Statistics and Probability Letters 235 (2026) 110714

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Kac–Stroock type approximations for the Brownian motion from renewal processes ☆

Xavier Bardina *, Salim Boukfal

Departament de Matemàtiques, Edifici C, Universitat Autònoma de Barcelona, Bellaterra, 08193, Barcelona, Spain

ARTICLE INFO

MSC:
60F05
60F17
60G50
60K05

Keywords:
Brownian motion
Renewal process
Weak convergence
Kac–Stroock approximations

ABSTRACT

In the present paper we show that the processes $X_n = \{X_n(t) : t \in [0,1]\}$, $n \in \mathbb{N}$, defined by $X_n(t) = \sqrt{n}C\int_0^t (-1)^{L(nu)}du$, where $L = \{L(t) : t \geq 0\}$ is a renewal process whose inter-arrival times satisfy some integrability conditions and $C > 0$ is some normalizing constant, weakly converge, in the space of continuous functions over $[0,1]$, $C([0,1])$, to the Brownian motion as n approaches infinity. This result thus generalizes the well-known result of D. W. Stroock (1982), where L is taken to be a standard Poisson process. In particular, we see that these results are a mere consequence of Donsker's invariance principle.

1. Introduction

It is well known (see Stroock and Karmakar (1982)) that, if $N = \{N(t) : t \geq 0\}$ is a standard Poisson process, the processes X_n defined by

$$X_n(t) = \sqrt{n}\int_0^t (-1)^{N(nu)}du, \quad t \in [0,1], \tag{1}$$

which were introduced by M. Kac in Kac (1974) in order to solve the telegrapher's equation, weakly converge, in the space of continuous functions over $[0,1]$ (from now on, $\mathcal{C}$) towards the standard Brownian motion. Since then, several generalizations of this result have been seen. For instance:

1. By noting that $-1 = e^{i\theta}$ with $\theta = \pi$ and consider any other angle $\theta \in (0,\pi) \cup (\pi, 2\pi)$, showing convergence towards the Brownian sheet (see Bardina (2001)).
2. Replacing the Poisson process N for any other Lévy process, showing convergence towards the Brownian sheet (see Bardina and Rovira (2016)).
3. By considering the multiparameter analogs of the Poisson process and, more generally, of the Lévy Process and showing that the corresponding processes converge towards a Brownian sheet or a complex Brownian sheet (see Bardina et al. (2020, 2000)).

☆ Supported by the grant PID2021-123733NB-I00 funded by MCIN/AEI/10.13039/501100011033. Salim Boukfal is also supported by the program of predoctoral grants AGAUR-FI (2025 FI-1 00119). Joan Oró from the Department of Research and Universities of the Government of Catalonia and the co-funding of the European Social Fund Plus (ESF+).

* Corresponding author.
E-mail addresses: xavier.bardina@uab.cat (X. Bardina), salim.boukfal@uab.cat (S. Boukfal).

https://doi.org/10.1016/j.spl.2026.110714
Received 24 November 2025; Received in revised form 4 March 2026; Accepted 4 March 2026
Available online 11 March 2026

4. By considering integrals of functions with respect to such processes and showing convergence towards the stochastic integrals with respect to the Brownian motion or Brownian sheet (see Delgado and Jolis (2000), Bardina et al. (2003), Bardina and Boukfal (2026, 2025)).
5. By modifying the Poisson process or by considering a renewal–reward process (with a very specific reward) in such a way that the obtained convergence is in the strong sense (see Griego et al. (1971), Bardina and Rovira (2023)).

In order to show the weak convergence in the first four extensions, the authors show that the sequence of stochastic processes is tight and that the finite dimensional distributions, or any weak limit of any subsequence, converge as desired.

As for the strong convergence in the fifth point, the authors show that, via Skorokhod's embedding theorem, there are some realizations of the processes converging almost surely towards the Brownian motion.

Inspired by the results exhibited in Bardina and Rovira (2023), in this paper we provide a further extension of the original theorem where the Poisson process N is replaced by a general renewal process whose inter-arrival times verify some integrability condition. Hence, providing a larger class of processes which, unlike Brownian motion, are suitable for modeling transport and diffusion-like processes whose velocity of propagation is finite. Moreover, as done in Kac (1974) and Masoliver et al. (1989), these processes can be used to solve some PDEs by considering expected values of functionals of these.

As the proof of the main result of this paper shows, the obtained convergence (and thus, the convergence when a Poisson process is considered) can be seen as a mere consequence of Donsker's invariance principle since, as we will see, these processes are closely related to the random walk.

2. Statement and proof of the main result

Let $\{U_k : k \in \mathbb{N}\}$ be a sequence of i.i.d. non-negative random variables such that $\mathbb{P}\{U_1 = 0\} < 1$, $\mathrm{Var}(U_1) > 0$ and $\mathbb{E}[U_1^p] < \infty$ for some $p > 2$. Define $S_0 = 0$ and, for $n \in \mathbb{N}$ and $t \geq 0$,

$$S_n = \sum_{k=1}^{n} U_k, \quad L(t) = \sum_{n=1}^{\infty} \mathbb{I}_{[0,t]}(S_n),$$

where $\mathbb{I}_A$ is the indicator of the set A (that is, $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise). $L = \{L(t) : t \geq 0\}$ is usually called a renewal process with arrival times $\{S_n : n \in \mathbb{N}\}$ or, equivalently, with inter-arrival times $U_n = S_n - S_{n-1}$, $n \in \mathbb{N}$.

The main result of this paper reads as follows.

Theorem 2.1. *The processes $X_n = \{X_n(t) : t \in [0,1]\}$, $n \in \mathbb{N}$, defined by*

$$X_n(t) = C\sqrt{n} \int_0^t (-1)^{L(nu)} du, \tag{2}$$

where $C^2 = \mathbb{E}[U_1]/Var(U_1)$, weakly converge, in $\mathcal{C}$, towards a standard Brownian motion as n approaches infinity.

Remark 2.1. The hypotheses required for the proof of Theorem 2.1 are less restrictive than the ones seen in Bardina and Rovira (2023) since these are not enough (not a priori, at least) to show the strong convergence of these processes (or a slight modification of them by considering some specific reward) towards the Brownian motion.

Indeed, we start by noticing that, when $\mathbb{P}\{U_1 = 0\} = 0$ (which is the case of the Poisson process, where U_1 is taken to be exponentially distributed), the renewal process L almost surely vanishes in a neighborhood of 0, meaning that the sample paths of the processes X_n always start increasing. In order to avoid this, in Bardina and Rovira (2023) the authors introduce a sequence of i.i.d. Bernoulli random variables $\{\eta_k : k \in \mathbb{N} \cup 0\}$ of parameter $1/2$ and define the renewal–reward process

$$T(t) = \sum_{k=0}^{L(t)} \eta_k,$$

which can take the value 1 at $t = 0$ implying that the processes

$$\tilde{X}_n(t) = \frac{C}{\sqrt{\beta(n)}} \int_0^t (-1)^{T(u/\beta(n))} du, \quad t \in [0,1],$$

where $\beta : \mathbb{N} \to \mathbb{R}$ is some strictly positive sequence (in our case, $\beta(n) = 1/n$) converging to 0 and C is some normalizing constant different from the one in (2), need not be increasing at $t = 0$.

In addition, one can see that the strong convergence displayed in Griego et al. (1971), Bardina and Rovira (2023) is a consequence of the Borel–Cantelli lemma, for which one needs to impose further conditions on the sequence $\{\beta(n) : n \in \mathbb{N}\}$. More precisely, one requires this sequence to be summable, this in particular implies that, contrary to what we will be seeing, the results seen in Bardina and Rovira (2023) need not be valid, for instance, for $\beta(n) = 1/n$ or for some general $\beta(n)$ approaching 0 from above. Furthermore, in order to apply Borel–Cantelli's lemma, the authors also need to make use of some of the bounds provided by the Skorokhod embedding theorem, for which one requires the random variables U_k to have fourth order moments, whilst we only need them to have moments of order $p > 2$.

Proof of Theorem 2.1. Without any loss of generality, we shall assume that $\mu = \mathbb{E}[U_1] > 1$, the general case is a matter of scaling the random variables U_k. Indeed, let us assume that we have shown the result for $\mu > 1$, and consider a general sequence $\{U_k : k \in \mathbb{N}\}$ verifying the hypotheses for which the theorem holds. For any $\rho \in (0,1)$, the variables $\tilde{U}_k = U_k/(\rho\mu)$ have mean $1/\rho > 1$, so that the theorem can be applied to the renewal process

$$\tilde{L}(t) = \sum_{k=1}^{\infty} \mathbb{I}_{[0,t]}(\tilde{U}_1 + \cdots + \tilde{U}_k) = L(t\rho\mu).$$

Hence, the processes

$$\tilde{X}_n(t) = \sqrt{\frac{n\mathbb{E}[\tilde{U}_1]}{\mathrm{Var}(\tilde{U}_1)}} \int_0^t (-1)^{\tilde{L}(nu)} du = \sqrt{\frac{n}{\rho \mathrm{Var}(U_1)}} \int_0^{t\rho\mu} (-1)^{L(nu)} du, \quad t \in [0,1],$$

will converge to a standard Brownian motion, meaning that

$$X_n(t) = \sqrt{\frac{n\mathbb{E}[U_1]}{\mathrm{Var}(U_1)}} \int_0^t (-1)^{L(nu)} du = \sqrt{\rho\mu}\tilde{X}_n\left(\frac{t}{\rho\mu}\right), \quad t \in [0,1],$$

will converge towards a standard Brownian motion as well.

In order to prove the result for $\mu > 1$, we start by noticing that X_n can be written as

$$X_n(t) = \frac{C}{\sqrt{n}} \sum_{j=1}^{L(nt)} (-1)^{j-1} U_j + \frac{C}{\sqrt{n}} (-1)^{L(nt)} \left(nt - S_{L(nt)}\right). \tag{3}$$

We will show that

$$W_n(t) = \frac{C}{\sqrt{n}} \sum_{j=1}^{L(nt)} (-1)^{j-1} U_j$$

and

$$R_n(t) = \frac{C}{\sqrt{n}} (-1)^{L(nt)} \left(nt - S_{L(nt)}\right)$$

converge, respectively, towards a Brownian motion and the null process in the space of càdlàg functions over $[0,1]$ (which we denote by $\mathcal{D}$) with respect to the Skorokhod topology (unless stated otherwise, weak convergence in $\mathcal{D}$ will mean convergence with respect to this topology). If we manage to show this, since R_n will converge to 0, we will have that X_n will converge towards the Brownian motion in $\mathcal{D}$. Since X_n has continuous paths and the convergence in $\mathcal{D}$ relativized to $\mathcal{C}$ coincides with the weak convergence in $\mathcal{C}$ (with the uniform topology), we will obtain the weak convergence in $\mathcal{C}$ of the processes X_n.

Now, in order to see that R_n converges to 0 in $\mathcal{D}$, we first note that, for any $t \geq 0$ and $n \in \mathbb{N}$, $S_{L(nt)} \leq nt < S_{L(nt)+1}$, so that

$$|R_n(t)| \leq \tilde{R}_n(t) := \frac{C}{\sqrt{n}} U_{L(nt)+1}.$$

In particular, $\sup_t |R_n(t)| \leq \sup_t \tilde{R}_n(t)$. Observe that if we show that $\tilde{R}_n$ goes to 0 in $\mathcal{D}$, then so will R_n. Indeed, recall that the map $\mathcal{D} \ni x \mapsto \sup_t |x(t)|$ is continuous with respect to the Skorokhod topology, meaning that, by the continuous mapping theorem, $\sup_t \tilde{R}_n(t)$ will weakly converge (in $\mathbb{R}$) towards 0 as well and thus, since 0 is a constant, we will have convergence in probability, giving us that, for any $\varepsilon > 0$

$$\mathbb{P}\left\{\sup_{0\leq t\leq 1} |R_n(t)| \geq \varepsilon\right\} \leq \mathbb{P}\left\{\sup_{0\leq t\leq 1} \tilde{R}_n(t) \geq \varepsilon\right\} \xrightarrow{n\to\infty} 0.$$

That is, R_n will converge towards 0 in probability and in the space $\mathcal{D}$ with the uniform topology, which, in turn, will imply convergence in law in the same space with the same topology. However, in $\mathcal{D}$, the topology induced by the uniform metric is finer than the topology induced by the Skorokhod metric (see, for instance, Billingsley (1968), pp. 150–151), implying that R_n will converge to 0 in $\mathcal{D}$ with the Skorokhod topology.

All in all, what we have seen is that it suffices to show that $\tilde{R}_n$ converges to 0 as a sequence of processes in $\mathcal{D}$ (with the uniform or the Skorokhod topology).

To do so, consider the processes $\{D_n : n \in \mathbb{N}\}$ defined by

$$D_n(t) = \frac{C}{\sqrt{n}} U_{[nt]+1}$$

and note that

$$\sup_{0\leq t\leq 1} D_n = \frac{C}{\sqrt{n}} \max_{1\leq i\leq n+1} U_i.$$

Next, by Jensen's inequality,

$$\mathbb{E}\left[\max_{1\leq i\leq n+1} U_i\right] = \mathbb{E}\left[\left(\max_{1\leq i\leq n+1} U_i^p\right)^{\frac{1}{p}}\right] \leq \left(\mathbb{E}\left[\max_{1\leq i\leq n+1} U_i^p\right]\right)^{\frac{1}{p}}$$

$$\leq \left(\mathbb{E}\left[\sum_{i=1}^{n+1} U_i^p \right] \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n+1} \mathbb{E}\left[U_i^p \right] \right)^{\frac{1}{p}} = (n+1)^{\frac{1}{p}} \left(\mathbb{E}[U_1^p] \right)^{\frac{1}{p}},$$

where, in the last step, we have used that the random variables U_k are identically distributed. Since $p > 2$, this means that D_n converges in L^1 to 0 in $\mathcal{D}$ with the uniform topology. In particular, it will weakly converge in the same space with the same topology.

Now let us define

$$\Phi_n(t) = \begin{cases} \frac{L(nt)}{n}, & \text{if } \frac{L(n)}{n} \leq 1, \\ \frac{t}{\mu}, & \text{otherwise.} \end{cases} \tag{4}$$

As a consequence of the functional law of large numbers for the renewal process L, it can be shown, (see the proof of Theorem 17.3 in Billingsley (1968), p. 149), that

$$\sup_{0 \leq t \leq 1} \left| \frac{L(nt)}{n} - \frac{t}{\mu} \right| \xrightarrow{n\to\infty} 0$$

in probability and therefore, that Φ_n weakly converges in $\mathcal{D}_0$ to ϕ, where $\phi(t) = t/\mu$ and where $\mathcal{D}_0 \subset \mathcal{D}$ is the set of non-decreasing càdlàg functions whose image is contained in $[0,1]$ (in other words, the càdlàg changes of time of the unit interval) with the subspace topology. Under the event $L(n)/n \leq 1$ (the probability of which goes to 1), we have that $\tilde{R}_n = D_n \circ \Phi_n$. Thus, both $\tilde{R}_n$ and $D_n \circ \Phi_n$ have the same limit. Since the random time change mapping $\mathcal{D} \times \mathcal{D}_0 \ni (X, \Phi) \mapsto X \circ \Phi$ is continuous (for more details, see Billingsley (1968, p. 145)) and, by Slutsky's theorem, (D_n, Φ_n) converges in distribution towards $(0, \phi)$, $D_n \circ \Phi_n$ will converge in $\mathcal{D}$ to $0 \circ \phi = 0$, showing that $\tilde{R}_n$ converges to 0 in $\mathcal{D}$ and that so does R_n.

The only thing left to do is to show that W_n converges to a standard Brownian motion. Again, this will be a consequence of the convergence in $\mathcal{D}$ of

$$\tilde{W}_n(t) = \frac{C}{\sqrt{n}} \sum_{j=1}^{[nt]} (-1)^{j-1} U_j$$

towards the Brownian motion alongside the fact that W_n and $\tilde{W}_n \circ \Phi_n$ (where Φ_n the same time change in (4)) have the same limit.

Even though the random variables $(-1)^{j-1} U_j$ are independent, they are not, in general, identically distributed, so that we cannot apply directly the invariance principle. Nevertheless, observe that

$$\tilde{W}_n(t) = \frac{C}{\sqrt{n}} \sum_{j=1}^{\left[\frac{[nt]}{2}\right]} \left(U_{2j-1} - U_{2j} \right) + C \frac{1 - (-1)^{[nt]}}{2\sqrt{n}} U_{[nt]},$$

where the first sum is null if $[nt] < 2$. A similar reasoning to the one used to show that D_n converges to 0 shows that the second term in the latter expression for $\tilde{W}_n$ converges to 0 in $\mathcal{D}$ as well. Thus, it remains to show that the processes $\tilde{B}_n$ defined by

$$\tilde{B}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\left[\frac{[nt]}{2}\right]} \left(U_{2j-1} - U_{2j} \right), \quad t \in [0,1],$$

converge towards a Brownian motion. But the latter is an (almost) immediate consequence of Donsker's invariance principle. Indeed, we first note that the random variables $\{U_{2j-1} - U_{2j} : j \in \mathbb{N}\}$ are i.i.d. $U_1 - U_2$ and that

$$\mathbb{E}[U_1 - U_2] = 0, \quad \mathbb{E}\left[\left(U_1 - U_2 \right)^2 \right] = 2\mathrm{Var}(U_1).$$

Hence, the random walks

$$B_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (U_{2j-1} - U_{2j})$$

will weakly converge in $\mathcal{D}$ towards a Brownian motion with variance $2\mathrm{Var}(U_1)$. However, $\tilde{B}_n = B_n \circ \Psi_n$, where $\Psi_n(t) = \frac{[nt]}{2n}$ verifies

$$\sup_{0 \leq t \leq 1} \left| \Psi_n(t) - \psi(t) \right| \leq \frac{1}{2n} \xrightarrow{n\to\infty} 0,$$

where $\psi(t) = t/2$. Therefore, by the same results in Billingsley (1968) previously employed and by Slutsky's theorem, $\tilde{B}_n$ weakly converges in $\mathcal{D}$ towards a Brownian motion with variance $\mathrm{Var}(U_1)$ and hence, $\tilde{W}_n$ will converge to a Brownian motion with variance $C^2\mathrm{Var}(U_1)$. Since W_n has the same limit as $\tilde{W}_n \circ \Phi_n$ and the latter converges towards a Brownian motion with variance $C^2\mathrm{Var}(U_1)/\mu$, which by the definition of C, yields unit diffusion coefficient and hence, we obtain the convergence in distribution of W_n in $\mathcal{D}$ towards a standard Brownian motion and, all in all, that X_n weakly converges, in $\mathcal{C}$ with the uniform topology, to the very same process, as was to be shown. □

Acknowledgments

The authors are grateful to the anonymous referees for their very helpful comments and suggestions leading to an improvement of the quality and presentation of the original manuscript.

Data availability

No data was used for the research described in the article.

References

Bardina, X., 2001. The complex Brownian motion as a weak limit of processes constructed from a Poisson process. In: Decreusefond, L., Øksendal, B.K., Üstünel, A.S. (Eds.), Stochastic Analysis and Related Topics VII. Birkhäuser Boston, Boston, MA, pp. 149–158.

Bardina, X., Boukfal, S., 2025. Weak convergence of stochastic integrals with applications to SPDEs. arXiv:2504.08317. URL: https://arxiv.org/abs/2504.08317.

Bardina, X., Boukfal, S., 2026. On the weak convergence of multiparameter stochastic integrals. Open Math. http://dx.doi.org/10.1515/math-2025-0234, (In press).

Bardina, X., Jolis, M., Rovira, C., 2000. Weak approximation of the Wiener process from a Poisson process: the multidimensional parameter set case. Statist. Probab. Lett. 50 (3), 245–255. http://dx.doi.org/10.1016/S0167-7152(00)00113-9, URL: https://www.sciencedirect.com/science/article/pii/S0167715200001139.

Bardina, X., Jolis, M., Tudor, C.A., 2003. Weak convergence to the fractional Brownian sheet and other two-parameter Gaussian processes. Statist. Probab. Lett. 65 (4), 317–329. http://dx.doi.org/10.1016/j.spl.2003.09.001, URL: https://www.sciencedirect.com/science/article/pii/S0167715203002578.

Bardina, X., Márquez, J.P., Quer-Sardanyons, L., 2020. Weak approximation of the complex Brownian sheet from a Lévy sheet and applications to SPDEs. Stochastic Process. Appl. 130 (9), 5735–5767. http://dx.doi.org/10.1016/j.spa.2020.04.006, URL: https://www.sciencedirect.com/science/article/pii/S0304414919304478.

Bardina, X., Rovira, C., 2016. Approximations of a complex Brownian motion by processes constructed from a Lévy process. Mediterr. J. Math. 13 (1), 469–482. http://dx.doi.org/10.1007/s00009-014-0472-4.

Bardina, X., Rovira, C., 2023. Strong limit of processes constructed from a renewal process. Open Math. 21 (1), 20230112. http://dx.doi.org/10.1515/math-2023-0112.

Billingsley, P., 1968. Convergence of Probability Measures. In: Wiley series in probability and mathematical statistics, Wiley, New York.

Delgado, R., Jolis, M., 2000. Weak approximation for a class of Gaussian processes. J. Appl. Probab. 37 (2), 400–407, URL: http://www.jstor.org/stable/3215716.

Griego, R.J., Heath, D., Ruiz-Moncayo, A., 1971. Almost Sure Convergence of Uniform Transport Processes to Brownian Motion. Ann. Math. Stat. 42 (3), 1129–1131. http://dx.doi.org/10.1214/aoms/1177693346.

Kac, M., 1974. A stochastic model related to the telegrapher's equation. Rocky Mountain J. Math. 4 (3), 497–510. http://dx.doi.org/10.1216/RMJ-1974-4-3-497.

Masoliver, J., Lindenberg, K., Weiss, G.H., 1989. A continuous-time generalization of the persistent random walk. Phys. A 157 (2), 891–898. http://dx.doi.org/10.1016/0378-4371(89)90071-X, URL: https://www.sciencedirect.com/science/article/pii/037843718990071X.

Stroock, D.W., Karmakar, S., 1982. Lectures on Topics in Stochastic Differential Equations, vol. 68, Springer Berlin.