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# The stratification by automorphism groups of smooth plane sextic curves

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**Abstract**

We obtain the list of automorphism groups for smooth plane sextic curves over an algebraically closed field $K$ of characteristic $p = 0$ or $p > 21$. Moreover, we assign to each group a *geometrically complete family over* $K$ that describe the corresponding stratum, that is, a generic polynomial equation with parameters such that any curve in the stratum is $K$-isomorphic to a smooth plane model obtained by specializing the values of those parameters in $K$. Additionally, we explore the connection with K3 surfaces of degree 2.

**Keywords** Plane curves · Automorphism groups · K3 surfaces

**Mathematics Subject Classification** 14H37 · 14H10 · 14H45 · 14H50 · 14J28

## 1 Introduction

Smooth plane curves of degree $d \geq 4$ with non-trivial automorphism groups hold significant importance across various mathematical disciplines. For instance, in algebraic geometry, they are pivotal in the study of the Cremona group $\mathrm{Bir}(\mathbb{P}^2(\mathbb{C}))$ of the complex projective plane $\mathbb{P}^2(\mathbb{C})$. Understanding the dynamics of Cremona transformations defines them as dynamical systems. Notably, the finite subgroups of $\mathrm{Bir}(\mathbb{P}^2(\mathbb{C}))$ are classified by the set of birational classes of curves of fixed genus, which has been crucial in identifying infinitely many conjugacy classes of elements of order $2n$ for any integer $n$. See [9, 10] for further details.

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Smooth plane sextics also play a crucial role in the theory of $K3$-surfaces, which are closely tied to physics. In [16], the conjecture that the maximal number of lines on a smooth 2-polarized $K3$-surface is 144 was explored, exemplified by the double plane $X \to \mathbb{P}^2$ ramified over the smooth sextic curve

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + X^3Z^3) = 0.$$

The conjecture was proved in [17] in terms of tritangents to the ramification locus $C \subset \mathbb{P}^2$ (a smooth sextic curve) rather than lines in the surface $X \to \mathbb{P}^2$. Additionally, smooth plane sextics are rich subjects in arithmetic geometry. Notably, they were the first instances where counterexamples were found demonstrating that not every twist $C'$ of a smooth curve $C$ over a field $K$ is given by a smooth plane model over $K$, even if $C$ itself is. Further elaboration on this topic can be found in [5, 6].

The study of the automorphism groups of smooth curves of genus $g \geq 2$, defined over an algebraically closed field $K$ has long been a subject of interest in algebraic geometry. The celebrated Hurwitz bound, established by Hurwitz [37], serves as a universal limit that is sharp for infinitely many genera. Oikawa [54] and Arakawa [1] improved upon this bound by considering cases where the automorphism group fixes finite subsets of points on the curve. These results are particularly useful in the study of automorphism groups. While the structure of automorphism groups is well-understood for hyperelliptic curves, see [12, 13, 55, 56], much remains to be explored for non-hyperelliptic curves, except in some special cases such as low genus and Hurwitz curves, see [11, 33, 44–46]. This knowledge gap motivates further investigation into smooth plane curves of degree $d \geq 4$.

In the moduli space $\mathcal{M}_g^{\mathrm{Pl}}$ of smooth plane curves of degree $d \geq 4$ (thus, genus $g = \frac{1}{2}(d-1)(d-2)$) over the field $K$, each isomorphism class $C$ can be represented by a defining equation $C : F(X, Y, Z) = 0$ of degree $d$ of a smooth curve in $\mathbb{P}^2_K$. The automorphism group $\mathrm{Aut}(C)$ can be viewed as the finite subgroup of $\mathrm{PGL}_3(K)$ that preserves this smooth plane model $C : F(X, Y, Z) = 0$ in $\mathbb{P}^2_K$. Additionally, for a finite non-trivial group $G$, we consider the stratum $\mathcal{M}_g^{\mathrm{Pl}}(G)$ consisting of $K$-isomorphism classes of smooth plane curves $C$ of genus $g$ such that $\mathrm{Aut}(C)$ contains a subgroup isomorphic to $G$. In the case where $\mathrm{Aut}(C)$ itself is isomorphic to $G$, we use $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G)$, so that $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G) \subseteq \mathcal{M}_g^{\mathrm{Pl}}(G)$.

This leads to the following two natural questions.

**Question 1** *Let $G$ be a finite non-trivial group. What are the values of $d$ such that the corresponding stratum $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G) \neq \emptyset$, that is, there exists a smooth plane curve $C$ of degree $d$ over $K$ whose* $\mathrm{Aut}(C)$ *isomorphic to $G$?*

As far as we are aware, a comprehensive answer to the aforementioned question remains elusive, with only a handful of special cases addressed. For example, through the work of S. Crass in [15, p.28] we know that $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(\mathrm{A}_6) \neq \emptyset$ exactly when $d = 6$, $d = 12$ and $d = 30$, where $\mathrm{A}_6$ is the alternating group on six letters. The recent work of Y. Yoshida in [59] provides similar results when $G$ is the alternating group $\mathrm{A}_5$ or the Klein group $\mathrm{PSL}(2, 7)$.

Building upon the research of P. Henn in [33] and Komiya–Kuribayashi in [43] for degree 4 curves, and extending the work of Badr–Bars [2] for degree 5 curves, the following question arises.

**Question 2** *Fix an integer $g = \frac{1}{2}(d-1)(d-2) \geq 3$. What does the stratification of the $K$-isomorphism classes of smooth plane curve $C$ of degree $d$ by automorphism groups over $K$ look like? Equivalently, determine the list of finite groups $G$ so that $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G) \neq \emptyset$.*

In this paper, our objective is to address Question 2 specifically for degree 6 curves, which correspond to genus 10 curves. Apart from targeted computations to establish or refute the existence of specific cases, our approach relies significantly on the contributions of Badr–Bars in [3] and Harui in [30] for characteristic $p = 0$. We also guarantee that the results we present for $p = 0$ remain valid for characteristic $p > 2g + 1 = 21$, as generally established, for instance in [4, §6].

Additionally, it is noteworthy that Doi–Idei–Kaneta in [18] demonstrated that the maximum order of the automorphism group of smooth plane sextics is 360. Furthermore, they established that the most symmetric smooth plane sextic curve is $K$-isomorphic to the Wiman sextic curve:

$$\mathcal{W}_6 : 27X^6 + 9X(Y^5 + Z^5) - 135X^4YZ - 45X^2Y^2Z^2 + 10Y^3Z^3 = 0$$

whose automorphism group equals $\mathrm{A}_6$. They investigated the presence of smooth plane sextics invariant under a $p$-group $G$, where $G$ has an order dividing 360 and $p$ is a prime integer. The classification presented in our paper, specifically Theorem 2.1, represents a significant extension of their findings. For example, we establish the existence of infinitely many smooth plane sextic curves that are $\mathrm{Q}_8$ invariant. Here, "infinitely many" denotes curves that are not projectively equivalent, which diverges from [18, Lemma 2.11]. We elucidate the precise relationship between our results and those of [18] immediately following Theorem 2.1; please refer to Corollary 2.3 for a detailed discussion.

## 2 Statement of the main result

**Notations.** Throughout the paper, $X$, $Y$ and $Z$ are the homogeneous coordinates of the projective plane $\mathbb{P}^2(K)$ over $K$. For $B \in \{X, Y, Z\}$, $L_{i,B}$ denotes the generic homogeneous polynomial of degree $i$ in the variables $\{X, Y, Z\} \setminus \{B\}$. Additionally, a projective linear transformation $A = (a_{i,j}) \in \mathrm{PGL}_3(K)$ is sometimes written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z].$$

Moreover, we adopt the standard indexing convention of the atlas for small finite groups [26]. Here "SmallGroup$(n, m)$" refers to the finite group of order $n$ that appears in the $m$-th position of that atlas. See also GroupNames.

Regarding Question 2 for smooth plane sextic curves, we obtain:

**Theorem 2.1** *Let $K$ be an algebraically closed field of characteristic $p$, where $p = 0$ or $p > 21$. In Table 1, we provide a listing of automorphism groups of smooth plane sextics over $K$, accompanied by geometrically complete defining polynomial equation $F(X, Y, Z) = 0$ over $K$ for each stratum. Specifically, $\widetilde{\mathcal{M}_g^{\mathrm{Pl}}}(G) \neq \emptyset$ if and only if $G$ corresponds to one of the groups that appear in Table 1.*

*Some of the groups $G$ in the list do not have a unique representation in $\mathrm{PGL}_3(K)$; specifically, this occurs for $G$ being $(\mathbb{Z}/3\mathbb{Z})^2$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$, or $\mathbb{Z}/3\mathbb{Z}$. For these cases, we use the $\varrho$ prefix to distinguish between the different representations of the same group. For simplicity, we also omit the algebraic conditions on the parameters ensuring that each family remains smooth and does not admit a larger automorphism group.*

Secondly, the following diagram illustrates the structure of the stratification of smooth plane sextic curves by their automorphism groups.

**Table 1** Automorphism Groups and Defining Equations

| *ID* | Aut$(C)$ | Generators | $F(X,Y,Z)$ |
|---|---|---|---|
| (360,118) | $A_6$ | $R^{-1}T_iR$ for $i=1,2,3,4$<br>See Appendix 11.1.2 | $27X^6+9X(Y^5+Z^5)-135X^4YZ-45X^2Y^2Z^2+10Y^3Z^3$ |
| (216,92) | $(\mathbb{Z}/6\mathbb{Z})^2 \rtimes S_3$ | $\mathrm{diag}(\zeta_6,1,1),\ \mathrm{diag}(1,\zeta_6,1)$,<br>$[X:Z:Y],\ [Y:Z:X]$ | $X^6+Y^6+Z^6$ |
| (216,153) | $\mathrm{Hess}_{216}$ | $S=\mathrm{diag}(1,\zeta_3,\zeta_3^{-1}),\ U=\mathrm{diag}(1,1,\zeta_3)$,<br>$T=[Y:Z:X]$,<br>$V=\begin{pmatrix}1&1&1\\1&\zeta_3&\zeta_3^{-1}\\1&\zeta_3^{-1}&\zeta_3\end{pmatrix}$ | $X^6+Y^6+Z^6-10(X^3Y^3+Y^3Z^3+Z^3X^3)$ |
| (168,42) | $\mathrm{PSL}(2,7)$ | $\mathrm{diag}(1,\zeta_7,\zeta_7^3),\ [Y:Z:X]$,<br>$\begin{pmatrix}\zeta_7-\zeta_7^6&\zeta_7^2-\zeta_7^5&\zeta_7^4-\zeta_7^3\\\zeta_7^2-\zeta_7^5&\zeta_7^4-\zeta_7^3&\zeta_7-\zeta_7^6\\\zeta_7^4-\zeta_7^3&\zeta_7-\zeta_7^6&\zeta_7^2-\zeta_7^5\end{pmatrix}$ | $X^5Y+Y^5Z+XZ^5-5X^2Y^2Z^2$ |
| (144,122) | $\mathbb{Z}/3\mathbb{Z}\times \mathrm{GL}_2(\mathbb{F}_3)$ | $\mathrm{diag}(1,\zeta_{24},\zeta_{24}^{19}),\ [X:Z:Y]$,<br>$[X:c(\zeta_8^5Z-Y):\zeta_8c(\zeta_8Z-Y)]$<br>$c:=(1+i)/2$ | $X^6+Y^5Z+YZ^5$ |
| (72,43) | $\mathbb{Z}/3\mathbb{Z}\rtimes S_4$ | $\mathrm{diag}(1,\zeta_6,\zeta_6^2),\ \mathrm{diag}(1,1,-1)$,<br>$[Z:Y:X],\ [Y:Z:X]$ | $X^6+Y^6+Z^6+\beta_{2,2}X^2Y^2Z^2$ |
| (63,3) | $\mathbb{Z}/21\mathbb{Z}\rtimes\mathbb{Z}/3\mathbb{Z}$ | $\mathrm{diag}(1,\zeta_{21},\zeta_{21}^{17}),\ [Z:X:Y]$, | $X^5Y+Y^5Z+XZ^5$ |

**Table 1** continued

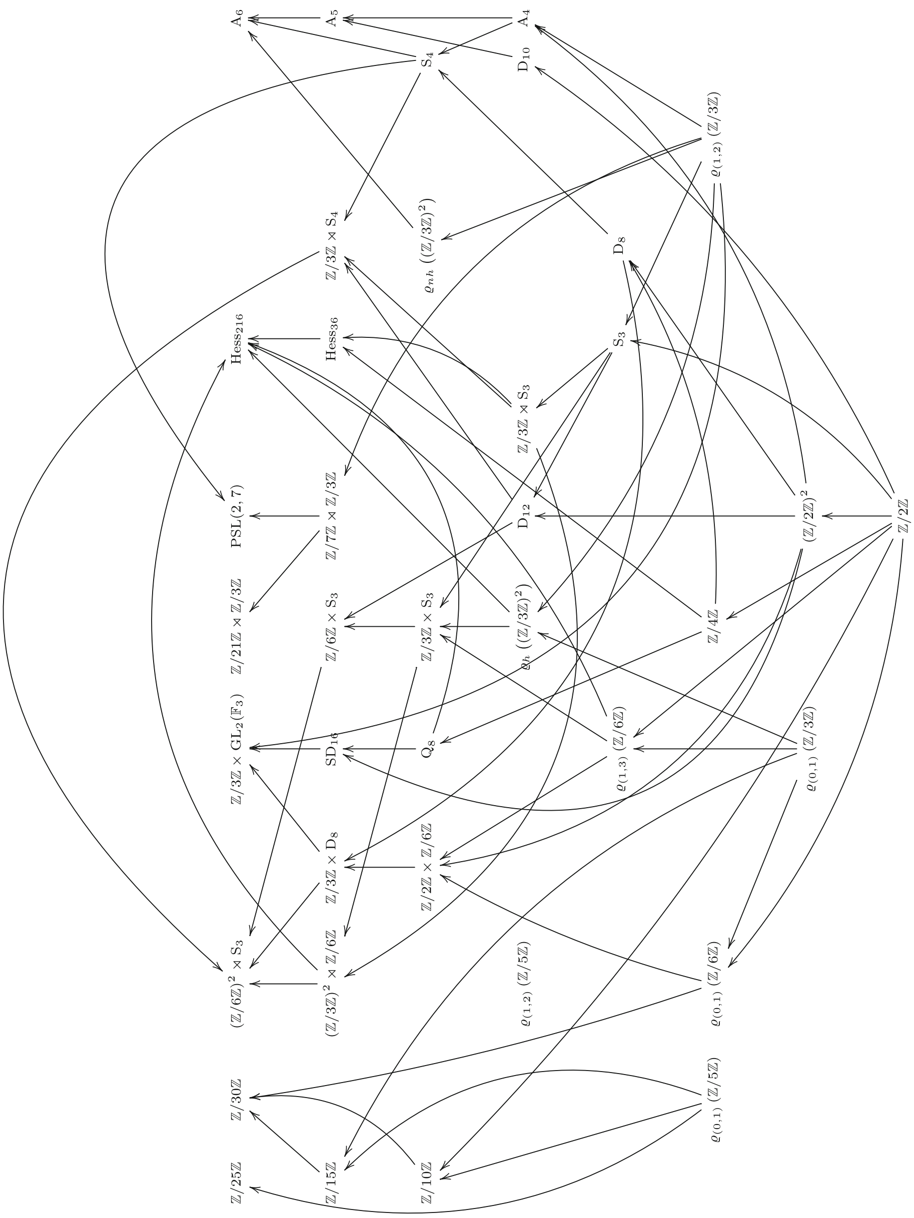
| *ID* | Aut($C$) | Generators | $F(X, Y, Z)$ |
|---|---|---|---|
| (60,5) | $\mathrm{A}_5$ | $\mathrm{diag}(1, \zeta_5, \zeta_5^{-1}),\ [X : Z : Y],$ | $32X^6 + \gamma^5 X(Y^5 + Z^5) + 8(12 - \gamma^5)X^4YZ+$ |
| | | $\begin{pmatrix} 1 & 1 & 1 \\ 2 & (-1+\sqrt{5})/2 & (-1-\sqrt{5})/2 \\ 2 & (-1-\sqrt{5})/2 & (-1+\sqrt{5})/2 \end{pmatrix}$ | $+2(48 + \gamma^5)X^2Y^2Z^2 + (32 - \gamma^5)Y^3Z^3$ |
| (54,5) | $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/6\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_3, 1),\ \mathrm{diag}(1, 1, \zeta_3), [X : Z : Y]$ | $X^6 + Y^6 + Z^6 + \beta_{3,3}\left(X^3Y^3 + Y^3Z^3 + X^3Z^3\right)$ |
| | | $[Z : X : Y]$ | |
| (36,12) | $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$ | $\mathrm{diag}(\zeta_6, 1, 1),\ \mathrm{diag}(1, \zeta_3, 1), [X : Z : Y]$ | $X^6 + Y^6 + Z^6 + \beta_{0,3}Y^3Z^3,$ |
| (36,9) | $\mathrm{Hess}_{36}$ | $S,\ T,\ V,\ UVU^{-1}$ | $X^6 + Y^6 + Z^6 + \beta_{4,1}XYZ(X^3 + Y^3 + Z^3)$ |
| | | | $+3\beta_{4,1}X^2Y^2Z^2 - 2(\beta_{4,1} + 5)(X^3Y^3 + Y^3Z^3 + X^3Z^3)$ |
| (30,4) | $\mathbb{Z}/30\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{30}^5, \zeta_{30}^6)$ | $X^6 + Y^6 + XZ^5$ |
| (25,1) | $\mathbb{Z}/25\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{25}, \zeta_{25}^{20})$ | $X^6 + Y^5Z + XZ^5$ |
| (24,12) | $\mathrm{S}_4$ | $\mathrm{diag}(1, -1, 1),\ \mathrm{diag}(1, 1, -1),$ | $X^6 + Y^6 + Z^6 + \beta_{2,2}X^2Y^2Z^2$ |
| | | $[X : Z : Y], [Y : Z : X]$ | $+\beta_{2,4}(X^2Y^4 + Y^2Z^4 + X^4Z^2 + X^2Z^4 + X^4Y^2 + Y^4Z^2)$ |
| (24,10) | $\mathbb{Z}/3\mathbb{Z} \times \mathrm{D}_8$ | $\mathrm{diag}(1, \zeta_{12}, \zeta_{12}^7), [X : Z : Y]$ | $X^6 + Y^5Z + YZ^5 + \beta_{3,3}Y^3Z^3$ |
| (21,1) | $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_7, \zeta_7^3),\ [Y : Z : X]$ | $X^5Y + Y^5Z + XZ^5 + \beta_{4,2}X^2Y^2Z^2$ |
| (18,3) | $\mathbb{Z}/3\mathbb{Z} \times \mathrm{S}_3$ | $\mathrm{diag}(1, \zeta_3, 1),\ \mathrm{diag}(1, 1, \zeta_3), [X : Z : Y]$ | $X^6 + Y^6 + Z^6 + \beta_{0,3}Y^3Z^3 + \beta_{3,3}X^3\left(Y^3 + Z^3\right)$ |
| (18,4) | $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3$ | $S,\ T,\ V^2$ | $X^6 + Y^6 + Z^6 + \beta_{4,1}XYZ(X^3 + Y^3 + Z^3) + \beta_{2,2}X^2Y^2Z^2$ |
| | | | $+\beta_{3,3}(X^3Y^3 + Y^3Z^3 + X^3Z^3)$ |
| (16,8) | $\mathrm{SD}_{16}$ | $\mathrm{diag}(1, \zeta_8, \zeta_8^3),\ [X : Z : Y]$ | $X^6 + Y^5Z + YZ^5 + \beta_{4,2}X^2Y^2Z^2$ |
| (15,1) | $\mathbb{Z}/15\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{15}^5, \zeta_{15}^6)$ | $X^6 + Y^6 + XZ^5 + \beta_{3,3}X^3Y^3$ |

**Table 1** continued

| *ID* | Aut($C$) | Generators | $F(X,Y,Z)$ |
|---|---|---|---|
| (12,5) | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ | $\mathrm{diag}(\zeta_6, 1, 1),\ \mathrm{diag}(1, 1, -1)$ | $X^6 + Y^6 + Z^6 + \beta_{2,4} Y^2 Z^4 + \beta_{4,2} Y^4 Z^2$ |
| (12,4) | $\mathrm{D}_{12}$ | $\mathrm{diag}(1, \zeta_6, \zeta_6^2),\ [Z : Y : X]$ | $X^6 + Y^6 + Z^6 + \beta_{3,0} X^3 Z^3 + \beta_{2,2} X^2 Y^2 Z^2 + \beta_{1,4} X Y^4 Z$ |
| (12,3) | $\mathrm{A}_4$ | $\mathrm{diag}(1, 1, -1),\ \mathrm{diag}(1, -1, 1)$ | $X^6 + Y^6 + Z^6 + \beta_{4,2}(X^4Y^2 + Y^4Z^2 + X^2Z^4) + \beta_{2,2} X^2 Y^2 Z^2$ |
| | | $[Y : Z : X]$ | $+\beta_{2,4}(X^2Y^4 + Y^2Z^4 + X^4Z^2)$ |
| | | | $X^6 + \zeta_3^{-1} Y^6 + \zeta_3 Z^6 + \beta_{4,2}(\zeta_3^{-1} X^4 Y^2 + \zeta_3 Y^4 Z^2 + X^2 Z^4)$ |
| | | | $+\beta_{2,4}(\zeta_3^{-1} X^2 Y^4 + \zeta_3 Y^2 Z^4 + X^4 Z^2)$ |
| (10,1) | $\mathrm{D}_{10}$ | $\mathrm{diag}(1, \zeta_5, \zeta_5^{-1}),\ [X : Z : Y]$ | $X^6 + X(Y^5 + Z^5) + \beta_{4,1} X^4 Y Z + \beta_{2,2} X^2 Y^2 Z^2 + \beta_{0,3} Y^3 Z^3$ |
| (10,2) | $\mathbb{Z}/10\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{10}^5, \zeta_{10}^6)$ | $X^6 + Y^6 + X Z^5 + \beta_{4,2} X^4 Y^2 + \beta_{2,4} X^2 Y^4$ |
| (9,2) | $\varrho_h\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ | $\mathrm{diag}(1, \zeta_3, 1), \mathrm{diag}(1, 1, \zeta_3)$ | $X^6 + Y^6 + Z^6 + Z^3\left(\beta_{3,0} X^3 + \beta_{0,3} Y^3\right) + \beta_{3,3} X^3 Y^3$ |
| (9,2) | $\varrho_{nh}\left((\mathbb{Z}/3\mathbb{Z})^2\right)$ | $S,\ T$ | $X^5 Y + Y^5 Z + X Z^5 + \beta_{2,4}(X^2Y^4 + Y^2Z^4 + X^4Z^2)$ |
| | | | $+\beta_{1,3}(XY^3Z^2 + X^2YZ^3 + X^3Y^2Z)$ |
| | | | $X^6 + \zeta_6^{2\ell} Y^6 + \zeta_6^{-2\ell} Z^6 + \beta_{1,1} XYZ(X^3 + \zeta_6^{2\ell} Y^3 + \zeta_6^{-2\ell} Z^3)$ |
| | | | $+\beta_{3,0}(X^3Y^3 + \zeta_6^{-2\ell} X^3 Z^3 + \zeta_6^{2\ell} Y^3 Z^3)$ |
| (8,4) | $\mathrm{Q}_8$ | $\mathrm{diag}(1, \zeta_4, \zeta_4^{-1}), [X : \zeta_8 Z : -\zeta_8^{-1} Y]$ | $X^6 + Y^5 Z + Y Z^5 + \beta_{2,0} X^2 (Z^4 - Y^4) + \beta_{2,2} X^2 Y^2 Z^2$ |
| (8,3) | $\mathrm{D}_8$ | $\mathrm{diag}(1, \zeta_4, \zeta_4^{-1}), [X : Z : Y]$ | $X^6 + Y^5 Z + Y Z^5 + \beta_{0,3} Y^3 Z^3 + \beta_{4,1} X^4 Y Z$ |
| | | | $+X^2\left(\beta_{2,0} Z^4 + \beta_{2,2} Y^2 Z^2 + \beta_{2,0} Y^4\right)$ |

**Table 1** continued

| *ID* | Aut($C$) | Generators | $F(X, Y, Z)$ |
|---|---|---|---|
| (6,1) | $S_3$ | $\mathrm{diag}(1, \zeta_3, \zeta_3^{-1}), [X : Z : Y]$ | $X^6 + Y^6 + Z^6 + \beta_{4,1}X^4YZ + \beta_{3,3}X^3(Y^3 + Z^3) + \beta_{0,3}Y^3Z^3$ |
| | | | $+\beta_{2,2}X^2Y^2Z^2 + \beta_{1,2}XYZ(Y^3 + Z^3)$ |
| (6,2) | $\varrho_{(0,1)}$ $(\mathbb{Z}/6\mathbb{Z})$ | $\mathrm{diag}(\zeta_6, 1, 1)$ | $X^6 + L_{6,X}$ |
| (6,2) | $\varrho_{(1,3)}$ $(\mathbb{Z}/6\mathbb{Z})$ | $\mathrm{diag}(1, \zeta_6, -1)$ | $X^6 + Y^6 + Z^6 + \beta_{2,0}X^4Z^2 + X^2\left(\beta_{4,0}Z^4 + \beta_{4,3}Y^3Z\right)$ |
| | | | $+\beta_{0,3}Y^3Z^3$ |
| (5,1) | $\varrho_{(1,2)}$ $(\mathbb{Z}/5\mathbb{Z})$ | $\mathrm{diag}(1, \zeta_5, \zeta_5^2)$ | $X^6 + X(Y^5 + Z^5) + \beta_{3,1}X^3YZ^2 + \beta_{2,3}X^2Y^3Z + \beta_{0,2}Y^2Z^4$ |
| (5,1) | $\varrho_{(0,1)}$ $(\mathbb{Z}/5\mathbb{Z})$ | $\mathrm{diag}(1, 1, \zeta_5)$ | $Z^5Y + L_{6,Z}$ |
| (4,1) | $\mathbb{Z}/4\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_4, \zeta_4^{-1})$ | $X^6 + Y^5Z + YZ^5 + \beta_{0,3}Y^3Z^3 + \beta_{4,1}X^4YZ$ |
| | | | $+X^2\left(\beta_{2,0}Z^4 + \beta_{2,2}Y^2Z^2 + \beta_{2,4}Y^4\right)$ |
| (4,2) | $(\mathbb{Z}/2\mathbb{Z})^2$ | $\mathrm{diag}(1, 1, -1),\ \mathrm{diag}(1, -1, 1)$ | $Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z}$ |
| | | | $L_{i,Z} \in K[X^2, Y^2]$ |
| (3,1) | $\varrho_{(0,1)}$ $(\mathbb{Z}/3\mathbb{Z})$ | $\mathrm{diag}(1, 1, \zeta_3)$ | $Z^6 + Z^3L_{3,Z} + L_{6,Z}$ |
| (3,1) | $\varrho_{(1,2)}$ $(\mathbb{Z}/3\mathbb{Z})$ | $\mathrm{diag}(1, \zeta_3, \zeta_3^{-1})$ | $X^5Y + Y^5Z + XZ^5 + \beta_{2,4}X^2Y^4 + \beta_{0,2}Y^2Z^4 + \beta_{4,0}X^4Z^2$ |
| | | | $+XYZ(\beta_{3,2}X^2Y + \beta_{1,3}Y^2Z + \beta_{2,1}XZ^2)$ |
| | | | $X^6 + Y^6 + Z^6 + XYZ\left(\beta_{4,1}X^3 + \beta_{1,4}Y^3 + \beta_{1,2}Z^3\right)$ |
| | | | $+\beta_{2,2}X^2Y^2Z^2 + \beta_{3,3}X^3Y^3 + \beta_{3,0}X^3Z^3 + \beta_{0,3}Y^3Z^3$ |
| (2,1) | $\mathbb{Z}/2\mathbb{Z}$ | $\mathrm{diag}(1, 1, -1)$ | $Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z}$ |


$\mathrm{A}_6$
$\mathrm{A}_5$
$\mathrm{A}_4$
$\mathrm{S}_4$
$\mathrm{D}_{10}$
$\varrho_{(1,2)}(\mathbb{Z}/3\mathbb{Z})$
$\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_4$
$\varrho_{nh}((\mathbb{Z}/3\mathbb{Z})^2)$
$\mathrm{D}_8$
$\mathrm{Hess}_{216}$
$\mathrm{Hess}_{36}$
$\mathrm{S}_3$
$\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3$
$\mathrm{PSL}(2,7)$
$\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$
$\mathrm{D}_{12}$
$(\mathbb{Z}/2\mathbb{Z})^2$
$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$
$\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$
$\mathbb{Z}/3\mathbb{Z} \times \mathrm{S}_3$
$\varrho_h((\mathbb{Z}/3\mathbb{Z})^2)$
$\mathbb{Z}/4\mathbb{Z}$
$\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)$
$\mathrm{SD}_{16}$
$\mathrm{Q}_8$
$\varrho_{(1,3)}(\mathbb{Z}/6\mathbb{Z})$
$\varrho_{(0,1)}(\mathbb{Z}/3\mathbb{Z})$
$\mathbb{Z}/3\mathbb{Z} \times \mathrm{D}_8$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$
$(\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$
$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/6\mathbb{Z}$
$\varrho_{(1,2)}(\mathbb{Z}/5\mathbb{Z})$
$\varrho_{(0,1)}(\mathbb{Z}/6\mathbb{Z})$
$\mathbb{Z}/30\mathbb{Z}$
$\varrho_{(0,1)}(\mathbb{Z}/5\mathbb{Z})$
$\mathbb{Z}/25\mathbb{Z}$
$\mathbb{Z}/15\mathbb{Z}$
$\mathbb{Z}/10\mathbb{Z}$

***Remark 2.2*** It is important to note that some conjugacy classes of subgroups $H \subseteq G \subset \mathrm{PGL}_3(K)$ are represented by larger strata. For instance, the subgroup $\mathbb{Z}/5\mathbb{Z}$, which is embedded in $\mathrm{PGL}_3(K)$ as $H = \langle \mathrm{diag}(1, \zeta_5, \zeta_5^{-1})\rangle$, corresponds to the stratum $\mathrm{D}_{10}$. This means that any smooth plane sextic curve with an automorphism group containing this $\mathbb{Z}/5\mathbb{Z}$ subgroup will always have an automorphism group larger than just $\mathbb{Z}/5\mathbb{Z}$ itself.

Also, not all conjugacy classes of subgroups $H \subseteq G \subset \mathrm{PGL}_3(K)$ are realized by the adjacencies of the respective strata, e.g. for example the group $\mathbb{Z}/15\mathbb{Z}$, which is represented in $\mathrm{PGL}_3(K)$ as $H = \langle \mathrm{diag}(1, 1, \zeta_{15})\rangle$, does not appear as automorphism group of a smooth plane curve of degree 6, but it appears $\langle \mathrm{diag}(1, \zeta_{15}^5, \zeta_{15}^6)$ which is not conjugate of $H$ in $\mathrm{PGL}_3(K)$.

**Corollary 2.3** *We align the following observations with the findings in* [18] *concerning the subgroups of automorphisms for smooth plane sextics of orders 5, 8, 9, and 27, respectively.*

1. *We find that* $\mathcal{M}_{10}^{\mathrm{Pl}}\left(\varrho_{(0,1)}(\mathbb{Z}/5\mathbb{Z})\right) \cap \mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{A}_6) = \emptyset$*, consistent with* [18, Proposition 2.13]. *Also,* $\mathcal{M}_{10}^{\mathrm{Pl}}\left(\varrho_{(1,2)}(\mathbb{Z}/5\mathbb{Z})\right)$ *exhibits two disjoint ES-irreducible components:* $\widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}\left(\varrho_{(1,2)}(\mathbb{Z}/5\mathbb{Z})\right)$ *and* $\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{D}_{10}) = \widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}(\mathrm{D}_{10}) \sqcup \widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}(\mathrm{A}_6)$*, in agreement with* [18, Lemma 2.14].
2. *We find that* $\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{Q}_8) \neq \emptyset$*, in contrast to* [18, Lemma 2.11]. *Specifically, this stratum is two-dimensional and decomposes as follows:*
$$\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{Q}_8) = \widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}(\mathrm{Q}_8) \sqcup \widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}(\mathrm{SD}_{16}) \sqcup \widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}(\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)) \sqcup \widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}(\mathrm{Hess}_{216}).$$
*However, the key finding from* [18] *asserting the Wiman sextic as the most symmetric smooth plane sextic remains valid.*
3. *We observe that* $\mathcal{M}_{10}^{\mathrm{Pl}}(G) = \emptyset$ *when* $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ *or* $(\mathbb{Z}/2\mathbb{Z})^3$*, consistent with* [18, Lemmas 2.8 and 2.9]. *Also,* $\mathcal{M}_{10}^{\mathrm{Pl}}(G) \neq \emptyset$ *for* $G = \mathbb{Z}/8\mathbb{Z}$ *and* $\mathrm{D}_8$*, each possessing a single ES-irreducible component. Specifically,*
$$\mathcal{M}_{10}^{\mathrm{Pl}}(\mathbb{Z}/8\mathbb{Z}) = \mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{SD}_{16}) = \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{SD}_{16}) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)),$$
*and*
$$\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{D}_8) = \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{D}_8) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{S}_4) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{A}_6) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_4) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}\left((\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3\right).$$
4. *The strata* $\mathcal{M}_{10}^{\mathrm{Pl}}(G) = \emptyset$ *for* $G = \mathbb{Z}/9\mathbb{Z}$ *and* $(\mathbb{Z}/3\mathbb{Z})^3$*, which follows* [18, Lemmas 1.4 and 1.5].
5. *For the Heisenberg group* $\mathrm{He}_3$ *of order* 27*, generated by* $S, U, T$*, we obtain that* $\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{He}_3) \neq \emptyset$*, and also* $\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{He}_3) \cap \mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{A}_6) = \emptyset$*. This is again consistent with* [18, Lemmas 1.8 and 1.9]. *Moreover, we refine this result by demonstrating that*
$$\mathcal{M}_{10}^{\mathrm{Pl}}(\mathrm{He}_3) = \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}\left((\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/6\mathbb{Z}\right) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}\left((\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3\right) \sqcup \widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\mathrm{Hess}_{216}).$$

In the stratification, we observe another instance of a phenomenon absent in quartics but emerging in quintics for the first time and now in sextics. In [8], Badr–Lorenzo introduced the concept of a *final stratum* defined by automorphism groups, which means a non-zero dimensional stratum that does not properly contain any other stratum. This might seem counterintuitive, as one might expect additional conditions on parameters to result in larger automorphism groups. However, they demonstrated in [8] that this pattern is typical for higher odd degrees $d$ such that $d = 1 \bmod 4$.

For even degrees, we present the following example:

**Corollary 2.4** *In the class of smooth sextics there is but a single final stratum, namely* $\widetilde{\mathcal{M}^{\mathrm{Pl}}}_{10}\left(\varrho_{(1,2)}\left(\mathbb{Z}/5\mathbb{Z}\right)\right)$.

***Remark 2.5*** We have confirmed the uniqueness property of the final stratum for smooth quintic and sextic curves. While this behavior holds for these two cases, it is not clear whether the same property persists for curves of higher degree ($d \geq 7$). Further investigation would be needed to determine whether a similar pattern arises in higher degrees, or if additional factors influence the structure of the strata.

### 2.1 Connection with *K*3 surfaces

Given a $K3$ surface $\mathcal{X}$, there exists an unique (up to scaling) holomorphic 2-form $\omega_{\mathcal{X}}$ associated with it. Automorphisms of $\mathcal{X}$ that preserve $\omega_{\mathcal{X}}$ are called *symplectic*. We denote by $\mathrm{Aut}_{sym}(\mathcal{X})$ the normal subgroup of $\mathrm{Aut}(\mathcal{X})$ given by the symplectic automorphisms of $\mathcal{X}$.

Assume that the full automorphism group $\mathrm{Aut}(\mathcal{X})$ of $\mathcal{X}$ is finite, meaning that $\mathcal{X}$ admits an ample class invariant under the action of $\mathrm{Aut}(\mathcal{X})$. Let $\rho : \mathrm{Aut}(\mathcal{X}) \to \mathbb{C}^*$ be the group homomorphism defined by:

$$\sigma^*(\omega_{\mathcal{X}}) = \rho(\sigma) \cdot \omega_{\mathcal{X}} \quad \text{for} \quad \sigma \in \mathrm{Aut}(\mathcal{X}).$$

This gives rise to the exact sequence:

$$1 \to \mathrm{Aut}_{sym}(\mathcal{X}) \to \mathrm{Aut}(\mathcal{X}) \xrightarrow{\rho} \mu_n \to 1$$

Concerning the non-symplectic part $\mu_n$ of $\mathrm{Aut}(\mathcal{X})$, numerous studies have sought to classify the cyclic groups that can appear in this context. For instance, we have $\varphi(n) \leq 20$, where $\varphi$ denotes Euler's totient function, see the pioneering work in the subject by Nikulin [52]. Notable references include works by Kondo [40], Xiao [57], Machida–Oguiso [47], among others, though we acknowledge possible omissions. For symplectic automorphisms, Nikulin's result [52, 53] shows that finite-order symplectic automorphisms have orders $\leq 8$, and he classified the finite abelian groups that can act as symplectic automorphism groups of $K3$ surfaces. Later, Mukai [51] showed that $\mathrm{Aut}_{sym}(\mathcal{X})$ can be embedded in the Mathieu group $M_{23}$ as a subgroup of one of the following maximal symplectic groups:

$$\begin{array}{llrl}
\mathrm{PSL}(2,7) & (\text{order } 168) & M_{20} & (\text{order } 960)\\
F_{384} = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{S}_4 & (\text{order } 384) & \mathrm{A}_6 & (\text{order } 360)\\
A_{4,4} = (\mathrm{S}_4 \times \mathrm{S}_4) \cap \mathrm{A}_8 & (\text{order } 288) & T_{192} = (\mathrm{Q}_8 * \mathrm{Q}_8) \rtimes \mathrm{S}_3 & (\text{order } 192)\\
H_{192} = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes D_{12} & (\text{order } 192) & \mathrm{S}_5 & (\text{order } 120)\\
N_{72} = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes D_8 & (\text{order } 72) & M_9 = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathrm{Q}_8 & (\text{order } 72)\\
T_{48} = \mathrm{Q}_8 \rtimes \mathrm{S}_3 & (\text{order } 48). & &
\end{array}$$

Further work is focused on determining which subgroups of these 11 groups can appear as the symplectic automorphism group of a $K3$ surface. Consequently, it has been proven that each of these maximal groups can indeed serve as the symplectic automorphism group for some K3 surface. The existence of these groups was established by Mukai himself, who provided explicit equations for each of the 11 groups. Additionally, Xiao provides a complete list of finite groups that can act as symplectic automorphisms of $K3$ surfaces, detailing 81 distinct cases in [58, Theorem 3]. Other contributions, such as those by Garbagnati et al. [27–29], Hashimoto [32], and Kondo [42], further investigate the possible subgroups in this context.

In most cases, the study of symplectic automorphisms focuses on the action of $G = \mathrm{Aut}_{sym}(\mathcal{X})$ on the $K3$ lattice $\Lambda := \mathrm{H}^2(\mathcal{X}, \mathbb{Z})$, and the isometry class of the invariant lattice $\Lambda_G$, also known as the orthogonal group of the $K3$ lattice $\Lambda$. Nikulin observed that for abelian groups, this class is unique up to isomorphism, allowing for the description of moduli spaces of $K3$ surfaces with finite abelian symplectic automorphisms. However, for non-abelian groups, unicity is not always guaranteed, although it holds in most cases [32]. For cases in positive characteristic, the techniques discussed here do not extend, as shown by Dolgachev and Keum in [19].

Returning to the case of smooth plane sextics, a $K3$ surface $\mathcal{X}$ of degree 2 over $\mathbb{C}$ can be viewed as a double cover of the projective plane $\mathbb{P}^2(\mathbb{C})$, branched along a sextic plane curve $C$ defined by $F(X, Y, Z) = 0$. In a more concrete construction, $\mathcal{X}$ can be realized in the weighted projective space $\mathbb{P}^{3,1,1,1}$ as the set of points satisfying:

$$\mathcal{X} = \{(t, X, Y, Z) \in \mathbb{P}^{3,1,1,1} : t^2 = F(X, Y, Z)\}$$

The converse also holds, as a given plane sextic curve $C$ defines a unique double cover $\mathcal{V}(C)$ of the projective plane, whose desingularization is a $K3$ surface of degree 2 if and only if the singularities of $\mathcal{V}(C)$ are isolated rational double points, as clearly stated in [38]. This condition is trivially satisfied when $C : F(X, Y, Z) = 0$ is smooth. On the other hand, the involution $g : (t, X, Y, Z) \mapsto (-t, X, Y, Z)$ on $\mathcal{X}$ has infinitely many fixed points, namely, the points on the smooth degree 6 curve $C$. Consequently, by [41, Lemma 8.25], we conclude that $g$ is a non-symplectic automorphism. Moreover, some of the automorphisms of $C$ can be symplectic, but others are not as will be illustrated as we go.

Our results, particularly in Theorem 2.1, offer a list of finite-order automorphisms for $K3$ surfaces of degree 2, though not necessarily symplectic, building on the work of Xiao [58]. We take an ad-hoc approach by constructing automorphisms of smooth plane sextics to describe a moduli framework, touching upon the question of uniqueness of the action of a fixed symplectic group on $\mathcal{X}$. However, our techniques do not delve into the more advanced study of the automorphism group's action on the lattice $\Lambda$ or its representation within the Mathieu group $M_{23}$. Specifically, we observe the following:

(i) Given that the finite-order symplectic automorphisms of $K_3$ surfaces have order at most 8, we can rule out the following groups in Theorem 2.1 as possible symplectic subgroups.

| | | |
|---|---|---|
| $(\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$ | $\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)$ | $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ |
| $\mathbb{Z}/30\mathbb{Z}$ | $\mathbb{Z}/25\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z} \times \mathrm{D}_8$ |
| $\mathbb{Z}/15\mathbb{Z}$ | $\mathbb{Z}/10\mathbb{Z}$ | |

(ii) Although, $\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)$ does not act symplectically, it includes the maximal symplectic group $T_{48}$ (also known as $\mathrm{GL}_2(\mathbb{F}_3)$), which is realized by a degree two $K3$ surface (see [51, Example 0.4]). Thus, we conclude that the following subgroups of $\mathrm{GL}_2(\mathbb{F}_3)$ act symplectically:

$$\mathrm{SD}_{16} \qquad \mathrm{Q}_8 \qquad \mathbb{Z}/4\mathbb{Z} \qquad (\mathbb{Z}/2\mathbb{Z})^2 \qquad \mathbb{Z}/2\mathbb{Z}$$

Similarly, the Hessian group $\mathrm{Hess}_{216}$ does not act symplectically, since its order is not a divisor of any of the eleven maximal symplectic groups. However, it includes the Mathieu maximal group $M_9$ (also known as the Hessian group $\mathrm{Hess}_{72}$), which can be realized by a degree two $K3$ surfaces. Therefore, we can also deduce that the following subgroups of $M_9$ acts symplectically:

$$\mathrm{Hess}_{36} \qquad \mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_3 \qquad \mathrm{S}_3 \qquad \varrho_{(1,2)}\,(\mathbb{Z}/3\mathbb{Z})$$

## 3 Automorphisms of maximal order and defining equations

Suppose that $C : F(X, Y, Z) = 0$ is a smooth plane sextic curve with non-trivial automorphism group $\mathrm{Aut}(C) \subseteq \mathrm{PGL}_3(K)$ over an algebraically closed field $K$ of characteristic 0, and let $\sigma \in \mathrm{Aut}(C)$ be of maximal order $m$. Up to projective equivalence, we can assume that $\sigma$ acts as $(X : Y : Z) \mapsto (X : \zeta_m^a Y : \zeta_m^b Z)$, where $\zeta_m$ denotes a fixed primitive $m$-th root of unity in $K$, and $a, b$ are integers with $0 \leq a < b \leq m-1$. In this case, we say that $C$ is of *Type* $m, (a, b)$.

Our findings in [3] enable us to enumerate all possible Types $m, (a, b)$ for a fixed degree $d$. In particular, we know that the positive integer $m$ should divide 21, 24, 25 or 30, see [3, Corollary 33]. On the other hand, we can we construct for each Type $m, (a, b)$ a *geometrically complete family over* $K$. That is, a defining polynomial equation $F_{m,(a,b)}(X, Y, Z) = 0$ with parameters $\beta_{i,j} \in K$ as its coefficients, where any smooth plane degree $d$ curve of Type $m, (a, b)$ is defined over $K$ by a specialization of those parameters.

As a consequence, we obtain:

**Proposition 3.1** *Let $C$ be a smooth plane curve of degree 6 over $K$. Then, $C$ falls into one of the following specified types. Particularly, we can see that $\mathcal{M}_{10}^{\mathrm{Pl}}(\mathbb{Z}/m\mathbb{Z}) \neq \emptyset$ if and only if $m \in \{2, 3, 4, 5, 6, 7, 8, 10, 12, 15, 21, 24, 25, 30\}$.*

## 4 The automorphism groups for very large Types $m, (a, b)$

In this section we determine the automorphism group $\mathrm{Aut}(C)$ in cases where there is $\sigma \in \mathrm{Aut}(C)$ of order $m = d(d-1)$, $(d-1)^2$, $d(d-2)$, $d^2 - 3d + 3$, or $q(d-1)$ for some $q \geq 2$. To do so, the following result is needed, see [3, 30]:

**Theorem 4.1** *Let $C$ be a smooth plane degree $d$ curve of Type $m, (a, b)$.*

1. *If $m = d(d-1)$, then $\mathrm{Aut}(C)$ is cyclic of order $d(d-1)$. In this scenario, $C$ is $K$-isomorphic to $C : X^d + Y^d + XZ^{d-1} = 0$, where $\mathrm{Aut}(C) = \langle\sigma\rangle$ with $\sigma = \mathrm{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d})$.*
2. *If $m = (d-1)^2$, then $\mathrm{Aut}(C)$ is cyclic of order $(d-1)^2$. In this scenario, $C$ is $K$-isomorphic to $C : X^d + Y^{d-1}Z + XZ^{d-1} = 0$, where $\mathrm{Aut}(C) = \langle\sigma\rangle$ with $\sigma = \mathrm{diag}(1, \zeta_{(d-1)^2}, \zeta_{(d-1)^2}^{-(d-1)})$.*
3. *If $m = d(d-2)$, then $C$ is $K$-isomorphic to $C : X^d + Y^{d-1}Z + YZ^{d-1} = 0$. For $d \neq 4, 6$, $\mathrm{Aut}(C)$ is a central extension of the dihedral group $\mathrm{D}_{2(d-2)}$ by $\mathbb{Z}/d\mathbb{Z}$. More precisely,*

   $$\mathrm{Aut}(C) = \langle \sigma,\ \tau \mid \sigma^{d(d-2)} = \tau^2 = 1,\ \tau\sigma\tau = \sigma^{-(d-1)}, \ldots \rangle,$$

   *with $\sigma = \mathrm{diag}(1, \zeta_{d(d-2)}, \zeta_{d(d-2)}^{-(d-1)})$ and $\tau = [X : Z : Y]$. Thus it has order $2d(d-2)$. For $d = 6$, $C$ admits an extra automorphism of order 3 namely, $\tau' = [X : c(\zeta_8^5 Z - Y) : \zeta_8 c(\zeta_8 Z - Y)]$ with $c = \frac{1+\zeta_4}{2}$. So $\mathrm{Aut}(C)$ is a central extension of $\mathrm{S}_4$ by $\mathbb{Z}/6$. Lastly, for $d = 4$, $C$ is $K$-isomorphic to the Fermat quartic curve $\mathcal{F}_4 : X^4 + Y^4 + Z^4 = 0$. Hence, $\mathrm{Aut}(C)$ would be isomorphic to $(\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathrm{S}_3$ of order 96.*

**Table 2** Maximal Cyclic Subgroups and Defining Equations

| | Type: $m$, $(a,\ b)$ | $F(X, Y, Z)$ |
|---|---|---|
| 1 | 30, (5, 6) | $X^6 + Y^6 + XZ^5$ |
| 2 | 25, (1, 20) | $X^6 + Y^5Z + XZ^5$ |
| 3 | 24, (1, 19) | $X^6 + Y^5Z + YZ^5$ |
| 4 | 21, (1, 17) | $X^5Y + Y^5Z + XZ^5$ |
| 5 | 15, (5, 6) | $X^6 + Y^6 + XZ^5 + \beta_{3,3}X^3Y^3$ |
| 6 | 12, (1, 7) | $X^6 + Y^5Z + YZ^5 + \beta_{3,3}Y^3Z^3$ |
| 7 | 10, (5, 6) | $X^6 + Y^6 + XZ^5 + \beta_{4,2}X^4Y^2 + \beta_{2,4}X^2Y^4$ |
| 8 | 8, (1, 3) | $X^6 + Y^5Z + YZ^5 + \beta_{4,2}X^2Y^2Z^2$ |
| 9 | 7, (1, 3) | $X^5Y + Y^5Z + XZ^5 + \beta_{4,2}X^2Y^2Z^2$ |
| 10 | 6, (1, 2) | $X^6 + Y^6 + Z^6 + \beta_{3,0}X^3Z^3 + \beta_{2,2}X^2Y^2Z^2 + \beta_{1,4}XY^4Z$ |
| 11 | 6, (1, 3) | $X^6 + Y^6 + Z^6 + \beta_{2,0}X^4Z^2 + \beta_{0,3}Y^3Z^3 + X^2\left(\beta_{4,0}Z^4 + \beta_{4,3}Y^3Z\right)$ |
| 12 | 6, (0, 1) | $Z^6 + L_{6,Z}$ |
| 13 | 5, (1, 2) | $X^6 + XZ^5 + XY^5 + \beta_{3,1}X^3YZ^2 + \beta_{2,3}X^2Y^3Z + \beta_{0,2}Y^2Z^4$ |
| 14 | 5, (1, 4) | $X^6 + XZ^5 + XY^5 + \beta_{4,1}X^4YZ + \beta_{2,2}X^2Y^2Z^2 + \beta_{0,3}Y^3Z^3$ |
| 15 | 5, (0, 1) | $Z^5Y + L_{6,Z}$ |
| 16 | 4, (1, 3) | $X^6 + Y^5Z + YZ^5 + \beta_{0,3}Y^3Z^3 + \beta_{4,1}X^4YZ + X^2\left(\beta_{2,0}Z^4 + \beta_{2,2}Y^2Z^2 + \beta_{2,4}Y^4\right)$ |
| 17 | 3, (1, 2) | $X^5Y + Y^5Z + XZ^5 + XYZ\left(\beta_{3,2}X^2Y + \beta_{1,3}Y^2Z + \beta_{2,1}XZ^2\right) + \beta_{2,4}X^2Y^4 + \beta_{0,2}Y^2Z^4 + \beta_{4,0}X^4Z^2$ |
| 18 | 3, (1, 2) | $X^6 + Y^6 + Z^6 + XYZ\left(\beta_{4,1}X^3 + \beta_{1,4}Y^3 + \beta_{1,2}Z^3\right) + \beta_{2,2}X^2Y^2Z^2 + \beta_{3,3}X^3Y^3 + \beta_{3,0}X^3Z^3 + \beta_{0,3}Y^3Z^3$ |
| 19 | 3, (0, 1) | $Z^6 + Z^3L_{3,Z} + L_{6,Z}$ |
| 20 | 2, (0, 1) | $Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z}$ |

4. *If* $m = d^2 - 3d + 3$*, then* $C$ *is* $K$*-isomorphic to the Klein curve defined by* $\mathcal{K}_d : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$*. Moreover, for* $d \geq 5$*, we have*

$$\mathrm{Aut}(C) = \langle \sigma, \tau | \sigma^{d^2-3d+3} = \tau^3 = 1\, \tau^{-1}\sigma\tau = \sigma^{-(d-1)} \rangle,$$

*with* $\sigma = \mathrm{diag}(1, \zeta_{d^2-3d+3}, \zeta_{d^2-3d+3}^{-(d-2)})$ *and* $\tau = [Y : Z : X]$*. Consequently,* $\mathrm{Aut}(C)$ *has order* $3(d^2 - 3d + 3)$.

5. *If* $m = q(d-1)$ *for some* $q \geq 2$*, then we guarantee that* $\mathrm{Aut}(C)$ *is cyclic.*

Substituting $d = 6$ in Theorem 4.1 yields:

**Corollary 4.2** *The stratum* $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(G) \neq \emptyset$ *if* $G = \mathbb{Z}/30\mathbb{Z}$, $\mathbb{Z}/25\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)$, $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/15\mathbb{Z}$*, or* $\mathbb{Z}/10\mathbb{Z}$*. Moreover, the table below presents the generators and the defining equations corresponding to each types up to projective equivalence.*

***Proof*** Cases (1), (2), and (4) are established by applying Theorem 4.1-(1), (2), and (4) to Table 2-(1), (2), and (4), respectively. For the Klein sextic curve, with $\sigma_1 = \sigma^7 = \mathrm{diag}(1, \zeta_3, \zeta_3^2)$, $\sigma_2 = \sigma^3 = \mathrm{diag}(1, \zeta_7, \zeta_7^3)$, and $\sigma_3 = [Z : X : Y]$ we have:

$$\mathrm{Aut}(\mathcal{K}_6) = \langle \sigma_1, \sigma_2, \sigma_3 : \sigma_1^3 = \sigma_2^7 = \sigma_3^3 = 1,\ \sigma_1\sigma_2 = \sigma_2\sigma_1,\ \sigma_1\sigma_3 = \sigma_3\sigma_1,\ \sigma_3\sigma_2\sigma_3^{-1} = \sigma_2^4 \rangle$$

This is precisely SmallGroup(63, 3) $= \mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ as claimed.

On the other hand, if $C$ is of Type 24, (1, 19), then Theorem 4.1-(3) implies that $\mathrm{Aut}(C)$ contains the subgroup $H = \langle \sigma, \tau \rangle$, which is isomorphic to SmallGroup(48, 26) Among all central extensions $G$ of $\mathrm{S}_4$ by $\mathbb{Z}/6\mathbb{Z}$, SmallGroup(144, 122) is the unique one containing such a subgroup. For further details refer to Extensions of $S_4$ by $\mathbb{Z}/6\mathbb{Z}$.

Regarding a curve $C$ of Type 15, (5, 6), Theorem 4.1-(5) with $q = 3$ guarantees that $\mathrm{Aut}(C)$ is always cyclic. According to [3, Proposition 22, Appendix A], the curve $C$ is defined by an equation of the form:

$$X^6 + Y^6 + XZ^5 + \beta_{3,3}(XY)^3 = 0.$$

Moreover, $\mathrm{Aut}(C)$ equals $\mathbb{Z}/30\mathbb{Z}$ if and only if $\beta_{3,3} = 0$. The additional condition $\beta_{3,3} \neq \pm 2$ ensures the smoothness of $C$.

Similarly, a smooth plane curve $C$ of Type 10, (5, 6) is given by an equation of the form:

$$X^6 + Y^6 + XZ^5 + \beta_{4,2}X^4Y^2 + \beta_{2,4}X^2Y^4 = 0,$$

where $\mathrm{Aut}(C)$ is also cyclic. If it is larger than $\mathbb{Z}/10\mathbb{Z}$, then it should be $\mathbb{Z}/30\mathbb{Z}$, which occurs if and only if $\beta_{4,2} = \beta_{2,4} = 0$, a condition excluded by assumption. □

## 5 Preliminaries about automorphism groups

The classification of finite subgroups $G$ of $\mathrm{PGL}_3(K)$ is well-established in the field. For instance, Mitchell [50] provided a comprehensive classification using geometric methods. Mitchell demonstrated that unless is primitive and conjugate to specific groups listed, it either fixes a point, a line, or a triangle. Notably, Maschke's theorem in group representation theory establishes equivalence between fixing a point and fixing a line (and vice versa), implying that if $G$ fixes a point (respectively, a line), it also fixes a line not passing through the point (respectively, a point not lying on the line).

**Notations.** For a non-zero monomial $cX^{i_1}Y^{i_2}Z^{i_3}$ with $c \in K^*$, its exponent is defined to be $\max\{i_1, i_2, i_3\}$. For a homogenous polynomial $F(X, Y, Z)$, the core of it is defined to be

**Table 3** Automorphism Groups For Very Large Types $m$, $(a, b)$

| Type $m$, $(a, b)$ | $\mathrm{Aut}(C)$ | Generators | $F(X, Y, Z)$ |
|---|---|---|---|
| 30, (5, 6) | $\mathbb{Z}/30\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{30}^5, \zeta_{30}^6)$ | $X^6 + Y^6 + XZ^5$ |
| 25, (1, 20) | $\mathbb{Z}/25\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{25}, \zeta_{25}^{20})$ | $X^6 + Y^5Z + XZ^5$ |
| 24, (1, 19) | $\mathbb{Z}/3\mathbb{Z} \times \mathrm{GL}_2(\mathbb{F}_3)$ | $\mathrm{diag}(1, \zeta_{24}, \zeta_{24}^{19})$, $[X : Z : Y]$, | $X^6 + Y^5Z + YZ^5$ |
| | | $[X : c(\zeta_8^5 Z - Y) : \zeta_8 c(\zeta_8 Z - Y)]$ | |
| 21, (1, 17) | $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{21}, \zeta_{21}^{17})$, $[Z : X : Y]$ | $X^5Y + Y^5Z + Z^5X$ |
| 15, (5, 6) | $\mathbb{Z}/15\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{15}^5, \zeta_{15}^6)$ | $X^6 + Y^6 + XZ^5 + \beta_{3,3}X^3Y^3$ |
| | | | $\beta_{3,3} \neq 0, \pm 2$ |
| 10, (5, 6) | $\mathbb{Z}/10\mathbb{Z}$ | $\mathrm{diag}(1, \zeta_{10}^5, \zeta_{10}^6)$ | $X^6 + Y^6 + XZ^5 + \beta_{4,2}X^4Y^2 + \beta_{2,4}X^2Y^4$ |
| | | | $\beta_{4,2}, \beta_{2,4} \neq 0$ |

the sum of all terms of $F$ with the greatest exponent. Now, let $C_0$ be a smooth plane curve over $K$, a pair $(C, G)$ with $G \leq \mathrm{Aut}(C)$ is is termed a descendant of $C_0$ if $C$ is defined by a homogenous polynomial whose core is a defining polynomial of $C_0$, and $G$ acts on $C_0$ under an appropriate change of variables, i.e. $G$ is $\mathrm{PGL}_3(K)$-conjugate to a subgroup of $\mathrm{Aut}(C_0)$.

The elements of $\mathrm{PGL}_3(K)$ having the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

are termed *intransitive*. The subgroup of $\mathrm{PGL}_3(K)$ consisting of all intransitive elements is denoted by $\mathrm{PBD}(2, 1)$. Obviously, there is a natural map $\Lambda : \mathrm{PBD}(2, 1) \to \mathrm{PGL}_2(K)$ given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{PBD}(2, 1) \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \mathrm{PGL}_2(K).$$

In order to determine the complete automorphism groups of other types of smooth plane sextic curves, Theorem 5.1 below proves invaluable. It can be viewed as a projection of Mitchell's classification onto smooth plane curves. For further insights, refer to Harui's work [30, Theorem 2.1].

**Theorem 5.1** *Let $C$ be a smooth plane curve of degree $d \geq 4$ defined over an algebraically closed field $K$ of characteristic 0. Then, one of the following scenarios applies*

1. $\mathrm{Aut}(C)$ *fixes a point on $C$, so it is cyclic.*
2. $\mathrm{Aut}(C)$ *fixes a point not lying on $C$. This situation can be understood through the following commutative diagram, with exact rows and vertical injective morphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^* & \longrightarrow & \mathrm{PBD}(2,1) & \xrightarrow{\Lambda} & \mathrm{PGL}_2(K) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & N & \longrightarrow & \mathrm{Aut}(C) & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

*where $N$ is a cyclic group such that $|N| \mid d$, and $G'$, a subgroup of* $\mathrm{PGL}_2(K)$*, can be*

(i) *A cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order $m$ with $m \leq d - 1$,*
(ii) *A Dihedral group $\mathrm{D}_{2m}$ of order $2m$ such that $|N| = 1$ or $m \mid (d - 2)$,*
(iii) *One of the alternating groups $\mathrm{A}_4$, $\mathrm{A}_5$, or the symmetry group $\mathrm{S}_4$.*

3. $\mathrm{Aut}(C)$ *is conjugate to a subgroup $G$ of $\mathrm{Aut}(\mathcal{F}_d)$, where $\mathcal{F}_d$ is the Fermat curve $X^d + Y^d + Z^d = 0$. Here, $|G|$ divides $|\mathrm{Aut}(\mathcal{F}_d)| = 6d^2$, and $(C, G)$ is a descendant of $\mathcal{F}_d$.*
4. $\mathrm{Aut}(C)$ *is conjugate to a subgroup $G$ of $\mathrm{Aut}(\mathcal{K}_d)$, where $\mathcal{K}_d$ is the Klein curve $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$. In this instance, $|\mathrm{Aut}(C)|$ divides $|\mathrm{Aut}(\mathcal{K}_d)| = 3(d^2 - 3d + 3)$, and $(C, G)$ is a descendant of $\mathcal{K}_d$.*
5. $\mathrm{Aut}(C)$ *is conjugate to one of the finite primitive subgroup of* $\mathrm{PGL}_3(K)$ *namely, the Klein group* $\mathrm{PSL}(2, 7)$*, the icosahedral group* $\mathrm{A}_5$*, the alternating group* $\mathrm{A}_6$*, or to one of the Hessian groups* $\mathrm{Hess}_*$ *with* $* \in \{36, 72, 216\}$.

***Remark 5.2*** In Theorem 5.1 $N$ can be viewed as the part of $\mathrm{Aut}(C)$ acting on the variable $B \in \{X, Y, Z\}$ and fixing the other two variables, while $G'$ is the part acting on $\{X, Y, Z\}\backslash\{B\}$ and fixing $B$. For example, if $B = X$, then every automorphism in $N$ takes the form $\mathrm{diag}(\zeta_n, 1, 1)$ for some $n$th root of unity $\zeta_n$.

An homology of period $n$ is a projective linear transformation of the plane $\mathbb{P}^2(K)$, $\mathrm{PGL}_3(K)$-conjugate to $\mathrm{diag}(1, 1, \zeta_n)$, where $\zeta_n$ is a primitive $n$th root of unity. Such a transformation fixes pointwise a line $\mathcal{L}$ (its axis) and a point $P$ off this line (its center). In its canonical form, $\mathcal{L} : Z = 0$ and center $P = (0 : 0 : 1)$.

The following fact due to Mitchell [50] is pivotal for determining the automorphism group of smooth plane curves over an algebraically closed field $K$ of characteristic 0.

**Theorem 5.3** *Let $G$ be a finite group of* $\mathrm{PGL}_3(K)$. *The following statements holds:*

1. *If $G$ contains an homology of period $n \geq 4$, then it fixes a point, a line or a triangle.*
2. *The Hessian group* $\mathrm{Hess}_{216}$ *is the unique finite subgroup of* $\mathrm{PGL}_3(K)$ *that contains homologies of period $n = 3$ but does not leave invariant a point, a line or a triangle.*
3. *Inside $G$, a transformation that leaves invariant the center of an homology must leave invariant its axis and vice versa.*

Furthermore, there exists a significant relationship between the presence of homologies within $\mathrm{Aut}(C)$ and the concept of *Galois points*. Initially introduced by Yoshihara in 1996, Galois points have since been extensively investigated by various mathematicians; see, for instance, [21–24, 34, 49, 60] for comprehensive studies.

**Definition 5.4** A point $P \in \mathbb{P}^2(K)$ is said to be a Galois point for $C$ if the natural projection $\pi_P$ from $C$ to a line $\mathcal{L}$ with center $P$ is a Galois covering. Moreover, if $P \in C$ (respectively $P \notin C$), then $P$ is an inner (respectively outer) Galois point for $C$.

In particular, a crucial result for smooth plane curves, due to Harui [30, Lemma 3.7], states:

**Proposition 5.5** *Let $C$ be a smooth plane curve of degree $d \geq 5$ over an algebraically closed field $K$ of characteristic* 0. *If $\sigma \in \mathrm{Aut}(C)$ is an homology with center $P$, then $|\langle\sigma\rangle|$ divides $d$ when $P \notin C$ and $|\langle\sigma\rangle|$ divides $d-1$ when $P \in C$. The equality $|\langle\sigma\rangle| = d$ (or $|\langle\sigma\rangle| = d-1$ respectively) if and only if $P$ is an outer (or inner) Galois point for $C$.*

## 6 The automorphism group for Type 12, (1, 7): The stratum $\mathcal{M}_g^{\mathrm{Pl}}(\mathbb{Z}/12\mathbb{Z})$

In this section, we investigate the full automorphism group of smooth plane sextic curves $C$ of Type 12, (1, 7), defined by an equation of the form:

$$C : X^6 + Y^5Z + YZ^5 + \beta_{3,3}Y^3Z^3 = 0, \tag{6.1}$$

for some $\beta_{3,3} \in K^*$. In this scenario, an element $\sigma = \mathrm{diag}(1, \zeta_{12}, \zeta_{12}^7) \in \mathrm{Aut}(C)$ of order 12 is present. It's assumed that $\sigma$ achieves maximal order in $\mathrm{Aut}(C)$, particularly when $\beta_{3,3} \neq 0$. If $\beta_{3,3} = 0$, the curve $C$ would fall under Type 24, (1, 19), which has been previously discussed.

We characterize the full automorphism group $\mathrm{Aut}(C)$ as follows:

**Proposition 6.1** *Let $C$ be a smooth plane sextic curves $C$ of Type* $12, (1, 7)$ *as described above (Eq. 6.1). Then, one of the following situations holds:*

1. *If* $\beta_{3,3} \neq \pm\frac{10}{3}$, *then* $\mathrm{Aut}(C) = \langle\sigma,\ \tau\rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathrm{D}_8$ *with* $\tau = [X : Z : Y]$.
2. *Otherwise, $C$ is $K$-isomorphic to the Fermat curve* $\mathcal{F}_6 : X^6 + Y^6 + Z^6 = 0$. *In this case,* $\mathrm{Aut}(\mathcal{F}_6) = \langle\tau,\ \eta_2,\ \eta_3,\ \eta_4\rangle \cong (\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$ *with* $\eta_2 = [Y : Z : X],\ \eta_3 = \mathrm{diag}(\zeta_6, 1, 1)$ *and* $\eta_4 = \mathrm{diag}(1, \zeta_6, 1)$.

***Proof*** Non of the finite primitive subgroups of $\mathrm{PGL}_3(K)$ mentioned in Theorem 5.1-(5) has elements of order 12. Specifically, except for the Klein group $\mathrm{PSL}(2, 7)$, which includes elements of order 7, none of these groups have elements of order greater than 6. Additionally, $12 \nmid \mathrm{Aut}(\mathcal{K}_6)$, indicating that $C$ is not a descendant of the Klein sextic curve $\mathcal{K}_6$. Moreover, $\mathrm{Aut}(C)$ is not cyclic as shown by the group structure

$$\langle\sigma, \tau : \sigma^{12} = \tau^2 = 1,\ \tau\sigma\tau = \sigma^7\rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathrm{D}_8 = \mathrm{SmallGroup}(24, 10),$$

which is non-cyclic subgroup of automorphisms. Therefore, $\mathrm{Aut}(C)$ fixes a point $P$ not lying on $C$, or $C$ is a descendant of the Fermat sextic curve $\mathcal{F}_6$.

Suppose first that $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ in $\mathbb{P}_2(K)$ and a point $P \notin C$ off this line. Given that $\sigma$ and $\tau$ are automorphisms, $\mathcal{L}$ can be taken as the reference line $X = 0$ and $P$ the reference point $(1 : 0 : 0)$. By Theorem 5.1-(2), all automorphisms of $C$ are intransitive of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. Moreover, $\mathrm{Aut}(C)$ fits into a short exact sequence $1 \to N \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1$, where $N = \langle\sigma^2\rangle$ by Remark 5.2, and $\Lambda(\mathrm{Aut}(C))$ contains the Klein 4-group $(\mathbb{Z}/2\mathbb{Z})^2$ that is generated by $\Lambda(\sigma) = \mathrm{diag}(\zeta_{12}, -\zeta_{12})$ and $\Lambda(\tau) = [Z : Y]$. Thus $\Lambda(\mathrm{Aut}(C))$ should be $(\mathbb{Z}/2\mathbb{Z})^2,\ \mathrm{A}_4,\ \mathrm{A}_5$ or $\mathrm{S}_4$. If $\Lambda(\mathrm{Aut}(C)) = \mathrm{A}_4,\ \mathrm{A}_5$ or $\mathrm{S}_4$, then the group structure of $\mathrm{A}_4$ implies that there must exist an automorphism $\eta \notin \langle\sigma, \tau\rangle$ of $C$ such that $\Lambda(\eta)$ has order 3, $\Lambda(\eta)\Lambda(\sigma)\Lambda(\eta)^{-1} = \Lambda(\sigma)\Lambda(\tau)$ and $\Lambda(\eta)\Lambda(\tau)\Lambda(\eta)^{-1} = \Lambda(\sigma)$. This implies that $\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & \pm\nu \\ 0 & -i\nu & \pm i\nu \end{pmatrix}$ or $\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & \pm\nu \\ 0 & i\nu & \mp i\nu \end{pmatrix}$ for some $\nu \in K^*$, however, it does not preserve the defining equation for $C$. Consequently, no more automorphisms for $C$ appear in this case.

Secondly, assume that $C$ is a descendant of $\mathcal{F}_6$. Notably, $\mathrm{Aut}(\mathcal{F}_6)$ is generated by $\tau,\ \eta_2,\ \eta_3$, and $\eta_4$ of orders 2, 3, 6, 6 respectively, satisfying relations such as

$$(\eta_1\eta_2)^2 = (\eta_1\eta_3)(\eta_3\eta_1)^{-1} = (\eta_3\eta_4)(\eta_4\eta_3)^{-1} = \eta_1\eta_4\eta_1(\eta_3\eta_4)^{-5} = \eta_2\eta_3\eta_2^{-1}(\eta_3\eta_4)^{-5} = 1.$$

These relations correspond to the group $\mathrm{SmallGroup}(216, 92) = (\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$, semidirect products of $(\mathbb{Z}/6\mathbb{Z})^2$ and $\mathrm{S}_3$ Under this assumption, $\phi^{-1}\ \mathrm{Aut}(C)\,\phi$ forms a subgroup of $\mathrm{Aut}(\mathcal{F}_6)$ for some $\phi \in \mathrm{PGL}_3(K)$. It is reasonable to consider $\phi^{-1}\,\sigma\,\phi = [X : \zeta_6 Z : Y]$ because all automorphisms of $\mathcal{F}_6$ of order 12 are conjugates within $\mathrm{Aut}(\mathcal{F}_6)$. This constraints $\phi$ to be of the form:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & e\,\zeta_{12} \\ 0 & r & -r\,\zeta_{12} \end{pmatrix},$$

for some $e,\ r \in K^*$ such that $\beta_{3,3} = \dfrac{1 - er\left(e^4 + r^4\right)}{(er)^3}$. Since $\phi^{-1}\tau\phi$ is an involution for both ${}^{\phi}C$ and $\mathcal{F}_6$, it follows that $e^4 - r^4 = 0$ (indicating $r = ce$ for some $c$ such that $c^4 = 1$)

and the transformed equation ${}^{\phi}C$ is

$$X^6 + Y^6 + Z^6 + (3 - 16ce^6)\zeta_{12}^4 \left(Y^2Z^4 + \zeta_{12}^4 Y^4Z^2\right) = 0,$$

Thus ${}^{\phi}C$ admits a larger automorphism group than $\phi^{-1}\langle\sigma, \tau\rangle\phi$ if and only if $3-16ce^6 = 0$, which equivalently leads to $\beta_{0,3} = \frac{10}{3c^2} = \pm\frac{10}{3}$), as required to be shown. □

## 7 The automorphism group for Type 8, (1, 3): The stratum $\mathcal{M}_g^{\text{Pl}}(\mathbb{Z}/8\mathbb{Z})$

In this section, we describe the full automorphism group of smooth plane sextic curves $C$ of Type 8, (1, 3), defined by an equation of the form:

$$C : X^6 + Y^5Z + YZ^5 + \beta_{4,2}X^2Y^2Z^2 = 0, \tag{7.1}$$

for some $\beta_{4,2} \in K$. Here, $\sigma = \text{diag}(1, \zeta_8, \zeta_8^3)$ is an element in $\text{Aut}(C)$ of order 8, and we assume that $\beta_{4,2} \neq 0$ to avoid $C$ being of Type 24, (1, 19).

The proposition concerning $\text{Aut}(C)$ is as follows:

**Proposition 7.1** *Let $C$ be a non-singular plane sextic curves $C$ of Type* 8, (1, 3) *as above (Eq.* 7.1*). Then,* $\text{Aut}(C) = \langle\sigma, \tau\rangle \cong \text{SD}_{16}$*, a quasidihedral group, where* $\tau = [X : Z : Y]$.

***Proof*** The group $\langle\sigma, \tau : \sigma^8 = \tau^2 = 1, \tau\sigma\tau = \sigma^3\rangle \cong \text{SmallGroup}(16, 8) = \text{SD}_{16}$ is always a subgroup of automorphisms of order 16. This implies that $\text{Aut}(C)$ does not belong to the finite primitive subgroups list of $\text{PGL}_3(K)$ mentioned in Theorem 5.1-(5), as it contains elements of order $8 > 7$, and is not cyclic (since $\text{SD}_{16}$ non-cyclic). Additionally, $C$ is not a descendant of either the Klein curve $\mathcal{K}_6$ or the Fermat curve $\mathcal{F}_6$, since $16 \nmid |\text{Aut}(\mathcal{K}_6)|, |\text{Aut}(\mathcal{F}_6)|$. Therefore, $\text{Aut}(C)$ fixes the line $\mathcal{L} : X = 0$ in $\mathbb{P}_2(K)$ and the point $(1 : 0 : 0)$ off this line, which does not lie on $C$. Moreover, there is a short exact sequence $1 \to N \to \text{Aut}(C) \to \Lambda(\text{Aut}(C)) \to 1$, where $N = \langle\sigma^4\rangle$ (by Remark 5.2), and $\Lambda(\text{Aut}(C))$ contains the dihedral group $\text{D}_8$, generated by $\Lambda(\sigma) = \text{diag}(\zeta_8, \zeta_8^3)$ and $\Lambda(\tau) = [Z : Y]$. Thus, if $\Lambda(\text{Aut}(C))$ is strictly bigger than $\text{D}_8$, it must be $\text{S}_4$ according to Theorem 9.10-(2). Since $\Lambda(H) := \langle\text{diag}(\zeta_4, -\zeta_4), [Z : Y]\rangle = (\mathbb{Z}/2\mathbb{Z})^2$ is normal in $\text{S}_4$, we need to consider conditions under which $C$ has an automorphism $\eta$ such that $\Lambda(\eta)$ is of order 3 and belongs to the normalizer of $\Lambda(H)$. By [35, Lemma 2.2.3-(b)], we find

$$\Lambda(\eta) = \begin{pmatrix} \zeta_4^{\nu} & -\zeta_4^{\nu+\nu'} \\ 1 & \zeta_4^{\nu'} \end{pmatrix},$$

for some $\nu, \nu' \in \{0, 1, 2, 3\}$. However, the polynomial $Y^5Z + YZ^5 + \beta_{4,2}X^2Y^2Z^2$ is not invariant under the action of $\Lambda(\eta)$. For example, the transformed equation for $C$ would include $Z^6$, which contradicts the form of $C$. Consequently, $C$ does not admit automorphisms beyond those discussed. □

## 8 The automorphism group for Type 7, (1, 3): The stratum $\mathcal{M}_g^{\text{Pl}}(\mathbb{Z}/7\mathbb{Z})$

In this section, we explore the full automorphism group of smooth plane sextic curves $C$ of Type 7, (1, 3), defined by an equation of the form:

$$C : X^5Y + Y^5Z + XZ^5 + \beta_{4,2}X^2Y^2Z^2 = 0, \tag{8.1}$$

for some $\beta_{4,2} \in K$. Here, $\sigma := \text{diag}(1, \zeta_7, \zeta_7^3)$ is an automorphism of maximal order 8. We assume $\beta_{4,2} \neq 0$ to avoid $C$ falling into Type 21, (1, 17).

The automorphism group $\text{Aut}(C)$ is detailed by the following result:

**Proposition 8.1** *Let C be a smooth plane sextic curves C of Type* 7, (1, 3) *as above (Eq.* 8.1*). Then, the full automorphism group* $\text{Aut}(C)$ *is classified as follows:*

1. *If* $\beta_{4,2} \neq -5$*, then* $\text{Aut}(C) = \langle \sigma, \tau \rangle \cong \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$*, the semidirect product of* $\mathbb{Z}/7\mathbb{Z}$ *and* $\mathbb{Z}/3\mathbb{Z}$ *acting faithfully, where* $\tau = [Y : Z : X]$.
2. *If* $\beta_{4,2} = -5$*, then* $\text{Aut}(C) = \langle \sigma, \tau, \eta \rangle \cong \text{PSL}(2, 7)$*, where* $\eta$ *is the involution*

$$\begin{pmatrix} \zeta_7 - \zeta_7^6 & \zeta_7^2 - \zeta_7^5 & \zeta_7^4 - \zeta_7^3 \\ \zeta_7^2 - \zeta_7^5 & \zeta_7^4 - \zeta_7^3 & \zeta_7 - \zeta_7^6 \\ \zeta_7^4 - \zeta_7^3 & \zeta_7 - \zeta_7^6 & \zeta_7^2 - \zeta_7^5 \end{pmatrix}.$$

***Proof*** Since $\langle \sigma, \tau : \sigma^7 = \tau^3 = 1,\ \tau\sigma\tau^{-1} = \sigma^4 \rangle \cong= \text{SmallGroup}(21, 1) = \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ is a subgroup of automorphisms of order 21, $\text{Aut}(C)$ is not cyclic. It does not fix any point in $\mathbb{P}_2(K)$ because $\sigma$ and $\tau$ do not share any fixed points. Moreover, $\text{Aut}(C)$ is not one of the finite primitive subgroups of $\text{PGL}_3(K)$ except possibly the Klein group $\text{PSL}(2, 7)$, and $C$ is not a descendant of the Fermat curve $\mathcal{F}_6$. If $C$ were a descendant of the Klein curve $\mathcal{K}_6$, then $\text{Aut}(C)$ could not be a larger group unless $C$ is $K$-isomorphic to the Klein curve itself, because $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ is a maximal subgroup of $\text{Aut}(\mathcal{K}_6)$. This is prohibited because the assumption is that automorphisms of $C$ have orders $\leq 7$, while $\mathcal{K}_6$ has automorphisms of order 21.

Now, assume that $\text{Aut}(C)$ is $\text{PGL}_3(K)$-conjugate to $\text{PSL}(2, 7)$ for certain specializations of the parameter $\beta_{4,2}$. Since the centralizer of $\langle \tau \rangle$ inside $\text{PSL}(2, 7)$ is $\mathbb{Z}/3\mathbb{Z}$ and its normalizer is $S_3$, there must exist an involution $\eta$ of $C$ such that $\eta\tau\eta = \tau^{-1}$. Write $\eta = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & r \end{pmatrix}$, then the condition $\eta\tau\eta = \tau^{-1}$ reads as $\begin{pmatrix} c & a & b \\ f & d & e \\ r & g & h \end{pmatrix} = \lambda \begin{pmatrix} g & h & r \\ a & b & c \\ d & e & f \end{pmatrix}$ for some $\lambda \in K^*$. This holds only if $\lambda^3 = 1$,

$$\eta = \begin{pmatrix} a & b & c \\ \zeta_3^\nu b & \zeta_3^\nu c & \zeta_3^\nu a \\ \zeta_3^{2\nu} c & \zeta_3^{2\nu} a & \zeta_3^{2\nu} b \end{pmatrix},$$

for some $\nu \in \{0, 1, 2\}$. Because $\eta^2 = 1$, we further obtain $c = -ab\zeta_3^{2\nu}/(\zeta_3^\nu a + b)$ such that $ab(\zeta_3^\nu a + b) \neq 0$ (otherwise, $\eta$ would reduce to $X \leftrightarrow Z$ which is never an automorphism for $C$). Thus,

$$\eta_\nu = \begin{pmatrix} a & b & -ab\zeta_3^{2\nu}/(\zeta_3^\nu a + b) \\ \zeta_3^\nu b & -ab/(\zeta_3^\nu a + b) & a\zeta_3^\nu \\ -ab\zeta_3^\nu/(\zeta_3^\nu a + b) & \zeta_3^{2\nu} a & \zeta_3^{2\nu} b \end{pmatrix},$$

For invertibility, we need to impose $a^2 + \zeta_3^\nu ab + \zeta_3^{2\nu} b^2 \neq 0$. If $\eta = \eta_0$, then the transformed equation ${}^{\eta_0}C$ would contain $Z^6$ unless $\beta_{4,2} = \dfrac{a^5b^4 + b^5(a+b)^4 - a^4(a+b)^5}{a^3b^3(a+b)^3}$. Moreover, we must eliminate the coefficients of $X^5Z,\ XY^5$, and $Y^2Z^4$ in ${}^{\eta_0}C$. Accordingly,

$$a^6 + a^5b - 5a^4b^2 - 5a^3b^3 + 5a^2b^4 + 5ab^5 + b^6 = 0,$$
$$(a^3 - 3ab^2 - b^3)(a^3 + a^2b - 2ab^2 - b^3) = 0,$$
$$(a^6 - a^5b - 12a^4b^2 - 9a^3b^3 + 8a^2b^4 + 7ab^5 + b^6)(a^3 + a^2b - 2ab^2 - b^3) = 0.$$

Using MATHEMATICA, we find that the above system is consistent only if $\beta_{4,2} = -5$. The work by Klein [39] (also see [20]) assures that we can take $a = \zeta_7 - \zeta_7^6$ and $b = \zeta_7^2 - \zeta_7^5$.
Similar considerations apply for ${}^{\eta_1}C$ and ${}^{\eta_2}C$. □

# 9 The automorphism group that contain homologies of period ≥ 3

In this section, we will analyze $\mathrm{Aut}(C)$ under the assumption that $C$ admits an homology of period $\geq 3$. According to Proposition 3.1, $C$ falls into the types 6, (0, 1), , 5, (0, 1), 6, (1, 3) or 3, (0, 1).

## 9.1 Type 5, (0, 1)

In this case, $C$ is defined by an equation of the form:

$$C : Z^5Y + L_{6,Z} = 0, \tag{9.1}$$

where $\sigma = \mathrm{diag}(1, 1, \zeta_5)$ is an automorphism of maximal order 5.

Since $\mathrm{Aut}(C)$ contains an homology of period $d - 1 = 5$ with center $P = (0 : 0 : 1)$ on $C$. By Proposition 5.5, $P \in C$ is an inner Galois point, and it is the unique one for $C$ as per [60, Theorem 4]. Therefore, $P$ must be invariant under the action of $\mathrm{Aut}(C)$. According to [36, Lemma 11.44 and Theorem 11.49], it follows that $\mathrm{Aut}(C)$ is cyclic of order 5.

Thus, we state the following proposition:

**Proposition 9.1** *Let C be a smooth plane sextic curves C of Type* 5, (0, 1) *as above (Eq.* 9.1*). Then,* $\mathrm{Aut}(C) = \langle\sigma\rangle \cong \mathbb{Z}/5\mathbb{Z}$.

## 9.2 Type 6, (0, 1)

In this scenario, $C$ is defined by an equation of the form:

$$C : X^6 + L_{6,X} = 0, \tag{9.2}$$

where $\sigma = \mathrm{diag}(\zeta_6, 1, 1)$ is an automorphism of maximal order 6.

Since $\sigma$ acts as an homology of period $d = 6$ with center $P = (1 : 0 : 0)$ not on $C$, and $P$ is an outer Galois point by Proposition 5.5, it implies that $\mathrm{Aut}(C)$ cannot have automorphisms of order greater than 6. Therefore, $C$ cannot be $K$-isomorphic to the Fermat curve $\mathcal{F}_6$. According to [60, Theorem 4', Proposition 5'], $P$ is the unique outer Galois point for $C$. Hence, $P$ remains invariant under the action of $\mathrm{Aut}(C)$. As stated in Theorem 5.3-(1), $\mathrm{Aut}(C)$ also fixes the axis $\mathcal{L} : X = 0$ of $\sigma$. Thus all automorphisms of $C$ take the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

By Theorem 5.1, either $C$ is a descendant of the Fermat curve $\mathcal{F}_6$, or $\mathrm{Aut}(C)$ fits into a short exact sequence $1 \to N = \langle\sigma\rangle \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1$, where $\Lambda(\mathrm{Aut}(C))$ is a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order $m \leq 5$, a dihedral group $\mathrm{D}_{2m}$ of order $2m$ with $m = 1, 2$, or $4$, one of the alternating groups $\mathrm{A}_4$, $\mathrm{A}_5$, or the symmetric group $\mathrm{S}_4$.

We will now explore these possibilities further.

### 9.2.1 When $C$ is a descendant of $\mathcal{F}_6$

In this scenario, there exists a $\phi \in \mathrm{PGL}_3(K)$ such that $\phi^{-1}\,\mathrm{Aut}(C)\phi \leq \mathrm{Aut}(\mathcal{F}_6)$. We can further assume that $\phi^{-1}\sigma^2\phi = \sigma^2$, as homologies of order 3 within $\mathrm{Aut}(\mathcal{F}_6)$ form two non-conjugate classes represented by $\sigma^2$ and $\sigma^{-2}$. Consequently, $\phi$ exhibits the same structure as the automorphisms of $C$, allowing us to assume without loss of generality that $\mathrm{Aut}(C) \leq \mathrm{Aut}(\mathcal{F}_6)$.

The automorphisms of $C$ are characterized by the following forms:

$$\mathrm{Aut}(C) \subseteq \left\{[X : \zeta_6^r Y : \zeta_6^{r'} Z],\ [X : \zeta_6^{r'} Z : \zeta_6^r Y] : 0 \leq r, r' \leq 5\right\}.$$

According to [30, Lemma 6.5], $\mathrm{Aut}(C)$ fits into the short exact sequence:

$$1 \to H \times \langle\sigma\rangle \to \mathrm{Aut}(C) \to G \to 1,$$

where $H = 1$, $\langle \mathrm{diag}(1, -1, 1)\rangle$ or $\langle \mathrm{diag}(1, \zeta_3, 1)\rangle$, and $G = 1$ or $\langle [X : Z : Y]\rangle$.

**Claim 9.2** *If $G = 1$, then $\mathrm{Aut}(C)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.*

***Proof*** If $H = 1$, then $\mathrm{Aut}(C) = \langle\sigma\rangle \cong \mathbb{Z}/6\mathbb{Z}$, and $C$ is defined by an equation of the form: $X^6 + Y^6 + Z^6 +$ *lower order terms in* $Y, Z = 0$.

If $H = \langle \mathrm{diag}(1, -1, 1)\rangle$, then $\mathrm{Aut}(C) = \langle\sigma, \mathrm{diag}(1, -1, 1)\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, and $C$ is defined by: $X^6 + Y^6 + Z^6 + \beta_{4,2}Y^4Z^2 + \beta_{2,4}Y^2Z^4 = 0$, where $\beta_{4,2} \neq \beta_{2,4}$.

If $H = \langle \mathrm{diag}(1, \zeta_3, 1)\rangle$, then $\mathrm{Aut}(C) = \langle\sigma, \mathrm{diag}(1, \zeta_3, 1)\rangle$, and $C$ is defined by an equation of the form: $X^6 + Y^6 + Z^6 + \beta_{3,3}Y^3Z^3 = 0$. We reject this case since $[X : Z : Y]$ would be an automorphism contradicting the assumption that $G = 1$. □

**Claim 9.3** *If $G = \langle [X : Z : Y]\rangle$, then $\mathrm{Aut}(C)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$.*

***Proof*** If $H = 1$, then $\mathrm{Aut}(C) = \langle\sigma, [X : Z : Y]\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, and $C$ is defined by an equation involving terms like:

$$X^6 + Y^6 + Z^6 + \beta_{5,1}(Y^5Z + YZ^5) + \beta_{4,2}(Y^4Z^2 + Y^2Z^4) + \beta_{3,3}Y^3Z^3 = 0,$$

where $\beta_{5,1} \neq 0$ or $\beta_{4,2} \neq 0$.

If $H = \langle \mathrm{diag}(1, -1, 1)\rangle$, this case is rejected since $[\zeta_6 X : -Z : Y]$ would be an automorphism of order $12 > 6$, contradicting the assumption $G = \langle [X : Z]\rangle$.

If $H = \langle \mathrm{diag}(1, \zeta_3, 1)\rangle$, then $\mathrm{Aut}(C) = \langle\sigma, \mathrm{diag}(1, \zeta_3, 1), [X : Z : Y]\rangle$, and $C$ is described by an equation of the form: $X^6 + Y^6 + Z^6 + \beta_{3,3}Y^3Z^3 = 0$, for some $\beta_{3,3} \neq 0$. It can be verified that

$$\begin{aligned}\mathrm{Aut}(C) &= \langle \sigma, \tau, \eta : \sigma^6 = \tau^2 = \eta^3 = 1,\ \sigma\eta = \eta\sigma,\ \sigma\tau = \tau\sigma,\ \tau\eta\tau = \sigma^4\eta^{-1}\rangle \\ &\cong \mathrm{SmallGroup}(36, 12) = \mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3\,.\end{aligned}$$

with $\tau = [X : Z : Y]$ and $\eta = \mathrm{diag}(1, \zeta_3, 1)$. □

### 9.2.2 When $C$ is not a descendant of $\mathcal{F}_6$

As previously mentioned, $\mathrm{Aut}(C)$ fits into a short exact sequence: $1 \to \langle\sigma\rangle \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1$, where $\Lambda(\mathrm{Aut}(C))$ can take one of the following forms: (i) $\mathbb{Z}/m\mathbb{Z}$ for some $m \leq 5$, (ii) $\mathrm{D}_{2m}$ for some $m \mid 4$, (iii) $\mathrm{A}_4$, $\mathrm{A}_5$, or $\mathrm{S}_4$.

**Claim 9.4** $\Lambda(\mathrm{Aut}(C)) = 1$ *or* $\mathbb{Z}/2\mathbb{Z}$.

***Proof*** If $\Lambda(\mathrm{Aut}(C)) \cong \mathbb{Z}/4\mathbb{Z}$, $\mathrm{D}_8$, or $\mathrm{S}_4$, according to [35, Lemma 2.2.1, I], we can assume $\langle\Lambda(\eta) = \mathrm{diag}(1, \zeta_4)\rangle \leq \Lambda(\mathrm{Aut}(C))$ for some $\eta \in \mathrm{Aut}(C)$. Since $L_{6,X}$ is invariant under the action of $\Lambda(\eta)$, $C$ reduces up to $K$-isomorphism to $X^6 + \beta 2, 4Y^2Z^4 = 0$, which is singular at the point $(0 : 0 : 1)$, a contradiction.

If $\Lambda(\mathrm{Aut}(C)) \cong \mathbb{Z}/3\mathbb{Z}$, a similar argument shows that $C$ has an automorphism $\eta$ with $\Lambda(\eta) = \mathrm{diag}(1, \zeta_3)$ leaving $L_{6,X}$ invariant. This reduces $C$, up to $K$-isomorphism, to $X^6 + Y^6 + Z^6 + \beta_{3,4}Y^3Z^3 = 0$. Clearly, $[Z : Y]$ is another element in $\Lambda(\mathrm{Aut}(C))$ of order $2 \nmid 3$, a contradiction.

If $\Lambda(\mathrm{Aut}(C)) \cong \mathbb{Z}/5\mathbb{Z}$ or $\mathrm{A}_5$, then $C$ would necessarily contain an automorphism of order $15 > 5$, since $N = \langle\sigma\rangle$ is a normal subgroup in $\mathrm{Aut}(C)$ and $\mathrm{Aut}(C)$ includes elements of order 5 where $\gcd(|N|, 5) = 1$. However, this directly contradicts the assumption that all automorphisms of $C$ have orders $\leq 6$.

Finally, if $\Lambda(\mathrm{Aut}(C)) \cong \mathrm{D}_4$ or $\mathrm{A}_4$, then we can assume, without loss of generality using [35, Lemma 2.2.1, I], that $L_{6,X}$ is invariant under the action of $\Lambda(\tau) = \mathrm{diag}(1, -1)$ and $\Lambda(\eta) = [Z : Y]$ for some $\tau, \eta \in \mathrm{Aut}(C)$. This reduction of $C$ up to $K$-isomorphism implies that $C$ can be represented by the equation:

$$C : X^6 + Y^6 + Z^6 + \beta_{2,4}\left(Y^2Z^4 + Y^4Z^2\right) = 0,$$

where $\tau = \mathrm{diag}(1, \lambda, -\lambda)$ and $\eta = [X : \mu Z : \mu Y]$, with $\lambda^6 = \mu^6 = 1$. Consequently, $\mathrm{Aut}(C)$ would contain the subgroup

$$\langle\sigma, \mathrm{diag}(1, 1, -1), [X : Z : Y]\rangle \cong \mathrm{SmallGroup}(24, 10) = \mathbb{Z}/3\mathbb{Z} \times \mathrm{D}_8$$

Thus, $C$ would possess automorphisms of order $12 > 6$, which contradicts the initial assumption. □

**Claim 9.5** *If* $\Lambda(\mathrm{Aut}(C))$ *is* $\mathbb{Z}/2\mathbb{Z}$, *then* $\mathrm{Aut}(C)$ *should be* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

***Proof*** Utilizing [35, Lemma 2.2.1, I], we can establish that $L_{6,X}$ remains invariant under the action of $\Lambda(\tau) = \mathrm{diag}(1, -1)$ for some $\tau \in \mathrm{Aut}(C)$. This reduction of $C$, up to $K$-isomorphism, simplifies to the equation:

$$C : X^6 + Y^6 + Z^6 + \beta_{2,4}Y^2Z^4 + \beta_{4,2}Y^4Z^2 = 0,$$

where $\eta = \mathrm{diag}(1, \lambda, -\lambda)$ for some $\lambda \in K$ satisfying $\lambda^6 = 1$. Consequently, all automorphisms of $C$ are diagonal, specifically,

$$\mathrm{Aut}(C) = \langle\sigma, \tau : \sigma^6 = \tau^2 = 1,\ \sigma\tau = \tau\sigma\rangle \cong \mathrm{SmallGroup}(12, 5) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z},$$

where $\tau := \mathrm{diag}(1, 1, -1)$. □

Based on the preceding discussion, we can formulate the following proposition:

**Proposition 9.6** *Let $C$ be a smooth plane sextic curve of Type* 6, (0, 1) *as described above (Eq. 9.2), and let* $\sigma = \mathrm{diag}(\zeta_6, 1, 1)$, $\tau = \mathrm{diag}(1, 1, -1)$, $\tau' = [X : Z : Y]$ *and* $\eta = \mathrm{diag}(1, \zeta_3, 1)$. *Then,* $\mathrm{Aut}(C)$ *is cyclic of order* 6 *generated by* $\sigma$ *unless one of the following situations holds:*

1. *In the case where $C$ is $K$-isomorphic to*

$$X^6 + Y^6 + Z^6 + \beta_{2,4}Y^2Z^4 + \beta_{4,2}Y^4Z^2 = 0$$

*for some $\beta_{2,4} \neq \beta_{4,2}$, the automorphism group* $\mathrm{Aut}(C) = \langle \sigma, \tau \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

2. *In the case where $C$ is $K$-isomorphic to*

$$X^6 + Y^6 + Z^6 + \beta_{5,1}(Y^5Z + YZ^5) + \beta_{4,2}(Y^4Z^2 + Y^2Z^4) + \beta_{3,3}Y^3Z^3 = 0,$$

*and assuming $\beta_{5,1} \neq 0$ or $\beta_{4,2} \neq 0$,* $\mathrm{Aut}(C) = \langle \sigma, \tau' \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

3. *In the case where $C$ is $K$-isomorphic to*

$$X^6 + Y^6 + Z^6 + \beta_{3,3}Y^3Z^3 = 0,$$

*where $\beta_{3,3} \neq 0$, the automorphism group* $\mathrm{Aut}(C) = \langle \sigma, \tau', \eta \rangle \cong \mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$.

***Remark 9.7*** The families described in Proposition 9.6-(1) and (2) are related by a change of variables in $\mathrm{PGL}_3(K)$ of the form

$$\phi = [\lambda X : \mu(Y + Z) : Y - Z],$$

where $\lambda, \mu \in K$ such that $\lambda^6 = 1$ and $\mu^4 + \beta_{4,2}\mu^2 + \beta_{2,4} = 0$. This implies that any curve belonging to either family can be $K$-isomorphic to a member of the other family, and vice versa.

### 9.3 Type 6, (1, 3)

In this scenario, the curve $C$ is defined by an equation of the form:

$$C : X^6 + Y^6 + Z^6 + \beta_{2,0}X^4Z^2 + \beta_{0,3}Y^3Z^3 + X^2\left(\beta_{4,0}Z^4 + \beta_{4,3}Y^3Z\right), \qquad (9.3)$$

where $\sigma = \mathrm{diag}(1, \zeta_6, -1)$ is an automorphism of maximal order 6. In particular, $\sigma^2 = \mathrm{diag}(1, \zeta_3, 1)$ represents an homology of period 3 in $\mathrm{Aut}(C)$.

According to Theorem 5.3, $\mathrm{Aut}(C)$ either fixes a line and a point not on this line, fixes a triangle, or is ither fixes a line and a point off this line, fixes a triangle, or $\mathrm{Aut}(C)$ is $\mathrm{PGL}_3(K)$-conjugate to $\mathrm{Hess}_{216}$.

**Claim 9.8** *If* $\mathrm{Aut}(C)$*, where $C$ is given as Eq 9.3, fixes a triangle and and no line or point is left invariant then* $\mathrm{Aut}(C)$ *is isomorphic to* $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathrm{S}_3$, $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$, *or* $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/6\mathbb{Z}$.

***Proof*** If $\mathrm{Aut}(C)$ fixes a triangle and no line or point is left invariant, then $C$ is a descendant of either the Fermat curve $\mathcal{F}_6$ or the Klein curve $\mathcal{K}_6$. However, $C$ cannot be a descendant of $\mathcal{K}_6$ because $6 \nmid |\mathrm{Aut}(\mathcal{K}_6)| (= 63)$. On the other hand, if $C$ is a descendant of $\mathcal{F}_6$, then according to Proposition 6.1, we can assert that $\phi^{-1}\,\mathrm{Aut}(C)\phi$ is a subgroup of $\mathrm{Aut}(\mathcal{F}_6) \cong (\mathbb{Z}/6\mathbb{Z})^2 \rtimes \mathrm{S}_3$ for some $\phi \in \mathrm{PGL}_3(K)$. Here, $\phi^{-1}\sigma^2\phi = \sigma^2$ since $\sigma^2$ and $\sigma^{-2}$ represent homologies of period 3 in $\mathrm{Aut}(\mathcal{F}_6)$, forming two conjugacy classes in $\mathrm{PGL}_3(K)$. If $\phi^{-1}\sigma\phi$ is an element of order 6 inside $\mathrm{Aut}(\mathcal{F}_6)$, then $\phi$ can be expressed in one of the forms: $\mathrm{diag}(a, 1, b)$, $[aZ : Y : bX]$, or $[aX + bZ : Y : cX - \frac{bc}{a}Z]$. In the following discussion, we address each of these cases.

**Case** $\phi = \mathrm{diag}(a, 1, b)$ or $[aZ : Y : bX]$. Thus we have $a^6 = b^6 = 1$ and $C$ is $K$-isomorphic to $C'$ of the form:

$$C' : X^6 + Y^6 + Z^6 + \beta'_{2,0}X^4Z^2 + \beta'_{0,3}Y^3Z^3 + X^2\left(\beta'_{4,0}Z^4 + \beta'_{4,3}Y^3Z\right) = 0$$

Let us revisit the automorphisms of $\mathcal{F}_6$:

$$\tau_1=[X:\zeta_6^rY:\zeta_6^{r'}Z],\ \tau_2=[\zeta_6^{r'}Z:\zeta_6^rY:X],\ \tau_3=[X:\zeta_6^{r'}Z:\zeta_6^rY],$$
$$\tau_4=[\zeta_6^rY:X:\zeta_6^{r'}Z],\ \tau_5=[\zeta_6^rY:\zeta_6^{r'}Z:X],\ \tau_6=[\zeta_6^{r'}Z:X:\zeta_6^rY],$$

for $0\leq r\leq 5$.

Firstly, it's evident that $\tau_4,\tau_5$ and $\tau_6$ are never automorphisms of $C'$. Secondly, $\tau_3\in\mathrm{Aut}(C')$ only if $\beta'_{2,0}=\beta'_{4,0}=\beta'_{4,3}=0$. In this case, $C$ is $K$-isomorphic to $X^6+Y^6+Z^6+\beta'_{0,3}Y^3Z^3=0$, hence $\mathrm{Aut}(C)\cong\mathbb{Z}/6\mathbb{Z}\times\mathrm{S}_3$ by Proposition 9.6. Thirdly, $\tau_2\in\mathrm{Aut}(C')$ only if $\beta'_{0,3}=\beta'_{4,3}=0$ and $\beta'_{4,0}=\beta'_{2,0}\zeta_6^{2(r'-r)}$. In such scenario, $C$ is $K$-isomorphic, via a transformation of the form $\mathrm{diag}(1,\lambda,\mu)$, where $\lambda^4\mu'^2=\lambda^2\mu^4\zeta_6^{2(r'-r)}$ and $\lambda^6=1$, to $X^6+Y^6+Z^6+\beta_{2,0}X^2Z^2(X^2+Z^2)=0$. However, in this case, $[Z:\zeta_6Y:-X]$ is an automorphism of order $12>6$, which leads to a contradiction.

**Case** $\phi=[aX+bZ:Y:cX-\frac{bc}{a}Z]$. For this to hold, we have, $\beta_{2,0}=\frac{1-a^6-c^6-a^2c^4}{a^4c^2}$, $b=\zeta_6^na$ and $-2ac=\zeta_6^{n'}$. Therefore, $C$ is $K$-equivalent via $\phi$, followed by a rescaling $Z\mapsto\zeta_6^{-n}Z$, to $C''$ of the form:

$$\begin{aligned}C'':X^6+Y^6+Z^6+\alpha'_1XZ\left(X^4+Z^4\right)+\alpha'_2X^2Z^2\left(X^2+Z^2\right)+\alpha'_3X^3Z^3\\+\alpha'_4\left(X^3+Z^3\right)Y^3+\alpha'_5XY^3Z\left(X-Z\right)=0,\end{aligned}$$

where

$$\begin{aligned}\alpha'_1&:=\frac{(-2a^4\beta_{4,0}\zeta_6^{4n'}+32a^{12}+16a^6-1)}{8a^6},\ \alpha'_2:=\frac{(64a^{12}-4a^6+1)}{4a^6},\\ \alpha'_3&:=\frac{(2a^4\beta_{4,0}\zeta_6^{4n'}+96a^{12}-16a^6-1)}{4a^6},\ \alpha'_4:=-\frac{(4a^4\beta_{4,3}+\beta_{6,3}\zeta^{2n'})\zeta_6^{n'}}{8a^3},\\ \alpha'_5&:=-\frac{(4a^4\beta_{4,3}-3\beta_{6,3}\zeta^{2n'})\zeta_6^{n'}}{8a^3},\end{aligned}$$

Here $[Z:\zeta_6^{1-n-n'}Y:X]$ must be an automorphism for $C'$ of order 6. This implies that $\alpha'_4=0$ when $2\mid n+n'$ and $\alpha'_5=0$ when $2\nmid n+n'$. Moreover,

(i) Suppose that $\left(\alpha'_1,\alpha'_2,\alpha'_5\right)=(0,0,0)$. If $\alpha'_4=0$, then $\mathrm{Aut}(C)$ is again $\mathbb{Z}/6\mathbb{Z}\times\mathrm{S}_3$ according to Proposition 9.6-(3). But if $\alpha'_4\neq0$, then $\mathrm{diag}(1,\zeta_3,1)$, $\mathrm{diag}(1,1,\zeta_3)$, $[Z:Y:X]$ would be automprhisms for $C''$. This means that $\mathrm{SmallGroup}(18,3)=\mathbb{Z}/3\mathbb{Z}\times\mathrm{S}_3$ is always a subgroup of $\mathrm{Aut}(C'')$. On the other hand, $\tau_3,\tau_4,\tau_5,\tau_6$ do not qualify as automorphisms for $C''$ if $\alpha'_3=0$, implying that $\mathrm{Aut}(C'')$ exactly is $\mathbb{Z}/3\mathbb{Z}\times\mathrm{S}_3$. Conversely, if $\alpha'_3\neq0$, then it can be verified easily that $\mathrm{Aut}(C'')$ is contained in $\langle\mathrm{diag}(1,\zeta_3,1),\mathrm{diag}(1,1,\zeta_3),[Z:Y:X],[Z:X:Y]\rangle$, which is isomprhic to $\mathrm{SmallGroup}(54,5)=(\mathbb{Z}/3\mathbb{Z})^2\rtimes\mathbb{Z}/6\mathbb{Z}$. Since $\mathbb{Z}/3\mathbb{Z}\times\mathrm{S}_3$ is maximal in $(\mathbb{Z}/3\mathbb{Z})^2\rtimes\mathbb{Z}/6\mathbb{Z}$, $\mathrm{Aut}(C'')$ can either be $\mathbb{Z}/3\mathbb{Z}\times\mathrm{S}_3$ or $(\mathbb{Z}/3\mathbb{Z})^2\rtimes\mathbb{Z}/6\mathbb{Z}$. More concretely, $\mathrm{Aut}(C'')$ is $(\mathbb{Z}/3\mathbb{Z})^2\rtimes\mathbb{Z}/6\mathbb{Z}$ if and only if $[Z:X:Y]\in\mathrm{Aut}(C'')$, which occurs precisely when $\alpha'_3=\alpha'_4$.

(ii) Suppose that $\left(\alpha'_1,\alpha'_2,\alpha'_5\right)\neq(0,0,0)$. Then, $\tau_3,\tau_4,\tau_5$ and $\tau_6$ are never automorphisms for $C''$. Exploring different scenarios:

 (a) If $\alpha'_4\neq0$ (so $\alpha'_5=0$), then $\mathrm{Aut}(C'')$ is contained within

$$\langle\mathrm{diag}(1,\zeta_3,1),\mathrm{diag}(1,1,\zeta_3),[Z:Y:X]\rangle\cong\mathbb{Z}/3\mathbb{Z}\times\mathrm{S}_3\,.$$

Since $\mathbb{Z}/6\mathbb{Z}$ is maximal in $\mathbb{Z}/3\mathbb{Z} \times \mathrm{S}_3$, and $\mathrm{diag}(1, 1, \zeta_3)$, $[Z : Y : X]$ are in $\mathrm{Aut}(C'')$ only if $\alpha'_1 = \alpha'_2 = 0$, we conclude

$$\mathrm{Aut}(C'') = \langle [Z : \zeta_3 Y : X] \rangle \cong \mathbb{Z}/6\mathbb{Z}$$

in this situation.

(b) Secondly, if $\alpha'_5 \neq 0$ (so $\alpha'_4 = 0$), then the automrphisms of $C''$ are elements of

$$\left\{ I, \mathrm{diag}(1, \zeta_6^{\pm 2}, 1), [Z : sY : X] : s = -1, \zeta_6, -\zeta_6^2 \right\} = \langle [Z : \zeta_6 Y : X] \rangle.$$

Therefore, $\mathrm{Aut}(C'')$ is again $\mathbb{Z}/6\mathbb{Z}$ generated by $[Z : \zeta_6 Y : X]$.

(c) Thirdly, if $\alpha'_4 = \alpha'_5 = 0$ and $(\alpha'_1, \alpha'_3) \neq (0, 0)$, then $\mathrm{diag}(1, \zeta_6, 1)$ serves as an additional automorphism for $C''$, that is,

$$\mathrm{Aut}(C'') = \langle \mathrm{diag}(1, \zeta_6, 1), [Z : Y : X] \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

(d) Lastly, the case $\alpha'_4 = \alpha'_5 = \alpha'_1 = \alpha'_3 = 0$ leads to inconsistency since $[Z : \zeta_6 Y : -X]$ would be an automorphism of order $12 > 6$.

Thus, the various possibilities for $\mathrm{Aut}(C'')$ have been elucidated, confirming Claim 9.8, equivalently, Proposition 9.10-(2), (3) below. □

**Claim 9.9** *If* $\mathrm{Aut}(C)$*, where* $C$ *is given as Eq.* 9.3*, fixes a line* $\mathcal{L}$ *and a point* $P$ *not lying on* $\mathcal{L}$*, then* $\mathrm{Aut}(Z) \cong \mathbb{Z}/6\mathbb{Z}$*.*

***Proof*** Since $\sigma$ is a non-homology inside $\mathrm{Aut}(C)$ in its canonical form, then $\mathcal{L}$ must be one of the reference lines $B = 0$ where $B = X, Y$ or $Z$, and $P$ must be one of the reference points $(1 : 0 : 0)$, $(0 : 1 : 0)$, or $(0 : 0 : 1)$ respectively. However, a projective permutation enables us to confine $B = X$ and $P = (1 : 0 : 0)$. Consequently, all automorphisms of $C$ fall into the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Also, $\mathrm{Aut}(C)$ fits in a short exact sequence $1 \to N \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1$ where $N = \mathbb{Z}/n\mathbb{Z}$ with $n \,|\, 6$, and $\Lambda(\mathrm{Aut}(C))$ is either (i) $\mathbb{Z}/m\mathbb{Z}$ with $m \leq 5$, (ii) $\mathrm{D}_{2m}$ such that $n = 1$ or $m \,|\, 4$, or (iii) $\mathrm{A}_4$, $\mathrm{A}_5$ or $\mathrm{S}_4$.

We observe that $\langle \sigma^3 \rangle$ is a subgroup of $N$ of order 2, and $\Lambda(\mathrm{Aut}(C))$ includes $\Lambda(\sigma) = [\zeta_3^{-1} Y, Z]$ of order 3. Thus $\Lambda(\mathrm{Aut}(C))$ cannot be $\mathrm{D}_{2m}$ since $3 \nmid 2m$ when $m|4$. If $\Lambda(\mathrm{Aut}(C)) = \mathbb{Z}/m\mathbb{Z}$, then $m = 3$ and $\mathrm{Aut}(C) = \langle \sigma \rangle = \mathbb{Z}/6\mathbb{Z}$, as previously claimed. Moreover, if $\Lambda(\mathrm{Aut}(C)) = \mathrm{A}_5$, then $\mathrm{Aut}(C)$ would contain an automorphism of order $10 > 6$, contradicting the maximality of the order of $\sigma$ in $\mathrm{Aut}(C)$.

To conclude, we need to demonstrate that $\Lambda(\mathrm{Aut}(C))$ cannot be $\mathrm{S}_4$ or $\mathrm{A}_4$.

**Case** $\Lambda(\mathrm{Aut}(C)) = \mathrm{S}_4$. Specifically, $\Lambda(\mathrm{Aut}(C))$ must include an element $\Lambda(\tau)$ of order 2 that satisfies the relation $\Lambda(\tau)\Lambda(\sigma)\Lambda(\tau) = \Lambda(\sigma)^{-1}$. This is because the group $\mathbb{Z}/3\mathbb{Z}$ within $\mathrm{S}_4$ is equal to its centralizer and has an $\mathrm{S}_3$ as its normalizer. Therefore, $\Lambda(\tau)$ must be of the form $[cZ : bY]$ for some $b, c \in K^*$, which occurs if and only if $\beta_{2,0} = \beta_{4,0} = \beta_{4,3} = 0$, making $C$ $K$-isomorphic to $C : X^6 + Y^6 + Z^6 + \beta_{0,3} Y^3 Z^3 = 0$ (in particular, with $b^6 = c^6 = (bc)^3 = 1$). Additionally, the centralizer of $\langle \Lambda(\tau) \rangle$ in $\mathrm{S}_4$ is $(\mathbb{Z}/2\mathbb{Z})^2$, which necessitates the existence of another element $\Lambda(\tau') \notin \langle \Lambda(\sigma), \Lambda(\tau) \rangle$ of order 2 that commutes with $\Lambda(\tau)$. This implies $\Lambda(\tau') = [Y : -Z]$, which leaves the polynomial $Y^6 + Z^6 + \beta_{0,3} Y^3 Z^3$ invariant only if $\beta_{0,3} = 0$. Consequently, $C$ would be the Fermat curve $\mathcal{F}_6$, leading to a contradiction.

**Case** $\Lambda(\mathrm{Aut}(C)) = \mathrm{A}_4$. In this situation, $\mathrm{Aut}(C)$ includes a subgroup $G$ of order 24 that has $H = \langle\sigma^3\rangle$ as a normal subgroup, with the quotient $G/H$ isomorphic to $\mathrm{A}_4$. Referring to the list of Group of order 24, we find that $G$ must be either $\mathrm{SL}_2(\mathbb{F}_3) = \mathrm{SmallGroup}(24, 3)$ or $\mathbb{Z}/2\mathbb{Z} \times \mathrm{A}_4 = \mathrm{SmallGroup}(24, 13)$.

(i) For $G$ to be $\mathbb{Z}/2\mathbb{Z} \times \mathrm{A}_4$, there must be an involution $\tau$ that commutes with $\sigma^3$, as any $\mathbb{Z}/2\mathbb{Z}$ subgroup in $\mathbb{Z}/2\mathbb{Z} \times \mathrm{A}_4$ is part of a $(\mathbb{Z}/2\mathbb{Z})^2$. Therefore, $\tau$ must be of one of the following forms: $\mathrm{diag}(1, \pm 1, \mp 1)$, $[X : \pm Y : sY \mp Z]$, or $[X : sY + tZ : \frac{1-s^2}{t} Y - tZ]$ for some $s, t \in K$. If $\tau = \mathrm{diag}(1, \pm 1, \mp 1)$ is an automorphism, then $\beta_{0,3}$ and $\beta_{4,3}$ must be zero. In this case, $\mathrm{Aut}(C)$ would include $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, not $\mathbb{Z}/2\mathbb{Z} \times \mathrm{A}_4$, according to Proposition 9.6, which is a contradiction. For $\tau = [X : \pm Y : sY \mp Z]$ to be an automorphism, $s$ must be zero; otherwise, the term $YZ^5$ would appear in the transformed equation under the action of $\tau$. This leads us back to the scenario where $\mathrm{diag}(1, \pm 1, \mp 1)$ is in $\mathrm{Aut}(C)$. The case $\tau = [X : sY + tZ : \frac{1-s^2}{t} Y - tZ]$ is also invalid. The transformed equation would include the term $\frac{\beta_{2,0}}{t^2}\left((s^2-1)Y + t^2 Z\right)^2 X^4$, implying $\beta_{2,0}$ must be zero, or $s^2 = t^2 = 1$. However, the latter case is excluded because removing $YZ^5$ would also eliminate $Z^6$ from the transformed equation (both terms appear with coefficient $\pm(\beta_{0,3} - 2)$). Thus, $\beta_{2,0}$ must be zero, and the coefficients of $YZ^5$ and $Z^6$ imply $s \neq 0$ and $t = \frac{1-s^2}{s}$. The action does not produce the monomial $Y^3Z^3$, so $\beta_{0,3} = 0$, and the transformed equation would have $Y^2Z^4$ with a non-zero coefficient, leading to a contradiction.

(ii) For $G$ to be $\mathrm{SL}_2(\mathbb{F}_3)$, $C$ must have an automorphism $\tau$ of order 4 such that $\tau^2 = \sigma^3$. This implies that $\tau$ could be of the form $\mathrm{diag}(\pm\zeta_4, 1, 1)$, $[X : \pm\zeta_4 Y : sY \mp \zeta_4 Z]$, or $[X : sY + tZ : -\frac{s^2+1}{t} Y - tZ]$ for some $s, t \in K$. However, neither $\mathrm{diag}(\pm\zeta_4, 1, 1)$ nor $[X : \pm\zeta_4 Y : sY \mp \zeta_4 Z]$ preserves the core polynomial $X^6 + Y^6 + Z^6$ of the defining equation for $C$. On the other hand, $\tau = [X : sY + tZ : -\frac{s^2+1}{t} Y - tZ]$ can be an automorphism of $C$ only if $s \neq 0$ and $t = -\frac{1+s^2}{s}$ (otherwise, $YZ^5$ would appear in the transformed equation). To eliminate the monomial $X^4Y^2$, we require $\beta_{2,0} = 0$, and to eliminate $X^2YZ^3$ and $X^2Z^4$, we need $\beta_{4,0} = \beta_{4,3} = 0$. Consequently, according to Proposition 9.6, $\mathrm{Aut}(C)$ would include $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$, not $\mathrm{SL}_2(\mathbb{F}_3)$, which is a contradiction.

Thus $\Lambda(\mathrm{Aut}(C)) \neq \mathrm{A}_4$ as well. □

Regarding the Hessian group $\mathrm{Hess}_{216}$, its representations within $\mathrm{PGL}\,3(\overline{k})$ are unique up to conjugation. For more information, refer to Mitchell [50, p. 217]. For instance, we can define $\mathrm{Hess}_{216}$ as $\langle S, T, U, V\rangle$, where:

$$S = \mathrm{diag}(1, \zeta_3, \zeta_3^{-1}), U = \mathrm{diag}(1, 1, \zeta_3),\ \ V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^{-1} \\ 1 & \zeta_3^{-1} & \zeta_3 \end{pmatrix},\ \ T = [Y : Z : X].$$

Additionally, we consider the primitive Hessian subgroups of orders 36 and 72 respectively: $\mathrm{Hess}_{36} = \langle S, T, V\rangle$ and $\mathrm{Hess}_{72} = \langle S, T, V, UVU^{-1}\rangle$.

Assume $\mathrm{Aut}(C)$ is conjugate to Hess 216. We can then assume $\phi^{-1}\,\mathrm{Aut}(C)\phi = \mathrm{Hess}_{216}$, where $\phi^{-1}\sigma\phi$ is either $[\zeta_3 Z : Y : \zeta_3 X]$ or $[\zeta_3^{-1} Z : Y : \zeta_3^{-1} X]$, since any group of order 6 within $\mathrm{Hess}_{216}$ is conjugate to $\langle[\zeta_3 Z : Y : \zeta_3 X]\rangle$. This implies

$$\phi = \begin{pmatrix} a & 0 & -a \\ 0 & 1 & 0 \\ b & 0 & b \end{pmatrix}$$

with $ab \neq 0$. The transformed equation under $\phi$ must be invariant under any permutation of the variables $X$, $Y$, and $Z$, as the subgroup $\langle [X : Z : Y], [Y : Z : X] \rangle$ is part of the automorphisms. For instance, the coefficients of $Z^6$, $X^6$, and $XZ^5$ in the transformed equation $C'$ must be 1, 1, and 0, respectively. Thus, $\beta_{2,0} = \frac{1-4a^6+2b^6}{2a^4b^2}$ and $\beta_{4,0} = \frac{1+2a^6-4b^6}{2a^2b^4}$. By comparing the coefficients of $X^3Z^3$ and $Y^3Z^3$, we find that $\beta_{4,3} = -\frac{32a^6-32b^6+b^3\beta_{0,3}}{a^2b}$. To ensure that the coefficients of $XY^3Z^2$ and $X^2Z^4$ are zero, we set $\beta_{0,3} = -\frac{8(a^6-b^6)}{b^3}$ and require that $16a^6 + 16b^6 = 1$. This simplifies $C'$ to:

$$X^6 + Y^6 + Z^6 - 2(32a^6 - 1)(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0.$$

To confirm that $V \in \mathrm{Aut}(C')$, we must have $16a^6 = 3$ and $16b^6 = -2$. This yields:

$$\beta_{0,3} = \frac{-5}{2b^3},\ \beta_{4,3} = \frac{-15}{2a^2b},\ \beta_{4,0} = \frac{15}{16a^2b^4},\ \beta_{2,0} = 0.$$

This supports Proposition 9.10-(1) below.

Summarizing, we obtain:

**Proposition 9.10** *Let $C$ be a smooth plane sextic curves $C$ of Type* 6, (1, 3) *as described by Eq.* 9.3. *Then,* $\mathrm{Aut}(C)$ *is classified as follows:*

1. *If* $\beta_{0,3} = \frac{-5}{2b^3}$, $\beta_{4,3} = \frac{-15}{2a^2b}$, $\beta_{4,0} = \frac{15}{16a^2b^4}$, $\beta_{2,0} = 0$ *with* $16a^2 = 3$ *and* $16b^2 = -2$, *then $C$ is $K$-isomorphic to*

$$C' : X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0,$$

   *whose automorphism group is* $\mathrm{Hess}_{216}$.
2. *If* $\beta_{2,0} = \beta_{4,0} = \beta_{4,3} = 0$, *then $C$ is defined by the equation:*

$$X^6 + Y^6 + Z^6 + \beta_{0,3}Y^3Z^3 = 0,$$

   *which is a descendant of the Fermat curve. In particular,* $\mathrm{Aut}(C)$ *is the group* $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$ *presented in Proposition 9.6.*
3. *If* $\beta_{2,0} = \dfrac{-64a^{12} + 60a^6 - 1}{16a^8}$, *then $C$ is $K$-isomorphic to*

$$\begin{aligned} C' \ :\ & X^6 + Y^6 + Z^6 + \alpha_1' XZ\left(X^4 + Z^4\right) + \alpha_2' X^2Z^2\left(X^2 + Z^2\right) + \alpha_3' X^3Z^3 \\ & + \alpha_4'\left(X^3 + Z^3\right)Y^3 + \alpha_5' XY^3Z\,(X - Z) = 0. \end{aligned}$$

   *In this context, the classification of* $\mathrm{Aut}(C')$ *is as follows:*

   - *(i) If* $\left(\alpha_1', \alpha_2', \alpha_5'\right) = (0, 0, 0)$ *and* $\alpha_4' = 0$, *then* $\mathrm{Aut}(C')$ *is* $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$, *as described in Proposition 9.6-(3).*
   - *(ii) If* $\left(\alpha_1', \alpha_2', \alpha_5'\right) = (0, 0, 0)$, $\alpha_4' \neq 0$, *and* $\alpha_3' = 0$ *or* $\alpha_3' \neq 0$ *with* $\alpha_3' \neq \alpha_4'$, *then* $\mathrm{Aut}(C')$ *is* $\mathbb{Z}/3\mathbb{Z} \times \mathrm{S}_3$, *generated by* $\mathrm{diag}(1, \zeta_3, 1)$, $\mathrm{diag}(1, 1, \zeta_3)$, *and* $[Z : Y : X]$.
   - *(iii) If* $\left(\alpha_1', \alpha_2', \alpha_5'\right) = (0, 0, 0)$ *and* $\alpha_4' = \alpha_3' \neq 0$, *then* $\mathrm{Aut}(C')$ *is* $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/6\mathbb{Z}$, *generated by* $\mathrm{diag}(1, \zeta_3, 1)$, $\mathrm{diag}(1, 1, \zeta_3)$, $[Z : Y : X]$ *and* $[Z : X : Y]$.
   - *(iv) If* $\left(\alpha_1', \alpha_2', \alpha_5'\right) \neq (0, 0, 0)$ *and* $\alpha_4' = \alpha_5' = 0$, *with* $\left(\alpha_1', \alpha_3'\right) \neq (0, 0)$, *then* $\mathrm{Aut}(C')$ *is* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, *generated by* $\mathrm{diag}(1, \zeta_6, 1)$ *and* $[Z : Y : X]$. *In this case, $C$ is $K$-isomorphic to a member of the family described in Proposition 9.6-(1).*
   - *(v) If* $\left(\alpha_1', \alpha_2', \alpha_5'\right) \neq (0, 0, 0)$ *and either* $\alpha_4' \neq 0$ *or* $\alpha_5' \neq 0$, *then* $\mathrm{Aut}(C')$ *is* $\mathbb{Z}/6\mathbb{Z}$, *generated by* $[Z : \zeta_3 Y : X]$ *if* $\alpha_4' \neq 0$ *and by* $[Z : \zeta_6 Y : X]$ *if* $\alpha_5' \neq 0$.
4. *Otherwise,* $\mathrm{Aut}(C)$ *is* $\mathbb{Z}/6\mathbb{Z}$, *generated by* $\mathrm{diag}(1, \zeta_6, -1)$.

***Remark 9.11*** It is known in the literature that the curve:

$$X^6 + Y^6 + Z^6 - 10(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$$

has $\mathrm{Hess}_{216}$ as its automorphism group (see [30, 2]). Our result, however, demonstrates that this is the only smooth plane sextic curve with such a property.

### 9.4 Type 3, (0, 1)

In this case, $C$ is defined by an equation of the form:

$$C : Z^6 + Z^3L_{3,Z} + L_{6,Z} = 0, \tag{9.4}$$

where $\sigma = \mathrm{diag}(1, 1, \zeta_3)$ is an automorphism of maximal order 3. In particular, $L_{3,Z} \neq 0$ or $\mathrm{diag}(1, 1, \zeta_6)$ will be an automorphism of order $6 > 3$. Moreover, due to the smoothness of the curve, $L_{6,Z}$ must have a degree at least 5 in both $X$ and $Y$.

The automorphism group of $C$ is characterized by the following result, which directly follows from [7, Theorem 2.4].

**Proposition 9.12** *Let $C$ be a smooth plane sextic curve of Type* 3, (0, 1) *as described above (Eq. 9.4). Then,* $\mathrm{Aut}(C)$ *is always cyclic and generated by $\sigma$, except in the case where $C$ is $K$-isomorphic to $C'$ of the form:*

$$C' : X^6 + Y^6 + Z^6 + Z^3\left(\beta_{3,0}X^3 + \beta_{0,3}Y^3\right) + \beta_{3,3}X^3Y^3 = 0,$$

*with $\beta_{3,0}$, $\beta_{0,3}$, and $\beta_{3,3}$ being pairwise distinct modulo $\pm 1$. In this specific case,* $\mathrm{Aut}(C')$ *is* $(\mathbb{Z}/3\mathbb{Z})^2$*, generated by $\sigma$ and* $\mathrm{diag}(1, \zeta_3, 1)$.

## 10 Remaining types

In this section, we explore the automorphism group of smooth plane curves $C$ of the following types: Type 6, (1, 2), Type 5, (1, 2), Type 5, (1, 4), Type 4, (1, 3), Type 3, (1, 2), and Type 2, (0, 1).

### 10.1 Type 6, (1, 2)

In this situation, $C$ is defined by an equation of the form:

$$C : X^6 + Y^6 + Z^6 + \beta_{3,0}X^3Z^3 + \beta_{2,2}X^2Y^2Z^2 + \beta_{1,4}XY^4Z = 0, \tag{10.1}$$

where $\sigma = \mathrm{diag}(1, \zeta_6, \zeta_6^2)$ is an automorphism of maximal order 6.

It is clear that $\tau = [Z : Y : X]$ is always an automorphism for $C$, implying that $\mathrm{Aut}(C)$ can never be cyclic. Additionally, $C$ is not a descendant of the Klein curve $\mathcal{K}_6$ 6 does not divide $|\mathrm{Aut}(\mathcal{K}_6)| = 63$. Moreover, $\mathrm{Aut}(C)$ cannot be $\mathrm{PGL}_3(K)$-conjugate to the Klein group $\mathrm{PSL}(2, 7)$, which includes elements of order 7, exceeding 6. Similarly, $\mathrm{Aut}(C)$ is not conjugate to $A_5$, $A_6$, $\mathrm{Hess}_{36}$, or $\mathrm{Hess}_{72}$ as none of these groups contain elements of order 6. Consequently, we conclude that $\mathrm{Aut}(C)$ either fixes a line and a point off this line, is conjugate to $\mathrm{Hess}_{216}$, or $C$ is a descendant of the Fermat curve $\mathcal{F}_6$.

**Claim 10.1** $\mathrm{Aut}(C)$ *can never be conjugate to* $\mathrm{Hess}_{216}$.

***Proof*** All copies of $\mathbb{Z}/6\mathbb{Z}$ within $\mathrm{Hess}_{216}$ are conjugate to each other (see Hessian group of order 216). Consider a specific copy generated by to $\rho = [X : Z : \zeta_3 Y]$. Notably, $\rho^2$ is an homology of period 3. Hence, $\langle\sigma\rangle$ can not be $\mathrm{PGL}_3(K)$-conjugate to $\langle\rho\rangle$, as $\sigma^2 = \mathrm{diag}(1, \zeta_3, \zeta_3^{-1})$ is a non-homology. □

**Claim 10.2** *Assume that $C$ is a descendant of the Fermat cur$\mathcal{F}_6$. Then,* $\mathrm{Aut}(C)$ *is* $\mathrm{D}_{12}$, $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_4$, *or* $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$.

***Proof*** We can assume that $\phi^{-1}\,\mathrm{Aut}(C)\phi \leq \mathrm{Aut}(\mathcal{F}_6)$ for some $\phi \in \mathrm{PGL}_3(K)$ such that $\phi^{-1}\sigma^2\phi = \sigma^2$, given that non-homologies of period 3 derived from an automorphism of order 6 inside $\mathrm{Aut}(\mathcal{F}_6)$ form a single conjugacy class represented by $\sigma^2$. Thus, $\phi$ can be of the form $\mathrm{diag}(1, a, b)$, $[bZ : X : aY]$, or $[aY : bZ : X]$ with $a^6 = b^6 = 1$. In all these cases, $C$ is $K$-isomorphic to:

$$C' : X^6 + Y^6 + Z^6 + \beta'_{3,0}X^3Z^3 + \beta'_{2,2}X^2Y^2Z^2 + \beta'_{1,4}XY^4Z = 0,$$

where $\left(\beta'_{3,0}, \beta'_{2,2}, \beta'_{1,4}\right) = \left(\beta_{3,0}b^3, \beta_{2,2}(ab)^2, \beta_{1,4}a^4b\right)$. Next, we use the notations from Sect. 9.3 to examine when $C'$ might have a larger automorphism group within $\mathrm{Aut}(\mathcal{F}_6)$ than $\langle\sigma, \tau\rangle \cong \mathrm{SmallGroup}(12, 4) = \mathrm{D}_{12}$.

**Case** $\beta'_{3,0} = \beta'_{1,4} = 0$ and $\beta'_{2,2} \neq 0$. In this scenario, $\tau_5$ and $\tau_6$ are automrphisms for $C'$, implying that $\mathrm{Aut}(C)$ is the semidirect product SmallGroup(72, 43) of $\mathbb{Z}/3\mathbb{Z}$ and $\mathrm{S}_4$, generated by $\sigma$, $\tau$, $\tau' = \mathrm{diag}(1, 1, -1)$, and $\eta := [Y : Z : X]$

**Case** $\beta'_{3,0} \neq 0$ or $\beta'_{1,4} \neq 0$. In this scenario, $\tau_3$, $\tau_4$ $\tau_5$ and $\tau_6$ cannot be automorphism for $C'$. Specifically, if $\beta'_{3,0} \neq 0$ and $\beta'_{1,4} = \beta'_{2,2} = 0$, then $C$ becomes $K$-isomorphic, through a permutation $X \leftrightarrow Y$, to a curve within the family described in Proposition 9.6-(3) (so $\mathrm{Aut}(C)$ is the group $\mathbb{Z}/6\mathbb{Z} \times \mathrm{S}_3$). Otherwise, if $\beta'_{1,4} \neq 0$ or $\beta'_{2,2} \neq 0$, then $\mathrm{Aut}(C) = \langle\sigma, \tau\rangle \cong \mathrm{D}_{12}$. □

**Claim 10.3** *If* $\mathrm{Aut}(C)$ *fixes a line $\mathcal{L}$ and a point $P$ not on $\mathcal{L}$, then* $\mathrm{Aut}(C)$ *must be either* $\mathrm{D}_{12}$ *or* $\mathbb{Z}/3\mathbb{Z} \rtimes \mathrm{S}_4$.

***Proof*** Since $\sigma, \tau \in \mathrm{Aut}(C)$, the line $\mathcal{L}$ can be taken as $Y = 0$ and the point $P$ as $(0 : 1 : 0)$. Consequently, all automorphisms of $C$ are of the form:

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

Also, we can describe $\mathrm{Aut}(C)$ in terms of a short exact sequence:

$$1 \to N \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1,$$

where $N$ is a cyclic group of order dividing 6, and $\Lambda(\mathrm{Aut}(C))$ can be one of the following: (i) A cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order $m \leq d-1$, (ii) A Dihedral group $\mathrm{D}_{2m}$ of order $2m$, where $|N| = 1$ or $m = 1, 2$ or 4, (iii) one of the groups $\mathrm{A}_4$, $\mathrm{A}_5$ and $\mathrm{S}_4$.

Given that $\langle\sigma^3\rangle \leq N$ has order 2, and $\Lambda(\mathrm{Aut}(C))$ contains an $\mathrm{S}_3$ generated by $\Lambda(\sigma) = [\zeta_3 X, \zeta_3^{-1}Z]$ of order 3 and $\Lambda(\tau) = [Z : X]$ of order 2, we we analyze the possible cases for $\Lambda(\mathrm{Aut}(C))$.

Clearly, $\Lambda(\mathrm{Aut}(C)) \neq \mathbb{Z}/m\mathbb{Z}$ since $\mathrm{S}_3$ is not cyclic, $\Lambda(\mathrm{Aut}(C)) \neq \mathrm{D}_{2m}$ as $3 \nmid 2m$ if $m|4$, $\Lambda(\mathrm{Aut}(C)) \neq A_4$ as $\mathrm{A}_4$ doesn't contain $\mathrm{S}_3$ as a subgroup, $\Lambda(\mathrm{Aut}(C)) \neq \mathrm{A}_5$ as $\mathrm{Aut}(C)$ would otherwise include an automorphism of order 10, which exceeds 6. On the other hand, if $\Lambda(\mathrm{Aut}(C)) = \mathrm{S}_4$, then $\Lambda(\mathrm{Aut}(C))$ must include an element $\Lambda(\tau') \notin \langle\Lambda(\sigma), \Lambda(\tau)\rangle$ of order 2 that commutes with $\Lambda(\tau)$, since $\langle\Lambda(\tau)\rangle = \mathbb{Z}/2\mathbb{Z}$ in $S_4$ has centralizer $(\mathbb{Z}/2\mathbb{Z})^2$ in $S_4$. This element must be $\Lambda(\tau') = [X : -Z]$, which corresponds to $\tau' = \mathrm{diag}(\lambda, 1, -\lambda)$. Such an automorphism exists if and only if $\beta_{3,0} = \beta_{1,4} = 0$. In this case, $\mathrm{Aut}(C)$ reduces to the previously considered scenario where it is a semidirect product of $\mathbb{Z}/3\mathbb{Z}$ and $S_4$. □

Based on the preceding analysis, we can summarize the classification of the automorphism groups of smooth plane sextic curves of Type 6, (1, 2) as follows:

**Proposition 10.4** *Let $C$ be a smooth plane sextic curves $C$ of Type* 6, (1, 2) *as described above (Eq. 10.1). Then,*

1. *If $\beta_{3,0} = \beta_{1,4} = 0$, then $C$ is given by $X^6 + Y^6 + Z^6 + \beta_{2,2}X^2Y^2Z^2 = 0$ for some $\beta_{2,2} \neq 0$, which is a descendant of the Fermat curve. In this case,* $\mathrm{Aut}(C) = \langle\sigma, \tau, \tau', \eta\rangle$ *is isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes S_4$, where $\tau' = \mathrm{diag}(1, 1, -1)$ and $\eta = [Y : Z : X]$.*
2. *If $\beta_{3,0} \neq 0$, $\beta_{1,4} = \beta_{2,2} = 0$, then* $\mathrm{Aut}(C) \cong \mathbb{Z}/6\mathbb{Z} \times S_3$*, as described in Proposition 9.6-(3).*
3. *Otherwise,* $\mathrm{Aut}(C) = \langle\sigma, \tau\rangle \cong D_{12}$.

## 10.2 Type 5, (1, 4)

In this situation, $C$ is defined by an equation of the form:

$$C : X^6 + XZ^5 + XY^5 + \beta_{4,1}X^4YZ + \beta_{2,2}X^2Y^2Z^2 + \beta_{0,3}Y^3Z^3 = 0, \qquad (10.2)$$

where $\sigma = \mathrm{diag}(1, \zeta_5, \zeta_5^{-1})$ is an automorphism of maximal order 5.

Clearly, $\langle\sigma, \tau\rangle \cong D_{10}$, with $\tau := [X : Z : Y]$, is always a subgroup of automorphisms for $C$, which means that $\mathrm{Aut}(C)$ is never cyclic. Additionally, $C$ is neither a descendant of the Fermat curve $\mathcal{F}_6$ nor the Klein curve $\mathcal{K}_6$, as 10 does not divide $|\mathrm{Aut}(\mathcal{F}_6)| = 216$ and 10 does not divide $|\mathrm{Aut}(\mathcal{K}6)| = 63$. For the same reasons, $\mathrm{Aut}(C)$ cannot be conjugate to Hess for $\in 36, 72, 216$ or to $\mathrm{PSL}(2, 7)$. Thus, $\mathrm{Aut}(C)$ either fixes a line and a point not on this line or it is $\mathrm{PGL}_3(K)$-conjugate to $A_5$ or $A_6$.

**Claim 10.5** *If* $\mathrm{Aut}(C)$ *fixes a line $\mathcal{L}$ and a point $P$ off this line, then* $\mathrm{Aut}(C)$ *is exactly* $\langle\sigma, \tau\rangle$.

***Proof*** Since $\sigma$ and $\tau$ are in $\mathrm{Aut}(C)$, we can fix $\mathcal{L} : X = 0$ and $P = (1 : 0 : 0)$. Consequently, all automorphisms of $C$ must be intransitive and of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Also, $\mathrm{Aut}(C)$ fits into a short exact sequence:

$$1 \to N = 1 \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1,$$

where $\Lambda(\mathrm{Aut}(C))$ could be $\mathbb{Z}/m\mathbb{Z}$ with $m \leq d - 1$, $D_{2m}$, $A_4$, $A_5$, or $S_4$.

Since $\Lambda(\mathrm{Aut}(C))$ includes $\langle\Lambda(\sigma), \Lambda(\tau)\rangle = D_{10}$, it follows that $\Lambda(\mathrm{Aut}(C))$ cannot be $\mathbb{Z}/m\mathbb{Z}$, $A_4$, or $S_4$. Additionally, if $\Lambda(\mathrm{Aut}(C))$ were $A_5$, then there would need to be an element $\Lambda(\tau') \notin \langle\Lambda(\sigma), \Lambda(\tau)\rangle$ of order 2 that commutes with $\Lambda(\tau)$, as every copy of $\mathbb{Z}/2\mathbb{Z}$ in $A_5$ has a centralizer isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This implies that $\Lambda(\tau') = [Y : -Z]$, or equivalently, $\tau' = \mathrm{diag}(1, \lambda, -\lambda)$ for some $\lambda \in K^*$. However, the presence of monomials like $XZ^5$ and $XY^5$ rules out such an automorphism, so $\Lambda(\mathrm{Aut}(C)) \neq A_5$. Finally, if $\Lambda(\mathrm{Aut}(C)) = D_{2m}$, then $m = 5$ due to the maximal order of $\sigma$. □

Now, it remains to consider $C$ described above (Eq. 10.2), where $\mathrm{Aut}(C)$ is conjugate to either $\mathrm{A}_5$ or $\mathrm{A}_6$. In this case, $\mathrm{Aut}(C)$ must include an additional involution $\tau'$ that commutes with $\tau$, since any $\mathbb{Z}/2\mathbb{Z}$ subgroup within $\mathrm{A}_5$ or $\mathrm{A}_6$ is part of a $(\mathbb{Z}/2\mathbb{Z})^2$ structure. According to [18, Lemma 2.16], this involution $\tau'$ must have the form:

$$\tau'_\gamma = \begin{pmatrix} 1 & 2/\gamma & 2/\gamma \\ \gamma & (-1+\sqrt{5})/2 & (-1-\sqrt{5})/2 \\ \gamma & (-1-\sqrt{5})/2 & (-1+\sqrt{5})/2 \end{pmatrix},$$

where $\beta_{4,1} = \dfrac{12-\gamma^5}{\gamma^2}$, $\beta_{2,2} = \dfrac{48+\gamma^5}{\gamma^4}$, and $\beta_{0,3} = \dfrac{64-2\gamma^5}{\gamma^6}$ for some $\gamma \in K^*$. As a consequence, $C$ is $K$-isomorphic to:

$$C_\gamma : X^6 + X(Y^5+Z^5) + \frac{12-\gamma^5}{\gamma^2}X^4YZ + \frac{48+\gamma^5}{\gamma^4}X^2Y^2Z^2 + \frac{64-2\gamma^5}{\gamma^6}Y^3Z^3 = 0,$$

Assume $\mathrm{Aut}(C)$ is $\mathrm{A}_6$. According to [18, Theorem 2.1], $C$ must be $K$-equivalent to the Wiman sextic curve $\mathcal{W}_6$, given by:

$$\mathcal{W}_6 : 27X^6 + 9XZ^5 + 9XY^5 - 135X^4YZ - 45X^2Y^2Z^2 + 10Y^3Z^3 = 0.$$

All instances of $\mathrm{D}_{10}$ within $\mathrm{A}_6$ are conjugate. Therefore, without loss of generality, we can assume that ${}^{\phi}C\gamma$ is the Wiman sextic curve, where $\phi \in \mathrm{PGL}_3(K)$ and $\phi^{-1}\langle\sigma,\tau\rangle\phi = \langle\sigma,\tau\rangle$ because $\langle\sigma,\tau\rangle \leq \mathrm{Aut}(\mathcal{W}_6)$. This implies that $\phi$ can be of the form $\mathrm{diag}(\lambda, \zeta_5^j, 1)$ or $[\lambda X : \zeta_5^j Z : Y]$ for some $j = 0, 1, 2, 3, 4$. In either case, the transformed equation of $C_\gamma$ becomes

$$\lambda^6X^6 + \lambda XZ^5 + \lambda XY^5 + \frac{12-\gamma^5}{\gamma^2}\lambda^4\zeta_5^jX^4YZ + \frac{48+\gamma^5}{\gamma^4}\lambda^2\zeta_5^{2j}X^2Y^2Z^2 + \frac{64-2\gamma^5}{\gamma^6}\zeta_5^{3j}Y^3Z^3 = 0.$$

In particular, we obtain:

$$\lambda^6 = 27\nu,\ \lambda = 9\nu,\ \frac{12-\gamma^5}{\gamma^2}\lambda^4\zeta_5^j = -135\nu,\ \frac{64-2\gamma^5}{\gamma^6}\zeta_5^{3j} = 10\nu,\ \frac{48+\gamma^5}{\gamma^4}\lambda^2\zeta_5^{2j} = -45\nu,$$

for some $\nu \in K^*$. Equivalently,

$$\frac{12-\gamma^5}{\gamma^2} = \frac{-15}{(\sqrt[5]{3})^3}\zeta_5^{-(3i+j)},\ \frac{48+\gamma^5}{\gamma^4} = \frac{-5}{\sqrt[5]{3}}\zeta_5^{-(i+2j)},\ \frac{64-2\gamma^5}{\gamma^6} = \frac{10\sqrt[5]{3}}{9}\zeta_5^{i-3j}$$

for $i, j = 0, 1, 2, 3, 4$. So the automorphism group is given by $R^{-1}\langle T_1, T_2, T_3, T_4\rangle R$, as detailed in Appendix 11.1.2. Lastly, we remark that

$$\tau' := \phi^{-1}\,\tau'_\gamma\,\phi = \begin{pmatrix} 1 & 1 & 1 \\ 2 & (-1+\sqrt{5})/2 & (-1-\sqrt{5})/2 \\ 2 & (-1-\sqrt{5})/2 & (-1+\sqrt{5})/2 \end{pmatrix},$$

with $\phi = \mathrm{diag}(2/\gamma, 1, 1)$. Moreover, that $\phi$ is in the normalizer of $\langle\sigma,\tau\rangle$ in $\mathrm{PGL}_3(K)$. In this case, $C$ is $K$-isomorphic to:

$$C_\gamma : 32X^6 + \gamma^5X(Y^5+Z^5) + 8(12-\gamma^5)X^4YZ + 2(48+\gamma^5)X^2Y^2Z^2 + (32-\gamma^5)Y^3Z^3 = 0,$$

where $\langle\sigma,\tau,\tau'\rangle$ is a subgroup of $\mathrm{Aut}(C_\gamma)$. Since $\mathrm{A}_5$ is the only proper subgroup of $\mathrm{A}_6$ that contains $\mathrm{D}_{10}$, it follows that $\langle\sigma,\tau,\tau'\rangle \cong \mathrm{A}_5$.

From this, we conclude:

**Proposition 10.6** *Let $C$ be a smooth plane sextic curves $C$ of Type* 5, (1, 4)*, as defined by Eq.* (10.2)*. Then,*

1. *If* $\beta_{4,1} = \frac{-15}{(\sqrt[5]{3})^3}\zeta_5^{-(3i+j)}$, $\beta_{2,2} = \frac{-5}{\sqrt[5]{3}}\zeta_5^{-(i+2j)}$, $\beta_{0,3} = \frac{10\sqrt[5]{3}}{9}\zeta_5^{i-3j}$ *for some* $i, j \in \{0, 1, 2, 3, 4\}$*, then $C$ is $K$-isomorphic to the Wiman sextic curve given by*

$$27X^6 + 9XZ^5 + 9XY^5 - 135X^4YZ - 45X^2Y^2Z^2 + 10Y^3Z^3 = 0.$$

   *In this case, the automorphism group* $\mathrm{Aut}(C)$ *is* $\mathrm{PGL}_3(K)$*-conjugate to* $\mathrm{A}_6$ *and is represented as* $R^{-1}\langle T_1, T_2, T_3, T_4\rangle R$.
2. *If* $\beta_{4,1} = \frac{12-\gamma^5}{\gamma^2}$, $\beta_{2,2} = \frac{48+\gamma^5}{\gamma^4}$, $\beta_{0,3} = \frac{64-2\gamma^5}{\gamma^6}$ *for some* $\gamma \in K^*$ *such that* $\beta_{2,1} \neq \frac{-15}{(\sqrt[5]{3})^3}\zeta_5^{-(3i+j)}$, $\beta_{4,2} \neq \frac{-5}{\sqrt[5]{3}}\zeta_5^{-(i+2j)}$ *or* $\beta_{0,3} \neq \frac{10\sqrt[5]{3}}{9}\zeta_5^{i-3j}$ *for some* $i, j \in \{0, 1, 2, 3, 4\}$*, then $C$ is $K$-isomorphic to*

$$32X^6 + \gamma^5X(Y^5 + Z^5) + 8(12 - \gamma^5)X^4YZ + 2(48 + \gamma^5)X^2Y^2Z^2 + (32 - \gamma^5)Y^3Z^3 = 0,$$

   *where the automorphism group* $\mathrm{Aut}(C)$ *is isomorphic to* $\mathrm{A}_5$ *and is represented as* $\langle\sigma, \tau, \tau'\rangle$.
3. *Otherwise,* $\mathrm{Aut}(C)$ *is precisely* $\mathrm{D}_{10}$ *and is represented as* $\langle\sigma, \tau\rangle$.

### 10.3 Type 5, (1, 2)

In this situation, $C$ is defined by an equation of the form:

$$C : X^6 + XZ^5 + XY^5 + \beta_{3,1}X^3YZ^2 + \beta_{2,3}X^2Y^3Z + \beta_{0,2}Y^2Z^4 = 0, \tag{10.3}$$

where $\sigma = \mathrm{diag}(1, \zeta_5, \zeta_5^2)$ is an automorphism of maximal order 5.

First, we observe that 5 does not divide 216 and 63, the curve $C$ cannot be a descendant of either the Fermat curve $\mathcal{F}_6$ or the Klein curve $\mathcal{K}_6$. Second, if $\mathrm{Aut}(C)$ is conjugate to one of the finite primitive subgroups of $\mathrm{PGL}_3(K)$, then it must be $\mathrm{A}_5$ or $\mathrm{A}_6$, as these are the only subgroups containing elements of order 5. Thus, the normalizer of $\langle\sigma\rangle$ within $\mathrm{Aut}(C)$ would be $\mathrm{D}_5$, implying the existence of an automorphism $\tau$ for $C$ of order 2 such that $\tau\sigma\tau$ lies in $\langle\sigma\rangle$. However, it is straightforward to verify that such an involution does not exist. Therefore, $\mathrm{Aut}(C)$ cannot be one of the primitive subgroups of $\mathrm{PGL}_3(K)$.

Now, suppose that $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ and a point $P$ not lying on $\mathcal{L}$. Without loss of generality, we can take $\mathcal{L} : X = 0$ and $P = (1 : 0 : 0)$, given that $\sigma$ is a non-homology in its canonical form inside $\mathrm{Aut}(C)$. Consequently, all automorphisms of $C$ are intransitive and have the forms:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Following the notations of Theorem 5.1, we obtain that $N = 1$ and the image $\Lambda(\mathrm{Aut}(C))$ of $\mathrm{Aut}(C)$ in $\mathrm{PGL}_2(K)$ includes a cyclic group $\mathbb{Z}/5\mathbb{Z}$ generated by $\Lambda(\sigma) = [\zeta_5Y : \zeta_5^2Z]$. This implies that $\Lambda(\mathrm{Aut}(C))$ cannot be $\mathrm{A}_4$ or $\mathrm{S}_4$. Therefore, $\Lambda(\mathrm{Aut}(C))$ must be either $\mathbb{Z}/5\mathbb{Z}$, $\mathrm{D}_{10}$, or $\mathrm{A}_5$ (noting that if $\Lambda(\mathrm{Aut})(C) = \mathrm{D}_{2m}$, then $m = 5$ due to the maximal order of $\sigma$). If $\Lambda(\mathrm{Aut}(C)) \neq \mathbb{Z}/5\mathbb{Z}$, then there must be an element $\Lambda(\tau)$ of order two in the normalizer of $\langle\Lambda(\sigma)\rangle$, as any copy of $\mathbb{Z}/5\mathbb{Z}$ inside $\mathrm{D}_{10}$ or $\mathrm{A}_5$ has $\mathrm{D}_{10}$ as its normalizer. According to [35, Lemma 2.2.3-(a)], $\Lambda(\tau) = [\lambda Z : Y]$ for some $\lambda \in K^*$, which translates to $\tau = [X : \lambda\mu Z :$

$\mu Y] \in \mathrm{Aut}(C)$ for some $\lambda, \mu \in K^*$. This holds only when $\beta_{3,1} = \beta_{4,3} = \beta_{6,2} = 0$, which contradicts the smoothness of $C$. Therefore, $\mathrm{Aut}(C)$ must be $\mathbb{Z}/5\mathbb{Z}$ in this scenario.

As a result of the above discussion, we obtain:

**Proposition 10.7** *Let $C$ be a smooth plane sextic curves $C$ of Type $5, (1,2)$, as defined by Eq. (10.3). Then, $\mathrm{Aut}(C)$ is always cyclic of order 5, generated by $\sigma = \mathrm{diag}(1, \zeta_5, \zeta_5^2)$.*

### 10.4 Type 4, (1, 3)

In this situation, $C$ is defined by an equation of the form:

$$C : X^6 + Y^5Z + YZ^5 + \beta_{0,3}Y^3Z^3 + \beta_{4,1}X^4YZ + X^2\left(\beta_{2,0}Z^4 + \beta_{2,2}Y^2Z^2 + \beta_{2,4}Y^4\right) = 0, \tag{10.4}$$

where $\sigma = \mathrm{diag}(1, \zeta_4, \zeta_4^{-1})$ is an automorphism of maximal order 4.

We notice the following properties of the curve $C$:

(1) $C$ cannot be a descendant of the Klein curve $\mathcal{K}_6$, as $4 \nmid |\mathrm{Aut}(\mathcal{K}_6)| = 63$.

(2) Assuming that $C$ is a descendant of the Fermat curve $\mathcal{F}_6$ with an automorphism group larger than $\langle\sigma\rangle$, it follows that $\mathrm{Aut}(C)$ must be either $\mathrm{D}_8$ or $\mathrm{S}_4$. This is because these are the only subgroups of $\mathrm{Aut}(\mathcal{F}_6)$ that contain $\mathbb{Z}/4\mathbb{Z}$ as a subgroup and have elements are of orders at most 4. In either case, $C$ has an involution $\tau$ such that $\tau\sigma\tau = \sigma^{-1}$. A straightforward calculation shows that $\tau$ can be represented as $\tau = [X : \zeta_4^i Z : \zeta_4^{-i} Y]$ for some $i \in \{0, 1, 2, 3\}$, requiring that $\beta_{2,0} = \beta_{2,4}$. Thus, if $\beta_{2,0} = \beta_{2,4}$, then $\mathrm{D}_8$ is a subgroup of automorphisms for $C$, generated by $\sigma$ and $\tau = [X : Z : Y]$. Furthermore, if if $\mathrm{Aut}(C)$ is exactly $\mathrm{S}_4$, the Subgroups Lattice of $\mathrm{S}_4$ ensures that any $\mathbb{Z}/4\mathbb{Z}$ inside $\mathrm{Aut}(\mathcal{F}_6)$ is $\mathrm{Aut}(\mathcal{F}_6)$-conjugate to $\langle[X : Z : -Y]\rangle$. Therefore, there exists a change of variables $\phi \in \mathrm{PGL}_3(K)$ that transforms $C$ (with $\beta_{2,0} = \beta_{2,4}$) to the form: $X^6 + Y^6 + Z^6 + \ldots$, such that $\phi^{-1}\sigma\phi = [X : Z : -Y]$. This results in two possible forms for $\phi$:

$$\phi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 s & s \\ 0 & \zeta_4^{-1}t & t \end{pmatrix} \text{ or } \phi_2 = \begin{pmatrix} s' & 1 & -1 \\ 0 & \zeta_4 s & s \\ 0 & \zeta_4^{-1}t & t \end{pmatrix}.$$

**Case** $\phi = \phi_1$. It can be shown that $\phi_1^{-1}\tau\phi_1$ is an order 2 automorphism for $\mathcal{F}_6$ only if $s = ct$ for some $c$ such that $c^4 = 1$. Additionally, to obtain the core equation $X^6 + Y^6 + Z^6$, we need to set $\beta_{0,3} = \dfrac{1 - 2ct^6}{\pm ct^6}$. Under these conditions, the transformed equation for $C$ becomes

$$\begin{aligned} X^6 + Y^6 + Z^6 &+ t^4\left(2\beta_{2,0} \pm \beta_{2,2}\right)X^2\left(Y^4 + Z^4\right) + \left(3 - 16ct^6\right)\left(Y^4Z^2 + Y^2Z^4\right) \\ &+ ct^2\beta_{4,1}X^4\left(Y^2 + Z^2\right) + 2t^4\left(\pm\beta_{2,2} - 6\beta_{2,0}\right)X^2Y^2Z^2 = 0 \end{aligned}$$

where $\langle[X : Z : -Y], \mathrm{diag}(1, 1, -1)\rangle = \mathrm{D}_8$ is a subgroup of automorphisms inside $\mathrm{Aut}(\mathcal{F}_6)$. To achieve an $\mathrm{S}_4$, it is sufficient to have an additional automorphism of order 3 from $\mathrm{Aut}(\mathcal{F}_6)$, since $\mathrm{D}_8$ is maximal in $\mathrm{S}_4$. Without loss of generality, we can assume such an element to be $[Y : Z : X]$, as any $\mathrm{S}_4$ subgroup inside $\mathrm{Aut}(\mathcal{F}_6)$ is conjugated to the one generated by $\mathrm{diag}(1, -1, 1)$, $\mathrm{diag}(1, 1, -1)$, $[X : Z : Y]$, and $[Y : Z : X]$. This condition holds only if $\beta_{4,1} = \dfrac{3 - 16ct^6}{ct^2}$ and $\beta_{2,0} = (\mp/2)\beta_{2,2} + \dfrac{3 - 16ct^6}{2t^4}$, which means that $C$ must be $K$-isomorphic to

$$X^6 + Y^6 + Z^6 + \beta_1(X^2Y^4 + Y^2Z^4 + X^4Z^2 + X^2Z^4 + X^4Y^2 + Y^4Z^2) + \beta_2X^2Y^2Z^2 = 0.$$

for some $\beta_1, \beta_2 \in K$.

**Case** $\phi = \phi_2$. It can be observed that

$$\phi_2^{-1}\tau\phi_2 = \begin{pmatrix} 1 & \frac{(1+\zeta_4)(t+s)(t-\zeta_4 s)}{2sts'} & -\frac{(1+\zeta_4)(s-t)(s+\zeta_4 t)}{2sts'} \\ 0 & -\frac{s^2+t^2}{2st} & \zeta_4\frac{s^2-t^2}{2st} \\ 0 & \zeta_4\frac{s^2-t^2}{2st} & \frac{s^2+t^2}{2st} \end{pmatrix},$$

and to ensure that $\phi_2^{-1}\tau\phi_2 \in \mathrm{Aut}(\mathcal{F}_6)$, we need to set $s = -\zeta_4 t$. Under this condition, $\phi_2^{-1}\tau\phi_2$ simplifies to $[X : Z : Y]$. To achieve the core form $X^6 + Y^6 + Z^6$, we need to set $\beta_{0,3} = (s'/t)^6\zeta_4 + 2$. Under these restrictions, the transformed equation of $C$ would be

$$X^6 + Y^6 + Z^6 + (t/s')^4\left(2\beta_{2,0} - \beta_{2,2}\right)X^2\left(Y^4 + Z^4\right) + \left(3 + 16(t/s')^6\zeta_4\right)\left(Y^4Z^2 + Y^2Z^4\right)$$
$$- \zeta_4(t/s')^2\beta_{4,1}X^4\left(Y^2 + Z^2\right) - 2(t/s')^4\left(\beta_{2,2} + 6\beta_{2,0}\right)X^2Y^2Z^2 = 0$$

where $\langle[X : Z : -Y], \mathrm{diag}(1, 1, -1)\rangle = \mathrm{D}_8$ is a subgroup of automorphisms inside $\mathrm{Aut}(\mathcal{F}_6)$. A similar argument can be used to reach an $\mathrm{S}_4$ by forcing $[Y : Z : X]$ to be an additional automorphism. This condition holds only if $\beta_{4,1} = -\frac{16t^4}{s_1^4} + \frac{3is_1^2}{t^2}$ and $\beta_{2,0} = \frac{1}{2}\left(\beta_{2,2} + \frac{3s_1^4}{t^4} + \frac{16it^2}{s_1^2}\right)$, which equivalently means that $C$ is $K$-isomorphic to

$$X^6 + Y^6 + Z^6 + \beta_1(X^2Y^4 + Y^2Z^4 + X^4Z^2 + X^2Z^4 + X^4Y^2 + Y^4Z^2) + \beta_2X^2Y^2Z^2 = 0.$$

for some $\beta_1, \beta_2 \in K$ as before.

(3) Now, suppose that $\mathrm{Aut}(C)$ fixes a line $\mathcal{L}$ and a point $P \notin C$ that is off this line $\mathcal{L}$, as in Theorem 5.1-(2). In this scenario, let $\mathcal{L}$ be $X = 0$ and $P$ be the point $(1 : 0 : 0)$, given that $\sigma \in \mathrm{Aut}(C)$ is a non-homology in its canonical form. Consequently, all automorphisms of $C$ are intransitive of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

This allows us to represent $\mathrm{Aut}(C)$ in a short exact sequence:

$$1 \to N = \langle\sigma^2\rangle = \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(C) \to \Lambda(\mathrm{Aut}(C)) \to 1,$$

where $\Lambda(\mathrm{Aut}(C))$ includes a $\mathbb{Z}/2\mathbb{Z}$ subgroup generated by $\Lambda(\sigma) = [\zeta_4 Y : \zeta_4^{-1}Z]$. Thus, according to Theorem 5.1, $\Lambda(\mathrm{Aut}(C))$ could be $\mathbb{Z}/m\mathbb{Z}$, $\mathrm{D}_{2m}$, with $m = 2$ or 4, or it could be $\mathrm{A}_4$, $\mathrm{A}_5$, or $\mathrm{S}_4$.

(i) First, we exclude the possibilities where $\mathrm{Aut}(C)$ is $\mathrm{A}_4$, $\mathrm{A}_5$, or $\mathrm{S}_4$. This is because such groups would have an order divisible by 3 and would include a normal subgroup $\mathbb{Z}/2\mathbb{Z} = N$. Consequently, they would contain elements of order 6, which exceeds 4, leading to a contradiction.

(ii) Next, if $\Lambda(\mathrm{Aut}(C)) = \mathrm{D}_{2m}$ with $m = 2$ or 4, then there exists an element $\Lambda(\tau) \notin \langle\Lambda(\sigma)\rangle$ such that $\Lambda(\tau)$ is an order 2 automorphism that commutes with $\Lambda(\sigma)$. According to [35, Lemma 2.2.3-(a)], $\Lambda(\tau) = [\lambda Z : Y]$ for some $\lambda \in K^*$. In this case, $\tau$ can be expressed as $[X : \lambda\mu Z : \mu Y]$ for some $\lambda, \mu \in K^*$. This is only valid if $\beta_{2,0} = \beta_{2,4}$, and $\lambda\mu^2$ must be $\pm 1$ to ensure that $\tau$ has an order of at most 4. Therefore, $\tau$ simplifies to $[X : \pm 1/\mu Z : \mu Y]$ where $\mu^4 = \pm 1$. In this scenario, $\mathrm{Aut}(C)$ contains a $\mathrm{D}_8$ subgroup generated by $\sigma$ and $[X : Z : Y]$ if and only if $\beta_{2,0} = \beta_{2,4}$. Moreover, if $\Lambda(\mathrm{Aut}(C)) = \mathrm{D}_8$, then $\mathrm{Aut}(C)$ must be either SmallGroup(16, 11) $= \mathbb{Z}/2\mathbb{Z} \times \mathrm{D}_8$ or SmallGroup(16, 13) $= \mathbb{Z}/4\mathbb{Z} \circ \mathrm{D}_8$. These are the only subgroups of order 16 containing a normal $\mathbb{Z}/2\mathbb{Z}$, a $\mathrm{D}_8$, and elements of order at most 4. For further details, see Groups of order 16. In these cases, every copy of $\mathbb{Z}/4\mathbb{Z}$ in $\mathrm{Aut}(C)$ (such as $\langle\sigma\rangle$) is contained within a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Therefore, there exists an involution $\tau' \notin \langle\sigma, \tau\rangle$ that commutes with $\sigma$. Simple calculations show that $\tau' = \mathrm{diag}(1, -1, 1)$ or $\mathrm{diag}(1, 1, -1)$, but neither of these can define an automorphism for $C$ due to the presence of the monomial $X^6$ in the defining equation of $C$.

(iii) Finally, if $\Lambda(\mathrm{Aut}(C)) = \mathbb{Z}/4\mathbb{Z}$, then $\mathrm{Aut}(C)$ must be one of the following groups: $\mathrm{Q}_8$, $\mathrm{D}_8$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. These are the only groups of order 8 with a normal $\mathbb{Z}/2\mathbb{Z}$ and elements of order at most 4. As previously discussed, $\mathrm{Aut}(C) = \mathrm{D}_8$ if and only if $\beta_{2,0} = \beta_{2,4}$. On the other hand, $\mathrm{Aut}(C) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ only if $\mathrm{Aut}(C)$ includes $\mathrm{diag}(1, -1, 1)$ or $\mathrm{diag}(1, 1, -1)$, which is not possible for any specialization of the parameters. If $\mathrm{Aut}(C)$ is the quaternion group $\mathrm{Q}_8$, it must contain an element $\sigma'$ of order 4 such that $\sigma'^2 = \sigma^2$ and $\sigma'\sigma\sigma'^{-1} = \sigma^{-1}$. This implies $\sigma' = [X : \lambda Z : -1/\lambda Y]$ for some $\lambda \in K^*$. Furthermore, such an element $\sigma'$ lies in $\mathrm{Aut}(C)$ only if $\lambda^4 = -1$ (contributing to the monomials $Y^5Z$ and $YZ^5$), $\beta_{4,1} = \beta_{0,3} = 0$ (related to the monomials $Y^3Z^3$ and $YZ$), and $\beta_{2,4} = -\beta_{2,0}$ (involving the monomials $Z^4$ and $Y^4$).

(4) If $\mathrm{Aut}(C)$ is conjugate to one of the finite primitive subgroups of $\mathrm{PGL}_3(K)$, then it must be either $\mathrm{Hess}_{36}$ or $\mathrm{Hess}_{72}$. These are the only subgroups that include elements of order 4 and do not have elements of order greater than 4.

Using the notation from Sect. 9.3, we can assume that $\phi^{-1}\,\mathrm{Aut}(C)\phi$ contains $\langle S, T, V\rangle \cong \mathrm{Hess}_{36}$ for some $\phi \in \mathrm{PGL}_3(K)$. Since all elements of order 2 within $\mathrm{Hess}_{36}$ are conjugate, we can force $\phi^{-1}\sigma^2\phi = V^2 = [X : Z : Y]$. Thus, we can choose $\phi$ to be of the form:

$$\phi = \begin{pmatrix} 0 & 1 & -1 \\ \lambda & \gamma & \gamma \\ \mu & \nu & \nu \end{pmatrix}$$

We may further assume that $\phi^{-1}\sigma\phi = V$ or $V^{-1}$, as all groups of order 4 within $\mathrm{Hess}_{36}$ are conjugate to $\langle V\rangle$. Hence, $\phi$ can be reduced to one of the following forms:

$$\phi_{\pm} = \begin{pmatrix} 0 & 1 & -1 \\ a & -a(1 \pm \sqrt{3})/2 & -a(1 \pm \sqrt{3})/2 \\ b & -b(1 \mp \sqrt{3})/2 & -b(1 \mp \sqrt{3})/2 \end{pmatrix}.$$

In this context, the monomials $X^5Y$, $X^5Z$, $X^4Y^2$, $X^4Z^2$, $X^3Y^2Z$, $X^3YZ^2$, $X^2Y^4$, $X^2Z^4$, $X^2YZ^3$, $YZ^5$, and $Z^5Y$ should not appear in the defining equation for ${}^{\phi_{\pm}}C$ since $S \in$

$\mathrm{Aut}({}^{\phi_\pm}C)$. Therefore, with respect to $\phi_\pm$, we must have

$$\beta_{0,3}=\pm\frac{(2\sqrt{3}\mp 3)b^4-(2\sqrt{3}\pm 3)a^4}{3a^2b^2},\ \beta_{2,0}=\frac{-a\left(9(12\pm 7\sqrt{3})a^4\pm\sqrt{3}b^4\right)}{8b^3},$$
$$\beta_{2,4}=\frac{-b\left(9(12\mp 7\sqrt{3})b^4\mp\sqrt{3}a^4\right)}{8a^3},\ \beta_{2,2}=\frac{\mp 15\left((\sqrt{3}\mp 2)b^4-(\sqrt{3}\pm 2)a^4\right)}{4ab},$$
$$\beta_{4,1}=\pm\frac{3}{2}((2\sqrt{3}\pm 3)a^4-(2\sqrt{3}\mp 3)b^4),\ 3ab\left((\sqrt{3}\pm 2)a^4-(\sqrt{3}\mp 2)b^4\right)=\pm 8,$$

implying that ${}^{\phi_\pm}C$ is defined by

$$-2(f_4(a,b)+5)(X^3Y^3+X^3Z^3+Y^3Z^3)=0,$$

with $f_4(a,b)=\pm\dfrac{12\left((\sqrt{3}\pm 2)a^4+(\sqrt{3}\mp 2)b^4\right)}{b^4-a^4}$.

Finally, $UVU^{-1}\in\mathrm{Aut}({}^{\phi_\pm}C)$ if and only if $f_4(a,b)=0$ or $-6$. However, Proposition 9.10-(2) tells us that $\mathrm{Aut}({}^{\phi_\pm}C)=\langle S,T,U,V\rangle\cong\mathrm{Hess}_{216}$ when $f_4(a,b)=0$, so this case is not possible. On the other hand, when $f_4(a,b)=-6$, the curve ${}^{\phi_\pm}C$ has a singularity at the point $(1:-1:0)$. Therefore, we conclude that $\mathrm{Aut}(C)$ cannot be conjugate to $\mathrm{Hess}_{72}$.

As a consequence of the previous discussion:

**Proposition 10.8** *Let $C$ be a smooth plane sextic curves $C$ of type $4,(1,3)$, as defined by Eq. 10.4. Then, $\mathrm{Aut}(C)$ is classified as follows.*

1. *If $\beta_{2,0}=\beta_{2,4}$, then $\mathrm{Aut}(C)=\langle\sigma,\tau\rangle\cong\mathrm{D}_8$, where $\sigma=\mathrm{diag}(1,\zeta_4,\zeta_4^{-1})$ and $\tau=[X:Z:Y]$, except in the following cases:*
   - *(i)* $\beta_{0,3}=\dfrac{1-2ct^6}{\pm ct^6}$, $\beta_{4,1}=\dfrac{3-16ct^6}{ct^2}$, $\beta_{2,0}=(\mp/2)\beta_{2,2}+\dfrac{3-16ct^6}{2t^4}$ *for some $t\in K^*$ such that $c^4=1$.*
   - *(ii)* $\beta_{0,3}=(s'/t)^6\zeta_4+2$, $\beta_{4,1}=3\zeta_4(s'/t)^2-(2t/s')^4$, $\beta_{2,0}=\frac{\beta_{2,2}+3(s'/t)^4+\zeta_4(4t/s')^2}{2}$ *for some $s',t\in K^*$*

   *In these cases, $C$ is $K$-isomorphic to*

   $$X^6+Y^6+Z^6+\alpha_{2,4}(X^2Y^4+Y^2Z^4+X^4Z^2+X^2Z^4+X^4Y^2+Y^4Z^2)+\alpha_{2,2}X^2Y^2Z^2=0.$$

   *where $\alpha_{2,4},\alpha_{2,2}\in K^*$. In particular, $\mathrm{Aut}(C)$ is conjugate to $\mathrm{S}_4$, generated by $\mathrm{diag}(1,-1,1)$, $\mathrm{diag}(1,1,-1)$, $[X:Z:Y]$ and $[Y:Z:X]$.*
2. *If $\beta_{2,4}=-\beta_{2,0}$ and $\beta_{4,1}=\beta_{0,3}=0$, then $\mathrm{Aut}(C)=\langle\sigma,\sigma'\rangle\cong\mathrm{Q}_8$, where $\sigma'=[X:\zeta_8Z:-\zeta_8^{-1}Y]$.*
3. *If there exist $a$ and $b$ satisfying $3ab\left((\sqrt{3}\pm 2)a^4-(\sqrt{3}\mp 2)b^4\right)=\pm 8$, such that*

   $$\beta_{0,3}=\pm\frac{(2\sqrt{3}\mp 3)b^4-(2\sqrt{3}\pm 3)a^4}{3a^2b^2},\ \beta_{2,0}=\frac{-a\left(9(12\pm 7\sqrt{3})a^4\pm\sqrt{3}b^4\right)}{8b^3},$$
   $$\beta_{2,4}=\frac{-b\left(9(12\mp 7\sqrt{3})b^4\mp\sqrt{3}a^4\right)}{8a^3},\ \beta_{2,2}=\frac{\mp 15\left((\sqrt{3}\mp 2)b^4-(\sqrt{3}\pm 2)a^4\right)}{4ab},$$
   $$\beta_{4,1}=\pm\frac{3\sqrt{3}((2\pm\sqrt{3})a^4-(2\mp\sqrt{3})b^4)}{2},$$

*then C is K-isomorphic to*

$$X^6+Y^6+Z^6+\ f_4(a,b)XYZ(X^3+Y^3+Z^3)+3f_4(a,b)X^2Y^2Z^2 \\ -\ 2(f_4(a,b)+5)(X^3Y^3+X^3Z^3+Y^3Z^3)=0,$$

*where* $f_4(a,b)=\pm\dfrac{12\left((\sqrt{3}\pm 2)a^4+(\sqrt{3}\mp 2)b^4\right)}{b^4-a^4}$. *In this case,* $\mathrm{Aut}(C)$ *would be conjugate to* $\mathrm{Hess}_{36}=\langle S,T,V\rangle$.

4. *Otherwise, the automorphism group* $\mathrm{Aut}(C)$ *is precisely* $\mathbb{Z}/4\mathbb{Z}$, *generated by* $\sigma$.

### 10.5 Type 3, (1, 2)

In this situation, $C$ belongs to one of the following families:

$$\mathcal{C}_1\ :\ X^6+Y^6+Z^6+XYZ\left(\beta_{4,1}X^3+\beta_{1,4}Y^3+\beta_{1,2}Z^3\right)+\beta_{2,2}X^2Y^2Z^2 \\ +\ \beta_{3,3}X^3Y^3+\beta_{3,0}X^3Z^3+\beta_{0,3}Y^3Z^3=0$$
$$\mathcal{C}_2\ :\ X^5Y+Y^5Z+XZ^5+XYZ\left(\beta_{3,2}X^2Y+\beta_{1,3}Y^2Z+\beta_{2,1}XZ^2\right) \\ +\ \beta_{2,4}X^2Y^4+\beta_{0,2}Y^2Z^4+\beta_{4,0}X^4Z^2=0,$$

where $\sigma=\mathrm{diag}(1,\zeta_3,\zeta_3^{-1})$ is an automorphism of maximal order 3.

The automorphism group of $\mathcal{C}_i$, for $i=1,2$, was analyzed in [7, Theorem 2.5]. For reference, we provide a summary of the classification here.

**Proposition 10.9** *The classification of the automorphism group of* $\mathrm{Aut}(\mathcal{C}_i)$ *for* $i=1,2$ *is as follows:*

*(1) The automorphism group* $\mathrm{Aut}(\mathcal{C}_1)$ *is generally cyclic and is generated by* $\sigma$, *except in the following cases:*

*(i) If* $\beta_{4,1}=\beta_{1,4}=\beta_{1,2}=\beta_{2,2}=0$, *then the curve* $\mathcal{C}_1$ *simplifies to*

$$X^6+Y^6+Z^6+X^3\left(\beta_{3,3}Y^3+\beta_{3,0}Z^3\right)+\beta_{0,3}Y^3Z^3=0,$$

*where* $\mathrm{Aut}(\mathcal{C}_1)$ *is isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^2$ *and is generated by* $\mathrm{diag}(1,\zeta_3,1)$ *and* $\mathrm{diag}(1,1,\zeta_3)$.

*(ii) If any of the following conditions holds:*

$$(a)\ \beta_{4,1}=\pm\beta_{1,4}\ \text{and}\ \beta_{3,0}=\pm\beta_{0,3},$$
$$(b)\ \beta_{1,4}=\pm\beta_{1,2}\ \text{and}\ \beta_{3,3}=\pm\beta_{3,0},$$
$$(c)\ \beta_{4,1}=\pm\beta_{1,2}\ \text{and}\ \beta_{3,3}=\pm\beta_{0,3},$$

*then the curve* $\mathcal{C}_1$ *is K-isomorphic to*

$$\mathcal{C}_1'\ :X^6+Y^6+Z^6+\beta_{4,1}'X^4YZ+\beta_{3,3}'X^3(Y^3+Z^3)+\beta_{2,2}'X^2Y^2Z^2 \\ +\ \beta_{1,2}'XYZ(Y^3+Z^3)+\beta_{0,3}'Y^3Z^3=0,$$

*where* $\mathrm{Aut}(\mathcal{C}_1')$ *is* $\mathrm{S}_3$ *generated by* $\sigma$ *and* $[X:Z:Y]$ *if* $\beta_{4,1}'\neq\beta_{1,2}'$ *or* $\beta_{3,3}'\neq\beta_{0,3}'$. *Otherwise,* $\mathrm{Aut}(\mathcal{C}_1')$ *is* $\mathbb{Z}/3\mathbb{Z}\rtimes\mathrm{S}_3$ *generated by* $\sigma$, $[X:Z:Y]$ *and* $[Y:Z:X]$.

*(iii)* *If there exists* $\ell \neq 0$ *or* 3 mod 6, *such that* $\beta_{4,1} = \zeta_6^{\ell}\beta_{1,1},\ \beta_{1,4} = \pm\zeta_6^{-\ell}\beta_{1,1},\ \beta_{3,3} = \pm(-1)^{\ell}\beta_{3,0},\ \beta_{0,3} = \pm\beta_{3,0}$, *then the curve* $\mathcal{C}_1$ *is* $K$*-isomorphic to*

$$\mathcal{C}_1'' : X^6 + \zeta_6^{2\ell}Y^6 + \zeta_6^{-2\ell}Z^6 + \beta_{1,1}'XYZ(X^3 + \zeta_6^{2\ell}Y^3 + \zeta_6^{-2\ell}Z^3) + \beta_{3,0}'(X^3Y^3 + \zeta_6^{-2\ell}X^3Z^3 + \zeta_6^{2\ell}Y^3Z^3) = 0.$$

*In this case,* $\mathrm{Aut}(\mathcal{C}_1'')$ *is isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^2$*, with the group generated by* $\sigma$ *and* $[Y : Z : X]$.

*(iv)* *If (a)* $(\beta_{4,1}, \beta_{1,2}, \beta_{1,4}),\ (\beta_{1,4}, \beta_{4,1}, \beta_{1,2})$*, or* $(\beta_{1,2}, \beta_{1,4}, \beta_{4,1})$ *equals*

$$\left(\frac{2\left(29 - 54\lambda^6 - 54\mu^6\right)}{27\lambda\mu}, \frac{2\left(27\mu^6 - 54\lambda^6 - 52\right)}{27\lambda\mu^4}, \frac{2\left(27\lambda^6 - 54\mu^6 - 52\right)}{27\lambda^4\mu}\right),$$

*(b)* $(\beta_{3,0}, \beta_{3,3}, \beta_{0,3}),\ (\beta_{3,3}, \beta_{0,3}, \beta_{3,0})$*, or* $(\beta_{0,3}, \beta_{3,0}, \beta_{3,3})$ *equals*

$$\left(\frac{2\left(81\lambda^6 - 27\mu^6 - 26\right)}{27\mu^3}, \frac{2\left(81\mu^6 - 27\lambda^6 - 26\right)}{27\lambda^3}, \frac{2\left(82 - 27\lambda^6 - 27\mu^6\right)}{27\lambda^3\mu^3}\right),$$

*and (c)* $\beta_{2,2} = \dfrac{9\lambda^6 + 9\mu^6 + 10}{3\lambda^2\mu^2}$ *for some* $\lambda, \mu \in K^*$*, then the curve* $\mathcal{C}_1$ *is* $K$*-isomorphic to* $\mathcal{C}_{1,\lambda,\mu}$*, defined by*

$$X^6 + Y^6 + Z^6 + f_{\lambda,\mu}X^2Y^2Z^2 + g_{\lambda,\mu}(X^4Y^2 + X^2Z^4 + Y^4Z^2) + g_{\mu,\lambda}(X^4Z^2 + X^2Y^4 + Y^2Z^4) = 0,$$

*where*

$$f_{\lambda,\mu} = 3(80 + 81\lambda^6 + 81\mu^6),$$
$$g_{\lambda,\mu} = 81\left(1 + \zeta_3\lambda^6 + \zeta_3^{-1}\mu^6\right).$$

*In this case,* $\mathrm{Aut}(\mathcal{C}_{1,\lambda,\mu})$ *is isomorphic to* $\mathrm{A}_4$*, with the group generated by* $\mathrm{diag}(1, -1, 1)$ *and* $\mathrm{diag}(1, 1, -1)$*, and* $[Y : Z : X]$.

*(2)* *The automorphism group* $\mathrm{Aut}(\mathcal{C}_2)$ *is generally cyclic and is generated by* $\sigma$*, except in the following cases:*

*(i)* *If* $\beta_{0,2} = \zeta_{21}^{-12r}\beta_{4,0},\ \beta_{2,4} = \zeta_{21}^{3r}\beta_{4,0},\ \beta_{1,3} = \zeta_{21}^{-6r}\beta_{3,2},\ \beta_{2,1} = \zeta_{21}^{3r}\beta_{3,2}$ *for some integer* $r$*, then* $\mathcal{C}_2$ *is* $K$*-isomorphic to*

$$\mathcal{C}_2' : X^5Y + Y^5Z + XZ^5 + \beta_{4,0}\zeta_{21}^{4r}\left(X^4Z^2 + X^2Y^4 + Y^2Z^4\right) + \beta_{3,2}\zeta_{21}^{-r}XYZ\left(X^2Y + XZ^2 + Y^2Z\right) = 0,$$

*where* $\mathrm{Aut}(\mathcal{C}_2')$ *is* $(\mathbb{Z}/3\mathbb{Z})^2$*, generated by* $\sigma$ *and* $[Y : Z : X]$.

*(ii)* *If (a)* $(\beta_{2,4}, \beta_{4,0}, \beta_{0,2}),\ (\beta_{0,2}, \beta_{2,4}, \beta_{4,0})$*, or* $(\beta_{4,0}, \beta_{0,2}, \beta_{2,4})$ *equals*

$$\left(\frac{\lambda^5\mu + 4\mu^5}{2\lambda^4}, \frac{\lambda + 4\lambda^5\mu}{2\mu^2}, \frac{4\lambda + \mu^5}{2\lambda^2\mu^4}\right),$$

*and (b)* $(\beta_{1,3}, \beta_{3,2}, \beta_{2,1}),\ (\beta_{2,1}, \beta_{1,3}, \beta_{3,2})$ *or* $(\beta_{3,2}, \beta_{2,1}, \beta_{1,3})$ *equals*

$$\left(\frac{2\left(2\lambda^5\mu + 2\lambda + \mu^5\right)}{\lambda^3\mu^2}, \frac{2\lambda^5\mu + 4\lambda + 4\mu^5}{\lambda^2\mu}, \frac{2\left(2\lambda^5\mu + \lambda + 2\mu^5\right)}{\lambda\mu^3}\right),$$

*then the curve $\mathcal{C}_2$ is $K$-isomorphic to $\mathcal{C}_{2,\lambda,\mu}$, defined by*

$$X^6 + Y^6 + Z^6 + h_{\lambda,\mu}(\zeta_3^{-1}X^4Y^2 + X^2Z^4 + Y^4Z^2) \\ + s_{\lambda,\mu}(X^4Z^2 + \zeta_3 X^2Y^4 + Y^2Z^4) = 0,$$

*where*

$$h_{\lambda,\mu} = \frac{\sqrt{3}\zeta_9\left(\zeta_4\lambda^5\mu + \zeta_{12}\lambda + \zeta_{12}^5\mu^5\right)}{\lambda^5\mu + \lambda + \mu^5},$$
$$s_{\lambda,\mu} = \frac{\sqrt{3}\zeta_{18}\left(\zeta_{12}^5\lambda^5\mu + \zeta_{12}\lambda + \zeta_4\mu^5\right)}{\lambda^5\mu + \lambda + \mu^5}.$$

*In this case,* $\mathrm{Aut}(\mathcal{C}_{2,\lambda,\mu})$ *is* $\mathrm{A}_4$*, with the group generated by* $\mathrm{diag}(1,-1,1)$, $\mathrm{diag}(1,1,-1)$*, and* $[\zeta_6^{-1}Y : Z : X]$.

***Remark 10.10*** Irreducibility of the stratum $\mathcal{M}_{10}^{\mathrm{Pl}}(\mathbb{Z}/3\mathbb{Z})$, including the existence of the two components $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_{(0,1)}(\mathbb{Z}/3\mathbb{Z}))$ and $\widetilde{\mathcal{M}_{10}^{\mathrm{Pl}}}(\varrho_{(1,2)}(\mathbb{Z}/3\mathbb{Z}))$, was initially addressed in [4].

Additionally, for Proposition 10.9 (1)-(ii), $\left(\beta'_{3,3}, \beta'_{1,2}\right)$ must not be $(0,0)$ or the curve will have automorphisms of order $> 3$ like $\mathrm{diag}(1, \zeta_6, \zeta_6^{-1})$. Similarly, in Proposition 10.9 (2)-(i), $\left(\beta_{2,4}, \beta_{1,3}\right) \neq (0,0)$ (Otherwise, $\mathrm{diag}(1, \zeta_{21}, \zeta_{21}^{-4})$ will be an automorphism of order $> 3$).

## 10.6 Type 2, (0, 1)

In this case, $C$ is defined by an equation of the form:

$$C : Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z} = 0$$

where $\sigma = \mathrm{diag}(1,1,-1)$ is an automorphism of maximal order 2.

Clearly, either $L_{2,Z} \neq 0$ or $L_{4,Z} \neq 0$, otherwise $\mathrm{diag}(1,1,\zeta_4)$ would act as an automorphism of order $4 > 2$. Also, due to smoothness, $L_{6,Z}$ must be of degree $\geq 5$ in both $X$ and $Y$.

The full automorphism group of such a curve $C$ was investigated in [7]. According to [7, Theorem 2.1], we have the following result:

**Proposition 10.11** *Let $C$ be a smooth plane sextic curves $C$ of Type* 2, (0, 1) *as above. Then,* $\mathrm{Aut}(C)$ *is always cyclic of order* 2*, unless $L_{2,Z}$, $L_{4,Z}$ and $L_{6,Z}$ are elements of the ring $K[X^2, Y^2]$. In that case,* $\mathrm{Aut}(C)$ *is* $(\mathbb{Z}/2\mathbb{Z})^2$*, generated by $\sigma$ and* $\mathrm{diag}(1,-1,1)$.

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## 11.1 Appendix

### 11.1.1 $S_3$ and $A_4$ inside $\mathrm{Aut}(\mathcal{F}_6)$

WWe have established in Proposition 6.1 that $\mathrm{Aut}(\mathcal{F}_6)$ is isomorphic to $(\mathbb{Z}/6\mathbb{Z})^2 \times S_3$.

Consider the following subgroups of $\mathrm{Aut}(\mathcal{F}_6)$:

$$S_{3,1} := \langle \mathrm{diag}(1, \zeta_3, \zeta_3^{-1}),\ [X : Z : Y]\rangle$$
$$S_{3,2} := \langle [Y : Z : X],\ [X : Z : Y]\rangle$$

Clearly each of these subgroups is isomorphic to $S_3 = \langle a, b : a^2 = b^3 = (ab)^2 = 1\rangle$ with $a = [X : Z : Y]$ and $a = \mathrm{diag}(1, \zeta_3, \zeta_3^{-1})$, $[Y : Z : X]$ respectively. Moreover, $\langle \mathrm{diag}(1, \zeta_3, \zeta_3^{-1})\rangle$ and $\langle [Y : Z : X]\rangle$ are not conjugate inside $\mathrm{Aut}(\mathcal{F}_6)$. Thus, any copy of $S_3$ inside $\mathrm{Aut}(\mathcal{F}_6)$ is $\mathrm{Aut}(\mathcal{F}_6)$-conjugate to either $S_{3,1}$ or $S_{3,2}$, since $\mathrm{Aut}(\mathcal{F}_6)$ contains exactly two conjugacy classes of $S_3$ as illustrated in the subgroups lattice of $\mathrm{Aut}(\mathcal{F}_6)$.

Similarly, any copy of $A_4$ inside $\mathrm{Aut}(\mathcal{F}_6)$ is $\mathrm{Aut}(\mathcal{F}_6)$-conjugate to $A_{4,1}$ or $A_{4,2}$, where

$$A_{4,1} := \langle [Y : Z : X],\ \mathrm{diag}(1, 1, -1),\ \mathrm{diag}(1, -1, 1)\rangle,$$
$$A_{4,2} := \langle [Y : \zeta_6 Z : \zeta_6 X],\ \mathrm{diag}(1, 1, -1),\ \mathrm{diag}(1, -1, 1)\rangle.$$

Each of these groups is isomorphic to

$$A_4 = \langle a, b, c : a^2 = b^2 = c^3 = 1,\ cac^{-1} = ab = ba, cbc^{-1} = a\rangle,$$

where $a = \mathrm{diag}(1, -1, 1)$, $b = \mathrm{diag}(1, 1, -1)$, and $c = [Y : Z : X]$ or $[\zeta_6^{-1} Y : Z : X]$ respectively. Moreover, if $\phi^{-1} A_{4,2}\, \phi = A_{4,1}$ for some $\phi \in \mathrm{Aut}(\mathcal{F}_6)$, then $\phi$ must lie in the normalizer of $\langle \mathrm{diag}(1, 1, -1), \mathrm{diag}(1, -1, 1)\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence $\phi$ can be expressed as:

$$\phi_1 = [aX : bY : Z],\ \phi_2 = [aY : bX : Z],\ \phi_3 = [aX : Z : bY],$$
$$\phi_4 = [aY : Z : bX],\ \phi_5 = [Z : aX : bY],\ \phi_6 = [aZ : Y : bX],$$

for some $a, b$ such that $a^6 = b^6 = 1$. This yields the 24 $\mathrm{Aut}(\mathcal{F}_6)$-conjugates of $[Y : Z : X]$ namely, $\{\phi_i [Y : Z : X] \phi_i^{-1} : i = 1, 2, 3, 4, 5, 6\}$ More precisely, the $\mathrm{Aut}(\mathcal{F}_6)$-conjugates of $[Y : Z : X]$ are given by the following sets:

$$\{[Y : \zeta_6^{\ell} Z : \zeta_6^{\ell'} X]\}_{(\ell,\ell')\in\{(0,0),(0,3),(3,0),(1,2),(2,1),(2,4),(4,2),(3,3),(4,5),(5,4),(5,1)\}}$$
$$\{[Z : \zeta_6^{\ell} X : \zeta_6^{\ell'} Y]\}_{(\ell,\ell')\in\{(0,0),(0,3),(3,0),(1,1),(1,4),(4,1),(2,2),(2,5),(5,2),(3,3),(4,4),(5,5)\}}$$

Because non of these conjugates lie in $A_{4,2}$, we deduce that $A_{4,1}$ and $A_{4,2}$ are non-conjugated inside $\mathrm{Aut}(\mathcal{F}_6)$. However, it is important to note that both groups are $\mathrm{PGL}_3(K)$ conjugated via a rescaling of the variables in the normalizer of $\mathrm{Aut}(\mathcal{F}_6)$, specifically through $X \to \lambda' X,\ Y \to \mu' Y,\ Z \to Z$ with $\dfrac{\mu'^2}{\lambda'} = \dfrac{\mu'}{\lambda'^2} = \zeta_6$.

### 11.1.2 The Wiman's sextic curve

The most symmetric smooth plane sextic curve is known to be the Wiman's sextic $\mathcal{W}_6$, defined by

$$\mathcal{W}_6 : 27X^6 + 9XZ^5 + 9XY^5 - 135X^4YZ - 45X^2Y^2Z^2 + 10Y^3Z^3 = 0.$$

Further details can be found in [18]. According to [59, Appendix A], $\mathrm{Aut}(\mathcal{W}_6)$ is given by $R^{-1}\langle T_1, T_2, T_3, T_4\rangle R$, where

$$\begin{aligned} R &:= \begin{pmatrix} a(1-b) & a(1-b) & \frac{1}{2}(1+2c+bc) \\ a\sqrt{b-3} & -a\sqrt{b-3} & 0 \\ a & a & \frac{-1}{2}(1+c+b)+bc \end{pmatrix} \\ T_1 &= \mathrm{diag}(1,-1,1), \\ T_2 &= [Z : X : Y] \\ T_3 &= [X : \zeta_3^{-1}Z : -\zeta_3 Y] \\ T_4 &= \begin{pmatrix} 1 & 1/b & -b \\ 1/b & b & 1 \\ b & -1 & 1/b \end{pmatrix} \end{aligned}$$

with $a = \frac{1}{6}(1+2c+2b+bc),\ b = \frac{1+\sqrt{5}}{2},\ c = \frac{-1+\sqrt{-3}}{2}$.

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