Research Article

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On the weak convergence of multiparameter stochastic integrals

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Abstract: In this paper we provide sufficient conditions for sequences of stochastic processes of the form $\int_{[0,t]} f_n(u)\theta_n(u)\mathrm{d}u$, to weakly converge, in the space of continuous functions over a closed interval, to integrals with respect to the Brownian motion, $\int_{[0,t]} f(u)W(\mathrm{d}u)$, where $\{f_n\}_n$ is a sequence of functions converging to f which verify some integrability conditions and $\{\theta_n\}_n$ is a sequence of stochastic processes whose integrals $\int_{[0,t]}\theta_n(u)\mathrm{d}u$ converge in law to the Brownian motion (in the sense of the finite dimensional distribution convergence), in the multiparameter case.

Keywords: Brownian sheet; random walk; Poisson process; Kac-Stroock kernels; Donsker kernels

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1 Introduction

In the literature one can find several examples of processes that approximate the Brownian motion (when dealing with one parameter processes) or the Brownian sheet (when dealing with several parameters). Among these examples, there are very well-known processes like the random walk or the Kac-Stroock process (see, for instance, Corollary 1 in refs. [1] or [2]), which have the characteristic that they are processes of bounded variation and thus, it makes sense to talk about integrals with respect to such processes in a pathwise sense by using the usual Lebesgue-Stieltjes theory of integration. This naturally raises the question of whether these integrals approximate the stochastic integral with respect to the Brownian motion/sheet defined in the Itô sense (for the one parameter case) or as in (2.1) (for the multidimensional parameter set case).

More precisely, for fixed $T > 0$, let $\{\zeta_n\}_{n\in\mathbb{N}}$, $\zeta_n = \{\zeta_n(t) : t \in [0,T]\}$, be a sequence of continuous processes weakly converging to a Brownian motion $W = \{W(t) : t \in [0,T]\}$ in the space of continuous functions over $[0,T]$, $C([0,T])$, and whose sample paths are of bounded variation and let $\{Y_n\}_{n\in\mathbb{N}}$, $Y_n = \{Y_n(t) : t \in [0,T]\}$, be a sequence of processes converging in some sense to another process $Y = \{Y(t) : t \in [0,T]\}$. Is it then true that the processes

$$X_n := \left\{ X_n(t) = \int_0^t Y_n(u)\mathrm{d}\zeta_n(u) : t \in [0,T] \right\},$$

converge (in some sense) to

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$$X := \left\{ X(t) = \int_0^t Y(u)\mathrm{d}W(u) : t \in [0,T] \right\}$$

as n approaches infinity?

This problem has already been addressed (in the one-parameter case) in, for instance [3], and [4] by considering càdlàg processes Y_n, Y and with $\{\zeta_n\}_{n\in\mathbb{N}}$ a sequence of càdlàg semimartingales such that the joint law of (Y_n, ζ_n) weakly converges, in the space of càdlàg functions, to (Y, ζ), where $\zeta = \{\zeta(t) : t \in [0,T]\}$ is some càdlàg process (ultimately, a semimartingale) for which the stochastic integral of Y with respect to ζ is well defined.

One can find some results in this direction as well, when the parameter set is of dimension 2, in ref. [5], where the problem of convergence of stochastic integrals is used to give approximations to solutions of the stochastic heat equation when the driving noise is approximated in distribution.

Mostly motivated by the results seen in the latter, the purpose of this paper will be to provide similar results in the multidimensional parameter set case by considering the random walk and the Kac-Stroock processes as approximating sequences, for which we have explicit expressions of ζ_n, and by replacing the processes Y_n, $n \in \mathbb{N}$, by functions satisfying some integrability condition, which we shall denote by f_n, converging to some other function f.

The article is organized in the following way, in Section 2 we introduce the involved processes and some preliminary results. In Section 3 we state and prove the main result. Finally, subsections 3.1 and 3.2 provide a couple of families of processes verifying the hypotheses of the main result.

2 Preliminaries

In this section we shall provide the main definitions and tools we will be working with.

Let $d \geq 1$ and consider $D = [0,T] = \prod_{i=1}^d [0,T_i] \subset \mathbb{R}_+^d$, $T = (T_1, \dots, T_d) \in \mathbb{R}_+^d$, with the usual partial order (total in the case $d = 1$). For $s = (s_1, \dots, s_d), t = (t_1, \dots, t_d) \in \mathbb{R}^d$, $s < t$, we write $(s,t] = \prod_{i=1}^d (s_i, t_i]$ (and analogously for open and closed rectangles) and denote by $\Delta_s X(t)$ the increment of the process over the rectangle $(s,t]$.

Let $(\Omega, \mathscr{F}, Q)$ be a complete probability space and let $\{\mathscr{F}_t : t \in D\}$ be a family of sub-σ-fields of $\mathscr{F}$ such that $\mathscr{F}_s \subset \mathscr{F}_t$ if $s \leq t$. For fixed $t \in D$, we also define $\mathscr{F}_t^T = \bigvee_{i=1}^d \mathscr{F}_{T_1,\dots,T_{i-1},t_i,T_{i+1},\dots,T_d}$.

To define the Brownian sheet and the stochastic integral with respect to such process, we will make use of the isonormal Gaussian process over a real separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$.

Definition 2.1. We say that a stochastic process $W = \{W(h) : h \in H\}$ defined in a complete probability space is an isonormal Gaussian process if it is a centered Gaussian process with covariance function $\mathrm{Cov}(W(f), W(g)) = \langle f, g \rangle_H$ for all $f, g \in H$.

From now on, we shall take $H = L^2(D)$ with the usual inner product. A Brownian sheet (or d-parameter Wiener process) is then defined as the process $\tilde{W} = \{\tilde{W}(t) : t \in D\}$ with

$$\tilde{W}(t) = W\left(\mathbb{I}_{[0,t]}\right),$$

where $\mathbb{I}_A$ is the indicator function of the set $A \subset \mathbb{R}^d$.

For a given function $f \in L^2(D)$ and $t \in D$, we then define the Wiener integral of f with respect to the Brownian sheet over $[0,t]$ as $W(f\mathbb{I}_{[0,t]})$ and denote it by

$$W(f\mathbb{I}_{[0,t]}) = \int_{[0,t]} f(u)\tilde{W}(\mathrm{d}u). \tag{2.1}$$

To simplify the notation, we will write $\int_{[0,t]} f(u)W(\mathrm{d}u)$ instead of $\int_{[0,t]} f(u)\tilde{W}(\mathrm{d}u)$.

One can easily check (via Kolmogorov's continuity Theorem), that the Brownian sheet and the integral of an $L^2(D)$ function with respect to it have a continuous version, so, when talking about these objects, we will be talking about the continuous versions.

We now introduce the two approximating sequences for the Brownian sheet, the random walk and the Kac-Stroock process, which will be defined via the Donsker kernels and the Kac-Stroock kernels, respectively.

From now on, for simplicity, we will consider the case $T = (T_1, \dots, T_d) = (1, \dots, 1)$, so that $D = [0,1]^d$.

Definition 2.2. Let $\{Z_k : k \in \mathbb{N}^d\}$ be a sequence of independent and identically distributed centered random variables with unit variance, the Donsker kernels are the processes $\{\theta_n\}_{n\in\mathbb{N}}$ defined by

$$\theta_n(t) = n^{\frac{d}{2}} \sum_{k=(k_1,\dots,k_d)\in\mathbb{N}^d} Z_k \mathbb{I}_{[k-1,k)}(nt), \quad t \in D, \tag{2.2}$$

where $k-1 = (k_1 - 1, \dots, k_d - 1)$.

The approximating sequence of random walks is then defined as the sequence of processes

$$\zeta_n(t) = \int_{[0,t]} \theta_n(u)\mathrm{d}u = n^{-\frac{d}{2}} \sum_{k\le[nt]} Z_k + n^{-\frac{d}{2}} \sum_{k\in\mathbb{N}^d} Z_k \left(\int_{[0,nt]\setminus[0,[nt]]} \mathbb{I}_{[k-1,k)}(u)\mathrm{d}u \right), \tag{2.3}$$

where $[s] = ([s_1], \dots, [s_d])$ for $s \in \mathbb{R}^d_+$ and $[x]$ is the integer part of $x \in \mathbb{R}_+$. The reader might note that this is not exactly a random walk, but the multiparameter analogous to the corresponding linear interpolation in the one parameter case.

As stated in the introduction, in ref. [1], Corollary 1, it is shown that the sums $n^{-\frac{d}{2}}\sum_{k\le[nt]} Z_k$ weakly converge towards a Brownian sheet in the space of càdlàg functions as n approaches infinity. For the sake of completeness, we will show in the Appendix that the processes ζ_n, as defined in (2.3), converge towards the same process in the space of continuous functions over D, which we will denote by $C(D)$.

Remark 2.3. Note that, if $W = \{W(t) : t \in D\}$ is a Brownian sheet on $D = [0,1]^d$ and $T = (T_1, \dots, T_d) \in \mathbb{R}^d_+$ is fixed (T not necessarily being the point $(1, \dots, 1)$), then

$$\left(\prod_{i=1}^d T_i\right)^{\frac{1}{2}} W\Big(\frac{t}{T}\Big), \quad t \in [0,T],$$

where $\frac{t}{T} = (t_1/T_1, \dots, t_d/T_d)$, defines a Brownian sheet over $[0,T]$. Hence, the processes $\{\tilde{\zeta}_n : n \in \mathbb{N}\}$ defined by

$$\tilde{\zeta}_n(t) = \left(\prod_{i=1}^d T_i\right)^{\frac{1}{2}} \zeta_n\Big(\frac{t}{T}\Big), \quad t \in [0,T]$$

constitute an approximating sequence for the Brownian sheet on $[0,T]$. A simple change of variables shows that

$$\tilde{\zeta}_n(t) = \int_{[0,t]} \left(\prod_{i=1}^d T_i\right)^{-\frac{1}{2}} \theta_n\Big(\frac{u}{T}\Big)\mathrm{d}u.$$

Hence, the Donsker kernels on $[0,T]$ are given by

$$\theta_n(t) = \prod_{i=1}^d \left(\frac{n}{T_i}\right)^{\frac{1}{2}} \sum_{k\in\mathbb{N}^d} Z_k \mathbb{I}_{[k-1,k)}\Big(n\frac{t}{T}\Big), \quad t \in [0,T].$$

Definition 2.4. A d-parameter càdlàg process $N_\mu = \{N_\mu(t) : t \geq 0\}$ is an $\mathscr{F}_t$-Poisson process with intensity $\mu > 0$ if it is null on the axes, adapted to $\{\mathscr{F}_t : t \geq 0\}$ and, for all $0 \leq s < t$, $\Delta_s N_\mu(t)$ is independent of $\mathscr{F}_s^T$ and has a Poisson law of parameter $\mu \prod_{i=1}^d (t_i - s_i)$.

If we do not specify the filtration, it will be the one generated by the process itself, completed with the necessary null sets.

The Poisson process defined above can be regarded as a Poisson random measure whose intensity measure is $\mu \cdot \lambda$ (being λ the Lebesgue measure on $\mathbb{R}_+^d$) restricted to rectangular sets of the form $(0, t]$. Hence, the existence of such processes can be seen as a consequence of the existence of Poisson random measures, which is proven, for instance, in ref. [6, Proposition 19.4].

Definition 2.5. The Kac-Stroock kernels are the processes $\{\theta_n : n \in \mathbb{N}\}$ defined by

$$\theta_n(t) = n^{\frac{d}{2}} \left(\prod_{i=1}^d t_i \right)^{\frac{d-1}{2}} (-1)^{N_n(t)}, \quad t = (t_1, \dots, t_d) \in D,$$

where N_n is a Poisson process of intensity n.

Similarly to the case of the random walks, the Kac-Stroock processes will be given by $\zeta_n(t) = \int_{[0,t]} \theta_n(u) \mathrm{d}u$. It is shown, in ref. [2], that these processes weakly converge, in the space $C(D)$, towards a Brownian sheet as n approaches infinity.

At this point, we observe that both approximating sequences treated in this paper (the random walk and the Kac-Stroock process), have a very specific form which allows us to easily formulate the multiparameter analogue of the weak convergence already studied in refs. [3] and [4] and mentioned in the introduction.

More precisely, the aim of this paper will be to establish the weak convergence of the processes

$$X_n := \left\{ X_n(t) = \int_{[0,t]} f_n(u) \theta_n(u) \mathrm{d}u : t \in D \right\} \tag{2.4}$$

towards the process

$$X := \left\{ X(t) = \int_{[0,t]} f(u) W(\mathrm{d}u) : t \in D \right\} \tag{2.5}$$

in the space $C(D)$ as n approaches infinity, where ζ_n weakly converges towards a Brownian sheet in $C(D)$ and $f_n \to f$ in $L^2(D)$.

Finally, we end this section by introducing the main tools used in order to prove the desired convergence in law. As it is customary, to prove such convergence, one needs to prove that the sequence of laws associated to the sequence of processes X_n is tight in $C(D)$ and that their finite dimensional distributions converge towards the ones of X. The first result (Theorem 2.6) is a tightness criteria which is an immediate consequence of Theorem 3 in ref. [7] and the discussion after its proof. The second one (Lemma 2.7) is a general result that will be used to prove the convergence of the finite dimensional distributions.

Theorem 2.6. *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of real valued continuous processes over D vanishing along the axes. Suppose that there exist $\beta > 1$, $\gamma > 0$ and finite nonnegative measures μ and $\{\mu_n\}_{n\in\mathbb{N}}$ on D with continuous marginals such that μ_n weakly converges to μ and, for each $0 \leq s \leq t \leq T$, $n \in \mathbb{N}$,*

$$\mathbb{E}\left[\left| \Delta_s Y_n(t) \right|^\gamma \right] \leq \left(\mu_n((s,t]) \right)^\beta.$$

Then the sequence of laws associated to the processes $\{Y_n\}_{n\in\mathbb{N}}$ is tight.

In the theorem above, continuity of the marginals of μ (and similarly, of those of μ_n) is understood as continuity of the map $t \mapsto \mu_p((0,t])$, where μ_p is the marginal of μ on the pth edge of D.

Lemma 2.7. *Let $(F, \|\cdot\|)$ be a normed vector space and $\{J^n\}_{n\in\mathbb{N}}$ and J be linear maps from F to $L^1(\Omega)$. Assume there exists a positive constant C such that, for any $f \in F$*

$$\sup_{n\geq 1} \mathbb{E}\big[|J^n(f)|\big] \leq C\|f\|, \quad \mathbb{E}\big[|J(f)|\big] \leq C\|f\|, \tag{2.6}$$

and that, for some dense subspace D of F, it holds that $J^n(f)$ converges in law to $J(f)$, as n tends to infinity, for all $f \in D$. Then, the sequence of random variables $\{J^n(f)\}_{n\in\mathbb{N}}$ converges in law to $J(f)$ for any $f \in F$.

If, in addition, we have $\{f_n\}_{n\in\mathbb{N}} \subset F$ and $f \in F$ such that $f_n \to f$ in $(F, \|\cdot\|)$ and

$$\mathbb{E}\big[|J^n(f_n - f)|\big] \leq C\|f_n - f\|,$$

where $C > 0$ is independent of n, $\{f_n\}_{n\in\mathbb{N}}$ and f, then the sequence $\{J^n(f_n)\}_{n\in\mathbb{N}}$ converges in law to $J(f)$.

Proof. For the first part, recall that a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ converges in law to a random variable X if, and only if, for any bounded Lipschitz function $g\colon \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\big[g(X_n)\big] \xrightarrow{n\to\infty} \mathbb{E}\big[g(X)\big],$$

(see, for instance, [8, Theorem 18.7]). Thus, we shall see that, for any $\varepsilon > 0$, there is $n \in \mathbb{N}$ large enough such that

$$\Big|\mathbb{E}\big[g\big(J^n(f)\big)\big] - \mathbb{E}\big[g\big(J(f)\big)\big]\Big| < \varepsilon, \tag{2.7}$$

where g is any Lipschitz function as before. Consider any $h \in D$ such that $\|f - h\| < \frac{\varepsilon}{3L_g C}$, where $L_g > 0$ is the Lipschitz constant of g, and apply the triangle inequality to obtain

$$\begin{aligned}\Big|\mathbb{E}\big[g\big(J^n(f)\big)\big] - \mathbb{E}\big[g\big(J(f)\big)\big]\Big| &\leq \Big|\mathbb{E}\big[g\big(J^n(f)\big)\big] - \mathbb{E}\big[g\big(J^n(h)\big)\big]\Big| + \Big|\mathbb{E}\big[g\big(J^n(h)\big)\big] - \mathbb{E}\big[g\big(J(h)\big)\big]\Big| \\ &\quad + \Big|\mathbb{E}\big[g\big(J(h)\big)\big] - \mathbb{E}\big[g\big(J(f)\big)\big]\Big|.\end{aligned}$$

Now observe that

$$\begin{aligned}\Big|\mathbb{E}\big[g\big(J^n(f)\big)\big] - \mathbb{E}\big[g\big(J^n(h)\big)\big]\Big| &\leq \mathbb{E}\Big[\Big|g\big(J^n(f)\big) - g\big(J^n(h)\big)\Big|\Big] \\ &\leq L_g \mathbb{E}\big[|J^n(f-h)|\big] \\ &\leq L_g C\|f - h\| < \frac{\varepsilon}{3}.\end{aligned}$$

Similarly,

$$\Big|\mathbb{E}\big[g\big(J(f)\big)\big] - \mathbb{E}\big[g\big(J(h)\big)\big]\Big| \leq L_g C\|f - h\| < \frac{\varepsilon}{3}.$$

Finally, given that $J^n(h)$ converges in law to $J(h)$ for $h \in D$, we have that, for n large enough,

$$\Big|\mathbb{E}\big[g\big(J^n(h)\big)\big] - \mathbb{E}\big[g\big(J(h)\big)\big]\Big| < \frac{\varepsilon}{3}.$$

Thus, for n large enough, we obtain (2.7) as desired.

For the second part, it suffices to show that the sequences $\{J^n(f_n)\}_{n\in\mathbb{N}}$ and $\{J^n(f)\}_{n\in\mathbb{N}}$ have the same limit in $L^1(\Omega)$ (and thus, the same limit in law) since, by the previous part, $J^n(f)$ converges in law to $J(f)$. But this is immediate from the additional hypothesis, since

$$\mathbb{E}\big[|J^n(f_n) - J^n(f)|\big] = \mathbb{E}\big[|J^n(f_n - f)|\big] \leq C\|f_n - f\| \xrightarrow{n\to\infty} 0. \qquad \square$$

3 Statement and proof of the main result

This section is devoted to proving the convergence in distribution of the processes X_n defined in (2.4) towards the process X defined by (2.5). As shown by the main result of this paper (Theorem 3.1), it turns out that this problem can be reduced to computing the moments of the increments over rectangles of the processes X_n.

Theorem 3.1. *Let $1 \leq q < \infty$ and suppose there is $m > 2q$ and a positive constant C independent of n such that*

$$\mathbb{E}\left[\left|\int_D g_n(u)\theta_n(u)\mathrm{d}u\right|^m\right] \leq C\left(\int_D |g_n(u)|^{2q}\mathrm{d}u\right)^{\frac{m}{2q}}, \tag{3.1}$$

for any sequence of functions $\{g_n\}_{n\in\mathbb{N}} \subset L^{2q}(D)$. Then, the processes X_n defined by (2.4) converge in law, as n approaches infinity, towards the process X defined by (2.5) in $C(D)$ whenever the sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^{2q}(D)$ converges in $L^{2q}(D)$ to $f \in L^{2q}(D)$.

Before we start with the proof, we shall make some remarks.

Remark 3.2. The introduction of the parameter q is due to the fact that, in the case of the Kac-Stroock kernels, when $d \geq 2$, we have only been able to check condition (3.1) for sequences in $L^{2q}(D)$ with $q > 1$.

Remark 3.3. As the proof shows, it turns out that, as long as condition (3.1) holds, it is only required that the finite dimensional distributions of the processes ζ_n converge to those of the Brownian sheet. There is no need for the convergence in law of the processes ζ_n to the Brownian sheet in the space $C(D)$ (this convergence will be a consequence of Theorem 3.1).

Remark 3.4. By inspecting the proofs of the results in Sections 3.1 and 3.2, one can see that, if m is an even integer and the Donsker or Kac-Stroock kernels are considered, condition (3.1) only needs to be checked for constant sequences of functions $\{g_n\}_{n\in\mathbb{N}} \subset L^{2q}(D)$ (that is, $g_n = g$ for some $g \in L^{2q}(D)$ and for all $n \in \mathbb{N}$), since the former is implied by the latter in this case.

Remark 3.5. One might note that, by taking $f_n = f \in L^{2q}(D)$ for all $n \in \mathbb{N}$, the integrals $\int f(u)\theta_n(u)\mathrm{d}u$ converge to the integral $\int f(u)W(\mathrm{d}u)$.

Remark 3.6. Given that $q \geq 1$ and $m > 2q$, condition (3.1) implies that, for each $0 \leq s \leq t \leq T$, the moments $\mathbb{E}\left[|\Delta_s X_n(t)|^r\right]$ are uniformly bounded in $n \in \mathbb{N}$ (because f_n converges in $L^{2q}(D)$, implying that the sequence of $L^{2q}(D)$-norms is bounded) for $r \leq m$. Thus, for each $0 \leq s \leq t \leq T$, $r < m$, the sequence $\{|\Delta_s X_n(t)|^r\}_{n\in\mathbb{N}}$ is uniformly integrable. On the other hand, since $|\Delta_s X_n(t)|^r$ is a continuous transformation of X_n and the latter converges in law towards X in $C(D)$, we have that $|\Delta_s X_n(t)|^r$ converges in law towards $|\Delta_s X(t)|^r$. This, alongside the uniform integrability of the sequence of random variables, implies the convergence of the expected values. That is,

$$\lim_{n\to\infty} \mathbb{E}\left[|\Delta_s X_n(t)|^r\right] = \mathbb{E}\left[|Z|^r\right]\left(\int_{(s,t]} f^2(u)\mathrm{d}u\right)^{\frac{r}{2}},$$

where Z is a standard Gaussian random variable (recall that $\Delta_s X(t)$ is normally distributed with zero mean and variance $\int_{(s,t]} f^2(u)\mathrm{d}u$).

In particular, for any even integer $0 \leq r < m$,

$$\lim_{n\to\infty} \mathbb{E}\left[(\Delta_s X_n(t))^r\right] = \frac{r!}{2^{\frac{r}{2}}\left(\frac{r}{2}\right)!}\left(\int_{[s,t]} f^2(u)\mathrm{d}u\right)^{\frac{r}{2}}$$

and $\lim_{n\to\infty}\mathbb{E}\left[(\Delta_s X_n(t))^r\right] = 0$ for any odd integer $0 < r < m$.

Proof of Theorem 3.1. As mentioned in the previous section, it suffices to show that the sequence is tight and that the finite dimensional distributions of X_n converge towards the ones of X.

Tightness

This is an immediate consequence of Theorem 2.6 by taking, up to some positive factor,

$$\mu_n(A) := \int_A |f_n(u)|^{2q}\mathrm{d}u, \quad \mu(A) := \int_A |f(u)|^{2q}\mathrm{d}u,$$

for all measurable sets $A \subset D$ and noticing that, for any $0 \leq s \leq t \leq T$,

$$\Delta_s X_n(t) = \int_{(s,t]} f_n(u)\theta_n(u)\mathrm{d}u = \int_D f_n(u)\mathbb{I}_{(s,t]}(u)\theta_n(u)\mathrm{d}u.$$

Indeed, μ_n, $n \in \mathbb{N}$, and μ defined as above are nonnegative finite measures with continuous marginals and, since $f_n \to f$ in $L^{2q}(D)$, we have that μ_n converges weakly towards μ. To see this, take any bounded continuous function $g : D \to \mathbb{R}$,

$$\left|\int_D g(u)\mu_n(\mathrm{d}u) - \int_D g(u)\mu(\mathrm{d}u)\right| \leq \sup_{u\in D}|g(u)| \int_D \left||f_n(u)|^{2q} - |f(u)|^{2q}\right|\mathrm{d}u.$$

If we show that this last integral converges to 0 as n approaches infinity then we are done. To show this, we will show that the functions $g_n(u) = \left||f_n(u)|^{2q} - |f(u)|^{2q}\right|$ converge to 0 in probability and that they are uniformly integrable (all with respect to the Lebesgue measure on $D = [0,1]^d$).

To see the convergence in probability, we first note that, since $f_n \to f$ in $L^{2q}(D)$, we have convergence in probability. Since convergence in probability is preserved by continuous transformations, we will have that g_n converges to 0 in probability. To see that the g_n are uniformly integrable, we first note that

$$|g_n(u)| \leq |f_n(u)|^{2q} + |f(u)|^{2q},$$

so it suffices to show that the $|f_n(u)|^{2q}$ are uniformly integrable, but this follows from the fact that the sequence converges in $L^{2q}(D)$.

Convergence of the finite dimensional distributions

To show that the finite dimensional distributions converge as desired, it suffices to show that, for any, $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{R}$ and any $t_1, \ldots, t_k \in D$, the sums $\sum_{j=1}^k a_j X_n(t_j)$ converge in law towards $\sum_{j=1}^k a_j X(t_j)$. In order to prove this, we first note that,

$$\sum_{j=1}^k a_j X_n(t_j) = J^n\left(f_n \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}\right), \quad \sum_{j=1}^k a_j X(t_j) = J\left(f \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}\right).$$

where J^n, J are the linear maps defined by

$$J^n(g) := \int_D g(u)\theta_n(u)\mathrm{d}u, \quad J(g) := \int_D g(u)W(\mathrm{d}u), \quad g \in L^2(D).$$

Since $f_n, f \in L^{2q}(D)$, the elements $f_n \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}$ and $f \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}$ are in the same space as well, so the maps can be defined in $(L^{2q}(D), \|\cdot\|_{2q})$, where $\|\cdot\|_{2q}$ is the standard norm in this space. By Hölder's inequality and (3.1), we have

$$\sup_{n\geq 1} \mathbb{E}\left[\left|J^n(g)\right|\right] \leq \sup_{n\geq 1} \left(\mathbb{E}\left[\left|J^n(g)\right|^m\right]\right)^{\frac{1}{m}} \leq C' \|g\|_{2q}.$$

Similarly, by Hölder's inequality again and the isometry property of the Wiener integral,

$$\mathbb{E}\left[|J(g)|\right] \leq \left(\mathbb{E}\left[|J(g)|^2\right]\right)^{\frac{1}{2}} = \|g\|_2 \leq C'' \|g\|_{2q},$$

for some positive constant C' which might be different from the one seen in the previous inequality. Hence, (2.6) is satisfied with $C = \max\{C', C''\}$. This in particular means that the maps J^n and J take values in $L^1(\Omega)$. Thus, it only remains to show that $J^n(g)$ converges in law towards $J(g)$ for simple functions g of the form

$$g(u) = \sum_{j=1}^{l} g_j \mathbb{I}_{(s_{j-1},s_j]}(u),$$

with $l \geq 1$, $g_j \in \mathbb{R}$ and $0 = s_0 < s_1 < \ldots < s_l = (1, \ldots, 1)$, which are dense in $L^{2q}(D)$. Indeed, if we manage to show this, then, by the first part of Lemma 2.7, we will have that $J^n\left(f \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}\right)$ converges in law towards $J\left(f \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}\right)$ and, by the second part and the fact that $f_n \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]} \to f \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}$ in $L^{2q}(D)$, we will have that $J^n\left(f_n \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}\right)$ will converge in law towards $J\left(f \sum_{j=1}^k a_j \mathbb{I}_{[0,t_j]}\right)$ as well.
To show the convergence in law for simple functions, we only need to notice that

$$J^n(g) = \int_D \left(\sum_{j=1}^{l} g_j \mathbb{I}_{(s_{j-1},s_j]}(u)\right) \theta_n(u) \mathrm{d}u = \sum_{j=1}^{l} g_j \int_{(s_{j-1},s_j]} \theta_n(u) \mathrm{d}u$$

and that this last sum converges in law towards

$$\sum_{j=1}^{l} g_j \int_{(s_{j-1},s_j]} W(\mathrm{d}u) = \int_D \left(\sum_{j=1}^{l} g_j \mathbb{I}_{(s_{j-1},s_j]}(u)\right) W(\mathrm{d}u) = J(g)$$

because the finite dimensional distributions of ζ_n (the Lebesgue integral of θ_n) converge to those of the Brownian sheet. □

3.1 Convergence for the Donsker kernels

This section, and the following one, is devoted to proving that condition (3.1) is satisfied for the Donsker kernels. This is the content of Proposition 3.7.

Proposition 3.7. *Let $\{\theta_n\}_{n\in\mathbb{N}}$ be the Donsker kernels defined in (2.2). Then, for any even integer $m \in \mathbb{N}$ and for any sequence of functions $\{g_n\}_{n\in\mathbb{N}} \subset L^2(D)$, if the random variables $\{Z_k\}_{k\in\mathbb{N}^d}$ have finite moments of order m, we have that*

$$\mathbb{E}\left[\left(\int_D g_n(u)\theta_n(u)\mathrm{d}u\right)^m\right] \leq C\left(\int_D g_n^2(u)\mathrm{d}u\right)^{\frac{m}{2}},$$

for some positive constant C independent of n and $\{g_n\}_{n\in\mathbb{N}}$.

Proof. We start by observing that

$$\mathbb{E}\left[\left(\int_D g_n(u)\theta_n(u)\mathrm{d}u\right)^m\right] = \int_{D^m}\left(\prod_{i=1}^m g_n(u^i)\right)\mathbb{E}\left[\prod_{i=1}^m \theta_n(u^i)\right]\mathrm{d}u^1\ldots\mathrm{d}u^m,$$

where $u^i = (u_1^i, \ldots, u_d^i) \in D$ and

$$\mathbb{E}\left[\prod_{i=1}^m \theta_n(u^i)\right] = n^{m\frac{d}{2}}\sum_{k^1,\ldots,k^m\in\mathbb{N}^d}\mathbb{E}\left[Z_{k^1}\cdot\ldots\cdot Z_{k^m}\right]\prod_{i=1}^m \mathbb{I}_{[k^i-1,k^i)}(nu^i).$$

Since the random variables $\{Z_k\}_k$ are independent and identically distributed with zero means, we have that

$$\mathbb{E}\left[Z_{k^1}\cdot\ldots\cdot Z_{k^m}\right] = 0$$

whenever there is some $j \in \{1,\ldots,m\}$ such that $k^j \neq k^i$ for all $i \in \{1,\ldots,m\}\backslash\{j\}$. Thus,

$$\mathbb{E}\left[\prod_{i=1}^m \theta_n(u^i)\right] = n^{m\frac{d}{2}}\sum_{(k^1,\ldots,k^m)\in A^m}\mathbb{E}\left[Z_{k^1}\cdot\ldots\cdot Z_{k^m}\right]\prod_{i=1}^m \mathbb{I}_{[k^i-1,k^i)}(nu^i),$$

where $A^m \subset \left(\mathbb{N}^d\right)^m$ is the set of points $(k^1,\ldots,k^d)$ such that for all $l \in \{1,\ldots,m\}$ there is some $j \in \{1,\ldots,m\}\backslash\{l\}$ such that $k^l = k^j$. Given that the random variables Z_k have finite moments of order m, we have that

$$\left|\mathbb{E}\left[\prod_{i=1}^m \theta_n(u^i)\right]\right| \leq Cn^{m\frac{d}{2}}\sum_{(k^1,\ldots,k^m)\in A^m}\prod_{i=1}^m \mathbb{I}_{[k^i-1,k^i)}(nu^i)$$

for some positive constant C independent of n. Now let us assume that in this last sum there is a non-zero summand. That is, there is some $(k^1,\ldots,k^m) \in A^m$ such that

$$\prod_{i=1}^m \mathbb{I}_{[k^i-1,k^i)}(nu^i) \neq 0,$$

which, in particular, implies that each factor in this product is non-zero and hence,

$$\mathbb{I}_{[k_j^i-1,k_j^i)}(nu_j^i) \neq 0$$

for all $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,d\}$ or, equivalently, $nu_j^i \in [k_j^i - 1, k_j^i)$ for all $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,d\}$. Given that we are in A^m, for each $i \in \{1,\ldots,m\}$ there will be some $l \in \{1,\ldots,m\}\backslash\{i\}$ such that $k^i = k^l = k = (k_1,\ldots,k_d)$. For this pair of indices, we will have $nu_j^i, nu_j^l \in [k_j - 1, k_j)$ for all $j \in \{1,\ldots,d\}$ and therefore, $|u_j^l - u_j^i| < \frac{1}{n}$, for all $j \in \{1,\ldots,d\}$. It can happen as well that there is some $r \in \{1,\ldots,m\}\backslash\{i,l\}$ for which $k^r = k^l = k^i$ and thus, verifying that $|u_j^l - u_j^r| < \frac{1}{n}$ and $|u_j^r - u_j^i| < \frac{1}{n}$ for all $j \in \{1,\ldots,d\}$ as well. Of course there might be cases where there are four or more variables nu^i whose components are not apart more than $\frac{1}{n}$, but this kind of situations can be reduced to the two previous ones. Bearing this in mind, we have

$$\sum_{(k^1,\ldots,k^m)\in A^m}\prod_{i=1}^m \mathbb{I}_{[k^i-1,k^i)}(nu^i) \leq \mathbb{I}_{B^m}(u_1^1, u_1^2, \ldots, u_1^m; \ldots; u_d^1, \ldots, u_d^m),$$

where B^m is the set of points

$$(u_1^1, u_1^2, \ldots, u_1^m; \ldots; u_d^1, \ldots, u_d^m) \in \prod_{j=1}^d (s_j, t_j]^m$$

such that for each $l \in \{1,\ldots,m\}$ there is some $j \in \{1,\ldots,m\}\backslash\{l\}$ verifying $|u_i^l - u_i^j| < \frac{1}{n}$ for every $i \in \{1,\ldots,d\}$ and that, if additionally, there is some $r \in \{1,\ldots,m\}\backslash\{j,l\}$ with $|u_i^r - u_i^l| < \frac{1}{n}$ for all $i \in \{1,\ldots,d\}$, then $|u_i^j - u_i^r| < \frac{1}{n}$ for all $i \in \{1,\ldots,d\}$.

However, the indicator $\mathbb{I}_{B^m}$ can be bounded by a finite sum (whose number of summands depends only on m) of products of indicators of the form

$$\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)$$

and

$$\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\cdot\mathbb{I}_{[0,n^{-1})^d}(|u_1^l-u_1^r|,\dots,|u_d^l-u_d^r|)\cdot\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^r|,\dots,|u_d^j-u_d^r|).$$

Moreover, in each of the products of indicators conforming each summand, all the variables u_i^j, $i\in\{1,\dots,d\}$, $j\in\{1,\dots,m\}$, appear in only one of the two types of indicators specified above. All in all, we have that $\left|\mathbb{E}\left[\left(\Delta_s X_n(t)\right)^m\right]\right|$ can be bounded (modulo some positive constant independent of n) by a finite sum of finite products of factors of the form

$$n^d\int\limits_{\prod_{i=1}^d[0,1]^2}|g_n(u^j)||g_n(u^l)|\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\mathrm{d}u_1^j\mathrm{d}u_1^l\dots\mathrm{d}u_d^j\mathrm{d}u_d^l,$$

and

$$n^{3\frac{d}{2}}\int\limits_{\prod_{i=1}^d[0,1]^3}|g_n(u^j)||g_n(u^l)||g_n(u^r)|\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)$$
$$\times\mathbb{I}_{[0,n^{-1})^d}(|u_1^l-u_1^r|,\dots,|u_d^l-u_d^r|)\cdot\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^r|,\dots,|u_d^j-u_d^r|)\mathrm{d}u_1^j\mathrm{d}u_1^l\mathrm{d}u_1^r\dots\mathrm{d}u_d^j\mathrm{d}u_d^l\mathrm{d}u_d^r.$$

Where the number of factors of each summand, say N, is such that

$$n^{\alpha_1\frac{d}{2}}\cdot\ldots\cdot n^{\alpha_N\frac{d}{2}}=n^{m\frac{d}{2}},\quad \alpha_i\in\{2,3\}.$$

So it only remains to show that the first kind of factors can be bounded, modulo some positive constant, by $\int_D g_n^2(u)\mathrm{d}u$ and that the last type of factors can be bounded by $\left(\int_D g_n^2(u)\mathrm{d}u\right)^{\frac{3}{2}}$.

Using that $2ab\le a^2+b^2$ with $a=|g_n(u^j)|$, $b=|g_n(u^l)|$, we have that

$$\begin{aligned}
&\int\limits_{\prod_{i=1}^d[0,1]^2}|g_n(u^j)||g_n(u^l)|\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\mathrm{d}u_1^j\mathrm{d}u_1^l\dots\mathrm{d}u_d^j\mathrm{d}u_d^l\\
&\quad\le\frac{1}{2}\int\limits_{\prod_{i=1}^d[0,1]^2}g_n^2(u^j)\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\mathrm{d}u_1^j\mathrm{d}u_1^l\dots\mathrm{d}u_d^j\mathrm{d}u_d^l\\
&\qquad+\frac{1}{2}\int\limits_{\prod_{i=1}^d[0,1]^2}g_n^2(u^l)\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\mathrm{d}u_1^j\mathrm{d}u_1^l\dots\mathrm{d}u_d^j\mathrm{d}u_d^l\\
&\quad=\int\limits_{\prod_{i=1}^d[0,1]^2}g_n^2(u^l)\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\mathrm{d}u_1^j\mathrm{d}u_1^l\dots\mathrm{d}u_d^j\mathrm{d}u_d^l\\
&\quad=\int_D g_n^2(u^l)\left(\int_D\mathbb{I}_{[0,n^{-1})^d}(|u_1^j-u_1^l|,\dots,|u_d^j-u_d^l|)\mathrm{d}u^j\right)\mathrm{d}u^l\\
&\quad\le n^{-d}\int_D g_n^2(u)\mathrm{d}u.
\end{aligned}\tag{3.2}$$

As for the last type of integrals, we will use the estimate $2abc \le ab^2 + ac^2$ with $a = |g_n(u^j)|$, $b = |g_n(u^l)|$ and $c = |g_n(u^r)|$, which leads to

$$
\begin{aligned}
&\int\limits_{\prod_{i=1}^{d}[0,1]^3} |g_n(u^j)||g_n(u^l)||g_n(u^r)| \mathbb{I}_{[0,n^{-1})^d}(|u_1^j - u_1^l|, \dots, |u_d^j - u_d^l|) \\
&\qquad \times \mathbb{I}_{[0,n^{-1})^d}(|u_1^l - u_1^r|, \dots, |u_d^l - u_d^r|) \cdot \mathbb{I}_{[0,n^{-1})^d}(|u_1^j - u_1^r|, \dots, |u_d^j - u_d^r|) \mathrm{d}u_1^j \mathrm{d}u_1^l \mathrm{d}u_1^r \dots \mathrm{d}u_d^j \mathrm{d}u_d^l \mathrm{d}u_d^r \\
&\le \int\limits_{\prod_{i=1}^{d}[0,1]^3} g_n^2(u^j)|g_n(u^l)| \left(\prod_{i=1}^{d} \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^l|) \mathbb{I}_{[0,n^{-1})}(|u_i^l - u_i^r|) \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^r|) \right) \\
&\qquad \times \mathrm{d}u_1^j \mathrm{d}u_1^l \mathrm{d}u_1^r \dots \mathrm{d}u_d^j \mathrm{d}u_d^l \mathrm{d}u_d^r \\
&\le \int\limits_{\prod_{i=1}^{d}[0,1]^3} g_n^2(u^j)|g_n(u^l)| \left(\prod_{i=1}^{d} \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^l|) \mathbb{I}_{[0,n^{-1})}(|u_i^l - u_i^r|) \right) \mathrm{d}u_1^j \mathrm{d}u_1^l \mathrm{d}u_1^r \dots \mathrm{d}u_d^j \mathrm{d}u_d^l \mathrm{d}u_d^r \\
&= \int\limits_{\prod_{i=1}^{d}[0,1]^2} g_n^2(u^j)|g_n(u^l)| \left(\prod_{i=1}^{d} \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^l|) \right) \left(\int_D \mathbb{I}_{[0,n^{-1})^d}(|u_1^l - u_1^r|, \dots, |u_d^l - u_d^r|) \mathrm{d}u^r \right) \\
&\qquad \times \mathrm{d}u_1^j \mathrm{d}u_1^l \dots \mathrm{d}u_d^j \mathrm{d}u_d^l \\
&\le n^{-d} \int_D |g_n(u^l)| \left(\int_D g_n^2(u^j) \mathbb{I}_{[0,n^{-1})^d}(|u_1^j - u_1^l|, \dots, |u_d^j - u_d^l|) \mathrm{d}u^j \right) \mathrm{d}u^l \\
&\le n^{-d} \left(\int_D g_n^2(u^l) \mathrm{d}u^l \right)^{\frac{1}{2}} (\Theta)^{\frac{1}{2}},
\end{aligned}
$$

where in the first inequality a single term appears (instead of the two we should expect from the inequality $2abc \le ab^2 + ac^2$) due to the fact that, similarly to (3.2), the resulting integrals are the same, in the last step we have used Cauchy-Schwarz's inequality and where

$$
\begin{aligned}
\Theta &= \int_D \left(\int_D g_n^2(u^j) \mathbb{I}_{[0,n^{-1})^d}(|u_1^j - u_1^l|, \dots, |u_d^j - u_d^l|) \mathrm{d}u^j \right)^2 \mathrm{d}u^l \\
&= \int\limits_{\prod_{i=1}^{d}[0,1]^3} g_n^2(u^j) g_n^2(u^p) \left(\prod_{i=1}^{d} \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^l|) \mathbb{I}_{[0,n^{-1})}(|u_i^p - u_i^l|) \right) \mathrm{d}u_1^j \mathrm{d}u_1^p \mathrm{d}u_1^l \dots \mathrm{d}u_d^j \mathrm{d}u_d^p \mathrm{d}u_d^l \\
&\le \int\limits_{\prod_{i=1}^{d}[0,1]^3} g_n^2(u^j) g_n^2(u^p) \left(\prod_{i=1}^{d} \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^l|) \right) \mathrm{d}u_1^j \mathrm{d}u_1^p \mathrm{d}u_1^l \dots \mathrm{d}u_d^j \mathrm{d}u_d^p \mathrm{d}u_d^l \\
&= \int\limits_{\prod_{i=1}^{d}[0,1]^2} g_n^2(u^j) g_n^2(u^p) \left(\prod_{i=1}^{d} \int_0^{T_i} \mathbb{I}_{[0,n^{-1})}(|u_i^j - u_i^l|) \mathrm{d}u_i^l \right) \mathrm{d}u_1^j \mathrm{d}u_1^p \dots \mathrm{d}u_d^j \mathrm{d}u_d^p
\end{aligned}
$$

$$\leq n^{-d} \int_{\prod_{i=1}^{d}[0,1]^2} g_n^2(u^j) g_n^2(u^p) \mathrm{d}u_1^j \mathrm{d}u_1^p \dots \mathrm{d}u_d^j \mathrm{d}u_d^p$$

$$= n^{-d} \left(\int_D g_n^2(u) \mathrm{d}u \right)^2.$$

Obtaining the desired result. □

Hence, in particular, condition (3.1) is satisfied with $q = 1$ and $m = 4$ (by requiring that the random variables Z_k have finite moments of order m).

3.2 Convergence for the Kac-Stroock kernels

The last section of this paper is devoted to verifying that condition (3.1) also holds for the Kac-Stroock kernels. As mentioned in Section 3, we have only been able to verify it for sequences in $L^{2q}(D)$ with $q > 1$ when $d \geq 2$. Moreover, and as we will see, the proof for these kernels is a bit more involved in the sense that we were not able to follow a direct approach as in the Donsker case. More precisely, we will check condition (3.1) for simple functions supported on rectangles $[s, t] \subset D$ such that $0 < s < t < 2s$ and then, by using Lemma 3.8 (which we state below without proof since it is a generalization of Lemma 3.2 in ref. [9] and a density argument, we will obtain the result for general sequences).

Lemma 3.8. *Let $Z = \{Z(t) : t \in D\}$ be a continuous process. Assume that for a fixed even $m \in \mathbb{N}$ and some $\delta_1, \dots, \delta_d \in (0, 1)$ there exists $C > 0$ such that*

$$\mathbb{E}\left[\left(\Delta_s Z(t) \right)^m \right] \leq C \prod_{i=1}^{d} (t_i - s_i)^{m\delta_i} \tag{3.3}$$

for any $0 < s < t < 2s$. Then there exists a constant $\tilde{C} > 0$ that only depends on m, $\delta_1, \dots, \delta_d$ such that Z enjoys (3.3) for any $0 \leq s \leq t \leq T$, with $\tilde{C} \cdot C$ instead of C.

Lemma 3.9. *If inequality*

$$\mathbb{E}\left[\left(\int_D g(u)\theta_n(u)\mathrm{d}u \right)^m \right] \leq C \left(\int_D |g(u)|^{2q} \mathrm{d}u \right)^{\frac{m}{2q}}, \tag{3.4}$$

holds for any simple function g, some positive constant C independent of n and g and some even integer $m \in \mathbb{N}$, then it also holds for any sequence of functions $\{g_n\}_{n\in\mathbb{N}} \subset L^{2q}(D)$ with the same values of m and C (that is, by replacing g by g_n).

Proof. For any $g_n \in L^{2q}(D)$, there is a sequence of simple functions $\{g_{n,p}\}_{p\in\mathbb{N}}$ converging to g in $L^{2q}(D)$. For each of these functions, inequality (3.4) holds, so we only need to show that

$$\lim_{p\to\infty} \mathbb{E}\left[\left(\int_D g_{n,p}(u)\theta_n(u)\mathrm{d}u \right)^m \right] = \mathbb{E}\left[\left(\int_D g_n(u)\theta_n(u)\mathrm{d}u \right)^m \right]$$

and that

$$\lim_{p\to\infty} \left(\int_D |g_{n,p}(u)|^{2q} \mathrm{d}u \right)^{\frac{m}{2q}} = \left(\int_D |g_n(u)|^{2q} \mathrm{d}u \right)^{\frac{m}{2q}}.$$

The second limit is an immediate consequence of the fact that convergence in $L^{2q}(D)$ implies convergence of the norms.
As for the first limit, we have that

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\int_D g_{n,p}(u)\theta_n(u)\mathrm{d}u\right)^m\right]-\mathbb{E}\left[\left(\int_D g_n(u)\theta_n(u)\mathrm{d}u\right)^m\right]\right| \\
&\quad=\left|\mathbb{E}\left[\int_{D^m}\left[\prod_{j=1}^m g_{n,p}(u_j)-\prod_{j=1}^m g_n(u_j)\right]\prod_{j=1}^m \theta_n(u_j)\mathrm{d}u_1\dots\mathrm{d}u_m\right]\right| \\
&\quad\le\int_{D^m}\left|\prod_{j=1}^m g_{n,p}(u_j)-\prod_{j=1}^m g_n(u_j)\right|\cdot\left|\mathbb{E}\left[\prod_{j=1}^m \theta_n(u_j)\right]\right|\mathrm{d}u_1\dots\mathrm{d}u_m \\
&\quad\le C\int_{D^m}\left|\prod_{j=1}^m g_{n,p}(u_j)-\prod_{j=1}^m g_n(u_j)\right|\mathrm{d}u_1\dots\mathrm{d}u_m \\
&\quad\le C\left(\int_{D^m}\left(\prod_{j=1}^m g_{n,p}(u_j)-\prod_{j=1}^m g_n(u_j)\right)^2\mathrm{d}u_1\dots\mathrm{d}u_m\right)^{\frac{1}{2}},
\end{aligned}\tag{3.5}$$

being C some constant depending on n, m, T and d, but independent of p. In the last step we have used Cauchy-Schwarz's inequality. Now observe that

$$\begin{aligned}
\int_{D^m}\left(\prod_{j=1}^m g_{n,p}(u_j)-\prod_{j=1}^m g_n(u_j)\right)^2\mathrm{d}u_1\dots\mathrm{d}u_m &= \int_{D^m}\prod_{j=1}^m g_{n,p}^2(u_j)\mathrm{d}u_1\dots\mathrm{d}u_m \\
&\quad+\int_{D^m}\prod_{j=1}^m g_n^2(u_j)\mathrm{d}u_1\dots\mathrm{d}u_m-2\int_{D^m}\prod_{j=1}^m g_{n,p}(u_j)g_n(u_j)\mathrm{d}u_1\dots\mathrm{d}u_m \\
&=\left(\int_D g_{n,p}^2(u)\mathrm{d}u\right)^m+\left(\int_D g_n^2(u)\mathrm{d}u\right)^m-2\left(\int_D g_{n,p}(u)g_n(u)\mathrm{d}u\right)^m,
\end{aligned}$$

and that

$$\begin{aligned}
\left|\int_D g_{n,p}(u)g_n(u)\mathrm{d}u-\int_D g_n^2(u)\mathrm{d}u\right| &\le \int_D |g_{n,p}(u)-g_n(u)|\cdot|g_n(u)|\mathrm{d}u \\
&\le\left(\int_D (g_{n,p}(u)-g_n(u))^2\mathrm{d}u\right)^{\frac{1}{2}}\left(\int_D g_n^2(u)\mathrm{d}u\right)^{\frac{1}{2}}.
\end{aligned}$$

Given that the inclusion $L^{2q}(D)\subset L^2(D)$ is continuous (that is, convergent sequences in $L^{2q}(D)$ will also converge to the same limit in $L^2(D)$), this last quantity goes to 0 as p approaches infinity and hence, since this convergence implies convergence of the $L^2(D)$ norms as well, we have that (3.5) converges to 0 as p approaches infinity as well. □

As the proof shows, the content of Lemma 3.9 remains true as long as $\left|\mathbb{E}\left[\prod_{j=1}^m \theta_n(u_j)\right]\right|\le C$ for some positive constant C which might depend on m, n, T and d.

With all this, it only remains to show that (3.4) holds for any simple function g.

Proposition 3.10. *Inequality* (3.4) *holds for any simple function g and for any $q > 1$.*

Proof. Let $g(u) = \sum_{j=1}^{k} a_k \mathbb{I}_{A_j}(u)$ with $a_1, \dots, a_k \in \mathbb{R}$ and $A_1, \dots, A_k \subset D$ are disjoint rectangles

$$A_j = (s_1^j, t_1^j] \times \cdots \times (s_d^j, t_d^j], \quad s_i^j < t_i^j.$$

Observe that if we fix $u^j = (u_1^j, \dots, u_d^j) \in D$, $j = 1, \dots, m$,

$$\prod_{j=1}^{m} g(u^j) = \sum_{j_1,\dots,j_m} a_{j_1} \dots a_{j_m} \mathbb{I}_{A_{j_1}}(u^1) \dots \mathbb{I}_{A_{j_m}}(u^m),$$

where, in the last sum, $1 \le j_l \le k$ for each $l \in \{1, \dots, m\}$. Moreover, for each $(j_1, \dots, j_m) \in \{1, \dots, k\}^m$, we have

$$\begin{aligned} \mathbb{I}_{A_{j_1}}(u^1) \dots \mathbb{I}_{A_{j_m}}(u^m) &= \left(\mathbb{I}_{(s_1^{j_1}, t_1^{j_1}]}(u_1^1) \dots \mathbb{I}_{(s_d^{j_1}, t_d^{j_1}]}(u_d^1) \right) \cdot \dots \cdot \left(\mathbb{I}_{(s_1^{j_m}, t_1^{j_m}]}(u_1^m) \dots \mathbb{I}_{(s_d^{j_m}, t_d^{j_m}]}(u_d^m) \right) \\ &= \left(\mathbb{I}_{(s_1^{j_1}, t_1^{j_1}]}(u_1^1) \dots \mathbb{I}_{(s_1^{j_m}, t_1^{j_m}]}(u_1^m) \right) \cdot \dots \cdot \left(\mathbb{I}_{(s_d^{j_1}, t_d^{j_1}]}(u_d^1) \dots \mathbb{I}_{(s_d^{j_m}, t_d^{j_m}]}(u_d^m) \right), \end{aligned} \tag{3.6}$$

where in the last step we have rearranged the factors by components.

Now let us assume that $0 < s < t < 2s$, then we have that, by recalling that the kernels θ_n are given by (2.5),

$$\begin{aligned} \mathbb{E}\left[\left(\int_{(s,t]} g(u)\theta_n(u)\mathrm{d}u \right)^m \right] &= \int_{(s,t]^m} \prod_{j=1}^{m} g(u^j) \mathbb{E}\left[\prod_{j=1}^{m} \theta_n(u^j) \right] \mathrm{d}u^1 \dots \mathrm{d}u^m \\ &= n^{m\frac{d}{2}} \int_{(s,t]^m} \prod_{j=1}^{m} g(u^j) \left(\prod_{\substack{1 \le j \le m \\ 1 \le i \le d}} u_i^j \right)^{\frac{d-1}{2}} \mathbb{E}\left[(-1)^{\sum_{j=1}^{m} N_n(u^j)} \right] \mathrm{d}u^1 \dots \mathrm{d}u^m \\ &= n^{m\frac{d}{2}} \sum_{j_1,\dots,j_m} a_{j_1} \dots a_{j_m} \int_{(s,t]^m} \prod_{l=1}^{m} \mathbb{I}_{A_{j_l}}(u^l) \left(\prod_{\substack{1 \le j \le m \\ 1 \le i \le d}} u_i^j \right)^{\frac{d-1}{2}} \mathbb{E}\left[(-1)^{\sum_{j=1}^{m} N_n(u^j)} \right] \mathrm{d}u^1 \dots \mathrm{d}u^m \\ &\le n^{m\frac{d}{2}} \sum_{j_1,\dots,j_m} |a_{j_1} \dots a_{j_m}| \int_{(s,t]^m} \prod_{l=1}^{m} \mathbb{I}_{A_{j_l}}(u^l) \left(\prod_{\substack{1 \le j \le m \\ 1 \le i \le d}} u_i^j \right)^{\frac{d-1}{2}} \left| \mathbb{E}\left[(-1)^{\sum_{j=1}^{m} N_n(u^j)} \right] \right| \mathrm{d}u^1 \dots \mathrm{d}u^m. \end{aligned} \tag{3.7}$$

As shown in the proof of Lemma 3.2 in ref. [2], one has the following estimate for the expected value inside the integral

$$\left| \mathbb{E}\left[(-1)^{\sum_{j=1}^{m} N_n(u^j)} \right] \right| \le \prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} \left(u_i^{(2j)} - u_i^{(2j-1)} \right) \right\}, \quad S_i = \prod_{l \ne i} s_l,$$

where, for each $i \in \{1, \dots, d\}$, the variables $u_i^{(j)}$, $j \in \{1, \dots, m\}$ are the variables u_i^j ordered increasingly. On the other hand, and due to the specific factorization shown in (3.6), one has that, for each $(j_1, \dots, j_m) \in \{1, \dots, k\}^m$,

$$\int_{(s,t]^m} \prod_{l=1}^{m} \mathbb{I}_{A_{j_l}}(u^l) \left(\prod_{\substack{1 \le j \le m \\ 1 \le i \le d}} u_i^j \right)^{\frac{d-1}{2}} \prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} \left(u_i^{(2j)} - u_i^{(2j-1)} \right) \right\} \mathrm{d}u^1 \dots \mathrm{d}u^m$$

$$= (m!)^d \int_{(s,t]^m} \prod_{l=1}^{m} \mathbb{I}_{A_{j_l}}(u^l) \left(\prod_{\substack{1\le j\le m\\ 1\le i\le d}} u_i^j \right)^{\frac{d-1}{2}} \prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} \left(u_i^{2j} - u_i^{2j-1} \right) \right\}$$

$$\times \prod_{i=1}^{d} \mathbb{I}_{\{u_i^1 \le \ldots \le u_i^m\}} \mathrm{d}u^1 \ldots \mathrm{d}u^m.$$

So, all in all, (3.7) can be bounded by

$$n^{m\frac{d}{2}} (m!)^d \sum_{j_1,\ldots,j_m} |a_{j_1} \ldots a_{j_m}| \int_{(s,t]^m} \prod_{l=1}^{m} \mathbb{I}_{A_{j_l}}(u^l) \left(\prod_{\substack{1\le j\le m\\ 1\le i\le d}} u_i^j \right)^{\frac{d-1}{2}} \prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} \left(u_i^{2j} - u_i^{2j-1} \right) \right\}$$

$$\times \prod_{i=1}^{d} \mathbb{I}_{\{u_i^1 \le \ldots \le u_i^m\}} \mathrm{d}u^1 \ldots \mathrm{d}u^m$$

$$= n^{m\frac{d}{2}} (m!)^d \int_{(s,t]^m} \left(\sum_{j_1,\ldots,j_m} \prod_{l=1}^{m} |a_{j_l}| \mathbb{I}_{A_{j_l}}(u^l) \right) \left(\prod_{\substack{1\le j\le m\\ 1\le i\le d}} u_i^j \right)^{\frac{d-1}{2}} \prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} \left(u_i^{2j} - u_i^{2j-1} \right) \right\}$$

$$\times \prod_{i=1}^{d} \mathbb{I}_{\{u_i^1 \le \ldots \le u_i^m\}} \mathrm{d}u^1 \ldots \mathrm{d}u^m$$

$$= n^{m\frac{d}{2}} (m!)^d \int_{(s,t]^m} \prod_{j=1}^{m} \tilde{g}(u^j) \left(\prod_{\substack{1\le j\le m\\ 1\le i\le d}} u_i^j \right)^{\frac{d-1}{2}} \prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} \left(u_i^{2j} - u_i^{2j-1} \right) \right\}$$

$$\times \prod_{i=1}^{d} \mathbb{I}_{\{u_i^1 \le \ldots \le u_i^m\}} \mathrm{d}u^1 \ldots \mathrm{d}u^m, \tag{3.8}$$

where $\tilde{g}(u) = \sum_{l=1}^{k} |a_l| \mathbb{I}_{A_l}(u)$.

The following step is to observe that

$$\prod_{i=1}^{d} \exp\left\{ -2nS_i \sum_{j=1}^{m/2} (u_i^{2j} - u_i^{2j-1}) \right\} = \prod_{j=1}^{m/2} \exp\left\{ -2n \sum_{i=1}^{d} S_i (u_i^{2j} - u_i^{2j-1}) \right\},$$

$$\prod_{j=1}^{m} |\tilde{g}(u^j)| = \prod_{j=1}^{m/2} |\tilde{g}(u^{2j-1})| |\tilde{g}(u^{2j})|,$$

$$\prod_{\substack{1\le i\le d\\ 1\le j\le m}} u_i^j = \prod_{j=1}^{m/2} \left(\prod_{i=1}^{d} u_i^{2j-1} u_i^{2j} \right),$$

$$\prod_{i=1}^{d} \mathbb{I}_{\{u_i^1 \le \ldots \le u_i^m\}} \le \prod_{j=1}^{m/2} \left(\prod_{i=1}^{d} \mathbb{I}_{\{u_i^{2j-1} \le u_i^{2j}\}} \right).$$

So (3.8) can be bounded by

$$n^{m\frac{d}{2}}(m!)^d\left(\int\limits_{\prod_{i=1}^d(s_i,t_i]^2}|\tilde{g}(x)||\tilde{g}(y)|\prod_{i=1}^d\left[\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\right]\mathrm{d}x\mathrm{d}y\right)^{\frac{m}{2}}$$

$$\le n^{m\frac{d}{2}}(m!)^d\left(\frac{1}{2}\int\limits_{\prod_{i=1}^d(s_i,t_i]^2}\tilde{g}^2(x)\prod_{i=1}^d\left[\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\right]\mathrm{d}x\mathrm{d}y\right.$$

$$\left.+\frac{1}{2}\int\limits_{\prod_{i=1}^d(s_i,t_i]^2}\tilde{g}^2(y)\prod_{i=1}^d\left[\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\right]\mathrm{d}x\mathrm{d}y\right)^{\frac{m}{2}},$$

where in the last step we have used that $2ab\le a^2+b^2$.

Now observe that

$$\int\limits_{\prod_{i=1}^d(s_i,t_i]^2}\tilde{g}^2(x)\prod_{i=1}^d\left[\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\right]\mathrm{d}x\mathrm{d}y$$

$$=\int\limits_{(s,t]}\tilde{g}^2(x)\left(\int\limits_{(s,t]}\prod_{i=1}^d\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\mathrm{d}y\right)\mathrm{d}x$$

$$=\int\limits_{(s,t]}\tilde{g}^2(x)\left(\prod_{i=1}^d\int\limits_{x_i}^{t_i}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\mathrm{d}y_i\right)\mathrm{d}x,$$

and that, for each $i\in\{1,\dots,m\}$

$$\int\limits_{x_i}^{t_i}(x_iy_i)^{\frac{d-1}{2}}e^{-2nS_i(y_i-x_i)}\mathrm{d}y_i\le 2^{d-1}s_i^{d-1}\int\limits_{x_i}^{t_i}e^{-2nS_i(y_i-x_i)}\mathrm{d}y_i$$

$$=\frac{2^{d-1}s_i^{d-1}}{2nS_i}\left(1-e^{-2nS_i(t_i-x_i)}\right)$$

$$\le\frac{2^{d-2}s_i^d}{n\prod_{l=1}^d s_l},$$

where we have used that $0<s_i\le x_i,y_i\le t_i<2s_i\le 2x_i,2y_i$ for all $i\in\{1,\dots,d\}$. Hence,

$$\int\limits_{\prod_{i=1}^d(s_i,t_i]^2}\tilde{g}^2(x)\prod_{i=1}^d\left[\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\right]\mathrm{d}x\mathrm{d}y\le n^{-d}2^{d(d-2)}\prod_{i=1}^d\frac{s_i^d}{\prod_{l=1}^d s_l}\int\limits_{(s,t]}\tilde{g}^2(x)\mathrm{d}x$$

$$=n^{-d}2^{d(d-2)}\int\limits_{(s,t]}f^2(x)\mathrm{d}x.$$

Similarly,

$$\int\limits_{\prod_{i=1}^{d}(s_i,t_i]^2} \tilde{g}^2(y)\prod_{i=1}^{d}\left[\mathbb{I}_{\{x_i\le y_i\}}(x_iy_i)^{\frac{d-1}{2}}\ \exp\{-2nS_i(y_i-x_i)\}\right]\mathrm{d}x\mathrm{d}y \le \frac{2^{d(d-2)}}{n^d}\int\limits_{(s,t]}\tilde{g}^2(y)\mathrm{d}y.$$

But, since the A_j in g (and $\tilde{g}$) are disjoint, we have that $\tilde{g}^2 = g^2$, giving us the desired result over rectangles $[s,t]$ with $0 < s < t < 2s$. Now, from Hölder's inequality, we will have that

$$\begin{aligned}\mathbb{E}\left[\left(\int\limits_{(s,t]} g(u)\theta_n(u)\mathrm{d}u\right)^m\right] &\le C\left(\int\limits_{(s,t]} g^2(u)\mathrm{d}u\right)^{\frac{m}{2}}\\ &\le C\left(\int\limits_{(s,t]} |g(u)|^{2q}\mathrm{d}u\right)^{\frac{m}{2q}}\prod_{i=1}^{d}(t_i-s_i)^{\frac{m(q-1)}{2q}}\\ &\le C\left(\int\limits_{D} |g(u)|^{2q}\mathrm{d}u\right)^{\frac{m}{2q}}\prod_{i=1}^{d}(t_i-s_i)^{\frac{m(q-1)}{2q}}\end{aligned}$$

for all $0 < s < t < 2s$. Since $0 < \frac{q-1}{2q} < 1$ for all $q > 1$, we can apply Lemma 3.8 to get the desired result. □

The reader might think that this can be extended to $q = 1$ by taking the limit q approaches to 1 from above. However, this fails due to the presence of the constant $\tilde{C}$ in Lemma 3.8, which diverges as $q \to 1^+$.

The proof shown above holds for any $q > 1$, $d \ge 1$. However, for $d = 1$, we can go a step further and see that condition (3.1) is verified for the Kac-Stroock kernels for any sequence $\{g_n\}_{n\in\mathbb{N}} \subset L^2(D)$ using a direct approach like in the Donsker case.

Proposition 3.11. *If $d = 1$, then condition* (3.1) *holds for any even integer $m \in \mathbb{N}$ and for any sequence of functions $\{g_n\}_{n\in\mathbb{N}} \subset L^2(D)$.*

Proof. As in the multiparameter case, we first note that

$$\mathbb{E}\left[\left(\int_s^t g_n(u)\theta_n(u)\mathrm{d}u\right)^m\right] = n^{\frac{m}{2}}m!\int\limits_{(s,t]^m} g_n(u_1)\cdot\ldots\cdot g_n(u_m)\mathbb{I}_{\{u_1\le\ldots\le u_m\}}\mathbb{E}\left[(-1)^{\sum_{j=1}^{m}N_n(u_j)}\right]\mathrm{d}u_1\ldots\mathrm{d}u_m. \tag{3.9}$$

Noticing that

$$(-1)^{\sum_{j=1}^{m}N_n(u_j)} = (-1)^{(N_n(u_m)-N_n(u_{m-1}))+\ldots+(N_n(u_2)-N_n(u_1))}$$

and using that $\mathbb{E}\left[(-1)^Z\right] = e^{-2\lambda}$ if Z is a Poisson random variable of parameter λ, we obtain that

$$\mathbb{E}\left[(-1)^{\sum_{j=1}^{m}N_n(u_j)}\right] = \exp\left\{-2n\sum_{j=1}^{m/2}(u_{2j}-u_{2j-1})\right\},$$

whenever $u_1 \le \ldots \le u_m$. Moreover, if we use the estimate

$$\mathbb{I}_{\{u_1\le\ldots\le u_m\}} \le \mathbb{I}_{\{u_1\le u_2\}}\cdot\ldots\cdot\mathbb{I}_{\{u_{m-1}\le u_m\}} = \prod_{j=1}^{m/2}\mathbb{I}_{\{u_{2j-1}\le u_{2j}\}},$$

we can see that, all in all, (3.9) can be bounded by

$$
\begin{aligned}
& n^{\frac{m}{2}} m! \int_{(s,t]^m} \prod_{j=1}^{m/2} \left[|g_n(u_{2j-1})\| g_n(u_{2j})| \mathbb{I}_{\{u_{2j-1} \le u_{2j}\}} e^{-2n(u_{2j}-u_{2j-1})} \right] \mathrm{d}u_1 \dots \mathrm{d}u_m \\
&= n^{\frac{m}{2}} m! \left(\int_{(s,t]^2} |g_n(x)\| g_n(y)| \mathbb{I}_{\{x \le y\}} e^{-2n(y-x)} \mathrm{d}x\mathrm{d}y \right)^{\frac{m}{2}} \\
&\le n^{\frac{m}{2}} m! \left(\frac{1}{2} \int_{(s,t]^2} g_n^2(x) \mathbb{I}_{\{x \le y\}} e^{-2n(y-x)} \mathrm{d}x\mathrm{d}y + \frac{1}{2} \int_{(s,t]^2} g_n^2(y) \mathbb{I}_{\{x \le y\}} e^{-2n(y-x)} \mathrm{d}x\mathrm{d}y \right)^{\frac{m}{2}},
\end{aligned}
$$

where, in the last step, we have used that $2ab \le a^2 + b^2$.

Finally, we see that

$$
\begin{aligned}
\int_{(s,t]^2} g_n^2(x) \mathbb{I}_{\{x \le y\}} e^{-2n(y-x)} \mathrm{d}x\mathrm{d}y &= \int_s^t g_n^2(x) \left(\int_s^t \mathbb{I}_{\{x \le y\}} e^{-2n(y-x)} \mathrm{d}y \right) \mathrm{d}x \\
&= \int_s^t g_n^2(x) \left(\int_x^t e^{-2n(y-x)} \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{2n} \int_s^t g_n^2(x) \left(1 - e^{-2n(t-x)}\right) \mathrm{d}x \\
&\le \frac{1}{2n} \int_s^t g_n^2(x) \mathrm{d}x
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\int_{(s,t]^2} g_n^2(y) \mathbb{I}_{\{x \le y\}} e^{-2n(y-x)} \mathrm{d}x\mathrm{d}y &= \int_s^t g_n^2(y) \left(\int_s^y e^{2n(x-y)} \mathrm{d}x \right) \mathrm{d}y \\
&= \frac{1}{2n} \int_s^t g_n^2(y) \left(1 - e^{-2n(y-s)}\right) \mathrm{d}y \\
&\le \frac{1}{2n} \int_s^t g_n^2(y) \mathrm{d}y,
\end{aligned}
$$

which finishes the proof. □

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A Appendix

In this appendix, we give a proof of the convergence in law of the processes ζ_n defined as in (2.3) towards the Brownian sheet in the space $C(D)$ when the random variables $\{Z_k\}_{k\in\mathbb{N}^d}$ are centered with unitary variances and have finite moments of order 4.

Theorem A.1. *If the centered random variables $\{Z_k\}_{k\in\mathbb{N}^d}$ are independent and identically distributed with unitary variances and finite moments of order 4, then the processes ζ_n defined as in (2.3) converge towards the Brownian sheet in $C(D)$ as n approaches infinity.*

Proof. Inspection of the proof of Theorem (3.1) shows that the sequence is tight by taking $f_n = 1$ for all $n \in \mathbb{N}$, $q = 1$ and $m = 4$. So it only remains to show that the finite dimensional distributions of the processes ζ_n converge to the ones of the Brownian sheet. To this purpose, for each $t \in D$, let

$$R_n(t) = n^{-\frac{d}{2}} \sum_{k\in\mathbb{N}^d} Z_k \left(\int_{[0,nt]\setminus[0,[nt]]} \mathbb{I}_{[k-1,k)}(u)\mathrm{d}u \right)$$

and note that $\mathbb{E}\left[R_n^2(t)\right]$ converges to 0 as n approaches infinity. Indeed, if $t_i = 0$ for some $i \in \{1,\dots,d\}$, then the result follows immediately. If $t > 0$, then we have that

$$\begin{aligned}
\mathbb{E}\left[R_n^2(t)\right] &= n^{-d} \sum_{k\in\mathbb{N}^d} \left(\int_{[0,nt]\setminus[0,[nt]]} \mathbb{I}_{[k-1,k)}(u)\mathrm{d}u \right)^2 \\
&\le n^{-d} \sum_{k\in\Delta_n} \left(\int_{[0,nt]\setminus[0,[nt]]} \mathbb{I}_{[k-1,k)}(u)\mathrm{d}u \right)^2 \\
&\le n^{-d} \sum_{k\in\Delta_n} 1 \\
&= n^{-d}|\Delta_n|,
\end{aligned}$$

where $\Delta_n \subset \mathbb{N}^d$ is the set of points $k \in \mathbb{N}^d$ such that $[k-1,k) \subset [0,[nt]+1]\setminus[0,[nt]]$, $|\Delta_n|$ is the number of elements in Δ_n and have used that $[k-1,k)$ has unitary Lebesgue measure. By definition of Δ_n, we have

$$|\Delta_n| = \prod_{j=1}^{d} \left([nt_j]+1\right) - \prod_{j=1}^{d} [nt_j],$$

so, for n large enough so that $nt_j \ge 1$ for each $j \in \{1,\dots,d\}$,

$$\mathbb{E}\left[R_n^2(t)\right] \le n^{-d}\left(\prod_{j=1}^{d} \left([nt_j]+1\right) - \prod_{j=1}^{d} [nt_j] \right)$$

$$= n^{-d}\prod_{j=1}^{d}[nt_j]\left(\prod_{j=1}^{d}\frac{[nt_j]+1}{[nt_j]}-1\right)$$
$$\leq \prod_{j=1}^{d}\frac{[nt_j]+1}{[nt_j]}-1,$$

where we have used that $[nt_j] \leq nt_j \leq n$ for each $j \in \{1,\dots,d\}$. Since $x \mapsto \frac{x+1}{x}$ decreases for $x \in (0,\infty)$, it follows that

$$\frac{[nt_j]+1}{[nt_j]} \leq \frac{[n\min_i t_i]+1}{[n\min_i t_i]} \leq \frac{n\min_i t_i}{n\min_i t_i - 1} \xrightarrow{n\to\infty} 1,$$

since $x-1 < [x] \leq x$ per a tot $x \geq 0$. Thus,

$$\mathbb{E}\left[R_n^2(t)\right] \leq \prod_{j=1}^{d}\frac{[nt_j]+1}{[nt_j]}-1 \leq \left(\frac{n\min_i t_i}{n\min_i t_i - 1}\right)^d - 1 \xrightarrow{n\to\infty} 0.$$

Hence, if we set $S_n(t) = n^{-\frac{d}{2}}\sum_{k\leq[nt]}Z_k$, Chebyshev's inequality will imply that the finite dimensional distributions of ζ_n and S_n have the same limit. So it only remains to show that the finite dimensional distributions of S_n converge to the ones of the Brownian sheet.

By the Cramér-Wold device, we only need to show that, for each $m \in \mathbb{N}$, $t^1,\dots,t^m \in D$ and each $\alpha_1,\dots,\alpha_m \in \mathbb{R}$, the sums $Y_n = \sum_{j=1}^m \alpha_j S_n(t^j)$ converge in law to the sum $\sum_{j=1}^m \alpha_j W(t^j)$. Given that W is a centered Gaussian process with covariance function $C(s,t) = \prod_{i=1}^d (t_i \wedge s_i)$, we have that the latter is a centered Gaussian random variable with variance

$$\mathbb{E}\left[\left(\sum_{j=1}^m \alpha_j W(t^j)\right)^2\right] = \sum_{i,j=1}^m \alpha_i\alpha_j \mathbb{E}\left[W(t^i)W(t^j)\right] = \sum_{i,j=1}^m \alpha_i\alpha_j \prod_{l=1}^d (t_l^i \wedge t_l^j). \tag{A.1}$$

Now let $a_n(s) = \sum_{j=1}^m \alpha_j \mathbb{I}_{[0,[nt^j]]}(s)$, $s \in \mathbb{N}^d$, and observe that

$$\sup_{n\geq 1}\sup_{s\in\mathbb{N}^d}|a_n(s)| \leq \sum_{j=1}^m |\alpha_j| < \infty,$$

and that

$$n^{-d}\sum_{s\leq[n]} a_n^2(s) = n^{-d}\sum_{s\leq[n]}\left(\sum_{i,j=1}^m \alpha_i\alpha_j \mathbb{I}_{[0,[nt^i]]}(s)\mathbb{I}_{[0,[nt^j]]}(s)\right)$$
$$= \sum_{i,j=1}^m \alpha_i\alpha_j\left(n^{-d}\sum_{s\leq[n]}\mathbb{I}_{[0,[nt^i]\wedge[nt^j]]}(s)\right)$$
$$= \sum_{i,j=1}^m \alpha_i\alpha_j \prod_{l=1}^d\left(\frac{1}{n}\sum_{s_l\leq[nt_l^i]\wedge[nt_l^j]}\mathbb{I}_{[0,1]}\left(\frac{s_l}{[nt_l^i]\wedge[nt_l^j]}\right)\right).$$

Since, for each $i,j \in \{1,\dots,m\}$ and $l \in \{1,\dots,d\}$,

$$\frac{1}{n}\sum_{s_l\leq[nt_l^i]\wedge[nt_l^j]}\mathbb{I}_{[0,1]}\left(\frac{s_l}{[nt_l^i]\wedge[nt_l^j]}\right) \xrightarrow{n\to\infty} \int_0^{t_l^i\wedge t_l^j} \mathrm{d}u = t_l^i \wedge t_l^j,$$

we can conclude, by Lemma 4.2.2 in ref. [10], that the finite dimensional distributions of S_n converge to the ones of the Brownian motion, finishing the proof. □

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