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On the critical points of planar polynomial Hamiltonian systems

Anna Cima, Armengol Gasull*, Francesc Mañosas

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Facultat de Ciències, Bellaterra, Barcelona, 08193, Spain

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It is well known that the critical points of planar polynomial Hamiltonian vector fields are either centers or points with an even number of hyperbolic sectors. We give a sharp upper bound of the number of centers that these systems can have in terms of the degrees of their components. We also prove that generically the critical points at infinity of their Poincaré compactification are either nodes or have indices $-1, 0$ or 1 and have at most two sectors: both hyperbolic, both elliptic or one of each type. These characteristics are no more true in the non generic situation. Although these results are known we revisit their proofs. The new proofs are shorter and based on a new approach.

1. Introduction and main results

Let $\mathcal{H}_{n,m}$ be the space of the polynomial Hamiltonian planar vector fields $X = (-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x})$ such that the degree of $\frac{\partial H}{\partial y}$ is n and the degree of $\frac{\partial H}{\partial x}$ is m. Our first result is the following theorem:

Theorem A. *Let $X \in \mathcal{H}_{n,m}$ and let C be the number of centers of X. Then*

$$C \leq E\left(\frac{nm+1}{2}\right),$$

where $E(z)$ denotes the integer part of z. Moreover this bound is optimal.

This result is already proved in Cima et al. [4] when the above vector fields have finitely many critical points and extended without this hypothesis in He et al. [9,10]. In fact the authors [10] communicated to us their extension and comment us about some gaps in our original proof. Motivated by their comments we decided to revisit and clarify our original proof of this theorem.

In this work we firstly give a shorter, self contained and new proof of Theorem A. It is different to the proof presented by He et al. [9,10] and although it follows the guidelines of our first work [4], it is based in a novel idea: *firstly prove the theorem in a generic subclass of $\mathcal{H}_{n,m}$ and afterwards obtain the general result by a perturbation argument.* As we will see, a simple but important property of the family of planar Hamiltonian systems is the one that allows us to use these perturbation type arguments. This property is that the fact of having a center is a robust property within this family, in contrast of what usually happens with centers in most families of systems, see Lemma 5.

A key point in our first proof in Cima et al. [4] of this theorem was to study the critical points at infinity of the Poincaré compactification of generic Hamiltonian polynomial planar vector fields, see Cima et al. [4, Thm 2.2]. Although the statement of that theorem was correct our proof had a gap, that we revisit here, see Remark 7 for more details. Our results on the critical points at infinity of planar polynomial Hamiltonian systems are given in forthcoming Theorem B. We state it after some definitions.

* Corresponding author.
E-mail addresses: anna.cima@uab.cat (A. Cima), armengol.gasull@uab.cat (A. Gasull), francesc.manosas@uab.cat (F. Mañosas).

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We want remark that the new proof of Theorem A is not based on Theorem B, see Remark 3 for more details.

Let us define which is the *generic* subclass of $\mathcal{G}_{n,m} \subset \mathcal{H}_{n,m}$ that we will deal with. Given $X = (P, Q) \in \mathcal{H}_{n,m}$ we denote by P_n and Q_m the homogeneous parts of maximum degree of P and Q respectively. We say that $X = (P, Q) \in \mathcal{H}_{n,m}$ belongs to $\mathcal{G}_{n,m}$ if P_n and Q_m do not share any factor (real or complex). It is clear that $\mathcal{G}_{n,m}$ is open in $\mathcal{H}_{n,m}$ and we will show that it is also dense in $\mathcal{H}_{n,m}$ in Lemma 4.

Notice that if $X = (P, Q) \in \mathcal{H}_{n,m}$ has infinitely many critical points, then P, Q share some common factor. This fact implies that P_n and Q_m share also some factor. Therefore if $X \in \mathcal{G}_{n,m}$ then from Bezout's Theorem it has at most nm critical points.

Recall that a characteristic orbit at a critical point p is an orbit tending to p in positive time (respectively in negative time) with a well defined slope. It is well known that an isolated critical point p of a planar analytical vector field X either does not have characteristic orbits or it is a finite union of hyperbolic, elliptic, and parabolic sectors, see Dumortier et al. [7, Sec. 1.5]. Moreover, its index is $i_X(p) = 1 + (\mathrm{e} - \mathrm{h})/2$, where e and h denote its number of elliptic and hyperbolic sectors, respectively. As we will see, along the paper we will use several properties of the index of critical points, see for instance the books [2,7] for an account of them. It is also known that isolated critical points that do not have characteristic orbits are necessarily monodromic, that is, locally admit a Poincaré return map.

It is also well known that when a system is Hamiltonian, it can not have limit cycles, and the critical points can not be focus and they have neither elliptic sectors, nor parabolic ones. In a few words, the reason is that since the flow of Hamiltonian systems preserves area, attracting or repelling behaviors are forbidden in bounded domains. Hence, an isolated critical point p of a Hamiltonian system X is either a center ($i_X(p) = 1$) or it has an even number $2k \geq 2$ of hyperbolic sectors ($i_X(p) = 1 - k$).

The first step in our proof of Theorem A will be based on controlling the total sum of the indices of the critical points of X when $X \in \mathcal{G}_{n,m}$, see Lemma 2. Afterwards, we will show that this total sum also controls the non generic situation.

Recall also that, via the stereographic projection and a rescalling of the time, a polynomial planar vector field $X = (P, Q)$ always admits an extension to the 2-dimensional sphere denoted by $p(X)$ and named the Poincaré compactification of X, see Cima and Llibre [5], Dumortier et al. [7], Sotomayor [13]. The vector field $p(X)$ allows us to study the behaviour of X in a neigbourhood of infinity, i.e., a neigbourhood of the equator of the sphere. To study the analytical expression of $p(X)$, the 2-sphere is taken as a differentiable manifold. The atlas that we consider is formed by the six coordinate neighbourhoods given by $U_i = \{x \in \mathbb{S}^2 \subset \mathbb{R}^3 \ : \ x_i > 0\}$ and $V_i = \{x \in \mathbb{S}^2 \subset \mathbb{R}^3 \ : \ x_i < 0\}$ for $i = 1, 2, 3$. Usually, in the charts U_1, U_2, U_3, the coordinates are taken as $(y, z) = \left(x_2/x_1, x_3/x_1\right)$, $(x, z) = \left(x_2/x_1, x_3/x_1\right)$, $(x, y) = \left(x_2/x_1, x_3/x_1\right)$, respectively.

From the expressions of $p(X)$ in these local charts it is easy to deduce that the infinity is invariant by the flow of $p(X)$ and that $p(X)$ has two copies of X on the northern and southern hemisphere of $\mathbb{S}^2$. Also the orbits of $p(X)$ in $\mathbb{S}^2$ are always symmetric with respect to the origin of $\mathbb{R}^3$. From this symmetry if $q \in \mathbb{S}^2$ is an infinite critical point then $-q$ is another one and the indices of $p(X)$ at both points coincide.

In the case that $p(X)$ has a finite number of critical points in $\mathbb{S}^2$, we can apply the Poincaré–Hopf's Theorem which asserts that

$$2 \sum_f i_X + \sum_\infty i_{p(X)} = 2,$$

where $\sum_f i_X$ denotes the sum of the indices of X at the critical points of X and $\sum_\infty i_{p(X)}$ denotes the sum of the indices of $p(X)$ at the critical points in the equator of the sphere.

As we have already said, our second result investigates the local structure of the infinite critical points when the vector field belongs to the generic subclass. To state it we need a further definition. Let q be an infinite critical point and let h be an hyperbolic sector associated to q. We say that h is *degenerated* if its two separatrices are contained in the equator of the Poincaré sphere.

Following Dumortier et al. [7, Sec. 1.5], we also recall some properties of the sectors associated to critical points. Given an isolated critical point with characteristic orbits, there exists a well-defined decomposition in sectors called *a minimal sectorial decomposition*. As its name indicates it is the decomposition with the minimal number of parabolic sectors (the number of elliptic and hyperbolic sectors is independent of the decomposition). In this decomposition the only parabolic sectors are either the global one (corresponding to a node), or the ones lying between two hyperbolic sectors. In fact, if you locally look at a neighborhood of a critical point with an elliptic sector, you can never know the global behavior of the trajectories near the boundaries of this sector. In the definition of minimal sectorial decomposition it is considered that they form part of the elliptic sector itself and they never give rise to parabolic sectors. Notice that the parabolic sectors that appear when people study global phase portraits, see for instance in Sun and Xiao [15], would not be counted as parabolic if the authors were used this minimal sectorial decomposition and looked only locally at the critical points.

In this paper, when we describe the sectors of a critical point, we always refer to its minimal sectorial decomposition.

From now on, we will assume without loss of generality that $n \geq m$.

Theorem B. *Let $X \in \mathcal{G}_{n,m}$. The following assertions hold*

(i) If $n = m$ then all infinite critical points of $p(X)$ are nodes.

(ii) If $n > m$ then there are only a couple of infinite critical points of $p(X)$ that correspond to the direction $y = 0$. Moreover if n is even these critical points are both nodes. If n is odd and m even they have one degenerated hyperbolic sector and one elliptic sector separated by the equator of the sphere. In the remaining case, n and m odd, there are two possibilities, either the critical points have two degenerated hyperbolic sectors or they have two elliptic sectors, and also the equator of the sphere separates these two sectors.

A similar version of the above theorem has also been proved in He et al. [9,10].

2. Preliminary results

Next lemmas will be useful to prove Theorem A.

Lemma 1. *Consider the planar differential system $\dot{x} = ay^n$, $\dot{y} = Q_m(x,y)$, where n and m are positive integers and Q_m is a homogeneous polynomial satisfying $Q_m(x,0) = bx^m$, with $ab \neq 0$. Then $(0,0)$ is an isolated critical point and its index is 0 when nm is even and it is $-\operatorname{sign}(ab)$ when nm is odd.*

Proof. Clearly $(0,0)$ is an isolated singularity. Write $X(x,y) = (ay^n, Q_m(x,y))$. Recall that its index $i = i_X(0)$ can be computed as

$$i = \sum_{p\in\mathbb{R}^2\cap X^{-1}(\varepsilon)} \operatorname{sgn}\left(\det(DX)_p\right),$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is any small enough regular value of X, that is every solution of $X(p) = \varepsilon$ is simple. By Sard's Theorem almost all values of ε are regular values.

When n is even, by taking a regular value ε, with $a\varepsilon_1 < 0$, we get that $\mathbb{R}^2 \cap X^{-1}(\varepsilon) = \emptyset$ and hence $i = 0$.

When n is odd and ε is a regular value, we have that

$$\mathbb{R}^2 \cap X^{-1}(\varepsilon) = \{(x_j^*, y^*),\ i = 1, 2, \dots, k\},$$

where $y^* = (\varepsilon_1/a)^{1/n}$, $k \le m$ is a non negative integer with the same parity that m and $x_1^* < x_2^* < \dots < x_k^*$ are the k real roots of $Q_m(x, y^*) - \varepsilon_2 = 0$, which all of them are simple. Easy computations give

$$\det(DX)_{(x_j^*,y^*)} = -an(y^*)^{n-1}\frac{\partial Q_m}{\partial x}(x_j^*, y^*).$$

Hence the values $\operatorname{sgn}\left(\det(DX)_{(x_j^*,y^*)}\right)$ are alternating their signs varying j. In particular, when m is even, then k is even, and their sum is 0, as we wanted to see. Otherwise, when m is odd, the sum of the k values coincides with the first value, that is,

$$i = \sum_{j=1}^{k} \operatorname{sgn}\left(\det(DX)_{(x_j^*,y^*)}\right) = \operatorname{sgn}\left(\det(DX)_{(x_1^*,y^*)}\right) = -\operatorname{sgn}\left(an(y^*)^{n-1}\frac{\partial Q_m}{\partial x}(x_1^*, y^*)\right) = -\operatorname{sgn}(ab).$$

Hence the lemma follows. □

Lemma 2. *Let $X = (-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}) \in \mathcal{G}_{n,m}$. Then $\sum_f i_X \le 1$. More specifically,*

$$\sum_f i_X = F(n,m) := \begin{cases} 1-k, & \text{when} \quad n = m,\\ 0, & \text{when} \quad n > m, \text{ and } nm \text{ even},\\ \pm 1, & \text{when} \quad n > m, \text{ and } nm \text{ odd}, \end{cases}$$

where k, with $0 \le k \le n+1$, is the number of real straight lines of $H_{n+1}(x,y) = 0$ and H_{n+1} is the homogeneous part of degree $n+1$ of H.

Proof. We start studying the case $n = m$. Write $P = P_0 + P_1 + \dots + P_n$, $Q = Q_0 + Q_1 + \dots + Q_n$, and $H = H_0 + H_1 + \dots + H_{n+1}$. Then, for $0 \le j \le n$,

$$P_j(x,y) = -\frac{\partial H_{j+1}(x,y)}{\partial y}, \quad Q_j(x,y) = \frac{\partial H_{j+1}(x,y)}{\partial x}.$$

Since H_{n+1} is homogeneous of degree $n+1$, by using Euler's identity,

$$-yP_n(x,y) + xQ_n(x,y) = x\frac{\partial H_{n+1}(x,y)}{\partial x} + y\frac{\partial H_{n+1}(x,y)}{\partial y} = (n+1)H_{n+1}(x,y).$$

Hence the number of critical points at infinity of $p(X)$ is the number of real straight lines of $H_{n+1}(x,y) = 0$, that is $k \le n+1$, see Dumortier et al. [7], Sotomayor [13]. Notice that in particular the genericity hypothesis implies that $H_{n+1}(x,y) \not\equiv 0$ and moreover that all its roots are simple. To see the character of these singularities at infinity, take one of them, say q. Without loss of generality we can assume that q lies in the local chart U_1 and has local coordinates $(y,z) = (0,0)$. The expression of $p(X)$ in the local chart U_1 given in the introduction, and after scaling the independent variable t, is

$$\begin{aligned} \dot{y} &= [-yP_n(1,y) + Q_n(1,y)] + z[-yP_{n-1}(1,y) + Q_{n-1}(1,y)] + \dots + z^n[-yP_0 + Q_0],\\ \dot{z} &= -zP_n(1,y) - z^2P_{n-1}(1,y) - \dots - z^{n+1}P_0, \end{aligned}$$

and the linear part of the vector field at $(0,0)$ has two eigenvalues with the same sign. So q is a node and has index $+1$. Thus $\sum_\infty i_{p(X)} = 2k \ge 0$ and from the Poincaré–Hopf's Theorem we obtain $2\sum_f i_X + 2k = 2$. Hence $\sum_f i_X = F(n,n) = 1-k$, as wanted to see.

If $n > m$ the two components of the vector field (P,Q) have the form $P(x,y) = ay^n + \alpha_{n-1}y^{n-1} + \dots + \alpha_{m+1}y^{m+1} + \bar{P}(x,y)$ with $\bar{P}$ of degree less or equal to m and $a \neq 0$ and $Q = Q_m + \bar{Q}$ with $\bar{Q}$ of degree less than m. Since P_n has only the factor y, from the generic assumption, $Q_m(x,0) = bx^m$ for some $b \neq 0$.

If we denote $\tilde{X} := (ay^n, Q_m(x,y))$, we claim that $\sum_f i_X = i_{\tilde{X}}(0,0)$. To see this we use the well-known fact that the index is invariant by a homotopy in the following sense. If Y_s, $s \in [0,1]$ is a continuous deformation between the vector fields Y_0 and Y_1 and $\mathcal{B}$ is a closed ball with the property that for any $s \in [0,1]$, Y_s never vanishes at the boundary of $\mathcal{B}$, then the sum of the indices at the critical points in $\mathcal{B}$ of Y_0 and Y_1 coincide, assuming that both vector fields have a finite number of critical points in $\mathcal{B}$. In our situation we consider

$$X_s = \big(ay^n + s(\alpha_{n-1}y^{n-1} + \dots + \alpha_{m+1}y^{m+1} + \bar{P}), Q_m + s\bar{Q}\big).$$

To prove the claim we only need to show the existence of a closed ball $\mathcal{B}$ that contains in its interior all the critical points of X_s for all $s \in [0,1]$. To see this we consider the change of variables

$$x = r^n \cos\theta, \quad y = r^m \sin\theta. \tag{1}$$

These coordinates are sometimes called *weighted polar coordinates.* Direct computations show that X_s writes as

$$X_s(x,y) = \left((a\sin^n\theta + s\,c\cos^m\theta)r^{nm} + \sum_{i=0}^{nm-1} f_i(\theta,s)r^i,\, b\cos^m\theta r^{nm} + \sum_{i=0}^{nm-1} g_i(\theta,s)r^i\right),$$

where c is the coefficient of x^m of $\bar{P}$ and f_i and g_i are continuous functions defined in $[0,2\pi]\times[0,1]$.

Thus if we denote by $\|\cdot\|$ the usual euclidean norm in $\mathbb{R}^2$ we get

$$\|X_s(x,y)\|^2 = \big((a\sin^n\theta + s\,c\cos^m\theta)^2 + b^2\cos^{2m}\theta\big)r^{2mn} + \sum_{i=0}^{2nm-1} h_i(\theta,s)r^i,$$

where h_i are continuous functions defined in $[0,2\pi]\times[0,1]$. We note that the coefficient of r^{2nm} is always strictly positive because it is a sum of squares that do not vanishes simultaneously. Since this coefficient is a continuous function on the compact $[0,2\pi]\times[0,1]$, it has a minimum value strictly positive. Then if r is large enough we get $\|X_s(x,y)\| \neq 0$ and hence we can consider a ball $\mathcal{B}$ with radius large enough in such a way all the critical points of X_s are in its interior for all $s \in [0,1]$. Then the claim is proved.

Applying Lemma 1 we get $\sum_f i_X = i_{\tilde{X}}(0,0) = F(n,m) \in \{-1,0,1\}$, as we want to prove. This ends the proof in this case. In short, for all $n \geq m$ we have proved

$$\sum_f i_X = F(n,m) \leq 1,$$

and the lemma follows. □

Remark 3. The proof of previous lemma that when $X \in \mathcal{G}_{n,m}$ then $F(n,n) = 1-k$ is based on the study of the critical points at infinity of $p(X)$, the Poincaré compactification of X. As we will see this is the only point of the proof of Theorem A where this compactification and the Poincaré–Hopf's Theorem are used. We want to remark that it is not difficult to get other proofs without using these two results. For instance, an application of the tools introduced by Argémi [3], would provide an alternative proof.

Lemma 4. *For any $X \in \mathcal{H}_{n,m}$ there exists a continuous deformation X_s, $s \in [0,\delta]$ such that $X_0 = X$ and $X_s \in \mathcal{G}_{n,m}$ for each $s \in (0,\delta]$.*

Proof. To prove the existence of a such deformation in the case $n = m$, we use the resultant of two 1-variable polynomials. This resultant is defined as the determinant of the Silvester's matrix associated to them. It provides a polynomial expression depending on the coefficients of the two polynomials and it has the property that it is zero if and only if both polynomials have a common zero (real or complex). See for instance [11,13,14].

Assume first that $n = m$. Fix $X = (P,Q) \in \mathcal{H}_{n,n}$ and consider $X_s = (P + sy^n, Q + sx^n)$. Clearly for each s, X_s belongs to $\mathcal{H}_{n,n}$. Write $P_n = a_0x^n + a_1x^{n-1}y + \dots + a_ny^n$ and $Q_n = b_0x^n + b_1x^{n-1}y + \dots + b_ny^n$. To prove that there exits $\delta > 0$ such that $X_s \in \mathcal{G}_{n,n}$ for each $s \in (0,\delta]$ it suffices to prove that the homogeneous polynomials $P_n(x,y) + sy^n$ and $Q_n(x,y) + sx^n$ do not have common factors.

For $s \neq 0$ small enough the coefficient of y^n of $P_n(x,y) + sy^n$ is different from zero and so it does not contain the factor x. Therefore its factors correspond with the roots of the one variable polynomial of degree n, $p_s(z) = a_0 + a_1z + \dots + (a_n + s)z^n$ where $z = y/x$. In an analogous way the factors of $Q_n(x,y) + sx^n$ correspond with the roots of the one variable polynomial $q_s(z) = (b_0 + s) + b_1z + \dots + b_nz^n$ taking into account that if q_s has degree $r \leq n$ then $Q_n(x,y) + sx^n$ has additionally x^{n-r} as one of its factor. In any case both polynomials share a common factor if and only if p_s and q_s have a common root. This fact does not hold if we prove that their resultant is not zero. Easy computations show that the determinant of the associated $(n+r)\times(n+r)$ Sylvester matrix is a monic polynomial in s of degree $n+r$. This shows that for s small enough it is different from zero and ends the proof of the lemma in this case.

Consider now the case $n > m$. Clearly we can assume that $X \notin \mathcal{G}_{n,m}$, because otherwise the result is trivial. Hence $Q_m(x,0) \equiv 0$ because y is a common factor of P_n and Q_m. For $X = (P,Q) \in \mathcal{H}_{n,m} \setminus \mathcal{G}_{n,m}$ we take $X_s = (P, Q + sx^m)$ which also belongs to $\mathcal{H}_{n,m}$ for each $s \in \mathbb{R}$. Moreover, for each $s \neq 0$ it also belongs to $\mathcal{G}_{n,m}$, because $P_n(x,y) = ay^n$, for some $a \neq 0$ and $Q_m(x,0) = sx^m$. □

Lemma 5. *Let $X = X_0 \in \mathcal{H}_{n,m}$ and assume that $q \in \mathbb{R}^2$ is a center of X. Let $X_s \in \mathcal{H}_{n,m}$, $s \in [0,\delta]$ a continuous perturbation of X_0 in $\mathcal{H}_{n,m}$. Let also $\mathcal{V}$ be an open ball centered at q that does not contain any other critical point of X. Then for any $\epsilon > 0$ small enough, X_ϵ has (at least) one center in $\mathcal{V}$.*

Proof. As we have already explained, the only critical points with positive index of planar Hamiltonian systems are centers and they have index $+1$. Also from the well-known stability properties of the index we already know that for ϵ small enough,

$$\sum_{\mathcal{V}} i_{X_\epsilon} = i_X(q) = 1,$$

where $\sum_{\mathcal{V}} i_{X_\epsilon}$ denotes the sum of the indices of the critical points of X_ϵ contained in $\mathcal{V}$. This implies that for ϵ small enough there is some critical point of X_ϵ with positive index contained in $\mathcal{V}$ and the result follows. □

3. Proof of Theorems A and B

Proof of Theorem A. First we prove the result for $X \in \mathcal{G}_{n,m}$. In this situation X has a finite number of critical points. Let C be the number of centers of X that coincides with the number of critical points with index $+1$. Let N be the sum of the indices of all critical points with negative index. Then $\sum_f i_X = C + N$, and from Lemma 2 we get $C + N \leq 1$.

On the other hand, it is well known that the index i of an isolated critical point of an analytic planar vector field with multiplicity μ, satisfies that $|i| \leq \mu$, see for instance [8]. Then, since the polynomial vector field X has a finite number of zeros, from Bezout's Theorem we have that $\sum_f |i_X| = C - N \leq nm$.

Adding the two inequalities obtained we get that $2C \leq nm + 1$, inequality that implies the desired result in this case.

Now consider the general case. Let $X \in \mathcal{H}_{n,m}$ and suppose to arrive a contradiction that $C > E\left(\frac{nm+1}{2}\right)$. From Lemma 4 we can consider a continuous deformation $X_s, s \in [0, \delta]$ such that $X_s \in \mathcal{G}_{n,m}$ for any $s \in (0, \delta]$. Therefore from Lemma 5 we get that the same inequality holds for the number of centers of X_s, when $s \neq 0$; a contradiction. This ends the proof of the inequality.

To show that the bound is the best possible we consider a vector field $X = (-P, Q)$ where $P = P(y), Q = Q(x)$ are monic polynomials of degrees m and n respectively and having all its roots real and simple. Clearly X is Hamiltonian, $X \in \mathcal{G}_{n,m}$ and has nm finite critical points, which are either saddles or centers, located in a $n \times m$ grid. By the same argument used in the proof of the Lemma 1 it follows that they indices must be ± 1 and they alternate in the grid like the colors in a chess board. So if nm is even we get $C = \frac{nm}{2} = E\left(\frac{nm+1}{2}\right)$. If n and m are odd and $n = m$ then, since the equation of the critical points at infinity is $-yP_n + xQ_n = y^{n+1} + x^{n+1}$, we see that there are no infinite critical points. Therefore $\sum_f i_X = 1$, and the number of centers exceeds the number of saddles by one. Hence $C = E\left(\frac{nm+1}{2}\right)$ also in this case. Lastly if n and m are odd and $n > m$, from the proof of Lemma 2 it follows that $\sum_f i_X = 1$, and we also obtain $C = E\left(\frac{nm+1}{2}\right)$. This ends the proof of the theorem. □

Remark 6. Notice that in the first part of the proof of the above theorem we simply have used that $C + N = \sum_f i_X \leq 1$. The bound obtained in Theorem A can be improved in some particular situations when $X \in \mathcal{G}_{n,m}$ if instead of this inequality we use the one given in Lemma 2, $C + N = \sum_f i_X \leq F(n, m)$. For instance, when $X \in \mathcal{G}_{n,n}$ and $H_{n+1}(x, y) = 0$ has k real invariant straight lines we obtain $C \leq E\left(\frac{n^2+1-k}{2}\right)$, result also proved in He et al. [9,10]. Similarly, if $X \in \mathcal{G}_{n,m}$, then $P_n(x, y) = ay^n$, $Q_m(x, 0) = bx^m$, with $ab \neq 0$. When $ab > 0$ we easily obtain that $C \leq E\left(\frac{nm-1}{2}\right)$.

Proof of Theorem B. As we have explained in the proof of the Lemma 2, statement (i) follows simply computing the differential of $p(X)$ at the infinite critical points.

To prove statement (ii) with $n > m$, consider $X = (-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}) \in \mathcal{G}_{n,m}$. Then

$$H(x, y) = Ay^{n+1} + \ldots + a_{m+2}\, y^{m+2} + H_{m+1}(x, y) + H_m(x, y) + \ldots + H_1(x, y) + H_0$$

where the polynomials H_i are homogeneous of degree i, $H_{m+1}(x, 0) = Bx^{m+1}$ with $B \neq 0$, and also $A \neq 0$. The system has only a pair of degenerate critical points at infinity, the ones corresponding to the direction $y = 0$. One of these points at infinity is in the local chart U_1, and it writes as $q = (y, z) = (0, 0)$. The other one is of the same type and is in the local chart V_1. Returning to the first point, the level curves $H(x, y) = h$ of the Hamiltonian in the chart U_1 are

$$Ay^{n+1} + a_n\, y^n z + \ldots + a_{m+1}\, y^{m+1}\, z^{n-m-1} + \sum_{i=0}^{m+1} H_i(1, y) z^{n+1-i} - hz^{n+1} = 0.$$

Note that this expression contains the monomial Bz^{n-m} which is the homogeneous part of minimum degree of the curve. From this observation, if a level curve arrives to q it must arrive tangent to $z = 0$. Now we investigate which level curves arrive to the critical point q. To do this we blow up this direction considering the following change of variables given by weighted polar coordinates:

$$y = r^{n-m} \cos\theta, \quad z = r^{n+1} \sin\theta. \tag{2}$$

Easy computations show that our level curve writes as

$$\left(A \cos^{n+1}\theta + B \sin^{n-m}\theta\right) r^{(n+1)(n-m)} + \sum_{i=(n+1)(n-m)+1}^{i=(n+1)^2} f_i(\theta) r^i = 0,$$

where f_i are trigonometric polynomials. Then our curve factorizes as

$$r = 0 \text{ and } \left(A \cos^{n+1}\theta + B \sin^{n-m}\theta\right) + \sum_{i=1}^{(n+1)(m+1)} h_i(\theta) r^i = 0$$

where $h_i(\theta) = f_{(n+1)(n-m)+i}(\theta)$. Therefore our level curve arrives to q if and only if

$$\frac{\sin^{n-m}\theta}{\cos^{n+1}\theta} = -\frac{A}{B}, \tag{3}$$

and since all its solutions $\theta = \theta^*$ are simple, by the implicit function theorem, for each of these solutions, there is a function $r = r(\theta)$ such that the corresponding points are on this level set and $r(\theta)$ tends to 0 when θ tends to θ^*. We stress the fact that this equation does not depend on h. Therefore, either all level curves arrive to q, or there are no levels that arrive to q. The solution of this equation depends on the parities of n and m. For instance, if n is even then, by studying the graphs of the functions $\frac{\sin^{n-m}\theta}{\cos^{n+1}\theta}$ for m even or odd, we see that there are exactly two values, θ_1, θ_2 with $\theta_1 \in (0, \pi)$ and $\theta_2 \in (\pi, 2\pi)$ satisfying Eq. (3). Then each level curve has two branches arriving to q, one through $z > 0$ and another one through $z < 0$. Then q is a node.

If n is odd and m is even then Eq. (3) has either, two solutions in $(0, \pi)$ and none in $(\pi, 2\pi)$, or vice-versa, depending on the sign of $-A/B$. Assume the first possibility and let θ_1, θ_2 be these two solutions. Then we have that $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (\pi/2, \pi)$, hence $\cos(\theta_1) > 0$ and $\cos(\theta_2) < 0$. Since $y = r^{n-m}\cos(\theta)$, the first orbit reaches q through $y > 0$ and the second through $y < 0$, both lying in $z > 0$. And since there are not level curves that arrive to q in $z < 0$ it follows that in $z < 0$ we have a degenerated hyperbolic sector. Moreover from Lemma 2 and Poincaré–Hopf's Theorem we know that $i_{p(X)}(q) = 0$. Since it has one hyperbolic sector it must have at least one elliptic sector. However by the previous observation any elliptic sector must intersect the line $y = 0$ and must arrive to $(0, 0)$ in the direction $z = 0$. Clearly this implies the uniqueness of this elliptic sector.

If n and m are odd the Eq. (3) does not have any solution if $A/B > 0$. So in this case, no level curves arrives to q and hence it has two degenerated hyperbolic sectors, one in $z > 0$ and the other in $z < 0$. When $A/B < 0$ the equation has four solutions, two in $(0, \pi)$ and two in $(\pi, 2\pi)$. Also from the Lemma 2 and Poincaré–Hopf's Theorem it follows that in this case $i_{p(X)}(q) = 2$. So q must have at least two elliptic sectors. As in the previous case each pair of solutions in $z > 0$ arrive to q from the first and second quadrants, so any elliptic sector must intersect $y = 0$ near $(0, 0)$. This implies the uniqueness of the elliptic sector in $z > 0$. The same situation occurs in $z < 0$. This ends the proof of the theorem. □

Remark 7. The results of Theorem B correspond with items (i) and (iii) of Cima et al. [4, Thm 2.2]. We want to comment that our proof of item (i) is the same, but the one dealing with the case $n > m$ and corresponding to item (iii) is quite different. Indeed in the proof of item (iii) of Theorem 2.2 of Cima et al. [4] it is used the fact that vector fields belonging to $\mathcal{G}_{nm}$ do not have non-degenerate hyperbolic sectors, which is stated in the last assertion of item (ii) of that theorem. Unfortunately the proof of this last assertion had a gap, because we forgot to discard the case of the existence of non-degenerated hyperbolic sectors tangent to the line of infinity. Fortunately, as we have seen in our new proof, these sectors do not exist. In short, the statement of that theorem was right but the proof of item (ii) had a gap. This gap is corrected in the proof of item (ii) of Theorem B.

We end this section with some more facts about the shape of the critical points at infinity for vector fields X in $\mathcal{H}_{n,m}$. The first one was proved in the first assertion of item (ii) of Cima et al. [4, Thm 2.2]. It ensures that if a critical point q of $p(X)$ at infinity has some non-degenerated sector then its two separatrices are tangent to the same direction and in a neighborhood of q this direction is not between them. The second one is that when the vector field $X \notin \mathcal{G}_{n,m}$ many other configurations can appear in the critical points at infinity. For instance in the papers [1,6] the authors show examples of $X \notin \mathcal{G}_{n,m}$ where some critical points at infinity exhibit non-degenerated hyperbolic sectors, with none of their separatrices contained in the equator of the Poincaré compatification.

The simplest one is given in Cima et al. [6, Ex. 3.22] and corresponds to the Hamiltonian

$$H(x, y) = \frac{x^2}{2} + x^2(x^2 + 1)\, y + \frac{1}{2}(x^2 + 1)^3\, y^2.$$

Clearly its associated $X \in \mathcal{H}_{7,7} \setminus \mathcal{G}_{7,7}$ because $P_7(x, y) = -x^6 y$ and $Q_7 = 3x^5 y$ share two factors. It has the critical points at infinity determined by the directions $x = 0, y = 0$ and while the critical points in the charts U_2, V_2 have two degenerated hyperbolic sectors, the ones in U_1, V_1 have two non-degenerated hyperbolic sectors. In [1] a couple of critical points at infinity with 4 non-degenerated hyperbolic do appear.

One of the reasons to be interested on the hyperbolic sectors at infinity of planar polynomial Hamiltonian systems goes back to the celebrated Jacobian conjecture in $\mathbb{R}^2$. Recall that if affirms that a polynomial map $(P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\partial P(x, y)}{\partial x}\frac{\partial Q(x, y)}{\partial y} - \frac{\partial P(x, y)}{\partial y}\frac{\partial Q(x, y)}{\partial x} \equiv c \neq 0$$

is bijective. The reason is that Sabatini proved in Sabatini [12] that when $P(0, 0) = Q(0, 0) = 0$ this conjecture holds if under its hypotheses it can be proved that the Hamiltonian system associated to

$$H(x, y) = \frac{P^2(x, y) + Q^2(x, y)}{2}$$

has a global center at the origin. In particular, notice that to have this property, all the critical points at infinity should have only degenerated hyperbolic sectors. See again the works [1,6] and their references for more details about this question.

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References

[1] J.C.A. Artés, F. Braun, J. Llibre, The phase portrait of the Hamiltonian system associated to a Pinchuk map, An. Acad. Bras. Ciênc. 90 (3) (2018) 2599–2616.
[2] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.L. Maier, Qualitative Theory of Second-order Dynamic Systems, John Wiley & Sons, New York, 1973.
[3] J.A. Argémi, Sur les points singuliers multiples de systémes dynamiques dans $\mathbb{R}^2$, Ann. Mat. Pura Appl. 79 (4) (1968) 35–69.

[4] A. Cima, A. Gasull, F.M. Mañosas, On polynomial Hamiltonian planar vector fields, J. Differ. Equ. 106 (1993) 367–383.
[5] A. Cima, J. Llibre, Bounded polynomial vector fields, Trans. Am. Math. Soc. 318 (1990) 557–579.
[6] A. Cima, F.M. Mañosas, J. Villadelprat, Isochronicity for several classes of Hamiltonian systems, J. Differ. Equ. 157 (1999) 373–413.
[7] F. Dumortier, J. Llibre, J.C.A. Artés, Qualitative theory of planar differential systems, Technical Report, Universitext, Berlin, 2006.
[8] D. Eisenbud, H. Levine, An algebraic formula for the degree of a C^∞ map germ, Ann. Math. 106 (1977) 19–44.
[9] H. He, C. Liu, D. Xiao, Centers and invariant straight lines of planar real polynomial vector fields and its configurations, 2023. https://arxiv.org/abs/2303.14403.
[10] H. He, C. Liu, D. Xiao, The maximal number of centers of planar polynomial Hamiltonian vector fields, Technical Report, Preprint, 2025.
[11] A. Kurosh, Higher Algebra. Translated from the Russian by George Yankovsky, Moscow, Mir Publishers, 1972.
[12] M. Sabatini, A connection between isochronous Hamiltonian centres and the Jacobian conjecture, Nonlinear Anal. 34(6) (1998) 829–838.
[13] J. Sotomayor, Curvas definidas por equaçoes diferenciais no plano. Instituto de Matematica Pura e Aplicada, Rio de Janeiro, 1981.
[14] B. Sturmfels, Solving Systems of Polynomial Equations, CBMS Regional Conference Series in Mathematics, 97 of Washington, DC, Published for the Conference Board of the Mathematical Sciences, 2002. by the AMS, Providence, RI
[15] X. Sun, D. Xiao, Topological classification of global dynamics of planar polynomial Hamiltonian systems with separable variables, J. Differ. Equ. 443 (2025) 113496.