Games and Economic Behavior 155 (2026) 149–166

Contents lists available at ScienceDirect

Games and Economic Behavior

journal homepage: www.elsevier.com/locate/geb

Sequential creation of surplus and the Shapley value ☆

Mikel Álvarez-Mozos [a], Inés Macho-Stadler [b], David Pérez-Castrillo [c,*]

[a] Universitat de Barcelona and BEAT, Spain
[b] Barcelona School of Economics, Spain
[c] Universitat Autònoma de Barcelona and Barcelona School of Economics, Spain

ARTICLE INFO

JEL classification:
C71
D62

Keywords:
Shapley value
Externalities
Sequential game
Equal treatment

ABSTRACT

We introduce the family of *games with intertemporal externalities*, where two disjoint sets of players play sequentially. Coalitions formed by the present players create worth today, but the way these players organize also affects the future: their partition imposes externalities that influence the worth of coalitions formed by future players. We adapt the classic Shapley axioms and explore their implications. They are not sufficient to uniquely determine a value. We propose two solution concepts based on interpreting the Shapley value as the players' expected contributions to coalitions: the *one-coalition externality value* and the *naive value*. Our main results show that adding a single axiom to the classical Shapley axioms yields a unique value: the one-coalition externality value arises adding a principle of equal treatment of direct and indirect contributions or an axiom on necessary players, while the naive value is characterized adding equal treatment of externalities.

1. Introduction

Our choices today may directly or indirectly affect the well-being of future generations. This is especially true for decisions with long-term consequences, such as the extraction of non-renewable resources, the efforts to reduce greenhouse gas (GHG) emissions, the management of nuclear waste, the construction of long-lasting infrastructures, or the investments in technological innovation.

From a normative perspective, if today's choices impact on the well-being of future generations, then it is essential to address how we consider future players (our children, grandchildren, and their successors) when determining the sharing of the wealth generated by these decisions.

Our paper considers this inter-generational situation by defining a new family of games, which we refer to as *games with intertemporal externalities*. It proposes cooperative solutions, acknowledging that one generation may be making decisions for people who cannot speak for their interests at the time.

Consider the case of global warming. This is a cooperative game with intertemporal externalities, where today's choices are represented by the coalitions formed by today's players. Today, players are aware that their choices will have long-term effects on the climate, and it is widely recognized that a substantial collective effort today is required to keep global warming below the 2ºC

☆ We thank Yukihiko Funaki, Chenghong Luo, Roberto Serrano, Chaoran Sun, David Wettstein, two anonymous reviewers, and participants in seminars in Nanjing, ISER in Osaka, Nagoya, Waseda, the SING conference in Messina, the ASSET meeting in Lisbon, REES in Castelló, UB Game Theory Workshop in Barcelona, SAEe in Salamanca, PET in Lyon, and CTN in Budapest, for many comments and discussions on this paper. We also thank MINECO and Feder (PID2023-150472NB-I00 and PID2021-122403NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europa), Generalitat de Catalunya (2021 SGR 00194 and 2021 SGR 00306), Severo Ochoa program (CEX2019-000915-S), and ICREA under the ICREA Academia program for their financial support.

* Corresponding author.
E-mail addresses: mikel.alvarez@ub.edu (M. Álvarez-Mozos), inesmacho@gmail.com (I. Macho-Stadler), david.perez@uab.cat (D. Pérez-Castrillo).

https://doi.org/10.1016/j.geb.2025.09.007
Received 1 November 2024

Available online 1 October 2025

threshold by the end of the 21st century (e.g., Paris 2015 agreement). While this poses a clear externality for future generations, the present generation may not feel a strong sense of urgency, as the most severe consequences lie ahead. As a result, they may fail to fully internalize the costs their actions impose on the next generation.

In a game with intertemporal externalities, two disjoint sets of players act in sequence across two periods. Coalitions formed by the present cohort generate worth today. Moreover, the partition of today's generation exerts an externality on the future cohort. As a result, the worth generated by future coalitions of players depends on the externality inherited from the past generation. Any value concept that aims to distribute the total worth in such a game needs to consider the two periods and the two sets of players.

We adapt the classic Shapley axioms to games with intertemporal externalities and study their implications. While these axioms provide structure, they are not sufficient to uniquely determine a value. We introduce two values using the common interpretation of the Shapley value as the players' expected contributions to coalitions: the *one-coalition externality value* and the *naive value*. We also show the relationship between these values and the Shapley value of an associated game in characteristic function form.

Our main results characterize the two values by adding one additional property to the classic Shapley axioms. We show that a property of equal treatment of direct and indirect contributions leads to characterizing the one-coalition externality value. Alternatively, a necessary player axiom can also be used. In contrast, a property of equal treatment of externalities characterizes the naive value.

Intertemporal externalities have previously been studied, as players' payoffs in most sequential games are inherently influenced by earlier decisions. In cooperative game theory, several models incorporate a temporal dimension. Related to our work is the literature on river sharing (e.g., Ambec and Sprumont, 2002; Ambec and Ehlers, 2008; Van den Brink et al., 2012; Béal et al., 2013; Steinmann and Winkler, 2019), which considers scenarios where agents are located sequentially along a river. These studies propose allocation mechanisms for both water and monetary transfers, with the aim of achieving stable proposals and efficient and equitable distributions that maximize welfare. Our model can be interpreted as a game where players are located along a river, with two distinct groups of agents: farmers positioned upstream and fishers located downstream. Each group owns a predetermined production technology that ties them to their respective locations. Coalitional behavior among farmers in terms of water use influences the quality of water received downstream, which in turn affects both the fish population and the fishers' activities. While there are no externalities within each group, inter-group externalities arise due to the spatial arrangement along the river and the directional flow of water.

Other models with a dynamic component have been proposed and studied in the literature. For instance, Rosenthal (1990) consider that players enter sequentially in a coalitional game. However, a key ingredient of our approach, the intertemporal externalities, is absent. A different strand of the literature has incorporated several stages in exchange economies and considered stable outcomes like the strong and weak sequential core (e.g., Predtetchinski et al., 2002; Herings et al., 2006).

The games with intertemporal externalities differ but share similarities with the "games with externalities," also called "partition function form games" (Thrall and Lucas, 1963). In this class of games, there is a unique set of players, and the worth of each coalition depends on the organization of the outside players. Recent literature studies extensions of the Shapley value for this class of games (see, e.g., Myerson, 1977; Bolger, 1989; Fujinaka, 2006; Macho-Stadler et al., 2007; De Clippel and Serrano, 2008; McQuillin, 2009; Grabisch and Funaki, 2012; Sánchez-Pérez, 2015; Alonso-Meijide et al., 2019).[1] Recently, Casajus et al. (2024) identify a unique natural way to generalize the potential to games with externalities. This one number summary is an average of the worth that a random partition generates according to a particular probability distribution and corresponds to the MPW-value (Macho-Stadler et al., 2007). This is in contrast to Dutta et al. (2010) who identify a whole class of restriction operators, each giving rise to a different generalization of the Shapley value. However, the family of games with intertemporal externalities is not included and does not include the family of games with externalities (see Section 8).

The rest of the paper is organized as follows. Section 2 introduces the family of games with intertemporal externalities. Section 3 adapts the Shapley axioms and describes the structure of any value that satisfies them. Section 4 intuitively introduces the one-coalition externality and the naive values and states their relationship with the Shapley value of an associated game in characteristic function form. Sections 5 and 6 axiomatically characterize the two values, respectively. Section 7 discusses the prescription of the values for games with intertemporal additive externalities. Section 8 discusses the relationship between values for games with intertemporal externalities and values for partition function form games. Section 9 concludes.

2. Framework

We introduce a new family of games called "games with intertemporal externalities." A game with intertemporal externalities is played by two disjoint sets of players, N_1 and N_2, with $N_1 \cap N_2 = \emptyset$. We think of players in N_1 interacting at period $t = 1$, while players in N_2 interact at $t = 2$.[2] We denote generic players of N_1 by i, i', generic players of N_2 by j, j', and generic players of $N_1 \cup N_2$ by h, h'.

A coalition S_1 of N_1 is a group of players of that set, that is, a non-empty subset of N_1, $S_1 \subseteq N_1$. If a coalition S_1 forms, the players jointly obtain a worth of $v_1(S_1) \in \mathbb{R}$. The worth $v_1(S_1)$ only depends on the coalition S_1 and not on how the other players in $N_1 \setminus S_1$ or N_2 are organized.

1 For reviews of the literature on values for games with externalities, see Kóczy (2018) and Macho-Stadler et al. (2019). Álvarez-Mozos and Ehlers (2024) and Bloch and Van den Nouweland (2014) propose extensions of the nucleolus and the core, respectively, for this class of games.

2 For convenience, we will refer to N_1 players as those in the present and N_2 players as those in the future. As we mentioned in the Introduction, these two disjoint sets can have alternative interpretations, such as agents upstream and downstream of a river.

A coalition S_2 of N_2 is a non-empty subset of N_2, $S_2 \subseteq N_2$. Contrary to what happens at $t=1$, the worth obtained by a coalition of N_2 depends not only on the identity of the players in the coalition but also on the past organization of the players in N_1; that is, there are intertemporal externalities between $t=1$ and $t=2$. To formally express these externalities, denote by $\mathcal{P}(M)$ the set of partitions of a finite set M. For technical convenience, we use the convention that $\emptyset \in P$ for every $P \in \mathcal{P}(M)$, while we do not write it explicitly.[3] Then, if coalition S_2 forms and players in N_1 are organized according to the partition $P_1 \in \mathcal{P}(N_1)$, the coalition S_2 generates a worth $v_2(S_2; P_1) \in \mathbb{R}$. For convenience, we denote by $[S] \equiv \{\{i\} : i \in S\}$ the partition of S where all the players are singletons.

The utility is transferable among all the players; that is, the cooperative game is a transferable utility game. In our two-period interpretation of the model, being a transferable utility game requires the existence of a perfect credit market that allows money to be transferred at zero interest rate (or at zero cost) in any direction between $t=1$ and $t=2$.

Therefore, a *game with intertemporal externalities*, or simply a *game*, is a pair (N, v) with $N = (N_1, N_2)$ and $v = (v_1, v_2)$, where $v_1 : 2^{N_1} \to \mathbb{R}$ and $v_2 : 2^{N_2} \times \mathcal{P}(N_1) \to \mathbb{R}$, with $v_1(\emptyset) = 0$ and $v_2(\emptyset; P_1) = 0$ for any $P_1 \in \mathcal{P}(N_1)$. We denote the set of all games by $\mathcal{G}$.

We look for proposals for the division of the worth created in games with intertemporal externalities. A *value* is a mapping $\Phi : \mathcal{G} \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ that assigns to every game (N, v) a unique payoff vector $\Phi(N, v)$.

3. The Shapley axioms

In this section, we introduce some reasonable requirements to impose on a value by extending those characterizing the Shapley value in the set of games in characteristic function form (*CFF games*). These are the axioms of efficiency, linearity, anonymity, and "null" player. We also analyze the implications of these axioms on the characteristics of a value.

We first define the operations of *addition* and *multiplication by a scalar*, and the notions of *permutation of a game* and *null player*.

Definition 1. (a) The *addition* of two games (N, v) and (N, v') is the game $(N, v + v')$ defined by $v + v' = (v_1 + v'_1, v_2 + v'_2)$, where $(v_1 + v'_1)(S_1) \equiv v_1(S_1) + v'_1(S_1)$ for all $S_1 \subseteq N_1$ and $(v_2 + v'_2)(S_2; P_1) \equiv v_2(S_2; P_1) + v'_2(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

(b) Given a game (N, v) and a scalar $\lambda \in \mathbb{R}$, the game $(N, \lambda v)$ is defined by $\lambda v = (\lambda v_1, \lambda v_2)$, where $(\lambda v_1)(S_1) \equiv \lambda v_1(S_1)$ for all $S_1 \subseteq N_1$ and $(\lambda v_2)(S_2; P_1) \equiv \lambda v_2(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

The permutation of a game uses the notion of a permutation of $N = (N_1, N_2)$. Given that N is composed of two disjoint sets, a permutation of N consists of a permutation of each set. That is, a *permutation of* $N = (N_1, N_2)$ is a pair $\sigma = (\sigma_1, \sigma_2)$, where σ_1 is a permutation of N_1 and σ_2 is a permutation of N_2.

Definition 2. Let $(N, v) \in \mathcal{G}$ and σ be a permutation of N. The permuted game $(N, \sigma v)$ is defined by $\sigma v = (\sigma v_1, \sigma v_2)$, where $\sigma v_1(S_1) \equiv v_1(\sigma_1(S_1))$ for all $S_1 \subseteq N_1$, and $\sigma v_2(S_2; P_1) \equiv v_2(\sigma_2(S_2); \sigma_1(P_1))$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

To define a null player, notice that a player in N_1 may influence the worth generated at both periods. On the other hand, a player in N_2 only affects the worth generated at $t=2$, although her influence may depend on the organization of the players at $t=1$. This is why the definition of a null player differs for the players in N_1 and N_2.

For every partition $P \in \mathcal{P}(N_1)$, and player $i \in N_1$, we define

$$P^{-i} \equiv \{S_1 \setminus \{i\} : S_1 \in P\} \cup \{\{i\}\}.$$

Then:

Definition 3. (a) Player $i \in N_1$ is a *null player* in the game (N, v) if

$$v_1(S_1) = v_1(S_1 \backslash \{i\}) \quad \text{for all } S_1 \subseteq N_1 \text{ and}$$
$$v_2(S_2; P_1) = v_2\left(S_2; P_1^{-i}\right) \quad \text{for all } S_2 \subseteq N_2 \text{ and all } P_1 \in \mathcal{P}(N_1).$$

(b) Player $j \in N_2$ is a *null player* in the game (N, v) if $v_2(S_2; P_1) = v_2(S_2 \backslash \{j\}; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

Note that there are two requirements for a player in the first period to be a null player. It should be a classic null player in the game of period 1, and it should not generate any externality in the coalitions of the second period when she leaves a coalition of the first period to remain singleton. Note that this implies that the worth in the second period is not affected by an arbitrary change of the affiliation of the null player. The second requirement of Definition 3(a) resembles a notion of null player in games with externalities (De Clippel and Serrano, 2008).[4]

[3] For example, we denote the partition where all the players in N_1 are together by $\{N_1\}$ instead of $\{N_1, \emptyset\}$. Similarly, if the players in N_1 are organized in two coalitions, S_1 and $N_1 \setminus S_1$, we denote the partition as $\{S_1, N_1 \setminus S_1\}$ instead of $\{S_1, N_1 \setminus S_1, \emptyset\}$.

[4] Note that PFF games feature two distinct definitions of a null player, depending on what happens when a player exits a coalition, whether she becomes a singleton or can join any other coalition (see Dutta et al., 2010; Skibski et al., 2018). However, in our framework, this difference vanishes because players in the first period only generate worth in v_1 and in v_2 they only generate externalities.

We now adapt the original Shapley (1953b) value axioms to our environment:

Axiom 1. *Efficiency: A value Φ is efficient if*

$$\sum_{h\in N_1\cup N_2}\Phi_h(N,v)=v_1\left(N_1\right)+v_2\left(N_2;\left\{N_1\right\}\right).$$

Note that we have in mind environments where it is efficient for the grand coalition to form in both periods.

Axiom 2. *Linearity: A value Φ is linear if*
1.1. $\Phi(N,v+v')=\Phi(N,v)+\Phi(N,v')$ *for any* $(N,v),(N,v')\in\mathcal{G}$, *and*
1.2. $\Phi(N,\lambda v)=\lambda\Phi(N,v)$ *for any* $\lambda\in\mathbb{R}$ *and* $(N,v)\in\mathcal{G}$.[5]

Axiom 3. *Anonymity: A value Φ satisfies anonymity if for any game $(N,v)\in\mathcal{G}$ and permutation σ of N,*

$$\Phi_i(N,\sigma v)=\Phi_{\sigma_1(i)}(N,v)\ \textit{for all } i\in N_1 \textit{ and}$$
$$\Phi_j(N,\sigma v)=\Phi_{\sigma_2(j)}(N,v)\ \textit{for all } j\in N_2.$$

Axiom 4. *Null player: A value Φ satisfies the null player axiom if, for any game $(N,v)\in\mathcal{G}$, $\Phi_h(N,v)=0$ if $h\in N_1\cup N_2$ is a null player in the game (N,v).*

The classic properties of efficiency, linearity, anonymity, and null player in which our axioms are inspired characterize a unique value (Shapley, 1953b) in CFF games. Let us denote by $\mathcal{G}^{CFF}$ the set of CFF games and $(M,\hat{w})\in\mathcal{G}^{CFF}$ a CFF game, i.e., M is the set of players and $\hat{w}:2^M\to\mathbb{R}$ is the characteristic function, with $\hat{w}(\emptyset)=0$. Hatted variables, such as $\hat{w}$, denote the characteristic function in CFF games, distinguishing them from the worth function in games with intertemporal externalities. The Shapley value *Sh* of a player $h\in M$ can be written as

$$Sh_h(M,\hat{w})=\sum_{S\subseteq M}\beta_h(M,S)\,\hat{w}(S)=\sum_{S\subseteq M,S\ni h}\beta_h(M,S)\left(\hat{w}(S)-\hat{w}(S\backslash\{h\})\right),$$

where the *Shapley coefficients*, $\beta_h(M,S)$, are defined for every $S\subseteq M$ by,[6]

$$\beta_h(M,S)=\begin{cases}\frac{(|S|-1)!(|M|-|S|)!)}{|M|!} & \text{if } h\in S\\ \frac{-(|S|!(|M|-|S|-1)!)}{|M|!} & \text{if } h\in M\backslash S.\end{cases}\tag{1}$$

Note that if $N_1=\emptyset$ or $N_2=\emptyset$, then the game with intertemporal externalities (N,v) is essentially a CFF game where the set of players is either N_2 or N_1, respectively. Therefore, any value that satisfies axioms 1 to 4 proposes the Shapley value for these games.

Moreover, consider a game (N,v) where both sets, N_1 and N_2, are non-empty, but there are no intertemporal externalities. That is, suppose that the worth generated by any coalition of N_2 does not depend on the organization of the players in $t=1$. Denote by $(N_1,\hat{v}_1)$ the CFF game where $\hat{v}_1(S_1)=v_1(S_1)$ for all $S_1\in N_1$.[7] Also, for a game without externalities, denote $\hat{v}_2(S_2)\equiv v_2(S_2;P_1)$ for any $S_2\in N_2$ and $P_1\in\mathcal{P}(N_1)$. Then, for the game (N,v), a value satisfying the four axioms allocates the Shapley value of $(N_1,\hat{v}_1)$ to the players of N_1 and the Shapley value of $(N_2,\hat{v}_2)$ to the players of N_2. We state and prove this result in Proposition 1.

Proposition 1. *Take a value Φ satisfying efficiency, linearity, anonymity, and the null player axiom. Also, consider a game (N,v) without externalities, that is, $v_2\left(S_2;P_1\right)=v_2\left(S_2;Q_1\right)$ for all $S_2\subseteq N_2$ and $P_1,Q_1\in\mathcal{P}(N_1)$. Then,*

$$\Phi_i(N,v)=Sh_i(N_1,\hat{v}_1)\ \textit{for all } i\in N_1 \textit{ and}$$
$$\Phi_j(N,v)=Sh_j\left(N_2,\hat{v}_2\right)\ \textit{for all } j\in N_2.$$

Proof. Define the games $(N,v^a),\left(N,v^b\right)\in\mathcal{G}$ as follows:

$$v_1^a(S_1)=v_1(S_1)\text{ for all } S_1\subseteq N_1,$$

[5] Linearity requires both additivity and multiplication by a real number. In games without externalities, it suffices to assume additivity, since the efficiency, anonymity, and null player axioms uniquely determine the value on basis games and their scalar multiples. However, in games with externalities, it is not possible to express every game as a linear combination of games in which the player set can be partitioned into anonymous and null players. In such settings, the value may be additive but not linear (i.e., it may satisfy additivity and the other Shapley axioms, yet fail to satisfy scalar multiplication). This is formally demonstrated in Appendix A of Macho-Stadler et al. (2007) for PFF games. Similar reasoning applies to the class of games with intertemporal externalities.

[6] We denote $|M|$ the number of players in M, for any finite set M.

[7] Note that we use v_1 to refer to the first component of the vector v in the game with intertemporal externalities (N,v); whereas $\hat{v}_1$ is the characteristic function of the CFF game without externalities $(N_1,\hat{v}_1)$.

$v_2^a(S_2; P_1) = 0$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$,

$v_1^b(S_1) = 0$ for all $S_1 \subseteq N_1$,

$v_2^b(S_2; P_1) = v_2(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

Note that $(N, v) = \left(N, v^a + v^b\right)$. Then, by linearity, $\Phi_h(N, v) = \Phi_h(N, v^a) + \Phi_h\left(N, v^b\right)$ for all $h \in N_1 \cup N_2$.

All the players in N_2 are null players in (N, v^a). Then, by the null player axiom $\Phi_j(N, v^a) = 0$ for every $j \in N_2$. Moreover, (N, v^a) is essentially a CFF game among the players in N_1 with a characteristic function $\hat{v}_1$, which is equal to the function v_1^a. Then, we can follow the same steps as in the original proof by Shapley (1953b) and conclude that $\Phi_i(N, v^a) = Sh_i(N_1, \hat{v}_1)$ for every $i \in N_1$.

Similarly, all the players in N_1 are null players in $\left(N, v^b\right)$: A player $i \in N_1$ does not generate any value in v_1^b, and her position in the partition formed at $t = 1$ does not affect the worth of any coalition $S_2 \subseteq N_2$, given that $v_2\left(S_2; P_1\right) = v_2\left(S_2; Q_1\right)$ for all $P_1, Q_1 \in \mathcal{P}(N_1)$. Hence, by the null player property, $\Phi_i\left(N, v^b\right) = 0$, for every $i \in N_1$. Then, $\left(N, v^b\right)$ is essentially a CFF game among the players in N_2 with characteristic function $\hat{v}_2$. The classic characterization of the Shapley value implies that $\Phi_j\left(N, v^b\right) = Sh_j\left(N_2, \hat{v}_2\right)$, for every $j \in N_2$. □

Proposition 1 states a desirable property: Our basic Shapley axioms lead to the Shapley value when applied to games without externalities.

Proposition 2 goes a step forward. It shows that because no externalities affect the function v_1, the worth generated at $t = 1$ should always be split only among the players in N_1, and the sharing should be done according to the Shapley value. On the other hand, the function v_2 receives the influence of players in N_1 and N_2; hence, all the players may share the worth obtained at $t = 2$.

Proposition 2. *Take a value Φ satisfying efficiency, linearity, anonymity, and the null player axiom. Then for every $(N, v) \in \mathcal{G}$ there exists a function f satisfying*

$$\sum_{h \in N_1 \cup N_2} f_h(N_1, N_2, v_2) = v_2(N_2; \{N_1\})$$

such that,

$$\Phi_i(N, v) = Sh_i(N_1, \hat{v}_1) + f_i(N_1, N_2, v_2) \text{ for all } i \in N_1 \text{ and}$$

$$\Phi_j(N, v) = f_j(N_1, N_2, v_2) \text{ for all } j \in N_2.$$

Proof. We define the games (N, v^a) and $\left(N, v^b\right)$ as in the proof of Proposition 1. The players in N_2 are null players in (N, v^a) hence, $\Phi_j(N, v^a) = 0$ for every $j \in N_2$. By the same argument as in the previous proof, $\Phi_i(N, v^a) = Sh_i\left(N_1, \hat{v}_1\right)$ for every $i \in N_1$.

On the other hand, $\left(N, v^b\right)$ is a game where players in N_1 do not generate value in $t = 1$, but they exert externalities in $t = 2$. The value obtained by the players in the game $\left(N, v^b\right)$ can depend on the sets N_1 and N_2 and on the function v_2, but not on v_1. That is, $\Phi_h\left(N, v^b\right)$ corresponds to a function $f_h\left(N_1, N_2, v_2\right)$, for every $h \in N_1 \cup N_2$.

The linearity of the value implies $\Phi_h(N, v) = \Phi_h(N, v^a) + \Phi_h\left(N, v^b\right)$ for all $h \in N_1 \cup N_2$, which leads to the expressions of $\Phi_h(N, v)$ stated in the proposition.

Finally, $\sum_{h \in N_1 \cup N_2} f_h(N_1, N_2, v_2) = \sum_{h \in N_1 \cup N_2} \Phi_h\left(N, v^b\right) = v_1^b\left(N_1\right) + v_2^b\left(N_2; \left\{N_1\right\}\right) = v_2(N_2; \{N_1\})$ by the efficiency of Φ. □

Proposition 2 provides the structure of the payoffs received by the players according to a value that satisfies the basic Shapley axioms of efficiency, linearity, anonymity, and null player. However, contrary to what happens in the set of CFF games, the four axioms do not characterize a unique value in the set of games with intertemporal externalities. The following sections first introduce and then characterize two values that satisfy the basic Shapley axioms together with additional properties.

4. The players' expected contribution for two random arrival processes

A common interpretation of the Shapley value of a player in a CFF game $(M, \hat{w}) \in \mathcal{G}^{CFF}$ is that it corresponds to her expected contribution to coalitions, where the distribution of coalitions arises in a particular way. Specifically, suppose the players enter a room in some order and that all $|M|!$ orderings of the players in M are equally likely. Then $Sh_h(M, \hat{w})$ is the expected contribution of the player h as she enters the room.

In the following two subsections, we propose two "natural" ways players can enter the room in a game with intertemporal externalities; each leads to a value on $\mathcal{G}$.

4.1. All orderings are feasible

We first consider a situation where, to compute the expected contribution of a player, we assume that the players can "arrive" in any order. Hence, we consider orders that intersperse players in N_1 and N_2. Given the temporal dimension of our games, one could view these orders as thought experiments of how players in N_2 may perceive what happened in period 1, that is, what would have happened if the grand coalition of period 1, N_1, had not formed.

Take a game $(N, v) \in \mathcal{G}$. An *ordering* of $N_1 \cup N_2$ is an injective mapping $\omega : N_1 \cup N_2 \to \{1, \dots, |N_1| + |N_2|\}$. Let $\Omega(N_1 \cup N_2)$ denote the set of orderings of $N_1 \cup N_2$. The set of players present at a given step k (that is, the set of predecessors together with the player who arrives at k), with $k \in \{1, \dots, |N_1| + |N_2|\}$, is $\omega^{-1}(\{1, \dots, k\})$. We divide this set in two:

$$B_1^{\omega}(k) = \omega^{-1}(\{1, \dots, k\}) \cap N_1,$$
$$B_2^{\omega}(k) = \omega^{-1}(\{1, \dots, k\}) \cap N_2,$$

and we define $B_1^{\omega}(0) = B_2^{\omega}(0) = \emptyset$. That is, $B_1^{\omega}(k)$ (respectively, $B_2^{\omega}(k)$) is the set of players who have arrived at step k who belong to N_1 (respectively, N_2).

We compute the contribution of a player given an ordering ω. Take the player who arrives in the k^{th} step, that is, player $\omega^{-1}(k)$. If she belongs to N_1, then she contributes to the worth obtained according to v_1 since the worth of the coalition $B_1^{\omega}(k)$ may be different from that of $B_1^{\omega}(k-1)$ due to the addition of $\omega^{-1}(k)$. Hence, the first contribution of player $\omega^{-1}(k)$ is $v_1(B_1^{\omega}(k)) - v_1(B_1^{\omega}(k-1))$. Moreover, player $\omega^{-1}(k)$ may also contribute by changing the externality that players in N_1 exert over the coalition of N_2 formed at this step, that is, $B_2^{\omega}(k)$ (that coincides with $B_2^{\omega}(k-1)$). In this logic, we assume that the players in N_1 who have not arrived yet, that is, those in $N_1 \setminus B_1^{\omega}(k)$, remain singletons. Hence, the contribution of player $\omega^{-1}(k)$ to the worth generated by the players in N_2 is

$$v_2(B_2^{\omega}(k); \{B_1^{\omega}(k)\} \cup [N_1 \setminus B_1^{\omega}(k)]) - v_2(B_2^{\omega}(k); \{B_1^{\omega}(k-1)\} \cup [N_1 \setminus B_1^{\omega}(k-1)]).$$

If the player $\omega^{-1}(k)$ is in N_2, she may only change the worth generated by the function v_2. This contribution depends on the set of players in N_1 who have already arrived. Following the same logic as before, the contribution of $\omega^{-1}(k)$, in this case, is

$$v_2\left(B_2^{\omega}(k); \{B_1^{\omega}(k)\} \cup [N_1 \setminus B_1^{\omega}(k)]\right) - v_2(B_2^{\omega}(k-1); \{B_1^{\omega}(k)\} \cup [N_1 \setminus B_1^{\omega}(k)]\}).$$

Therefore, using that $B_2^{\omega}(k) = B_2^{\omega}(k-1)$ if $\omega^{-1}(k) \in N_1$ and $B_1^{\omega}(k) = B_1^{\omega}(k-1)$ if $\omega^{-1}(k) \in N_2$, we can write the contribution to (N, v) of the player who arrives at step $k \in \{1, \dots, |N_1| + |N_2|\}$ of ω as:

$$\begin{aligned} m_k^{\omega}(N, v) = v_1(B_1^{\omega}(k)) - v_1(B_1^{\omega}(k-1)) + v_2(B_2^{\omega}(k); \{B_1^{\omega}(k)\} \cup [N_1 \setminus B_1^{\omega}(k)]) \\ - v_2(B_2^{\omega}(k-1); \{B_1^{\omega}(k-1)\} \cup [N_1 \setminus B_1^{\omega}(k-1)]). \end{aligned}$$

The *one-coalition externality value* Φ^{1c} allocates to every player $h \in N_1 \cup N_2$ in the game (N, v) her expected contribution to the game when all the orderings have the same probability, that is,[8]

$$\Phi_h^{1c}(N, v) = \frac{1}{(|N_1| + |N_2|)!} \sum_{\omega \in \Omega(N_1 \cup N_2)} m_{\omega(h)}^{\omega}(N, v). \tag{2}$$

We now relate the one-coalition externality value of a game with intertemporal externalities to the Shapley value of an associated CFF game. For any game $(N, v) \in \mathcal{G}$, define the *associated game* $(N_1 \cup N_2, \hat{v}^A) \in \mathcal{G}^{CFF}$ as follows:

$$\hat{v}^A(S) \equiv v_1\left(S \cap N_1\right) + v_2\left(S \cap N_2; \{S \cap N_1\} \cup [N_1 \setminus S]\right), \tag{3}$$

for every $S \subseteq N_1 \cup N_2$. Proposition 3 states that the one-coalition externality value of (N, v) and the Shapley value of $(N_1 \cup N_2, \hat{v}^A)$ coincide.

Proposition 3. *For any game with intertemporal externalities $(N, v) \in \mathcal{G}$,*

$$\Phi^{1c}(N, v) = Sh(N_1 \cup N_2, \hat{v}^A).$$

Proof. The set of the orderings that allow computing the Shapley value of the game $(N_1 \cup N_2, \hat{v}^A)$ is the same set that we have used to define the one-coalition externality value of (N, v). Moreover, it is immediate to check that, for any order, a player's contribution in both games is the same. Hence, the two values coincide. □

4.2. Players in N_1 go first

The existence of intertemporal externalities suggests that we may want only to consider orderings where the players in N_1 go before the players in N_2; we call them "constrained orderings." For a game $(N, v) \in \mathcal{G}$, a *constrained ordering* of $N_1 \cup N_2$ is an injective mapping $\theta : N_1 \cup N_2 \to \{1, \dots, |N_1| + |N_2|\}$ such that $\theta(i) < \theta(j)$, for all $i \in N_1$ and $j \in N_2$. We denote by $\Theta(N_1 \cup N_2)$ the set of constrained orderings of $N_1 \cup N_2$. As above, $B_1^{\theta}(k)$ and $B_2^{\theta}(k)$ are the sets of players who have reached step k and belong to N_1 and N_2, respectively.

We compute a player's contribution given a constrained ordering θ. When a player $j \in N_2$ arrives, all the players in N_1 are already in the room; hence, N_1 has been formed. Therefore, the order of arrival does not change the externality that the players in N_1 generate

[8] We call it the one-coalition externality value because it only considers the externalities exerted when, at most, one coalition of N_1 is formed.

on the worth of the coalitions in N_2. Thus, the contribution in (N,v) of the player who arrives at step $k \in \{1,\dots,|N_1|+|N_2|\}$ of θ is:

$$m_k^\theta(N,v) = \begin{cases} v_1(B_1^\theta(k)) - v_1(B_1^\theta(k-1)) & \text{if } \theta^{-1}(k) \in N_1 \\ v_2(B_2^\theta(k);\{N_1\}) - v_2(B_2^\theta(k-1);\{N_1\}) & \text{if } \theta^{-1}(k) \in N_2. \end{cases}$$

We define the *naive value* Φ^n as the players' expected contribution to constrained orderings when the probability of these orderings is the same. Considering the number of constrained orderings is $|N_1|!\,|N_2|!$, we have:

$$\Phi_h^n(N,v) = \frac{1}{|N_1|!\,|N_2|!} \sum_{\theta\in\Theta(N_1\cup N_2)} m_{\theta(h)}^\theta(N,v),$$

for any $h \in N_1 \cup N_2$.

There are similarities between the constrained orderings that we have used in the construction of the naive value and the "ordered partitions" used in the definition of the *weighted Shapley value* by Kalai and Samet (1987).[9] To see the relationship, we first recall one way to compute the weighted Shapley value for the "weight systems" where all the players' weights are the same, but the set of players M is partitioned in a non-singleton ordered set $\Sigma = (S_1,\dots,S_m)$.[10] Consider $(M,\hat{w}) \in \mathcal{G}^{CFF}$. The weighted Shapley value with the system Σ of player h, which we denote $Sh_h^\Sigma(M,\hat{w})$, corresponds to h's expected marginal contribution to $\hat{w}$ when the only feasible orderings of M have all the players of S_t precede those of S_{t+1} for $t=1,\dots,m-1$ and all the feasible orderings have the same probability.

Then, the naive value corresponds to the weighted Shapley value for the ordered partition $\Sigma^{12} = (N_1,N_2)$ of $N_1 \cup N_2$ of the CFF game $(N_1\cup N_2,\hat{v}^A)$, which we have used to characterize the one-coalition externality value; that is, for any game with intertemporal externalities $(N,v)\in\mathcal{G}$,

$$\Phi^n(N,v) = Sh^{\Sigma^{12}}(N_1\cup N_2,\hat{v}^A).$$

We can also relate the naive value of a game $(N,v)\in\mathcal{G}$ to the Shapley value of two CFF games, the first involving the players of N_1 and the second involving the players in N_2:

$$\Phi_h^n(N,v) = \begin{cases} Sh_h(N_1,\hat{v}_1) & \text{if } h\in N_1 \\ Sh_h\left(N_2,\hat{v}_2^{N_1}\right) & \text{if } h \in N_2, \end{cases} \tag{4}$$

where

$$\hat{v}_2^{N_1}(S_2) = v_2(S_2;\{N_1\}), \tag{5}$$

for every $S_2 \subseteq N_2$.

Equation (4) highlights that, according to the naive value, players in N_1 only receive the value they create at the first period. They do not enjoy or suffer the consequences of the externality generated in the second period by forming the grand coalition in the first period.

In this section, we have introduced the values Φ^{1c} and Φ^n for the set of games with intertemporal externalities $\mathcal{G}$. Each value is obtained as the expected contribution to coalitions for a particular arrival process. We have also shown that they correspond to the Shapley value and a weighted Shapley value, respectively, of the associated CFF game $(N_1\cup N_2,\hat{v}^A)$. In the following two sections, we propose new properties to complement the basic Shapley axioms described in Section 3 to characterize Φ^{1c} and Φ^n.

5. Characterization of the one-coalition externality value

This section offers two characterizations of the one-coalition externality value. The first uses an equal treatment axiom that applies to two players, one of each period, when the direct contributions of the player in N_2 are equal to the indirect contributions (or externality effect) of the player in N_1. The second one is based on a property that requires that players whose participation in a coalition is necessary for the creation of any worth get the highest payoff.

To present the first axiom, we first define the notion of *equally relevant* players.

Definition 4. Players $i\in N_1$ and $j\in N_2$ are equally relevant in (N,v) if

$v_1(S_1) = v_1(S_1\backslash\{i\})$ for all $S_1 \subseteq N_1$, and

$v_2(S_2;P_1) - v_2(S_2\setminus\{j\};P_1) = v_2(S_2;P_1) - v_2(S_2;P_1^{-i})$ for every $S_2\subseteq N_2$ and $P_1\in\mathcal{P}(N_1)$.

According to Definition 4, a player in N_1 and a player in N_2 are considered equally relevant if the player in N_1 has no effect on v_1 (since the player in N_2 cannot have any effect) and the contribution of the player in N_2 is the same as the externality effect

[9] The family of weighted Shapley value was introduced by Shapley (1953a).

[10] We use the notation Σ instead of P to indicate that the partition is ordered.

of the isolation of the player in N_1. Note that the latter condition has two important implications on how i and j can influence the creation of worth in the second stage. First, i generates no externality on coalitions that do not contain j, i.e., $v_2(S_2; P_1) = v_2\left(S_2; P_1^{-i}\right)$ whenever $j \notin S_2$. Second, j does not contribute to any coalition if i is isolated in the first period, i.e., $v_2(S_2; P_1) - v_2(S_2 \setminus \{j\}; P_1)$, as long as $\{i\} \in P_1$.

The axiom we propose, which we call "equal treatment of direct and indirect contributions," requires that two equally relevant players obtain the same payoff in the value. Being equally relevant is a demanding condition; hence, the axiom is weak.

Axiom 5. *Equal treatment of direct and indirect contributions: A value Φ satisfies equal treatment of direct and indirect contributions if, for any game $(N, v) \in \mathcal{G}$, $\Phi_i(N, v) = \Phi_j(N, v)$, for any equally relevant players $i \in N_1$ and $j \in N_2$.*

Theorem 1 states the characterization of the one-coalition externality value using Axiom 5.

Theorem 1. *The one-coalition externality value Φ^{1c} is the only value satisfying the axioms of efficiency, linearity, anonymity, null player, and equal treatment of direct and indirect contributions.*

Proof. We first show that Φ^{1c} satisfies all the properties. We use Proposition 3 and Shapley's original axioms for CFF games.

The value Φ^{1c} is efficient because, for each order, the contributions of all the players in $N_1 \cup N_2$ add up to $v_1(N_1) + v_2(N_2; \{N_1\})$: for any $\omega \in \Omega(N_1 \cup N_2)$,

$$\begin{aligned}\sum_{k=1}^{|N_1|+|N_2|} m_k^{\omega}(N, v) &= v_1(N_1) + v_2\left(N_2; \{N_1\}\right) - v_1(\emptyset) - v_2(\emptyset; [N_1]) \\ &= v_1(N_1) + v_2\left(N_2; \{N_1\}\right).\end{aligned}$$

Its linearity follows from (a) the associated CFF game of the sum of two games is the sum of the two corresponding associated CFF games, (b) the associated CFF game of the product of a game and a scalar is the product of the corresponding associated CFF game and the scalar, and (c) the linearity of the Shapley value.

Similarly, the anonymity of Φ^{1c} follows from the fact that the associated CFF game of a permuted game is a permuted game of the associated CFF game and the anonymity of the Shapley value.

For the null player property, let $i \in N_1$ be a null player in (N, v). Then, for every $S \subseteq N_1 \cup N_2$,

$$\begin{aligned}\hat{v}^A(S) =& v_1(S \cap N_1) + v_2\left(S \cap N_2; \{S \cap N_1\} \cup [N_1 \setminus S]\right) \\ =& v_1((S \setminus \{i\}) \cap N_1) + v_2\left(S \cap N_2; \{(S \setminus \{i\}) \cap N_1\} \cup [N_1 \setminus (S \setminus \{i\}]\right) \\ =& v_1((S \setminus \{i\}) \cap N_1) + v_2\left((S \setminus \{i\}) \cap N_2; \{(S \setminus \{i\}) \cap N_1\} \cup [N_1 \setminus (S \setminus \{i\}]\right) \\ =& \hat{v}^A(S \setminus \{i\}),\end{aligned}$$

where the first and last equalities follow the definition of $\hat{v}^A$, the second equality holds because $i \in N_1$ is a null player, and the third equality holds because $(S \setminus \{i\}) \cap N_2 = S \cap N_2$.

Similarly, if $j \in N_2$ is a null player in (N, v) then, for every $S \subseteq N_1 \cup N_2$,

$$\begin{aligned}\hat{v}^A(S) =& v_1(S \cap N_1) + v_2\left(S \cap N_2; \{S \cap N_1\} \cup [N_1 \setminus S]\right) \\ =& v_1(S \cap N_1) + v_2\left((S \setminus \{j\}) \cap N_2; \{S \cap N_1\} \cup [N_1 \setminus S]\right) \\ =& v_1((S \setminus \{j\}) \cap N_1) + v_2\left((S \setminus \{j\}) \cap N_2; \{(S \setminus \{j\}) \cap N_1\} \cup [N_1 \setminus (S \setminus \{j\}]\right) \\ =& \hat{v}^A(S \setminus \{j\}),\end{aligned}$$

where the second equality holds because $j \in N_2$ is a null player and the third because $(S \setminus \{j\}) \cap N_1 = S \cap N_1$.

Given that $\hat{v}^A(S) = \hat{v}^A(S \setminus \{h\})$ for every $S \subseteq N_1 \cup N_2$ if $h \in N_1 \cup N_2$ is a null player in $(N_1 \cup N_2, \hat{v}^A)$, the null player property of Φ^{1c} follows from the homonymous property of the Shapley value for CFF games.

We now prove that Φ^{1c} satisfies equal treatment of direct and indirect contributions. Let $(N, v) \in \mathcal{G}$ and $i \in N_1$ and $j \in N_2$ be equally relevant players in (N, v). We show that the two players obtain the same payoff in Φ^{1c} by proving that they are symmetric in the associated game $(N_1 \cup N_2, \hat{v}^A)$ and using the symmetry property of the Shapley value. Consider any $S \subseteq N_1 \cup N_2$ such that $i, j \in S$. Then,

$$\begin{aligned}\hat{v}^A(S \setminus \{i\}) =& v_1\left((S \setminus \{i\}) \cap N_1\right) + v_2\left(S \cap N_2; \{(S \setminus \{i\}) \cap N_1\} \cup [N_1 \setminus (S \setminus \{i\})]\right) \\ =& v_1\left((S \cap N_1) \setminus \{i\}\right) + v_2\left(S \cap N_2; \{(S \cap N_1) \setminus \{i\}\} \cup [N_1 \setminus S] \cup \{\{i\}\}\right) \\ =& v_1(S \cap N_1) + v_2\left((S \cap N_2) \setminus \{j\}; \{(S \cap N_1)\} \cup [N_1 \setminus S]\right) \\ =& \hat{v}^A(S \setminus \{j\})\end{aligned}$$

where the third equality holds because i and j are equally relevant players. Then, $\Phi_i^{1c}(N,v) = Sh_i(N_1 \cup N_2, \hat{v}^A) = Sh_j(N_1 \cup N_2, \hat{v}^A) = \Phi_i^{1c}(N,v)$.

For uniqueness, let Φ be a value on $\mathcal{G}$ satisfying the properties. By Proposition 1, we only need to prove that the value is uniquely determined for the games $(N, v^b) \in \mathcal{G}^b \equiv \{(N,v) \in \mathcal{G} : v_1(S_1) = 0 \text{ for all } S_1 \subseteq N_1\}$. To show it, we use a basis of the family of games $\mathcal{G}^b$. Let $S_2 \subseteq N_2$ be non-empty and $P_1 \in \mathcal{P}(N_1)$ and define $\left(N, v^{(S_2;P_1)}\right) \in \mathcal{G}^b$ as follows:

$$v_1^{(S_2;P_1)}(T_1) \equiv 0 \text{ for all } T_1 \subseteq N_1$$

$$v_2^{(S_2;P_1)}(T_2;Q_1) \equiv \begin{cases} 1 & \text{if } S_2 \subseteq T_2 \text{ and } P_1 \preceq Q_1 \\ 0 & \text{otherwise,} \end{cases}$$

where $P_1 \preceq Q_1$ denotes that P_1 is a finer partition than Q_1, which means that for every $S \in P_1$, there is a $T \in Q_1$ such that $S \subseteq T$.

We claim that $\left\{\left(N, v^{(S_2;P_1)}\right) : \emptyset \neq S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1)\right\}$ is a basis of $\mathcal{G}^b$. Clearly, $\mathcal{G}^b$ is a vector space of dimension $\left(2^{|N_2|} - 1\right)|\mathcal{P}(N_1)|$. Then, it is enough to check that the set of games is linearly independent. We do it by contradiction. Let $\{\lambda_{(S_2;P_1)} : \emptyset \neq S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1)\}$ be a set of scalars such that $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v^{(S_2;P_1)}$ is the null game. Suppose that not all scalars are equal to zero. Then, we choose one of them, $\lambda_{(T_2;Q_1)} \neq 0$, so that for every $T_2' \subsetneq T_2$ and $Q_1' \preceq Q_1$, $\lambda_{(T_2';Q_1')} = 0$. The worth of $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v^{(S_2;P_1)}$ evaluated in $(T_2;Q_1)$ is $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v_2^{(S_2;P_1)}(T_2;Q_1) = \lambda_{(T_2;Q_1)} \neq 0$, which is a contradiction and proves the claim.

By linearity, we only need to show that Φ is uniquely determined for every element of the basis. Consider $\left(N, v^{(S_2;P_1)}\right)$, for any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. For convenience, we write the partition P_1 as $P_1 = \{A_1, \ldots, A_k\} \cup [A_{k+1}]$, where $A_1, \ldots, A_k$ are non-singleton coalitions. That is, A_{k+1} includes all the players, if any, of the singleton coalitions of P_1. We prove that $\Phi_h\left(N, v^{(S_2;P_1)}\right)$ is uniquely determined for every $h \in N_1 \cup N_2$.

First, take $j \in N_2 \setminus S_2$. It is easy to check that j is a null player in $v^{(S_2;P_1)}$. Then, by the null player property, $\Phi_j\left(N, v^{(S_2;P_1)}\right) = 0$.

Second, consider $i \in A_{k+1}$. Then, i is a null player in $v^{(S_2;P_1)}$. In fact, $v_1^{(S_2;P_1)}(T_1) = v_1^{(S_2;P_1)}(T_1 \setminus \{i\}) = 0$ for all $T_1 \subseteq N_1$. Moreover, $P_1 \preceq Q_1$ if and only if $P_1 \preceq Q_1^{-i}$ for every $Q_1 \in \mathcal{P}(N_1)$; hence, $v_2^{(S_2;P_1)}(T_2;Q_1) = v_2^{(S_2;P_1)}(T_2;Q_1^{-i})$ for every $T_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. Then, by the null player property, $\Phi_i(N, v^{(S_2;P_1)}) = 0$.

Third, we show that the payoffs to the agents in $S_2 \cup A_1 \cup \cdots \cup A_k$ are also uniquely determined. Recall that S_2 is non-empty and let $j \in S_2$. Then, for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$,

$$v_2^{(S_2;P_1)}(T_2 \setminus \{j\}; Q_1) = 0. \tag{6}$$

Suppose that $N_1 \setminus A_{k+1} = \emptyset$, i.e., $P_1 = [N_1]$. If $S_2 = \{j\}$, then $\Phi_j\left(N, v^{(S_2;P_1)}\right) = 1$ by efficiency. Otherwise, anonymity implies that the payoffs to all agents in S_2 are equal. Indeed, let $j, j' \in S_2$ and $\sigma = (\sigma_1, \sigma_2)$ a permutation of N, with σ_1 the identity on N_1 and σ_2 the permutation on N_2 such that $\sigma_2(j) = j'$, $\sigma_2(j') = j$, and $\sigma_2(j'') = j''$, for every $j'' \in N_2 \setminus \{j, j'\}$. Then, it follows that $\sigma v^{(S_2;P_1)} = v^{(S_2;P_1)}$ and by anonymity $\Phi_j(N,v) = \Phi_{j'}(N,v)$. Using efficiency again, we obtain the uniqueness.

Otherwise, that is, if $N_1 \setminus A_{k+1} \neq \emptyset$, take $i \in N_1 \setminus A_{k+1}$. Observe that $P_1 \npreceq Q_1^{-i}$ for every $Q_1 \in \mathcal{P}(N_1)$, because player i forms a singleton coalition in Q_1^{-i} and belongs to a non-singleton coalition in P_1. Then, for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$,

$$v_2^{(S_2;P_1)}\left(T_2; Q_1^{-i}\right) = 0. \tag{7}$$

Moreover, recall that no worth is generated in the first period because $\left(N, v^{(S_2;P_1)}\right) \in \mathcal{G}^b$.

Finally, equations (6) and (7) together imply that every $i \in N_1 \setminus A_{k+1}$ and $j \in S_2$ are equally relevant players. Hence, by equal treatment of direct and indirect contributions, Φ allocates the same payoff to all of them. The proof concludes by calling efficiency. □

Theorem 1 provides a characterization of the one-coalition externality value based on an axiom that postulates that two equally relevant players obtain the same payoff. The definition of equal relevance takes into account not only the "direct" effect of a player in the worth of a coalition but also the "indirect" effect she may have through an externality. It gives a similar weight to both effects. In particular, if the contribution to a coalition of a player in N_2 is of the same magnitude as the externality generated by a player in N_1 then these players must obtain the same payoff, according to the axiom of equal treatment of direct and indirect contributions.

Moreover,[11] Axiom 5 can be replaced by a property concerning the payoffs to agents whose participation is essential for generating worth in all cases. Before introducing the new axiom, we first define the notions of a *necessary player* and a *monotone game*.

Definition 5. (a) Player $i \in N_1$ is necessary in (N,v) if

$$v_1\left(S_1 \setminus \{i\}\right) = 0, \text{ for every } S_1 \subseteq N_1 \text{ and}$$

$$v_2\left(S_2; P_1^{-i}\right) = 0, \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}\left(N_1\right).$$

[11] We thank a Referee for bringing this point to our attention.

(b) Player $j \in N_2$ is necessary in (N, v) if $v_1 = 0$ and

$$v_2\left(S_2 \setminus \{j\}; P_1\right) = 0, \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}\left(N_1\right).$$

In a game with intertemporal externalities, a necessary player is someone without whom no worth can be generated. If the player is from the first period, then no worth is created in this period without her participation. Moreover, if she is not a member of any (non-singleton) coalition, the coalitions in the second period also generate no worth. If the player is from the second period, then the player is necessary if the game in the first period is null, and no worth is generated in the second period unless the player participates in the coalition. Note that if $i \in N_1$ and $j \in N_2$ are necessary in (N, v), then they are equally relevant according to Definition 4 but not the other way around.

A game with intertemporal externalities is monotone if larger coalitions generate weakly more worth and the externalities are positive.

Definition 6. A game with intertemporal externalities (N, v) is monotone if:

$$v_1(S_1) \leq v_1(T_1) \text{ for every } S_1 \subseteq T_1 \subseteq N_1,$$

$$v_2(S_2; P_1) \leq v_2(T_2; P_1) \text{ for every } S_2 \subseteq T_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1), \text{ and}$$

$$v_2(S_2; P_1) \leq v_2(S_2; Q_1) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1, Q_1 \in \mathcal{P}(N_1) \text{ with } P_1 \preceq Q_1.$$

We denote the set of monotone games with intertemporal externalities by $\mathcal{G}^M$.

The next axiom is adapted from the corresponding axiom for CFF games introduced by Van den Brink and Gilles (1996). It states that any necessary player in a monotone game should receive a payoff at least as high as any other player.

Axiom 6. *Necessary player: A value Φ satisfies the necessary player property if, for every necessary player $h \in N_1 \cup N_2$ in $(N, v) \in \mathcal{G}^M$, $\Phi_h(N, v) \geq \Phi_{h'}(N, v)$ for any $h' \in N_1 \cup N_2$.*

The equal treatment of contributions axiom used in Theorem 1 can be replaced by the necessary player axiom to characterize the one-coalition externality value.

Theorem 2. *The one-coalition externality value Φ^{1c} is the only value satisfying the axioms of efficiency, linearity, anonymity, null player, and necessary player.*

Proof. We begin by showing that Φ^{1c} satisfies the necessary player axiom. Consider a game $(N, v) \in \mathcal{G}^M$. From (3), it follows that the associated game $\left(N_1 \cup N_2, \hat{v}^A\right)$ is a monotone CFF game; that is, $\hat{v}^A(S) \leq \hat{v}^A(T)$ for every $S \subseteq T \subseteq N_1 \cup N_2$. Then, applying Proposition 3 and using the fact that the Shapley value satisfies the necessary player axiom for monotone CFF games (Van den Brink and Gilles, 1996), the result follows.

To show uniqueness, let Φ be a value on $\mathcal{G}$ satisfying the five axioms. The proof follows the same lines as the one of Theorem 1 until the final step of checking that the payoffs to agents in $S_2 \cup A_1 \cup \cdots \cup A_k$ are unique for the game $\left(N, v^{(S_2;P_1)}\right)$, where $S_2 \subseteq N_2$ is non-empty and $P_1 = \{A_1, \ldots, A_k\} \cup [A_{k+1}] \in \mathcal{P}(N_1)$. Note that all players in $S_2 \cup A_1 \cup \cdots \cup A_k$ are necessary in $\left(N, v^{(S_2;P_1)}\right)$ and that $\left(N, v^{(S_2;P_1)}\right) \in \mathcal{G}^M$. Then, by the necessary player property, all these players should receive a payoff at least as high as any other player. Therefore, they all get the same payoff, which must be unique by efficiency. $\square$

6. Characterization of the naive value

To characterize the naive value, we use an axiom related to the equal treatment of the players in N_1 who generate similar externalities. The underlying idea is that if two players of N_1 have a similar role in generating externalities, their payoffs should be the same. We introduce this idea in some simple games, denoted $u^{(S_2;P_1)}$. These games are part of a basis for the set of games with intertemporal externalities $\mathcal{G}$.[12]

Consider a non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. We define the game $u^{(S_2;P_1)} \equiv \left(u_1^{(S_2;P_1)}, u_2^{(S_2;P_1)}\right)$, where $u_1^{(S_2;P_1)} : 2^{N_1} \to \mathbb{R}$ and $u_2^{(S_2;P_1)} : 2^{N_2} \times \mathcal{P}(N_1) \to \mathbb{R}$, by:

$$u_1^{(S_2;P_1)}(T_1) = 0 \quad \text{for all} \quad T_1 \subseteq N_1$$

$$u_2^{(S_2;P_1)}(T_2; Q_1) = \begin{cases} 1 & \text{if } (T_2; Q_1) = (S_2; P_1) \\ 0 & \text{otherwise.} \end{cases}$$

[12] In the proof of Theorem 1, we have used for convenience a different basis, which we denoted $\{v^{(S_2;P_1)}\}_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)}$, for the same set of games.

The set $\{u^{(S_2;P_1)}\}_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)}$ is a basis for the set of games $\mathcal{G}^b \equiv \{(N,v)\in\mathcal{G} : v_1(S_1)=0 \text{ for all } S_1\subseteq N_1\}$.[13,14] Indeed, for any game $(N,v^b)\in\mathcal{G}^b$, we have:

$$v^b = \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v_2^b(S_2;P_1)u^{(S_2;P_1)}.$$

In the game $u^{(S_2;P_1)}$, the role of all the players in N_1 is "similar": it is only when they form precisely the partition P_1 that they generate an externality on the coalition S_2. Our new axiom states that since the role of the players in N_1 in the game $u^{(S_2;P_1)}$ is similar, they should receive the same payoff in "compensation" for the externality that they generate. We call it the axiom of "equal treatment of externalities."

Axiom 7. *Equal Treatment of Externalities: A value Φ satisfies equal treatment of externalities if*

$$\Phi_i\left(N,u^{(S_2;P_1)}\right) = \Phi_{i'}\left(N,u^{(S_2;P_1)}\right) \text{ for all } i,i'\in N_1, \text{ non-empty } S_2\subseteq N_2, P_1\in\mathcal{P}(N_1). \tag{8}$$

Lemma 1 provides some information about the payoff obtained by the players in a value that satisfies equal treatment of externalities in addition to the basic Shapley axioms. It is a technical result, instrumental in the proof of our following theorem.

Lemma 1. *Consider a value Φ that satisfies efficiency, linearity, anonymity, null player, and equal treatment of externalities. Then, there exists weights $\{\gamma(S_2;P_1)\}_{\emptyset\neq S_2\subseteq N_2;P_1\in\mathcal{P}(N_1)}$ satisfying $\sum_{P_1\in\mathcal{P}(N_1)}\gamma(S_2;P_1)=1$ for all non-empty $S_2\subseteq N_2$, such that*

$$\Phi_i(N,v) = Sh_i(N_1,\hat{v}_1) + \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v_2(S_2;P_1)\Phi_k\left(N,u^{(S_2;P_1)}\right) \tag{9}$$

$$\Phi_j(N,v) = Sh_j(N_2,\hat{v}_2^{\gamma}) - \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} \frac{|N_1|}{|N_2\setminus S_2|} v_2(S_2;P_1)\Phi_k\left(N,u^{(S_2;P_1)}\right), \tag{10}$$

for any $i\in N_1$ and $j\in N_2$, where $\Phi_k\left(N,u^{(S_2;P_1)}\right)$ is the payoff obtained by any $k\in N_1$ in the basis game $\left(N,u^{(S_2;P_1)}\right)$ and $\left(N_2,\hat{v}_2^{\gamma}\right)$ is the CFF game defined by

$$\hat{v}_2^{\gamma}(S_2) \equiv \sum_{P_1\in\mathcal{P}(N_1)} \gamma(S_2;P_1)v_2(S_2;P_1) \tag{11}$$

for any non-empty $S_2\subseteq N_2$.

Proof. The proof is in the Appendix. □

Lemma 1 states that the axiom of equal treatment of externalities, together with efficiency, linearity, anonymity, and the null player axiom, restricts the set of values. However, it does not single out one value. To do it, we strengthen this axiom.

Equal treatment of externalities advocates that players in N_1 should receive the same payoff in a basis game $u^{(S_2;P_1)}$ because their role in creating the externality is similar. "Strong equal treatment of externalities" requires that, since the role of the players in N_1 in the games $u^{(S_2;P_1)}$ and $u^{(S_2;P_1')}$ is similar, for any $P_1, P_1'\in\mathcal{P}(N_1)$, their payoffs in these games should also be the same.

Axiom 8. *Strong Equal Treatment of Externalities: A value Φ satisfies strong equal treatment of externalities if*

$$\Phi_i\left(N,u^{(S_2;P_1)}\right) = \Phi_{i'}\left(N,u^{(S_2;P_1')}\right) \text{ for all } i,i'\in N_1, \text{ non-empty } S_2\subseteq N_2, P_1,P_1'\in\mathcal{P}(N_1).$$

Theorem 3 uses Lemma 1 to characterize the naive value through the basic Shapley axioms plus the strong equal treatment of externalities axiom.

[13] In the game $u^{(S_2;P_1)}$, forming the grand coalition in both periods is not efficient unless $(S_2;P_1)=(N_2,\{N_1\})$. We use these games for convenience. However, the same analysis can be done if we define a basis using the functions $w^{(S_2;P_1)}$, which are identical to $u^{(S_2;P_1)}$ except that $w^{(S_2;P_1)}(R_2;Q_1)=1$ if either $(R_2;Q_1)=(S_2;P_1)$ or $(R_2;Q_1)=(N_2;\{N_1\})$.

[14] Consider the game (N,u^{S_1}), where $u^{S_1}=\left(u_1^{S_1},u_2^{S_1}\right)$ is defined by:

$$u_1^{S_1}(R_1) \equiv \begin{cases} 1 & \text{if } R_1 = S_1 \\ 0 & \text{otherwise.}\end{cases}$$

$$u_2^{S_1}(R_2;Q_1) \equiv 0 \quad \text{for all} \quad (R_2;Q_1)\in 2^{N_2}\times\mathcal{P}(N_1).$$

Then, the set $\left\{\left(N,u^{S_1}\right)\right\}_{\emptyset\neq S_1\subseteq N_1} \cup \left\{\left(N,u^{(S_2;P_1)}\right)\right\}_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)}$ constitutes a basis of the whole set of games with intertemporal externalities $\mathcal{G}$.

Theorem 3. *The naive value Φ^n is the only value satisfying the axioms of efficiency, linearity, anonymity, null player, and strong equal treatment of externalities.*

Proof. We first show that Φ^n satisfies the five axioms. Given the characterization of Φ^n provided in equation (4), it is immediate to check that it satisfies efficiency, linearity, anonymity, and null player. It also satisfies the strong treatment of externalities because $\Phi_i^n\left(N, u^{(S_2;P_1)}\right) = 0$ for all $i \in N_1$, non-empty $S_2 \subseteq N_2$, and $P_1 \in \mathcal{P}_1$.

Notice that Φ^n corresponds to the value identified in Lemma 1 when the weights are $\gamma^n(S_2; P_1) \equiv 0$ and $\gamma^n(S_2; \{N_1\}) \equiv 1$, for all $S_2 \subseteq N_2$ and $P_1 \neq \{N_1\}$. For these weights, $\hat{v}_2^{N_1} = \hat{v}_2^{\gamma^n}$ (see equations (5) and (11)).

We now prove that Φ^n is the only value that satisfies all the axioms. Take Φ satisfying the axioms. We show that $\Phi(N, v) = \Phi^n(N, v)$ for all $(N, v) \in \mathcal{G}$.

First, take $i \in N_1$. Strong equal treatment of externalities requires that, for any non-empty $S_2 \subseteq N_2$, $\Phi_i\left(N, u^{(S_2;P_1)}\right)$ is the same for all $P_1 \in \mathcal{P}(N_1)$. Equation (18) (see Appendix) implies that $\sum_{P_1 \in \mathcal{P}(N_1)} \Phi_i\left(N, u^{(S_2;P_1)}\right) = 0$. Therefore, $\Phi_i\left(N, u^{(S_2;P_1)}\right) = 0$ for all $i \in N_1$, non-empty $S_2 \subseteq N_2$, and $P_1 \in \mathcal{P}(N_1)$. Then, using equation (9), $\Phi_i(N, v) = Sh_i(N_1, \hat{v}_1) = \Phi_i^n(N, v)$ for any $i \in N_1$.

Take now $j \in N_2$. Equation (10), together with $\Phi_k\left(N, u^{(S_2;P_1)}\right) = 0$ for all $k \in N_1$, implies that

$$\Phi_j(N, v) = Sh_j\left(N_2, \hat{v}_2^\gamma\right),$$

where $\hat{v}_2^\gamma$ is defined in (11), for some weight system γ. We prove that it is necessarily the case that $\gamma = \gamma^n$ by induction on the size of the coalition S_2. If $S_2 = N_2$, efficiency requires $\gamma(N_2; P_1) = 0$, for any $P_1 \neq \{N_1\}$. Otherwise, suppose $\gamma(N_2; P_1) \neq 0$ for some $P_1 \neq \{N_1\}$, and consider the game $v = u^{(N_2;P_1)}$. For this game, $\hat{v}_2^\gamma(N_2) = \gamma(N_2; P_1)$. Therefore, $Sh(N_2, \hat{v}_2^\gamma)$ shares $\gamma(N_2; P_1) \neq 0$ among the players in N_2, whereas the efficiency of Φ requires that the sum of the players' payoff be $v_2(N_2; \{N_1\}) = 0$. Moreover, $\gamma(N_2; P_1) = 0$ for any $P_1 \neq \{N_1\}$ implies $\gamma(N_2; \{N_1\}) = 1$. Hence, $\gamma(N_2; P_1) = \gamma^n(N_2; P_1)$ for all $P_1 \in \mathcal{P}_1$.

By the induction argument, assume that $\gamma(S_2; P_1) = \gamma^n(S_2; P_1)$ for all $P_1 \in \mathcal{P}\left(N_1\right)$ holds for all $S_2 \subseteq N_2$ with $\left|S_2\right| \geq m$, for $1 < m \leq \left|N_2\right|$.

Consider $S_2 \subseteq N_2$ with $\left|S_2\right| = m - 1$, $j \in N_2 \setminus S_2$, and $P_1 \in \mathcal{P}\left(N_1\right)$. Define the game (N, w) by $w \equiv u^{(S_2 \cup \{j\};P_1)} + u^{(S_2;P_1)}$. That is, the worth of the coalitions $S_2 \cup \{j\}$ and S_2 is 1 if the partition P_1 has been formed; the worth of a coalition is zero in any other case. The agent $j \in N_2$ is a null player in (N, w); hence, the null player axiom implies $\Phi_j(N, w) = 0$. Moreover, given the worth of the coalitions in w, the CFF game $(N_2, \hat{w}_2^\gamma)$ satisfies

$$\begin{aligned}\hat{w}_2^\gamma(S_2 \cup \{j\}) &= \gamma(S_2 \cup \{j\}; P_1)\\ \hat{w}_2^\gamma(S_2) &= \gamma(S_2; P_1)\\ \hat{w}_2^\gamma(T_2) &= 0 \text{ for all } T_2 \neq S_2, T_2 \neq S_2 \cup \{j\}.\end{aligned}$$

The contribution of j to any coalition in the game $(N_2, \hat{w}_2^\gamma)$ is zero, except possibly to S_2. Her contribution to S_2 is $\gamma(S_2 \cup \{j\}; P_1) - \gamma(S_2; P_1)$. Then, $0 = \Phi_j(N, w) = Sh_j\left(N_2, \hat{w}_2^\gamma\right)$ implies that this contribution must be zero; hence, $\gamma(S_2; P_1) = \gamma(S_2 \cup \{j\}; P_1)$ for all $P_1 \in \mathcal{P}_1$. Since $\left|S_2 \cup \{j\}\right| = m$, we use the induction argument and obtain $\gamma(S_2; P_1) = \gamma(S_2 \cup \{j\}; P_1) = \gamma^n(S_2 \cup \{j\}; P_1) = 0$ for all $P_1 \neq \{N_1\}$ and $\gamma(S_2; \{N_1\}) = \gamma(S_2 \cup \{j\}; \{N_1\}) = \gamma^n(S_2 \cup \{j\}; \{N_1\}) = 1$.

This completes the induction argument. We have shown that $\gamma = \gamma^n$; therefore, Φ^n is the only value that satisfies the five axioms. □

Theorem 3 identifies the naive value as the only one satisfying the basic Shapley axioms and rewarding the externalities generated by the players in N_1 in a strong symmetric way. It shows that the only way symmetric treatment of externalities (in a strong sense) is compatible with the Shapley axioms is to disregard the externalities completely.

7. Games with intertemporal additive externalities

In this section, we introduce a particular class of games in $\mathcal{G}$, which we call *games with intertemporal additive externalities*. They are games where the intertemporal externality remains constant across all coalitions formed in the second period, depending solely on the partition established in the first period. In this class of games, we first illustrate the form of any sharing rules that satisfy the basic Shapley axioms. Then, we highlight the differences in the distribution of the worth between the one-coalition externality value and the naive value.

Formally, a game with intertemporal additive externalities $(N, v) \in \mathcal{G}$ satisfies $v_2(\emptyset; P_1) = 0$ for any $P_1 \in \mathcal{P}(N_1)$ and, for every non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$,

$$v_2(S_2; P_1) = \hat{v}_2(S_2) + e(P_1).$$

The function $\hat{v}_2 : 2^{N_2} \setminus \emptyset \to \mathbb{R}$ provides the worth generated by any non-empty coalition of players in N_2, and the function $e : \mathcal{P}(N_1) \to \mathbb{R}$ measures the externality generated by the partition formed among the players in N_1, which is the same for every S_2. We normalize the function such that $e([N_1]) = 0$. This assumption is without loss of generality as we could subtract the worth of the partition of the singletons from all the externalities and add it to $v_2(S_2)$ for all non-empty $S_2 \subseteq N_2$.

Consider a value Φ satisfying efficiency, linearity, anonymity, and null player. We decompose any game with additive externalities $(N,v) \in \mathcal{G}$ as the sum of two games (N,v') and (N,v'') as follows. The game (N,v') satisfies $v'_1 = v_1$ and $v'_2(S_2;P_1) = \hat{v}_2(S_2)$ for any $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. The game (N,v'') is defined by (a) $v''_1 = 0$ and (b) $v''_2(\emptyset;P_1) = 0$ and $v''_2(S_2;P_1) = e(P_1)$ for any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

Note that (N,v') is a game without externalities. Then, by Proposition 1, $\Phi_i(N,v') = Sh_i(N_1,\hat{v}_1)$ for all $i \in N_1$ and $\Phi_j(N,v') = Sh_j(N_2,\hat{v}_2)$ for all $j \in N_2$. Therefore, if the externality is additive, the values that satisfy the basic Shapley axioms only differ in how they share the worth $e(\{N_1\})$ among the players in $N_1 \cup N_2$. Moreover, by anonymity, all the players in N_2 get the same payoff, hence $\Phi_j(N,v'') = \Phi_{j'}(N,v'')$ for every $j,j' \in N_2$.

We now describe the precise sharing proposed by Φ^n and Φ^{1c} for the class of additive games.

Consider the naive value, $\Phi = \Phi^n$. Equation (4) implies that $\Phi^n_i(N,v'') = Sh_i(N,\hat{v}''_1) = 0$ for all $i \in N_1$. Moreover, since all the players in N_2 must obtain the same payoff, $\Phi^n_j(N,v'') = \frac{e(N_1)}{|N_2|}$ for all $j \in N_2$. That is, the naive value divides equally the externality (positive or negative) generated by the formation of the grand coalition N_1 among the players in N_2.

Consider now the one-coalition externality value, $\Phi = \Phi^{1c}$. For this value, the externality generated by the formation of N_1 is shared among the players in $N_1 \cup N_2$ and not only among the players in N_2. Using equation (2), we compute the equal value assigned by Φ^{1c} to the players in N_2:

$$\Phi^{1c}_j(N,v'') = \sum_{\substack{S_1 \subseteq N_1 \\ |S_1| \geq 2}} \frac{|S_1|!(|N_1 \cup N_2| - |S_1| - 1)!}{|N_1 \cup N_2|!} e(\{S_1\} \cup [N_1 \setminus S_1]),$$

for all $j \in N_2$. On the other hand, the players in N_1 are not symmetric. Following also equation (2), the contributions of a player in N_1 determine the value that Φ^{1c} assigns to her:

$$\Phi^{1c}_i(N,v'') = \sum_{\substack{S_1 \subseteq N_1 \\ S_1 \supseteq \{i\}}} \left(\frac{(|S_1|-1)!(|N_1| - |S_1|)!}{|N_1|!} - \frac{(|S_1|-1)!(|N_1 \cup N_2| - |S_1|)!}{|N_1 \cup N_2|!} \right) \times \left(e(\{S_1\} \cup [N_1 \setminus S_1]) - e(\{S_1 \setminus \{i\}\} \cup [N_1 \setminus (S_1 \setminus \{i\}]) \right),$$

for all $i \in N_1$.

To illustrate the previous results on how the externality is shared among the players in N_1 and N_2, we consider two games (N,v'') with $N_1 = \{1,2,3\}$ and $N_2 = \{4\}$. Example 1 presents a game with positive externalities, meaning that forming larger coalitions in $t=1$ generates a higher worth in $t=2$. Example 2 presents a scenario with negative externalities, where forming larger coalitions at $t=1$ leads to a lower worth at $t=2$.

Example 1. We denote by (N,v^+) the game with positive externalities. In the first period, the worth of the coalitions is given by $v^+_1(S) = 3$ if $|S| = 1$, $v^+_1(S) = 1$ if $|S| = 2$, and $v^+_1(N_1) = 3$. In the second period, $\hat{v}^+_2(\{4\}) = 2$, and the externality is represented in the table below:

	{{1},{2},{3}}	{{1,2},{3}}	{{1,3},{2}}	{{2,3},{1}}	{{1,2,3}}
$e^+(P_1)$	0	0	3	6	9

The naive and one-coalition externality values for the game (N,v^+) are:

$$\Phi^n(N,v^+) = (1,1,1,11) \quad \text{and} \quad \Phi^{1c}(N,v^+) = (2,3,4,5).$$

For both values, the players in the set N_1 share $v^+_1(N_1)$ according to the Shapley value $Sh(N_1,v^+_1) = (1,1,1)$. Player 4 receives $Sh(N_2,\hat{v}^+_2) = 2$. The difference between the two values arises from the sharing of the externality worth $e^+(\{N_1\}) = 9$. The naive value allocates the positive externality generated in the first period solely to the agent in the second period. Hence, player 4 receives an additional payoff of 9. In contrast, the one-coalition externality value is more nuanced. It divides the externality among the players in the two periods. The players in N_1 receive a positive payoff of 6 in total. We may think of this payoff as a debt taken by the players in N_1 that must be paid by the player in N_2.[15] Since the externalities from the various partitions that might form in $t=1$ are asymmetric, the value received by the players in N_1 is not equal. In this game, the total positive payoff of 6 is shared according to the distribution $(1,2,3)$.

Example 2. In the game with negative externalities (N,v^-), the worth of the coalitions in the first period are $v^-_1(S) = 2$ if $|S| = 1$, $v^-_1(S) = 8$ if $|S| = 2$, and $v^-_1(N_1) = 21$. In the second period, $\hat{v}^-_2(\{4\}) = 5$, and the externality is defined by:

	{{1},{2},{3}}	{{1,2},{3}}	{{1,3},{2}}	{{2,3},{1}}	{{1,2,3}}
$e^-(P_1)$	0	-3	-3	-3	-9

[15] For example, players in N_1 may issue bonds to fund tax cuts or spending that benefit the present, while future taxpayers bear the repayment burden. In this way, the present generation benefits from the positive externality that materializes after they have exited the game.

The naive and one-coalition externality values for (N, v^-) are:

$$\Phi^n(N, v^-) = (7, 7, 7, -4) \quad \text{and} \quad \Phi^{1c}(N, v^-) = (5, 5, 5, 2).$$

In this game, we have $Sh(N_1, v_1^-) = (7, 7, 7)$ and $Sh(N_2, \hat{v}_2^-) = 5$. According to the naive value, players in N_1 do not face the consequences of the negative externalities. This results in player 4 having to assume this cost, receiving $5 - 9 = -4$. The one-coalition externality value takes into account that the formation of the grand coalition N_1 leads to the most negative externality. It assigns a total payoff of -6 to those agents. This payoff may take the form of investments made at $t = 1$ that benefit the players living at $t = 2$. The three players of N_1 share this burden symmetrically, as the externality of any partition of N_1 only depends on the sizes of the coalitions, not on the identities of the players.

8. Discussion on the relationship with values for partition function form games

Given the existence of externalities between two sets of players in a game with intertemporal externalities, there are similarities between the class of games we analyze in this paper and the set of games in partition function form (PFF games). In contrast with a CFF game, a PFF game considers that the worth of a coalition may depend on the organization of the rest of the players, that is, on the whole partition of players. As discussed in the Introduction, the literature has provided several values for PFF games that extend the Shapley value.

This section shows that a game with intertemporal externalities can be adapted into a "traditional" PFF game. Take a game with intertemporal externalities (N, v). The most intuitive way to transform it into a PFF game $(N_1 \cup N_2, \tilde{v})$ is by defining the worth function $\tilde{v}$ as follows:

$$\tilde{v}(S, P) \equiv v_1(S \cap N_1) + v_2(S \cap N_2; P \tilde{\cap} N_1), \tag{12}$$

for any $S \subseteq N_1 \cup N_2$ and $P \in \mathcal{P}(N_1 \cup N_2)$ with $S \in P$, where we denote $P \tilde{\cap} N_1 \equiv \{R \cap N_1 \mid R \in P\}$. That is, the game $(N_1 \cup N_2, \tilde{v})$ associates to a coalition S of $N_1 \cup N_2$ when the partition is P the sum of the worth in $t = 1$ of the players of S who are in N_1 plus the worth in $t = 2$ of the players in S who are in N_2. The worth of $S \cap N_2$ is computed taking into account that players in N_1 are organized according to the restriction of P to N_1.[16]

In the class of PFF games, several extensions of the Shapley value can be obtained through the "average approach" (Macho-Stadler et al., 2007). This approach consists of defining, for each PFF game, an "average" CFF game, where the worth of a coalition is a weighted average of the worth of the coalition for all the possible partitions that include it. Then, we obtain an extension of the Shapley value by applying this value to the resulting average CFF game. Each way of doing averages (i.e., each weight system) leads to a different extension of the Shapley value.

A weight system α is a function that associates a non-negative weight to each coalition and partition that contains it, with the condition that $\sum_{P \ni S, P \in \mathcal{P}(N_1 \cup N_2)} \alpha(S, P) = 1$, for all $S \subseteq N_1 \cup N_2$. Then, given the PFF game $(N_1 \cup N_2, \tilde{v})$ and the weight system α, the average approach constructs the CFF game $(N_1 \cup N_2, \hat{v}^\alpha)$ as follows:

$$\hat{v}^\alpha(S) = \sum_{\substack{P \in \mathcal{P}(N_1 \cup N_2) \\ P \ni S}} \alpha(S, P) \tilde{v}(S, P),$$

for any $S \subseteq N_1 \cup N_2$.

Using the two previous steps, we can go from a game $(N, v) \in \mathcal{G}$ to a PFF game $(N_1 \cup N_2, \tilde{v})$, and from this to a CFF game $(N_1 \cup N_2, \hat{v}^\alpha)$. Given the expression for $\tilde{v}$ in (12), we obtain

$$\hat{v}^\alpha(S) = v_1(S \cap N_1) + \sum_{\substack{P \in \mathcal{P}(N_1 \cup N_2) \\ P \ni S}} \alpha(S, P) v_2(S \cap N_2, P \tilde{\cap} N_1), \tag{13}$$

for any $S \subseteq N_1 \cup N_2$.

Therefore, we can propose an extension of the Shapley value for games with intertemporal externalities by computing $Sh(N_1 \cup N_2, \hat{v}^\alpha)$. Each vector of weights α that is symmetric, in the sense that it only depends on the sizes of the coalitions, and that satisfies a condition derived from the null player axiom (see Theorem 1 in Macho-Stadler et al., 2007) leads to an extension of the Shapley value to the class of games $\mathcal{G}$.

Pham Do and Norde (2007) and De Clippel and Serrano (2008) propose an extension, called the "externality-free" value, which corresponds to the weights $\alpha^{EF}(S, P) = 1$ if $P = \{S\} \cup [(N_1 \cup N_2) \setminus S]$ and $\alpha^{EF}(S, P) = 0$ otherwise. Using these weights, we obtain

$$\hat{v}^{\alpha^{EF}}(S) = v_1(S \cap N_1) + v_2(S \cap N_2; \{S \cap N_1\} \cup [N_1 \setminus S]) = \hat{v}^A(S),$$

[16] The previous expression of $\tilde{v}$ is also obtained if we consider the initial game (N, v) a game with two "issues," in the sense of Diamantoudi et al. (2015). Following the approach of that paper, each period can be considered an issue in which all the players participate; only the players in N_1 generate worth in the first issue, and only those in N_2 generate worth in the second issue, although the organization of all the players in N_1 matters for that worth. Diamantoudi et al. (2015) propose a way to convert a game with several issues into a PFF game. Easy calculations show that going from (N, v) to a game with two issues and from that game to a PFF game results in $(N_1 \cup N_2, \tilde{v})$.

for any $S \subseteq N_1 \cup N_2$. Since $\hat{v}^{\alpha^{EF}} = \hat{v}^A$, Proposition 3 implies that the use of the previous procedure and the externality-free value leads to the one-coalition externality value for games with intertemporal externalities.

McQuillin (2009) proposes the "extended Shapley" value for PFF games, which, in terms of the average approach, corresponds to the weights $\alpha^{ES}(S,P) = 1$ if $P = \{S, (N_1 \cup N_2) \setminus S\}$ and $\alpha^{ES}(S,P) = 0$ otherwise. Using these weights in (13), we obtain

$$\hat{v}^{\alpha^{ES}}(S) = v_1(S \cap N_1) + v_2(S \cap N_2; \{S \cap N_1, N_1 \setminus S\}),$$

for any $S \subseteq N_1 \cup N_2$. Let us denote $\Phi^{ES}(N,v) \equiv Sh\left(N_1 \cup N_2, \hat{v}^{\alpha^{ES}}\right)$ the value for games with intertemporal externalities obtained using the weights α^{ES}. It is worth noticing that this value corresponds to the players' expected contribution to a random arrival process similar to that in subsection 4.1, where all orderings are feasible, but with the key difference that all the players start off together rather than as individual singletons before entering the room.

Applying this value to the examples with additive externalities introduced in Section 7, we obtain $\Phi^{ES}(N,v^+) = (1.5, 2, 2.5, 8)$ and $\Phi^{ES}(N,v^-) = (6, 6, 6, -1)$.

We can also consider weight systems that assign a positive probability to all embedded coalitions, beyond those in which the outsiders are restricted to be either singletons or together. For instance, consider the weights proposed by Macho-Stadler et al. (2007):

$$\alpha^{MPW}(S,P) = \frac{\prod_{T \in N \setminus S}(|T|-1)!}{(|N|-|S|)!}.$$

If we apply this value to the examples with additive externalities in Section 7, we obtain $\Phi^{MPW}(N,v^+) = (1.75, 2.5, 3.25, 6.5)$ and $\Phi^{MPW}(N,v^-) = (5.5, 5.5, 5.5, 0.5)$.

As in the one-coalition externality value, under Φ^{ES} and Φ^{MPW}, players in N_1 enjoy or suffer part of the positive or negative externality they generate.

Can we obtain the naive value using this procedure? Specifically, does there exist a weight system α such that, when applied to the PFF game $(N_1 \cup N_2, \tilde{v})$ (see (12)), it defines a CFF game whose Shapley value coincides with the naive value? Note first that, using (4) and (5), we can write the naive value as $\Phi^n(N,v) = Sh(N_1 \cup N_2, \hat{v}^n)$, where $\hat{v}^n(S) \equiv v_1\left(S \cap N_1\right) + v_2\left(S \cap N_2; \{N_1\}\right)$, for every $S \subseteq N_1 \cup N_2$. Therefore, we can rephrase the question as follows: Are there weights that lead to $\hat{v}^\alpha(S) = \hat{v}^n(S) = v_1\left(S \cap N_1\right) + v_2\left(S \cap N_2; \{N_1\}\right)$ for any $S \subseteq N_1 \cup N_2$? The answer is negative. The reason is that, for any $S \subseteq N_1 \cup N_2$, $\hat{v}^\alpha(S)$ puts weights to the worth of the coalition $S \cap N_2$ when $S \cap N_1$ is an element of the partition P_1 (since $S \cap N_1 \in P\tilde{\cap}N_1$, for any (S,P) with $S \in P$), whereas $\hat{v}^n(S)$ only takes into account the worth of $S \cap N_2$ when $P_1 = N_1$. Therefore, $\hat{v}^\alpha(S)$ is typically different from $\hat{v}^n(S)$, for any $S \nsupseteq N_1$.

The fact that the naive value cannot be derived from an extension of the Shapley value for PFF games through the procedure previously described is not a weakness of the value. In PFF games, all the players have a priori equal possibilities to create worth and externalities with the other players. On the other hand, in a game with intertemporal externalities, the possibilities to create worth and externalities for players in N_1 are very different from those in N_2. A value for the set of games with intertemporal externalities may take into account these differences, as the naive value does.

9. Conclusion

We have introduced a new class of games in which the way coalitions are organized by one set of players creates externalities that affect the worth generated by coalitions formed by another set. This model fits environments with intergenerational externalities, where the decisions made (the coalitions formed) by organizations or individuals at a given moment strongly influence the worth that future organizations or individuals can derive. We apply a cooperative game-theoretic approach based on the original Shapley axioms to analyze these intertemporal externalities. However, these axioms alone do not suffice to single out a unique value for our class of games.

We study two extensions of the Shapley value, the one-coalition externality value and the naive value, defined as the players' expected contribution to coalitions. Each extension is based on a different probability distribution on the arrival of the players. We characterize the one-coalition externality value in two ways, using an equal treatment axiom and a necessary player axiom. Similarly, we characterize the naive value with the axiom of strong equal treatment of externalities. We numerically evaluate the values proposed by these two solutions within two simple games and discuss how other arrival protocols lead to different outcomes.

Several other extensions of the Shapley value can be proposed for games with intertemporal externalities. We can envision arrival processes different from those studied in this paper. We may also suggest new axioms that either characterize alternative appealing values for distributing gains across generations or provide additional properties for the values we have introduced.

Further research may also investigate alternative solution concepts, such as extensions of the nucleolus, to address intertemporal externalities. Moreover, our analysis has intentionally focused on games characterized solely by intertemporal externalities. However, in real-world settings, such as contemporary international negotiations on pollution abatement, externalities are not only intertemporal but also inter-coalitional at each date, with significant implications for the current welfare of different coalitions. Finally, we can also consider non-cooperative games that would implement the cooperative solutions proposed in this paper. These analyses, while interesting, fall beyond the scope of the current paper.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: David Perez-Castrillo reports financial support was provided by MINECO and Feder PID2021-122403NB-I00, Generalitat de Catalunya 2021 SGR 00194, Severo Ochoa program CEX2019-000915-S, and ICREA under the ICREA Academia. Ines Macho-Stadler reports financial support was provided by MINECO and Feder PID2021-122403NB-I00, Generalitat de Catalunya 2021 SGR 00194, Severo Ochoa program CEX2019-000915-S. Mikel Alvarez-Mozos reports financial support was provided by MINECO and Feder PID2023-150472NB-I00, Generalitat de Catalunya 2021 SGR 00194. The co-author Ines Macho-Stadler currently serves as Associate Editor for Games and Economic Behavior. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

Proof of Lemma 1. We decompose the game (N, v) in the games (N, v^a) and $\left(N, v^b\right)$, as in the proof of Proposition 1. We know that $\Phi(N, v^a)$ allocates $Sh\left(N_1, \hat{v}_1\right)$ to the players in N_1 and 0 to the players of N_2. We now focus on $\Phi\left(N, v^b\right)$.

Since Φ satisfies linearity, then

$$\Phi_h\left(N, v^b\right) = \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)} v^b(S_2; P_1)\Phi_h\left(N, u^{(S_2;P_1)}\right) \quad \text{for all} \quad h \in N_1 \cup N_2. \tag{14}$$

The anonymity of Φ implies that

$$\Phi_j\left(N, u^{(S_2;P_1)}\right) = \Phi_{j'}\left(N, u^{(S_2;P_1)}\right) \quad \text{if} \quad j, j' \in S_2, \text{ or } j, j' \in N_2 \setminus S_2, \tag{15}$$

and its efficiency implies that

$$\sum_{h \in N_1 \cup N_2} \Phi_h\left(N, u^{(S_2;P_1)}\right) = 0 \quad \text{if} \quad (S_2; P_1) \neq (N_2, \{N_1\}), \tag{16}$$

$$\sum_{h \in N_1 \cup N_2} \Phi_h\left(N, u^{(N_2,\{N_1\})}\right) = 1. \tag{17}$$

Moreover, because Φ satisfies efficiency, linearity, anonymity, and null player, then

$$\sum_{P_1 \in \mathcal{P}(N_1)} \Phi_h\left(N, u^{(S_2;P_1)}\right) = \begin{cases} 0 & \text{if} \quad h \in N_1 \\ \beta_h(N_2, S_2) & \text{if} \quad h \in N_2 \end{cases} \tag{18}$$

where $\beta_h(N_2, S_2)$ are the Shapley coefficients, see (1). Equation (18) follows from Proposition 1 because $\sum_{P_1 \in \mathcal{P}(N_1)}\left(N, u^{(S_2;P_1)}\right)$ is a game without externalities; hence, the worth $\sum_{P_1 \in \mathcal{P}(N_1)} u^{(S_2;P_1)}(N_2; \{N_1\})$ (which is equal to 0 unless $S_2 = N_2$, in which case the worth is 1) is shared among the players in N_2 according to their Shapley value.

Using equal treatment of externalities (see (8)) and (15), we can express equation (16) as follows:

$$\left|N_1\right| \Phi_k\left(N, u^{(S_2;P_1)}\right) + \left|S_2\right| \Phi_j\left(N, u^{(S_2;P_1)}\right) + \left|N_2 \setminus S_2\right| \Phi_{j'}\left(N, u^{(S_2;P_1)}\right) = 0 \tag{19}$$

for any $(S_2; P_1) \neq (N_2, \{N_1\})$, $k \in N_1$, $j \in S_2$.

When $S_2 \neq N_2$, that is, there is some $j' \in N_2 \setminus S_2$, we write equation (19) as:

$$\Phi_{j'}\left(N, u^{(S_2;P_1)}\right) = -\frac{\left|S_2\right|}{\left|N_2 \setminus S_2\right|}\Phi_j\left(N, u^{(S_2;P_1)}\right) - \frac{\left|N_1\right|}{\left|N_2 \setminus S_2\right|}\Phi_k\left(N, u^{(S_2;P_1)}\right), \tag{20}$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$, and we notice that the Shapley coefficients satisfy the following relation:

$$\left|S_2\right| \beta_j(N_2, S_2) + \left|N_2 \setminus S_2\right| \beta_{j'}(N_2, S_2) = 0 \tag{21}$$

for all $j \in S_2$ and $j' \in N_2 \setminus S_2$.

Using (21), we substitute $\left|N_2 \setminus S_2\right|$ in equation (20) to obtain:

$$\Phi_{j'}\left(N, u^{(S_2;P_1)}\right) = \beta_{j'}(N_2, S_2)\frac{1}{\beta_j(N_2, S_2)}\Phi_j\left(N, u^{(S_2;P_1)}\right) - \frac{\left|N_1\right|}{\left|N_2 \setminus S_2\right|}\Phi_k\left(N, u^{(S_2;P_1)}\right), \tag{22}$$

for any $S_2 \neq N_2$, $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$.

Define the "weights" $\gamma(S_2; P_1)$ as follows:

$$\gamma(S_2; P_1) \equiv \frac{1}{\beta_j(N_2, S_2)}\Phi_j\left(N, u^{(S_2;P_1)}\right), \tag{23}$$

where j is any player in S_2.

Notice that, using (18), $\sum_{P_1\in\mathcal{P}(N_1)}\gamma(S_2;P_1)=\frac{1}{\beta_j(N_2,S_2)}\sum_{P_1\in\mathcal{P}(N_1)}\Phi_j\left(N,u^{(S_2;P_1)}\right)=1$ (where j is any player in S_2), for all $S_2\subseteq N_2$.

Then, equations (23) and (22) lead to

$$\Phi_j(N,u^{(S_2;P_1)}) = \beta_j(N_2,S_2)\gamma(S_2;P_1), \tag{24}$$

$$\Phi_{j'}(N,u^{(S_2;P_1)}) = \beta_{j'}(N_2,S_2)\gamma(S_2;P_1) - \frac{|N_1|}{|N_2\setminus S_2|}\Phi_k\left(N,u^{(S_2;P_1)}\right), \tag{25}$$

for any $j\in S_2$, $j'\in N_2\setminus S_2$, and $k\in N_1$.

Using (14), (24), and (25), we can express the worth of any player $j\in N_2$ in a game $\left(N,v^b\right)$ according to a value Φ that satisfies efficiency, linearity, anonymity, and equal treatment of externalities as follows:

$$\begin{aligned}
\Phi_j\left(N,v^b\right) &= \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v^b(S_2;P_1)\Phi_j\left(N,u^{(S_2;P_1)}\right)\\
&= \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\supseteq\{j\}} v^b(S_2;P_1)\beta_j(N_2,S_2)\gamma(S_2;P_1)\\
&+ \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} v^b(S_2;P_1)\left(\beta_j(N_2,S_2)\gamma(S_2;P_1) - \frac{|N_1|}{|N_2\setminus S_2|}\Phi_k\left(N,u^{(S_2;P_1)}\right)\right)\\
&= \sum_{\emptyset\neq S_2\subseteq N_2}\beta_j(N_2,S_2)\sum_{P_1\in\mathcal{P}(N_1)}\gamma(S_2;P_1)v^b(S_2;P_1)\\
&- \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} v^b(S_2;P_1)\frac{|N_1|}{|N_2\setminus S_2|}\Phi_k\left(N,u^{(S_2;P_1)}\right)\\
&= Sh_j(N_2,\hat{v}_2^{\gamma}) - \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} \frac{|N_1|}{|N_2\setminus S_2|}\Phi_k\left(N,u^{(S_2;P_1)}\right)v^b(S_2;P_1),
\end{aligned}$$

where k is any player in N_1.

Similarly, using (14), we can express the worth of any player $i\in N_1$ as follows:

$$\Phi_i\left(N,v^b\right) = \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v^b(S_2;P_1)\Phi_i\left(N,u^{(S_2;P_1)}\right).$$

Given that $\Phi_k\left(N,v^{(S_2;P_1)}\right)$ is the same for every $k\in N_1$, linearity and equal treatment of externalities imply that all the players in N_1 obtain the same payoff in a game $\left(N,v^b\right)$.

Finally, the expression in the lemma follows from the linearity of Φ and the fact that $v=v^a+v^b$. □

Data availability

No data was used for the research described in the article.

References

Alonso-Meijide, J.M., Álvarez-Mozos, M., Fiestras-Janeiro, M.G., Jiménez-Losada, A., 2019. Complete null agent for games with externalities. Expert Syst. Appl. 135, 1–11.
Álvarez-Mozos, M., Ehlers, L., 2024. Externalities and the (pre) nucleolus in cooperative games. Math. Soc. Sci. 128, 10–15.
Ambec, S., Ehlers, L., 2008. Sharing a river among satiable agents. Games Econ. Behav. 64 (1), 35–50.
Ambec, S., Sprumont, Y., 2002. Sharing a river. J. Econ. Theory 107 (2), 453–462.
Béal, S., Ghintran, A., Rémila, E., Solal, P., 2013. The river sharing problem: a survey. Int. Game Theory Rev. 15 (3), 1340016.
Bloch, F., Van den Nouweland, A., 2014. Expectation formation rules and the core of partition function games. Games Econ. Behav. 88, 339–353.
Bolger, E.M., 1989. A set of axioms for a value for partition function games. Int. J. Game Theory 18 (1), 37–44.
Casajus, A., Funaki, Y., Huettner, F., 2024. Random partitions, potential, value, and externalities. Games Econ. Behav. 147, 88–106.
De Clippel, G., Serrano, R., 2008. Marginal contributions and externalities in the value. Econometrica 76 (6), 1413–1436.
Diamantoudi, E., Macho-Stadler, I., Pérez-Castrillo, D., Xue, L., 2015. Sharing the surplus in games with externalities within and across issues. Econ. Theory 60 (2), 315–343.
Dutta, B., Ehlers, L., Kar, A., 2010. Externalities, potential, value and consistency. J. Econ. Theory 145 (6), 2380–2411.
Fujinaka, Y., 2006. On the marginality principle in partition function form games. Mimeo.
Grabisch, M., Funaki, Y., 2012. A coalition formation value for games in partition function form. Eur. J. Oper. Res. 221 (1), 175–185.
Herings, P.J.J., Predtetchinski, A., Perea, A., 2006. The weak sequential core for two-period economies. Int. J. Game Theory 34 (1), 55–65.
Kalai, E., Samet, D., 1987. On weighted Shapley values. Int. J. Game Theory 16, 205–222.
Kóczy, L.Á., 2018. Partition function form games. Theory Decis. Libr., C 48, 312.
Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D., 2007. Sharing the surplus: an extension of the Shapley value for environments with externalities. J. Econ. Theory 135 (1), 339–356.
Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D., 2019. Extensions of the Shapley value for environments with externalities. In: Handbook of the Shapley Value. Chapman and Hall/CRC, pp. 131–155.

McQuillin, B., 2009. The extended and generalized Shapley value: simultaneous consideration of coalitional externalities and coalitional structure. J. Econ. Theory 144 (2), 696–721.
Myerson, R., 1977. Values of games in partition function form. Int. J. Game Theory 6, 23–31.
Pham Do, K.H., Norde, H., 2007. The Shapley value for partition function form games. Int. Game Theory Rev. 9 (2), 353–360.
Predtetchinski, A., Herings, P.J.J., Peters, H., 2002. The strong sequential core for two-period economies. J. Math. Econ. 38 (4), 465–482.
Rosenthal, E.C., 1990. Monotonicity of the core and value in dynamic cooperative games. Int. J. Game Theory 19 (1), 45–57.
Sánchez-Pérez, J., 2015. A note on a class of solutions for games with externalities generalizing the Shapley value. Int. Game Theory Rev. 17 (3), 1550003.
Shapley, L.S., 1953a. Additive and non-additive set functions. PhD Thesis. Department of Mathematics, Princeton University.
Shapley, L.S., 1953b. A value for n-person games. Ann. Math. Study 28, 307–317.
Skibski, O., Michalak, T.P., Wooldridge, M., 2018. The stochastic Shapley value for coalitional games with externalities. Games Econ. Behav. 108, 65–80.
Steinmann, S., Winkler, R., 2019. Sharing a river with downstream externalities. Games 10 (2), 23.
Thrall, R.M., Lucas, W.F., 1963. N-person games in partition function form. Nav. Res. Logist. Q. 10 (1), 281–298.
Van den Brink, R., Gilles, R.P., 1996. Axiomatizations of the conjunctive permission value for games with permission structures. Games Econ. Behav. 12 (1), 113–126.
Van den Brink, R., Van der Laan, G., Moes, N., 2012. Fair agreements for sharing international rivers with multiple springs and externalities. J. Environ. Econ. Manag. 63 (3), 388–403.