2t}}}{t^{(n+2\beta)/2}}.$$

Notice that for $t > 0$

$$e^{-\frac{1}{2t}} \le 3\min\left\{1, t^{1+\beta}\right\}, \qquad e^{-\frac{1}{2t}} \le C\min\left\{t^{(n+2\beta)/2}, t^{(n-2)/2}\right\}, \tag{36}$$

where C depends only on n and β, and the second estimate only holds for $n > 1$ (if $n = 1$, $e^{-\frac{1}{2t}} \le Ct^{(n+2\beta)/2}$ still holds). Therefore, we finally get

$$I_1 \lesssim_\beta \frac{\min\left\{1, t^{1+\beta}\right\}}{t^{1+\beta}} + \frac{\min\left\{t^{(n+2\beta)/2}, t^{(n-2)/2}\right\}}{t^{(n+2\beta)/2}} \simeq \min\left\{1, \frac{1}{t^{1+\beta}}\right\}.$$

Let us turn to I_2. Write

$$I_2 := \int_{-2t}^{-t/2} \frac{|f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u + \int_{t/2}^{2t} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u \lesssim \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{t/2}^{2t} \frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u, \tag{37}$$

where in the first integral we have used that $3t/2 \le |u-t| \le 3t$. For the second integral observe that

$$|f_1(u)-f_1(t)| \le \sup_{\xi\in[s,t]} |f_1'(\xi)||u-t|, \qquad \text{where} \qquad f_1'(\xi) = \left(1-\frac{n}{2}\xi\right)\frac{e^{-1/\xi}}{\xi^{n/2+2}}\,\chi_{\xi>0}.$$

Since $t/2 \leq \xi \leq 2t$, we have

$$|f_1'(\xi)| \lesssim \big(1+t\big)\frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{t>0} = \big(1+t\big)\frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{t>1} + \big(1+t\big)\frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{0<t\leq 1}$$
$$\lesssim \frac{e^{-\frac{1}{2t}}}{t^{n/2+1}}\,\chi_{t>1} + \frac{e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{0<t\leq 1}.$$

Combining the last two estimates we can bound the remaining integral of (37) by

$$\left(\frac{e^{-\frac{1}{2t}}}{t^{n/2+1}}\,\chi_{t>1} + \frac{2e^{-\frac{1}{2t}}}{t^{n/2+2}}\,\chi_{0<t\leq 1}\right)\int_{t/2}^{2t}\frac{\mathrm{d}u}{|u-t|^{\beta}} \lesssim_{\beta} \frac{e^{-\frac{1}{2t}}}{t^{(n+2\beta)/2}}\,\chi_{t>1} + \frac{e^{-\frac{1}{2t}}}{t^{(n+2+2\beta)/2}}\,\chi_{0<t\leq 1}.$$

Thus,

$$I_2 \lesssim_{\beta} \frac{2e^{-\frac{1}{2t}}}{t^{(n+2\beta)/2}} + \frac{e^{-\frac{1}{2t}}}{t^{(n+2+2\beta)/2}}.$$

If we now apply estimates

$$e^{-\frac{1}{2t}} \leq C_1 \min\left\{t^{(n+2\beta)/2}, t^{(n-2)/2}\right\}, \qquad e^{-\frac{1}{2t}} \leq C_2 \min\left\{t^{(n+2+2\beta)/2}, t^{n/2}\right\},$$

for some constants C_1, C_2 depending on n and β, we conclude

$$I_2 \lesssim_{\beta} \frac{\min\left\{t^{(n+2\beta)/2}, t^{(n-2)/2}\right\}}{t^{(n+2\beta)/2}} + \frac{\min\left\{t^{(n+2+2\beta)/2}, t^{n/2}\right\}}{t^{(n+2+2\beta)/2}} \simeq \min\left\{1, \frac{1}{t^{1+\beta}}\right\}.$$

Finally, for I_3, since $|u|/2 \leq |u-t| \leq 3|u|/2$, we have

$$I_3 := \int_{-\infty}^{-2t}\frac{|f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u + \int_{2t}^{\infty}\frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u \simeq \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{2t}^{\infty}\frac{|f_1(u)-f_1(t)|}{|u-t|^{1+\beta}}\mathrm{d}u$$
$$\leq \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\mathrm{d}u + \int_{2t}^{\infty}\frac{e^{-1/t}}{t^{n/2}u^{1+\beta}}\mathrm{d}u \simeq_{\beta} \frac{e^{-1/t}}{t^{(n+2\beta)/2}} + \int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\mathrm{d}u$$
$$\lesssim_{\beta} \min\left\{1, \frac{1}{t^{1+\beta}}\right\} + \int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\mathrm{d}u.$$

For the remaining integral observe that on the one hand

$$\int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\mathrm{d}u \leq \Gamma\left(\frac{n+2\beta}{2}\right) \lesssim_{\beta} 1,$$

while on the other hand, since $u > 2t$,

$$\int_{2t}^{\infty}\frac{e^{-1/u}}{u^{(n+2+2\beta)/2}}\mathrm{d}u \lesssim \frac{1}{t^{1+\beta}}\int_{2t}^{\infty}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u \leq \frac{1}{t^{1+\beta}}\int_{0}^{\infty}\frac{e^{-1/u}}{u^{n/2}}\mathrm{d}u \lesssim \frac{1}{t^{1+\beta}},$$

where the last inequality holds since $n > 2$. Therefore, combining the previous estimates we conclude that for $n > 2$, $|\partial_t^{\beta} f_1(t)| \lesssim_{\beta} \min\left\{1, t^{-1-\beta}\right\}$.

Before approaching the case $n = 2$, let us comment that the case $n = 1$ also follows from the above arguments. We also notice that the bounds for $|\partial_t^{\beta} f_2|$ and $|\partial_t^{\beta} f_3|$ are obtained by exactly the same computations.

So we are left to verify the following estimate

$$|\partial_t^{\beta} f_1(t)| \lesssim_{\beta,\alpha} \min\left\{1, |t|^{-1-\beta+\alpha/2}\right\}, \quad \forall\alpha \in (0,3], \quad n = 2,$$

that can be also obtained following the same scheme of proof.

□

Proof of Theorem 2.5 We write $K_\beta(\overline{x}) := \partial_t^\beta W(\overline{x})$. Regarding estimate *1*, by the same reasoning presented at the beginning of the proof of [14, Lemma 2.1] we get

$$K_\beta(\overline{x}) \simeq \frac{1}{|x|^{n+2\beta}} \partial_t^\beta f_1\left(\frac{4t}{|x|^2}\right).$$

Hence, if $n > 2$, by Lemma 6.1 we get

$$|K_\beta(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n+2\beta}} \min\left\{1, \frac{|x|^{2+2\beta}}{|t|^{1+\beta}}\right\} = \frac{1}{|x|^{n-2}} \min\left\{\frac{1}{|x|^{2+2\beta}}, \frac{1}{|t|^{1+\beta}}\right\} = \frac{1}{|x|^{n-2}|\overline{x}|_{p_1}^{2+2\beta}}.$$

For estimates *2* and *3* we follow the same procedure.

We move on to estimate *4*. First, observe that the expression $\nabla_x K$ is well-defined and that the operators ∇_x and ∂_t^β commute when applied to W.

We also observe that there is a constant C such that

$$\nabla_x W(x,t) = C\frac{x}{(4t)^{n/2+1}} e^{-|x|^2/(4t)} \chi_{t>0} = C\frac{x}{|x|^{n+2}} \left(\frac{|x|^2}{4t}\right)^{n/2+1} e^{-|x|^2/(4t)} \chi_{t>0},$$

so we can write

$$\nabla_x W(x,t) = C\frac{x}{|x|^{n+2}} f_2\left(\frac{4t}{|x|^2}\right),$$

with f_2 defined in Lemma 6.1. Since ∇_x and ∂_t^β commute,

$$\nabla_x K(x,t) = C\frac{x}{|x|^{n+2}} \partial_t^\beta \left[f_2\left(\frac{4\,\cdot}{|x|^2}\right)\right](t).$$

The previous fractional derivative can be written as follows

$$\partial_t^\beta \left[f_2\left(\frac{4\,\cdot}{|x|^2}\right)\right](t) = \int_{\mathbb{R}} \frac{f_2(4u/|x|^2) - f_2(4t/|x|^2)}{|u-t|^{1+\beta}} \mathrm{d}u \simeq \frac{1}{|x|^{2\beta}} \partial_t^\beta f_2\left(\frac{4t}{|x|^2}\right),$$

yielding the final equality

$$\nabla_x K(\overline{x}) = C\frac{x}{|x|^{n+2+2\beta}} \partial_t^\beta f_2\left(\frac{4t}{|x|^2}\right).$$

Applying Lemma 6.1 we finally deduce *3*:

$$|\nabla_x K(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n+1+2\beta}} \min\left\{1, \frac{|x|^{2+2\beta}}{|t|^{1+\beta}}\right\} = \frac{1}{|x|^{n-1}|\overline{x}|_{p_1}^{2+2\beta}}.$$

Concerning inequality *4*, since the operators ∂_t^β and ∂_t commute, we directly have

$$\partial_t K(\overline{x}) = \int_{\mathbb{R}} \frac{\partial_t W(x,u) - \partial_t W(x,t)}{|u-t|^{1+\beta}} \mathrm{d}u,$$

and this integral makes sense.

As done for $\nabla_x W$, we can also rewrite $\partial_t W$ as follows,

$$\begin{aligned}\partial_t W(x,t) &= C_1 \frac{e^{-|x|^2/(4t)}}{t^{n/2+1}} + C_2|x|^2 \frac{e^{-|x|^2/(4t)}}{t^{n/2+2}} \\ &= \left[\frac{C_1'}{|x|^{n+2}}\left(\frac{|x|^2}{4t}\right)^{n/2+1} + \frac{C_2'}{|x|^{n+2}}\left(\frac{|x|^2}{4t}\right)^{n/2+2}\right] e^{-|x|^2/(4t)}\end{aligned}$$

$$= \frac{C_1'}{|x|^{n+2}} f_2\left(\frac{4t}{|x|^2}\right) + \frac{C_2'}{|x|^{n+2}} f_3\left(\frac{4t}{|x|^2}\right),$$

where f_3 is defined in Lemma 6.1. By exactly the same change of variables as the one performed when studying $\nabla_x K$, we reach the identity

$$\partial_t K(\overline{x}) = \frac{C_1'}{|x|^{n+2+2\beta}} \partial_t^\beta f_2\left(\frac{4t}{|x|^2}\right) + \frac{C_2'}{|x|^{n+2+2\beta}} \partial_t^\beta f_3\left(\frac{4t}{|x|^2}\right).$$

By Lemma 6.1, we get inequality *4*:

$$|\partial_t K(\overline{x})| \lesssim_\beta \frac{1}{|x|^{n+2+2\beta}} \min\left\{1, \frac{|x|^{2+2\beta}}{|t|^{1+\beta}}\right\} = \frac{1}{|x|^n |\overline{x}|_{p_1}^{2+2\beta}}.$$

Finally, regarding *5*, we follow exactly the same proof as that of estimate *4* in Theorem 2.4. □

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence or bias the work reported in this paper.

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