



Cyclic completely regular codes $C_{1,5}$

Josep Rifa¹ · Victor Zinoviev²

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Abstract

We construct a family of cyclic completely regular codes (CRCs) of length $n = 2^m - 1$ and compute their intersection arrays. These codes, denoted $C_{1,5}$, are generated by the product $m_1(x)m_5(x)$, where $m_i(x)$ is the minimal polynomial of α^i , and α is a primitive element of the finite field $\mathbb{F}_{2^m}$. We consider two main cases: odd m and $m \equiv 2 \pmod{4}$. For odd m , these codes are known to be completely regular with covering radius $\rho = 3$ and minimum distance $d = 5$. We prove that, for any m , the codes $C_{1,3}$ and $C_{1,5}$ are non-equivalent despite sharing the same parameters and intersection array. For $m \equiv 2 \pmod{4}$, we demonstrate that $C_{1,5}$ forms a new family of completely regular codes with covering radius $\rho = 3$, minimum distance $d = 3$, and intersection array $\text{IA} = [n, n - 3, \frac{3n+7}{4}; 1, 4, \frac{n-3}{4}]$. Moreover, the corresponding extended cyclic codes $C_{1,5}^*$ are completely regular $[n+1, n-2m, 4; 4]$ -codes with intersection array $\text{IA} = [n+1, n, n-3, \frac{3n+7}{4}; 1, 4, \frac{n-3}{4}, n+1]$. We also describe the parameters and some properties of the coset graphs associated to these completely regular codes which form distance-regular graphs.

Keywords Completely regular codes · Cyclic codes · Distance regular graphs · Extended codes · Intersection array

1 Introduction

Let $\mathbb{F}_{2^m}$ be the finite field with 2^m elements, and let α be a primitive element generating $\mathbb{F}_{2^m}$. A binary linear $[n, k, d; \rho]$ -code C is a k -dimensional subspace of $\mathbb{F}_2^n$, where n is the length, d is the minimum distance, and ρ is the covering radius. The packing radius of C is $e = \lfloor (d-1)/2 \rfloor$.

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✉ Josep Rifa
josep.rifa@uab.cat
Victor Zinoviev
vazinov@iitp.ru

¹ Department of Information and Communication Engineering, Universitat Autònoma de Barcelona, Cerdanyola del Vallès, Catalonia, Spain

² A. A. Kharkevich Institute, Russian Academy of Sciences, Moscow, Russia

For any vector $\mathbf{v} \in \mathbb{F}_2^n$, its *distance to the code* C is defined as $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C} \{d(\mathbf{v}, \mathbf{x})\}$, and the covering radius of C is $\rho = \max_{\mathbf{v} \in \mathbb{F}_2^n} \{d(\mathbf{v}, C)\}$. It follows that $e \leq \rho$. If $e = \rho$, the code is called *perfect*; if $e + 1 = \rho$, it is called *quasi-perfect*.

Let $n = 2^m - 1$, the *support* of a vector $\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \in \mathbb{F}_2^n$ is defined as $\text{supp}(\mathbf{v}) = \{\alpha^i \in \mathbb{F}_{2^m} \mid v_i \neq 0\}$. A vector $\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \in \mathbb{F}_2^n$ can be identified with the polynomial $v(x) = v_0 + v_1x + \dots + v_{n-1}x^{n-1}$. A coset of C , denoted by $D = C + \mathbf{x}$, is the set formed by adding $\mathbf{x}$ to each element of C , where $+$ represents component-wise addition in $\mathbb{F}_2$.

For a given code C of length n and covering radius ρ , define

$$C(i) = \{\mathbf{x} \in \mathbb{F}_2^n \mid d(\mathbf{x}, C) = i\}, \quad i = 0, 1, \dots, \rho. \quad (1)$$

The sets $C(0) = C, C(1), \dots, C(\rho)$ are called the *sub-constituents* of C .

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called *neighbours* if $d(\mathbf{x}, \mathbf{y}) = 1$. We denote the all-zero vector by $\mathbf{0}$.

The cyclotomic coset of r modulo $2^m - 1$ is defined as

$$K_m(r) = \{r2^k \mid k = 0, 1, \dots, m-1\}.$$

For any integer i with $0 \leq i \leq 2^m - 1$, we say that i belongs to $K_m(r)$ if there exists $j \in \{0, 1, \dots, m-1\}$ such that $i2^j \equiv r \pmod{2^m - 1}$.

Definition 1.1 ([1]) A binary code C of length n and covering radius ρ is *completely regular* (CR) if every vector $\mathbf{x} \in C(l)$ has the same number c_l of neighbours in $C(l-1)$ and the same number b_l of neighbours in $C(l+1)$, for all $l \geq 0$. Define $a_l = n - b_l - c_l$ with $c_0 = b_\rho = 0$. The parameters a_l, b_l , and c_l ($0 \leq l \leq \rho$) are called *intersection numbers*, and the sequence $[b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho]$ is called the *intersection array* (IA) of C .

Let C be a linear completely regular code with covering radius ρ and intersection array $[b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho]$. The *coset graph* Γ_C is defined by taking all distinct cosets $B = C + \mathbf{x}$ as vertices. Two vertices $\gamma = \gamma(B)$ and $\gamma' = \gamma(B')$ are adjacent if and only if there exist $\mathbf{v} \in B$ and $\mathbf{v}' \in B'$ such that $d(\mathbf{v}, \mathbf{v}') = 1$.

Lemma 1.2 ([2]) If C is a linear completely regular code with covering radius ρ and intersection array $[b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho]$, then its coset graph Γ_C is distance-regular with diameter $D = \rho$ and the same intersection array.

In Ref. [3], the authors studied the minimum distance of binary cyclic codes of length $n = 2^m - 1$ generated by a product of two minimal polynomials. Throughout this paper, we focus on the binary case where these two minimal polynomials are not only irreducible but also primitive polynomials of degree m . Let $m_i(x)$ denote the minimal polynomial of α^i . Note that $m_1(x)$ is a primitive polynomial of degree m ; $m_3(x)$ is a primitive polynomial of degree m if and only if $\gcd(3, 2^m - 1) = 1$ (hence, when m is odd); and $m_5(x)$ is a primitive polynomial of degree m if and only if $\gcd(5, 2^m - 1) = 1$ (hence, when $m \not\equiv 0 \pmod{4}$).

Let C_i denote the binary cyclic code generated by $m_i(x)$, and let $C_{i,j}$ denote the binary cyclic code generated by the product $m_i(x)m_j(x)$.

Definition 1.3 ([4]) A quasi-perfect e -error-correcting code C is called *uniformly packed* if there exist natural numbers λ, μ such that for any vector $\mathbf{v} \in \mathbb{F}_2^n$:

$$B_{\mathbf{v}, e+1} = \begin{cases} \lambda & \text{if } d(\mathbf{v}, C) = e, \\ \mu & \text{if } d(\mathbf{v}, C) = e + 1, \end{cases}$$

where $B_{\mathbf{v}, i} = \#\{\mathbf{w} \in C \mid d(\mathbf{v}, \mathbf{w}) = i\}$.

For the case $\mu = \lambda + 1$, binary uniformly packed codes were introduced in Ref. [5]. In Ref. [4], it was proved that a uniformly packed code is completely regular.

Primitive binary BCH codes $C_{1,3}$ of length $n = 2^m - 1$, with $m \geq 5$ odd, generated by the polynomial $m_1(x)m_3(x)$, are quasi-perfect codes [6] and uniformly packed [7] with

$$\lambda + 1 = \mu = \frac{2^{m-1} - 1}{3}. \quad (2)$$

These codes are completely regular [5].

Codes $C_{1,5}$ with m odd are uniformly packed, as shown in Ref. [8]. The values of λ and μ are identical to those in (2), and, like the codes $C_{1,3}$, they are completely regular.

Using the values of λ and μ , the intersection array for the $C_{1,3}$ and $C_{1,5}$ codes can be computed, giving the following result.

Theorem 1.4 *Let $C \in \{C_{1,3}, C_{1,5}\}$ be a code of length $n = 2^m - 1$, where $m \geq 5$ is odd. Then C is a completely regular code with minimum distance $d = 5$, covering radius $\rho = 3$, and intersection array $\text{IA} = [n, n - 1, 2^{m-1} + 1; 1, 2, 2^{m-1} - 1]$.*

In Sect. 2, we show that $C_{1,5}$ codes with m odd share the same parameters and intersection array as their $C_{1,3}$ counterparts, despite being non-equivalent. At the end of the section, we discuss the coset graphs of the constructed codes, which are distance-regular graphs. The main contribution of this paper is the construction of the family of completely regular cyclic codes $C_{1,5}$ for $m \equiv 2 \pmod{4}$. In Sect. 3, we investigate $C_{1,5}$ codes for this case, proving that they are completely regular. We determine their minimum distance, covering radius, and intersection array. Additionally, we show that the extended codes $C_{1,5}^*$ are completely regular and compute their parameters. At the end of the section, we use a known concatenation technique to show the existence of additive quaternary codes with the same intersection array as the $C_{1,5}$ codes. Finally, we describe the distance-regular graphs obtained as coset graphs of the constructed codes.

2 The families of cyclic completely regular codes $C_{1,3}$ and $C_{1,5}$ of length $n = 2^m - 1$, for odd $m > 3$

We consider the class of codes $C_{i,j}$ generated by $m_i(x)m_j(x)$, where both $m_i(x)$ and $m_j(x)$ are primitive polynomials of degree m . For brevity, we refer to $C_{i,j}$ as *primitive codes*. We prove a sufficient condition for when two codes $C_{i,j}$ and $C_{k,s}$ are equivalent.

Proposition 2.1 *Primitive Codes $C_{i,j}$ and $C_{k,s}$ are equivalent if there exists an r such that $\gcd(r, n) = 1$, with $k \in K_m(ri)$ and $s \in K_m(rj)$ (or $k \in K_m(rj)$ and $s \in K_m(ri)$).*

Proof Assume there exists an r such that $\gcd(r, n) = 1$, with $k \in K_m(ri)$ and $s \in K_m(rj)$.

Suppose $k = ri$ and $s = rj$, since the minimal polynomial of α^k and α^s is the same for all elements in the same cyclotomic coset. The columns of the parity-check matrix of code $C_{k,s}$ are

$$\begin{pmatrix} (\alpha^k)^t \\ (\alpha^s)^t \end{pmatrix}, \quad \text{for } t = 1, \dots, n.$$

We have

$$\begin{pmatrix} (\alpha^k)^t \\ (\alpha^s)^t \end{pmatrix} = \begin{pmatrix} (\alpha^i)^{tr} \\ (\alpha^j)^{tr} \end{pmatrix},$$

which coincide with the columns in the parity-check matrix of the code $C_{i,j}$ after the permutation $t \mapsto tr$. Thus, the statement is proved. $\square$

We prove a converse of Proposition 2.1 for the case where one of the two codes is $C_{1,3}$. First, we present some preliminary results.

Definition 2.2 ([9]) The affine group on $\mathbb{F}_{2^m}$ is the set of all transformations $P_{u,v} : \alpha^j \mapsto u\alpha^j + v$ (and $0 \mapsto v$), where $u, v \in \mathbb{F}_{2^m}$, $u \neq 0$.

Proposition 2.3 ([9]) Let C be a primitive BCH code of length $n = 2^m - 1$, and let C^* be the extended code. Then the automorphism group of C^* contains the affine group on $\mathbb{F}_{2^m}$, and hence C^* is affine-invariant.

Proposition 2.4 ([10]) An affine-invariant code C^* and an extended cyclic code D^* are equivalent if and only if C^* is the image of D^* under a multiplier.

For the specific case of $C_{1,3}$ and $C_{1,5}$, we have:

Theorem 2.5 Let $C \in \{C_{1,3}, C_{1,5}\}$ be a primitive binary code of length $n = 2^m - 1$, with $m \geq 5$ odd. Then C is an $[n, n - 2m, 5; 3]$ -completely regular code with $\text{IA} = [n, n - 1, 2^{m-1} + 1; 1, 2, 2^{m-1} - 1]$. Moreover, we prove that the codes $C_{1,3}$ and $C_{1,5}$ are non-equivalent.

Proof Since $d = 5$ and $\rho = 3$, the values b_0, b_1, c_1, c_2 are $n, n - 1, 1, 2$, respectively. Thus, we only need to compute b_2 and c_3 (see Definition 1.1).

The subconstituents of C are defined in (1). For $\mathbf{v} \in C(3)$, the number of vectors in C at distance three from $\mathbf{v}$ is $B_{\mathbf{v},e+1} = \mu$ and also $\frac{c_3 c_2 c_1}{6}$. Hence, $\frac{c_3 c_2 c_1}{6} = \frac{2^{m-1}-1}{3}$, so $c_3 = 2^{m-1} - 1$.

Similarly, for $\mathbf{v} \in C(2)$, the number of vectors in C at distance three from $\mathbf{v}$ is $B_{\mathbf{v},e+1} = \lambda$ and also $\frac{a_2 c_2 c_1}{6}$. Thus, $\frac{a_2 c_2 c_1}{6} = \frac{2^{m-1}-4}{3}$. Substituting $a_2 = n - b_2 - c_2 = n - b_2 - 2$, we obtain $(n - b_2 - 2) \cdot 2 = 2(2^{m-1} - 4)$, so $b_2 = 2^{m-1} + 1$.

To show that $C_{1,3}$ and $C_{1,5}$ are non-equivalent, assume to the contrary that they are equivalent. Then, their extended codes $C_{1,3}^*$ and $C_{1,5}^*$ are also equivalent. Since $C_{1,3}^*$ is a primitive binary extended BCH code, it is affine-invariant by Proposition 2.3. Moreover, since $C_{1,5}^*$ is an extended cyclic code, by Proposition 2.4, there exists an r with $\gcd(r, n) = 1$ such that either $r \in K_m(1)$ and $3r \in K_m(5)$, or $r \in K_m(5)$ and $3r \in K_m(1)$. This implies $3 \in K_m(5)$ or $3 \cdot 5 \in K_m(1)$, which does not hold for $m > 3$.

Thus, the statement is proved. $\square$

It is obvious that if two binary codes are equivalent, then their corresponding coset graphs are also equivalent. However, in general, if two codes are non-equivalent, their coset graphs may still be equivalent. Nevertheless, in the specific case of the codes $C_{1,3}$ and $C_{1,5}$, the corresponding coset graphs must necessarily be non-equivalent.

Proposition 2.6 The coset graphs $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$ of $C_{1,3}$ and $C_{1,5}$, respectively, are non-equivalent.

Proof Assume, for the sake of contradiction, that $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$ are equivalent graphs. Then there exists a permutation π of the vertices of $\Gamma_{C_{1,3}}$ that preserves adjacency and maps $\Gamma_{C_{1,3}}$ to $\Gamma_{C_{1,5}}$. Let $C_{1,3} + \mathbf{v}$ be the vertex in $\Gamma_{C_{1,3}}$ that is mapped by π to the zero coset $C_{1,5}$ in $\Gamma_{C_{1,5}}$, so $\pi(C_{1,3} + \mathbf{v}) = C_{1,5}$. Since the minimum distance of the code is $d = 3$, the vertex $C_{1,3} + \mathbf{v}$ has exactly n neighbours in $\Gamma_{C_{1,3}}$ (where n is the length of the code), namely the cosets $C_{1,3} + \mathbf{v} + e_i$ for $i = 1, \dots, n$, where e_i denotes the i -th standard basis vector.

These n neighbours must be mapped by π to the n neighbours of $C_{1,5}$ in $\Gamma_{C_{1,5}}$, which are precisely the cosets $C_{1,5} + e_j$ for $j = 1, \dots, n$. Thus we can write

$$\pi(C_{1,3} + \mathbf{v} + e_i) = C_{1,5} + e_{\sigma(i)}$$

for some permutation σ of $\{1, \dots, n\}$. Equivalently, $\pi(e_i) = e_{\sigma(i)}$.

Now consider any vector

$\mathbf{w} = e_{i_1} + \dots + e_{i_k} \in C_{1,3}$. Hence,

$$C_{1,3} + \mathbf{v} + \mathbf{w} = C_{1,3} + \mathbf{v}.$$

Applying π we obtain

$$\pi(C_{1,3} + \mathbf{v} + \mathbf{w}) = C_{1,5} + \pi(\mathbf{w}) = C_{1,5},$$

and therefore $\pi(\mathbf{w}) \in C_{1,5}$.

Since π is induced by a coordinate permutation (in the sense that it maps weight-1 vectors to weight-1 vectors), it extends naturally to a permutation of the coordinates. Thus the map that sends each standard basis vector e_i to $e_{\sigma(i)}$ defines a permutation of the coordinates that maps codewords of $C_{1,3}$ to codewords of $C_{1,5}$. This shows that $C_{1,3}$ and $C_{1,5}$ are equivalent, which contradicts Theorem 2.5. This contradiction proves that the coset graphs $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$ cannot be equivalent. $\square$

For example, when $m = 5$, the length is $n = 31$. The coset graphs associated with $C_{1,5}$ and $C_{1,3}$ are distance-regular graphs with the same intersection array $\text{IA} = [31, 30, 17; 1, 2, 15]$ (Theorem 2.5). A graph with this IA was previously described in Ref. [11], which updates a table from [2]. Apart from Proposition 2.6, we can do an easy computation (e.g., using MAGMA [12]) and check that the coset graphs $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$, associated with $C_{1,3}$ and $C_{1,5}$, respectively, are two non-equivalent distance-regular graphs with the same intersection array.

From the binary cyclic completely regular codes $C_{1,3}$ and $C_{1,5}$, we construct their corresponding coset graphs. When m is odd, both coset graphs are distance-regular with the same intersection array given in Theorem 2.5, but are non-equivalent. Thus, we obtain two infinite families of non-equivalent distance-regular graphs with the same intersection array given in Theorem 2.5.

We summarise these results in the following theorem.

Theorem 2.7 *Let $C_{1,k}$ be the cyclic code of length $n = 2^m - 1$ generated by $m_1(x)m_k(x)$, where $m > 3$ is odd and $k \in \{3, 5\}$. The coset graphs $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$ are distance-regular graphs of diameter $\rho = 3$ with intersection array $\text{IA} = [n, n-1, 2^{m-1}+1; 1, 2, 2^{m-1}-1]$. However, $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$ are non-equivalent.*

Proof By Lemma 1.2 and Theorem 2.5, $C_{1,5}$ is a completely regular code, and its coset graph $\Gamma_{C_{1,5}}$ is distance-regular with the same intersection array as $C_{1,5}$, as computed in Theorem 2.5. Thus, $\Gamma_{C_{1,5}}$ has diameter $\rho = 3$ and the stated intersection array. The same holds for $C_{1,3}$ and its coset graph $\Gamma_{C_{1,3}}$ (Theorem 1.4). The graphs $\Gamma_{C_{1,3}}$ and $\Gamma_{C_{1,5}}$ are non-equivalent (Proposition 2.6). $\square$

3 The family of cyclic completely regular codes $C_{1,5}$ of length $n = 2^m - 1$, for $m \equiv 2 \pmod{4}$

In this case, the codes $C_{1,5}$ have a primitive generator polynomial $m_1(x)m_5(x)$, and $\mathbb{F}_{2^2} \subset \mathbb{F}_{2^m}$. If α is a primitive element of $\mathbb{F}_{2^m}$, then $\mathbb{F}_{2^2} = \{0, 1, \beta = \alpha^{(2^m-1)/3}, \gamma = \beta^2\}$.

Theorem 3.1 *Let $n = 2^m - 1$ with $m \equiv 2 \pmod{4}$. The cyclic code $C_{1,5}$ is an $[n, n - 2m, 3; 3]$ -code, and the supports of the minimum-weight codewords partition the set of coordinates.*

Proof By construction, $C_{1,5}$ has length n and dimension $n - 2m$. To determine the minimum distance, consider the system of equations:

$$\begin{aligned} x + y + z &= 0, \\ x^5 + y^5 + z^5 &= 0, \end{aligned} \quad (3)$$

where x, y, z are distinct non-zero elements in $\mathbb{F}_{2^m}$.

Since the code is cyclic, we may assume $x = 1$. Substituting $z = 1 + y$ from the first equation into the second, we obtain $1 + y^5 + (1 + y)^5 = y(y^3 + 1) = y(y + 1)(y^2 + y + 1) = 0$. The solutions are $y = 0$, $y = 1$, or $y \in \mathbb{F}_{2^2}$. Since $\mathbb{F}_{2^2} \subset \mathbb{F}_{2^m}$, there are two solutions $y \in \{\beta, \gamma\}$ for each x . Thus, the number of solutions to (3) is $2n$, and the number of codewords of weight 3 in $C_{1,5}$ is $\frac{2n}{3!} = \frac{n}{3}$. For each non-zero $x \in \mathbb{F}_{2^m}$, there is a codeword covering x , and the total number of weight-3 codewords is $\frac{n}{3}$. Hence, the supports of the minimum-weight codewords partition the set of coordinates.

To determine the covering radius ρ , we show that the following system has a solution:

$$\begin{aligned} x + y + z &= s_1, \\ x^5 + y^5 + z^5 &= s_5, \end{aligned} \quad (4)$$

for any $s_1, s_5 \in \mathbb{F}_{2^m}$.

Let $\bar{x} = x + s_1$, $\bar{y} = y + s_1$, $\bar{z} = z + s_1$. The system (4) has a solution if and only if the following system has:

$$\begin{aligned} \bar{x} + \bar{y} + \bar{z} &= 0, \\ \bar{x}^5 + \bar{y}^5 + \bar{z}^5 &= s_1^5 + s_5. \end{aligned} \quad (5)$$

Substituting $\bar{x} = \bar{y} + \bar{z}$ into the second equation of (5), we obtain:

$$\bar{y}\bar{z}(\bar{y}^3 + \bar{z}^3) = s_1^5 + s_5.$$

If $s_1^5 + s_5 \neq 0$, then $\bar{z} \neq 0$. Substituting $y^* = \bar{y}/\bar{z}$, we get:

$$(y^*)^4 + y^* = \frac{s_1^5 + s_5}{\bar{z}^5}. \quad (6)$$

Since $m \equiv 2 \pmod{4}$, every element in $\mathbb{F}_{2^m}$ is a fifth power, and (6) has a solution for $(y^*, \bar{z})$ for every $y^* \in \mathbb{F}_{2^m}$ such that $y^* \neq (y^*)^4$, i.e., $y^* \notin \mathbb{F}_{2^2}$. If $s_1^5 + s_5 = 0$, the values $x = s_1$, $y = 0$, $z = 0$ provide a solution to (5).

Thus, $C_{1,5}$ is an $[n, n - 2m, 3; 3]$ -code, proving the statement. $\square$

Next, we compute the parameters b_2, c_2, c_3 of a putative intersection array $\text{IA} = [b_0, b_1, b_2; c_1, c_2, c_3]$ for the $C_{1,5}$ code, without considering specific vectors. If these computations are feasible, we deduce that $C_{1,5}$ is completely regular and obtain its intersection array.

Proposition 3.2 Let $n = 2^m - 1$ with $m \equiv 2 \pmod{4}$. Let $\mathbf{v}$ be a vector of weight 2 at distance 2 from $C_{1,5}$, with non-zero elements $a \neq b$ in its support. The system of equations

$$\begin{aligned}x + y + a + b &= 0, \\x^5 + y^5 + a^5 + b^5 &= 0,\end{aligned}\tag{7}$$

has four distinct solutions.

Proof Substituting $x = y + a + b$, we obtain:

$$(y + a)(y + b)[(a + b)y^2 + (a + b)^2y + a^3 + b^3] = 0.$$

The solutions are $y = a$, $y = b$, and the solutions to $y^2 + (a + b)y + (a^2 + ab + b^2) = 0$, which are $y = \beta a + \gamma b$ and $y = \gamma a + \beta b$, where $\beta = \alpha^{(2^m-1)/3}$, $\gamma = \beta^2$.

Thus, the system admits exactly four distinct solutions: $(x = a, y = b)$, $(x = b, y = a)$, $(x = \gamma a + \beta b, y = \beta a + \gamma b)$, and $(x = \beta a + \gamma b, y = \gamma a + \beta b)$. $\square$

Theorem 3.3 Let $n = 2^m - 1$ with $m \equiv 2 \pmod{4}$. The cyclic code $C_{1,5}$ is an $[n, n - 2m, 3; 3]$ -completely regular code with

$$\text{IA} = \left[n, n - 3, \frac{3n + 7}{4}; 1, 4, \frac{n - 3}{4} \right].$$

Proof For $C_{1,5}$, we have $b_0 = n$, $c_1 = 1$, $a_1 = 2$, so $b_1 = n - 3$.

To compute c_2 , we take a vector of weight 2 and determine how many neighbors of weight 1 it has in $C(1)$. From Proposition 3.2, it follows that $c_2 = 4$.

To compute c_3 , consider a vector $\mathbf{v}$ of weight 6 containing three fixed values a, b, c in its support, at distance 3 from the code. We compute the number of solutions to:

$$\begin{aligned}x + y + z + a + b + c &= 0, \\x^5 + y^5 + z^5 + a^5 + b^5 + c^5 &= 0,\end{aligned}\tag{8}$$

equivalently:

$$\begin{aligned}\bar{x} + \bar{y} + \bar{z} &= 0, \\\bar{x}^5 + \bar{y}^5 + \bar{z}^5 &= s_1^5 + s_5,\end{aligned}\tag{9}$$

where $s_1 = a + b + c$, $s_5 = a^5 + b^5 + c^5$, $\bar{x} = x + s_1$, $\bar{y} = y + s_1$, $\bar{z} = z + s_1$.

Letting $y^* = \bar{y}/\bar{x}$, we count the solutions to:

$$(y^*)^4 + y^* = \frac{s_1^5 + s_5}{\bar{x}^5},\tag{10}$$

for $\bar{x}$, varying y^* , assuming $\bar{x} \neq 0$ (since otherwise, $x = s_1$, and $\mathbf{v}$ would not have weight 6). Equation (10) has a solution for $\bar{x}$ if $(y^*)^4 + y^* \neq 0$, i.e., $y^* \notin \{0, 1, \beta, \gamma\}$. Thus, $\bar{x}$ has $2^m - 4$ solutions. For each, there is a unique solution for x, y, z . However, the $2^m - 4$ solutions are not all distinct, as $(y_1^*)^4 + y_1^* = (y_2^*)^4 + y_2^*$ implies four values of y^* yield the same $\bar{x}$. Thus, $c_3 = \frac{2^m - 4}{4} = 2^{m-2} - 1$.

To compute b_2 , we use a_2 . Consider the system:

$$\begin{aligned}x + y + z + a + b &= 0, \\x^5 + y^5 + z^5 + a^5 + b^5 &= 0,\end{aligned}\tag{11}$$

where x, y, z are non-zero and distinct from each other and from a, b . Let $s_1 = a + b$, $s_5 = a^5 + b^5$. Note that $s_1^5 + s_5 \neq 0$, since $s_1^5 + s_5 = ab(a^3 + b^3) = ab(a+b)(a^2 + ab + b^2) = 0$ implies $a = 0$, $b = 0$, $a = b$, or $a, b \in \mathbb{F}_{2^2}$, none of which hold, as otherwise the vector with support (a, b) would not be at distance 2 from the code.

We count the solutions to:

$$(y^*)^4 + y^* = \frac{s_1^5 + s_5}{\bar{z}^5}, \quad (12)$$

where $s_1 = a + b$, $s_5 = a^5 + b^5$, $\bar{x} = x + s_1$, $\bar{y} = y + s_1$, $\bar{z} = z + s_1$, $y^* = \bar{y}/\bar{z}$. The constraints are $x \neq 0$, $y \neq 0$, $z \neq 0$, $x \neq z$, $y \neq z$, $x \neq y$. These correspond to $y^* \neq 0$ ($x = z$), $y^* \neq 1$ ($y = z$), $\bar{z} \neq 0$ ($x = y$), and other exclusions from Proposition 3.2. The set of exceptions for y^* is:

$$\left\{ 0, 1, \frac{b}{s_1}, \frac{a}{s_1}, \frac{\gamma a + \beta b}{s_1}, \frac{\beta a + \gamma b}{s_1}, \frac{s_1}{b}, \frac{s_1}{a}, \frac{s_1}{\gamma a + \beta b}, \frac{s_1}{\beta a + \gamma b}, \frac{b}{a}, \frac{a}{b}, \frac{\gamma a + \beta b}{\beta a + \gamma b}, \frac{\beta a + \gamma b}{\gamma a + \beta b} \right\}. \quad (13)$$

Moreover, y^* , $y^* + 1$, $y^* + \beta$, $y^* + \gamma$ yield the same value for $(y^*)^4 + y^*$. Thus, the set of exceptions is:

$$E(y^*) \cup E(y^* + 1) \cup E(y^* + \beta) \cup E(y^* + \gamma) = \left\{ 0, 1, \frac{b}{s_1}, \frac{a}{s_1}, \frac{\gamma a + \beta b}{s_1}, \frac{\beta a + \gamma b}{s_1}, \frac{s_1}{b}, \frac{s_1}{a}, \frac{s_1}{\gamma a + \beta b}, \frac{s_1}{\beta a + \gamma b}, \frac{b}{a}, \frac{a}{b}, \frac{\gamma a + \beta b}{\beta a + \gamma b}, \frac{\beta a + \gamma b}{\gamma a + \beta b}, \frac{\beta a + b}{a}, \frac{\gamma a + b}{a}, \frac{a + \beta b}{b}, \frac{a + \gamma b}{b}, \frac{\gamma b}{\beta a + \gamma b}, \frac{\gamma a}{\beta a + \gamma b}, \frac{\beta b}{\beta a + \gamma b}, \frac{\gamma b}{\gamma a + \beta b}, \frac{\gamma a}{\gamma a + \beta b}, \frac{\beta a}{\gamma a + \beta b} \right\}. \quad (14)$$

Thus, (12) has a solution for $\bar{z}$ for any y^* except these 24 values. Since four values of y^* yield the same $\bar{z}$, we have $a_2 = \frac{2^m - 24}{4} = \frac{n - 23}{4}$.

Finally, $b_2 = n - a_2 - c_2 = n - \frac{n - 23}{4} - 4 = \frac{3n + 7}{4}$. The statement is proved. $\square$

Remark 3.4 As an example (e.g., using MAGMA [12]), we see that for $m = 6$ the extended code $C_{1,5}^*$ turns out to be completely regular with $\text{IA} = [64, 63, 60, 49; 1, 4, 15, 64]$. This result can be generalized to any $m \equiv 2 \pmod{4}$.

For all codes $C_{1,5}^*$, the dual code $C_{1,5}^\perp$ has three non-zero weights w_1, w_2, w_3 such that $w_1 + w_3 = 2w_2 = n + 1$ [[13], Lemma 3.1]. Now, using a similar proof to [[14], Proposition 1], we obtain the following result.

Proposition 3.5 Let $n = 2^m - 1$ with $m \equiv 2 \pmod{4}$. The extended cyclic code $C_{1,5}^*$ is a completely regular $[n + 1, n - 2m, 4; 4]$ -code with

$$\text{IA} = \left[n + 1, n, n - 3, \frac{3n + 7}{4}; 1, 4, \frac{n - 3}{4}, n + 1 \right].$$

Remark 3.6 Using a known concatenation technique [15], the $[n, n - 2m, 3; 3]$ -code $C_{1,5}$ can be derived from a quaternary additive code of length $\frac{n}{3}$ and size $4^{\frac{n}{3} - m}$. This code is completely regular and shares the same intersection array, demonstrating the existence of such quaternary codes.

The code $C_{1,5}$ has minimum distance 3, and the supports of the minimum-weight code-words partition the set of coordinates (Theorem 3.1). We can reorder the coordinates such

that this partition is $(0, 1, 2), (3, 4, 5), \dots, (n-2, n-1, n)$. The corresponding columns h_i, h_{i+1}, h_{i+2} of the parity-check matrix, for $i = 0, 3, 6, \dots$, satisfy $h_i + h_{i+1} + h_{i+2} = 0$, meaning the rows of the parity-check matrix contain only 000, 011, 101, or 110 in these triples. Replacing these binary triples with the elements $0, 1, \alpha, \alpha^2$ of $\mathbb{F}_4$, we obtain a parity-check matrix (in the additive sense) of an additive quaternary code. By [[15], Theorem 6 (iii)], the coset graphs of the original binary and resulting quaternary codes are isomorphic. Thus, both codes are completely regular with the same intersection array.

To finish the section, we translate the results about completely regular codes to coset graphs of the constructed codes, which are distance-regular graphs.

Theorem 3.7 *Let $C_{1,5}$ be the cyclic code of length $n = 2^m - 1$ generated by $m_1(x)m_5(x)$, where $m \equiv 2 \pmod{4}$. The coset graph $\Gamma_{C_{1,5}}$ (respectively, $\Gamma_{C_{1,5}^*}$) is a distance-regular graph of diameter $\rho = 3$ (respectively, $\rho = 4$) with intersection array $\text{IA} = [n, n-3, \frac{3n+7}{4}; 1, 4, \frac{n-3}{4}]$ (respectively, $\text{IA} = [n+1, n, n-3, \frac{3n+7}{4}; 1, 4, \frac{n-3}{4}, n+1]$).*

These families of distance-regular graphs appear to be new. However, for the specific case $m = 6$, the distance-regular graph $\Gamma_{C_{1,5}}$ has the same intersection array as the distance-regular graph described in [[16], Proposition 3.1(i)].

Additionally, for the graph $\Gamma_{C_{1,5}^*}$, we have $a_i = 0$ for $i \in \{0, 1, 2, 3, 4\}$, so the graph is bipartite. By [[11], Proposition 2.13], the halved graphs of a bipartite distance-regular graph are distance-regular, and in this case, they are strongly regular graphs (since the diameter is 2). From [[17], Theorem 1.2.2] the intersection array for these halved graphs is:

$$\text{IA} = \left[\frac{n(n+1)}{4}, \frac{(n-3)(3n+7)}{16}; 1, \frac{(n-3)(n+1)}{16} \right].$$

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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