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Mixed Simultaneous Cyclicity for Piecewise Smooth Vector Fields Defined on an Invariant Sphere

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Abstract
In this work we consider the class of partially integrable 3-dimensional piecewise smooth vector fields $Y = (X^+, X^-)$ with separation set $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and first integral $H(x, y, z) = x^2 + y^2 + z^2$ that leaves invariant any sphere centered at the origin, $\mathbb{S}^2_\rho = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \rho^2\}$. We denote this class by $\mathcal{X}$ and by $\mathcal{X}_n$ when $X^\pm$ are polynomial vector fields of degree n. Our main goal is to study piecewise smooth vector fields in the class $\mathcal{X}_1$ presenting three periodic annuli on the invariant sphere $\mathbb{S}^2_1$ proving that there exists a mixed simultaneous configuration with at least five limit cycles bifurcating simultaneously of them, considering polynomial perturbations inside the class $\mathcal{X}_2$.

Keywords Piecewise smooth vector fields on an invariant sphere · Integrability · Mixed simultaneous cyclicity

Mathematics Subject Classification Primary 34C07 · Secondary 34C23 · 37C27

1 Introduction and Statement of the Main Result

The 16th Hilbert Problem aims to find an upper bound for the number of limit cycles that a planar polynomial vector field of degree n can present. This problem, as well as the local cyclicity problem, was initially proposed for smooth vector fields and later extended to the piecewise smooth case (see, for instance, [1, 12, 16, 18, 20, 28] and references therein). The same process occurs with several other problems that were originally studied for smooth fields

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and have recently been extended to piecewise smooth fields. For example, the Poincaré-Hopf Theorem and the structural stability problem, see [2, 5], for instance. In particular, these two last problems deal with smooth (or piecewise smooth) vector fields on compact manifolds.

Another problem related to the 16th Hilbert Problem and that follows the same approach as the cyclicity problem is the mixed simultaneous cyclicity problem (we refer the reader to [7, 8, 26] and references therein). In [7, 26], for example, the authors study the simultaneous cyclicity for smooth vector fields. Using Abelian integrals, the mixed simultaneous cyclicity of planar piecewise smooth vector fields was studied in [8], considering smooth centers with piecewise smooth perturbations. Observe that when we deal with piecewise smooth vector fields, the simultaneous cyclicity can also be studied considering monodromic invisible-invisible fold-fold points.

It is important to recall the two main bifurcation methods: the averaging and the degenerated Hopf bifurcation. The first is equivalent to the analysis of the zeros of the well-known Abelian integrals when the considered perturbation is autonomous. The second is used for studying the local cyclicity of a planar piecewise smooth vector field in a neighborhood of a monodromic equilibrium point of the center-focus or center-center type. See, for instance, [8, 12, 23]. These are extensions of the namesake tools initially proposed for the study of planar smooth vector fields (see, for example, [1, 6, 9, 29]). On the other hand, although the cyclicity of the invisible-invisible fold-fold point was studied in [14, 19], only recently have two different algorithms with minor computational cost become available. In fact, these two algorithms contribute to the local study being carried out in piecewise smooth vector fields with a monodromic fold-fold point. More specifically, they can be used in the calculation of the first return map defined in a neighborhood of such monodromic points. These two tools are presented in [10, 24]. Another important method used to study cyclicity problems is the higher-order Melnikov functions, which can be applied when dealing with Hamiltonian systems in both the smooth and piecewise cases (see [15, 22]). All these methods contribute to new results being obtained in studies related to the piecewise extended 16th Hilbert problems.

Recall that an m-dimensional system $X : \mathbb{R}^m \to \mathbb{R}^m$ is called *partially integrable* when X admits n independent first integrals with $n < m - 1$. When X has $m - 1$ independent first integrals, it is called *completely integrable*, or simply *integrable*. In the latter case, the trajectories of X are determined by the intersection of its $m - 1$ first integral level curves.

Denote by $\mathfrak{X}$ the class of partially integrable 3-dimensional smooth vector fields with first integral $H(x, y, z) = x^2 + y^2 + z^2$. This means that all the spheres centered at the origin, $\mathbb{S}^2_\rho = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \rho^2\}$, are invariant under the flow of $X \in \mathfrak{X}$. Denote by $\mathfrak{X}_n$ the restriction of the class $\mathfrak{X}$ to polynomial vector fields of degree n. We also consider the class $\mathcal{X}$ of partially integrable 3-dimensional piecewise smooth vector fields $Y = (X^+, X^-)$ with separation set $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and first integral $H(x, y, z) = x^2 + y^2 + z^2$, which leaves any sphere centered at the origin invariant. This means that $Y = (X^+, X^-) \in \mathcal{X}$ is given by

$$Y(x, y, z) = \begin{cases} X^+(x, y, z), & z \geq 0, \\ X^-(x, y, z), & z \leq 0, \end{cases} \tag{1}$$

with $X^+, X^- \in \mathfrak{X}$. We follow Filippov's convention as presented in [11] throughout this work. Consider also the restriction of the class $\mathcal{X}$ to piecewise polynomial vector fields of degree n, and denote it by $\mathcal{X}_n$. Thus, if $Y = (X^+, X^-) \in \mathcal{X}_n$, then $X^\pm \in \mathfrak{X}_n$ are both polynomial vector fields of maximum degree n. It is worth noting that if we restrict the study of $Y \in \mathcal{X}_1$ to an invariant sphere $\mathbb{S}^2_\rho$, we are working with a piecewise linear vector field with separation set $\Sigma = \{(x, y, z) \in \mathbb{S}^2_\rho : z = 0\}$. To simplify the notation, we continue calling

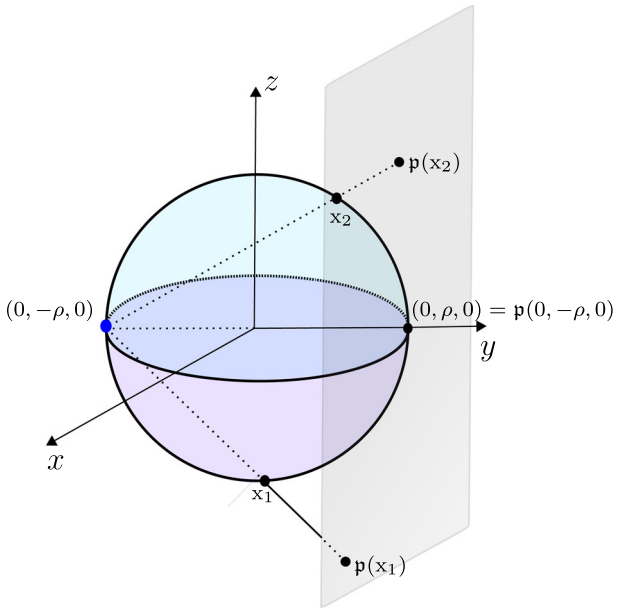


Fig. 1 The stereographic projection with respect to the point $(0, -\rho, 0)$ on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = \rho\}$.

the separation set Σ and the vector fields Y, $X^{\pm}$ restricted to the sphere $\mathbb{S}^2_\rho$ by Σ, Y, and $X^{\pm}$, respectively.

We use the stereographic projection with respect to the point $(0, -\rho, 0)$ on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = \rho\}$, defined by $\mathfrak{p} : \mathbb{S}^2_\rho \setminus \{(0, -\rho, 0)\} \to \mathbb{R}^2$, with $\mathfrak{p}(x, y, z) = 2\rho(x, z)/(y + \rho)$. See Figure 1. This projection allows us to apply planar tools in the study of the local behavior of a piecewise smooth vector field $Y \in \mathcal{X}$ restricted to the sphere $\mathbb{S}^2_\rho$, following the approach in [3, 4]. Using this projection, the projected vector field associated with $X \in \mathfrak{X}$ is defined by

$$\mathcal{P}_X(\mathrm{u}) = d\mathfrak{p}_{\mathfrak{p}^{-1}(\mathrm{u})} \circ X \circ \mathfrak{p}^{-1}(\mathrm{u}),$$

where $X = X_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u} = (u, v)$, and $\mathfrak{p}(\mathrm{x}) = \mathrm{u}$. Since $\mathfrak{p}$ maps $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ to $\mathfrak{p}(\Sigma) = \widetilde{\Sigma} = \{(u, v) \in \mathbb{R}^2 : v = 0\}$, the piecewise projected vector field $\mathcal{P}_Y : \mathbb{R}^2 \to \mathbb{R}^2$ from (1) is defined by

$$\mathcal{P}_Y(\mathrm{u}) = \begin{cases} \mathcal{P}_{X^+}(\mathrm{u}), & v \geq 0, \\ \mathcal{P}_{X^-}(\mathrm{u}), & v \leq 0, \end{cases} \tag{2}$$

where $X^{\pm} = X^{\pm}_{|_{\mathbb{S}^2_\rho}}$, and $\mathrm{u} = (u, v)$. Note that $\mathfrak{p}$ preserves both closed curves and the contact between curves contained in its domain of definition. This allows us to conclude that $p \in \mathbb{S}^2_\rho$ is a monodromic equilibrium point (or a fold-fold point) of (1) if $q = \mathfrak{p}(p)$ is a monodromic equilibrium point (or fold-fold point, respectively) of (2). We also say that $\Gamma \subset \mathbb{S}^2_\rho$ is a (crossing) closed trajectory of (1) if $\mathfrak{p}(\Gamma)$ is a (crossing) closed trajectory of (2). Of course, the definition of a (crossing) limit cycle can be considered in the same way. Moreover, the degree of (2) is always greater than the degree of (1), which increases the difficulty of this study.

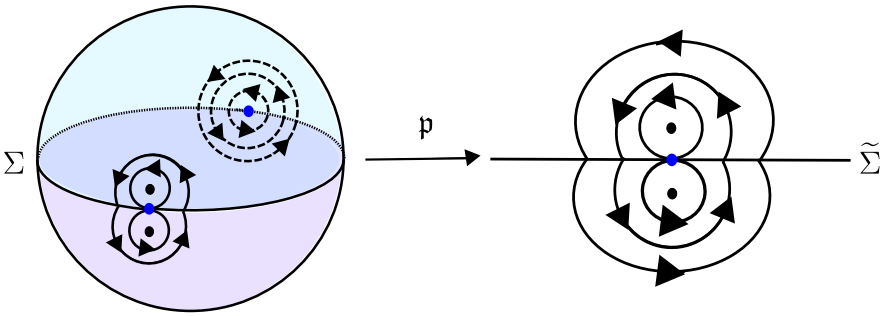


Fig. 2 Configuration of our case of interest and its projection.

In [4], the authors provide a qualitative classification of the phase portraits of $Y \in \mathcal{X}_1$, and some of their properties motivate the study presented in this paper. No piecewise smooth vector field $Y \in \mathcal{X}_1$ has limit cycles. There are configurations of piecewise linear vector fields in $\mathcal{X}_1$ with more than one periodic annulus. Hence, here we focus on piecewise linear vector fields $Y \in \mathcal{X}_1$ presenting two bounded smooth periodic annuli and one unbounded periodic annulus, forming the "figure-eight" configuration shown in Figure 2. Recall that a *periodic annulus* is a region in which all trajectories (crossing or smooth) are periodic. Such a study can be seen as a generalization of those done for analytic potential vector fields in [17, 21]. We point out that since $X \in \mathfrak{X}_1$ does not have saddles (in fact, all equilibrium points of $X \in \mathfrak{X}_1$ are centers; see [3] for more details), the only possible figure-eight configuration in $\mathcal{X}_1$ occurs at fold-fold points.

In this work, we establish lower bounds for the number of limit cycles that bifurcate simultaneously from the piecewise linear vector fields $Y \in \mathcal{X}_1$ under quadratic perturbations in the class $\mathcal{X}_2$. We show that these systems exhibit a *mixed simultaneous configuration of limit cycles* –that is, they produce both smooth and crossing limit cycles at the same time (see Definition 1 for details). To obtain these results, we apply the Lyapunov and Melnikov methods in tandem, even though these techniques are seldom used together; see [8] for a recent illustration. The motivation for combining them in this work is described in Section 4.3. Before summarizing our findings, we formalize the definition of a mixed simultaneous configuration of limit cycles. We also emphasize that, because the same perturbation parameters drive both smooth and crossing bifurcations, the analysis of the unfolding becomes considerably more intricate.

Definition 1 We call a *mixed simultaneous configuration of limit cycles*, denoted by (i, j, k), a configuration of limit cycles bifurcating from $Y = (X^+, X^-) \in \mathcal{X}_1$ restricted to the sphere $\mathbb{S}^2_\rho$, with $\rho > 0$ (see Figure 2), where: $i \in \mathbb{N}$ denotes the number of crossing limit cycles arising from the crossing periodic annulus; $j \in \mathbb{N}$ and $k \in \mathbb{N}$ denote the number of smooth limit cycles arising from the respective periodic annuli of $X^+_{|_{\Sigma^+}}$ and $X^-_{|_{\Sigma^-}}$.

Note that considering the configuration presented in Figure 2 and by choosing the projection point appropriately, the crossing periodic annulus of the projected vector field is unbounded in $\mathbb{R}^2$, while the both smooth periodic annulus are bounded. We use this notion in the following statement.

Theorem 2 *There exists a piecewise linear vector field in the class $\mathcal{X}_1$ that when restricted to the sphere $\mathbb{S}^2_1$ has 3 periodic annuli, two bounded and one unbounded, as in Figure 2,*

for which, with a piecewise quadratic perturbation in the family $\mathcal{X}_2$, one of the following statements holds: (a) At least one small-amplitude smooth limit cycle bifurcates from the center in each bounded periodic annulus; (b) At least four crossing limit cycles bifurcate from the unbounded crossing periodic annulus; (c) A mixed simultaneous configuration of type (3, 1, 1) *bifurcates from all three periodic annulus.*

To prove Theorem 2, we employ second-order Lyapunov quantities for statements (a) and (c), and first-order Melnikov functions for statements (b) and (c). We emphasize that statement (c) does not follow directly from (a) and (b). Moreover, as shown in Section 6, the configuration (4, 1, 1) cannot occur. This illustrates that, even though more limit cycles may bifurcate in each region separately, the analysis of simultaneous bifurcation is more intricate and requires a finer approach, especially in the absence of specialized tools for this problem.

We note that as the limit cycles and crossing limit cycles of Theorem 2 are hyperbolic on the invariant sphere $\mathbb{S}^2_1$, they are normally hyperbolic with respect to the radial direction. So for each limit cycle (crossing limit cycle, respectively) on the invariant sphere $\mathbb{S}^2_1$, the 3−dimensional vector field has a invariant surface foliated by closed trajectories (crossing closed trajectories, respectively), which are locally topologically equivalent to cylinders. This phenomenon was observed and explained in [4].

This paper is structured as follows. First, in Section 2, we present our case of interest and the type of piecewise polynomial perturbation we deal with. In Section 3, we recall the algorithm used to calculate Lyapunov quantities for the so-called degenerate Hopf bifurcation, and we apply it to our system, proving Theorem 2(a). After that, we recall Melnikov method (in Section 4) and apply it in Section 5, proving Theorem 2(b). Finally, in Section 6, we address the mixed simultaneity cyclicity problem, proving Theorem 2(c).

2 Configuration of Interest and its Normal Form

Firstly, we recall some properties of 3-dimensional smooth (piecewise smooth) vector fields in the class $\mathfrak{X}$ ($\mathcal{X}$, respectively) given in [3, 4]. After that, we introduce our object of study. Subsequently, we explain the type of quadratic perturbation we work with in Section 2.3. As mentioned, we follow Filippov's convention presented in [11] throughout this work. To avoid repetition, we do not recall it here.

2.1 Some Background

First, we recall that in [3] it was proved that if $X \in \mathfrak{X}_1$, then

$$X(\mathrm{x}; a_1, a_2, a_3) = (-a_1 y - a_2 z, a_1 x - a_3 z, a_2 x + a_3 y), \tag{3}$$

with $\mathrm{x} = (x, y, z)$. Moreover, $H_2(x, y, z) = a_3 x - a_2 y + a_1 z$ is also a first integral of $X \in \mathfrak{X}_1$, and $X \in \mathfrak{X}_1$ has two equilibrium points of center type on each sphere $\mathbb{S}^2_\rho$, with $\rho > 0$. Using (3), a piecewise linear vector field $Y = (X^+, X^-) \in \mathcal{X}_1$ is of the form

$$Y(x, y, z) = \begin{cases} X^+(\mathrm{x}; a_1^+, a_2^+, a_3^+), & z \geq 0, \\ X^-(\mathrm{x}; a_1^-, a_2^-, a_3^-), & z \leq 0, \end{cases} \tag{4}$$

where

$$X^\pm(\mathrm{x}; a_1^\pm, a_2^\pm, a_3^\pm) = (-a_1^\pm y - a_2^\pm z, a_1^\pm x - a_3^\pm z, a_2^\pm x + a_3^\pm y).$$

In [4] it was proved that when $a_2 \neq 0$, the tangency lines of $X^{\pm}$ are $S_{X^{\pm}} = \{(x, y, z) \in \mathbb{R}^3 : x = -(a_3^{\pm}/a_2^{\pm})y, z = 0\}$, and that $X^{\pm}$ have two fold-fold points on $\mathbb{S}^2_\rho$, for all $\rho \in \mathbb{R}$, $\rho \neq 0$. Moreover, if $a_1^{\pm} \neq 0$, then one of the tangency points of $Y = (X^+, X^-)$ on $\mathbb{S}^2_\rho$, with $\rho > 0$, is visible and the other one is invisible. Besides, when $S_{X^+} = S_{X^-}$, there exist two fold-fold points of Y on each sphere $\mathbb{S}^2_\rho$, with $\rho > 0$, and it occurs if, and only if, $a_2^+ a_3^- = a_2^- a_3^+$.

We also recall that in [4] it was proved that the projected vector field associated with (4), on the sphere $\mathbb{S}^2_\rho$, is given by (2), where

$$\begin{aligned}\mathcal{P}_{X^{\pm}}(u, v) = \big(& -4\rho^2 a_1^{\pm} - 4\rho a_2^{\pm} v - a_1^{\pm} u^2 + 2a_3^{\pm} uv + a_1^{\pm} v^2, \\ & 4\rho^2 a_3^{\pm} + 4\rho a_2^{\pm} u - a_3^{\pm} u^2 - 2a_1^{\pm} uv + a_3^{\pm} v^2\big).\end{aligned} \tag{5}$$

2.2 Our Configuration of Interest

In this section, we present our object of study, showing in the next lemma the projected vector field that we will deal with from now on.

Lemma 3 *The projected system*

$$\mathcal{P}_Y(u, v) = \begin{cases} \big(-4a^+ v + u^2 - v^2 + 4, 2u(2a^+ + v)\big), & v \geq 0, \\ \big(-4a^- v - u^2 + v^2 - 4, 2u(2a^- - v)\big), & v \leq 0, \end{cases} \tag{6}$$

with $a^+ a^- > 0$, restricted to the sphere $\mathbb{S}^2_1$, presents the configuration shown in Figure 2(a).

Proof System (6) is obtained from $Y \in \mathcal{X}_1$, defined in (4), taking $a_1^+ = -1$, $a_1^- = 1$, $a_2^+ = a^+, a_2^- = a^-, a_3^+ = a_3^- = 0$. We note that since $a^{\pm} = a_2^{\pm} \neq 0$, Σ is not invariant under the flow of $\mathcal{P}_Y$. Moreover, we have the extra conditions $a_1^+ = -1, a_1^- = 1$, and $a_3^{\pm} = 0$. Thus, $(a_2^{\pm})^2 + (a_3^{\pm})^2 \neq 0$ and (6) has a fold-fold point at the origin. Note that this fold-fold point is at the boundary of two crossing regions, because $(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f)(u, 0) = 16a^+ a^- u^2 > 0$, with $(u, 0)$ in the separation set $\widetilde{\Sigma}$ of (6). Moreover, $a^+ > 0$ ($a^+ < 0$) gives the (clockwise) counterclockwise orientation. □

Remark 4 Note that in (6) we have fixed the values of $a_3^+ = a_3^- = 0$ introduced in (4). Geometrically, this means that the admissible centers of $X^{\pm}$ belong to the v-axis after the projection. Besides the necessary conditions for system (5) to have the configuration of Figure 1, the condition $a_3^+ = a_3^- = 0$ is the only extra condition imposed in this study. It was added to decrease the number of free parameters of system (5) and, consequently, to simplify the computations to reach the desired results.

2.3 Piecewise Quadratic Perturbations

This section is devoted to introducing the type of perturbations that we work with in the following section and explaining why we need to impose some extra conditions on them.

We consider the following quadratic 3-dimensional perturbation of (6):

$$\widetilde{Y} = (\widetilde{X}^+, \widetilde{X}^-) = (X^+ + \varepsilon X_2^+, X^- + \varepsilon X_2^-), \tag{7}$$

where $\widetilde{X}^{\pm} \in \mathfrak{X}_2$. In other words, we consider a perturbation that keeps all the spheres centered at the origin invariant under the flow of $\widetilde{Y}$. With this kind of perturbation, we ensure that the limit cycles obtained are on the invariant sphere.

The admissible center points of (6) are $p_1 = (0, -2a^+ + 2\sqrt{(a^+)^2+1}) \in \widetilde{\Sigma}^+ = \{(u,v) \in \mathbb{R}^2 : v > 0\}$ and $p_2 = (0, 2a^- - 2\sqrt{(a^-)^2+1}) \in \widetilde{\Sigma}^- = \{(u,v) \in \mathbb{R}^2 : v < 0\}$. As we saw before, we use Lyapunov method to ensure the existence of limit cycles of small amplitude bifurcating from p_1 and p_2. Therefore, we impose that p_1 and p_2 are also equilibrium points of the projection of (7). Under these conditions, we obtain:

$$\mathcal{P}_{\widetilde{Y}}(u,v) = \begin{cases} \mathcal{P}_{\widetilde{Y}^+}(u,v) = (\mathcal{P}^{[1]}_{\widetilde{Y}^+}(u,v), \mathcal{P}^{[2]}_{\widetilde{Y}^+}(u,v)), & v \geq 0, \\ \mathcal{P}_{\widetilde{Y}^-}(u,v) = (\mathcal{P}^{[1]}_{\widetilde{Y}^-}(u,v), \mathcal{P}^{[2]}_{\widetilde{Y}^-}(u,v)), & v \leq 0, \end{cases}$$

where

$$\begin{aligned}
\mathcal{P}^{[1]}_{\widetilde{Y}^+}(u,v) &= (u^2+v^2+4)(-4a^+v + u^2 - v^2 + 4) + \varepsilon \mathcal{P}_1(u,v), \\
\mathcal{P}^{[2]}_{\widetilde{Y}^+}(u,v) &= 2(u^2+v^2+4)(2a^+ + v)u + \varepsilon \mathcal{P}_2(u,v), \\
\mathcal{P}^{[1]}_{\widetilde{Y}^-}(u,v) &= (u^2+v^2+4)(-4a^-v - u^2 + v^2 - 4) + \varepsilon \mathcal{P}_3(u,v), \\
\mathcal{P}^{[2]}_{\widetilde{Y}^-}(u,v) &= 2(u^2+v^2+4)(2a^- - v)u + \varepsilon \mathcal{P}_4(u,v).
\end{aligned}$$

Being the expressions of the projected components $\mathcal{P}_i(u,v), i = 1, \ldots, 4$ detailed as follows:

$$\begin{aligned}
\mathcal{P}_1 =& [((c_1^+ - c_2^+)v^4 + 2(c_3^+ u - 2(c_2^+ a^+ + c_4 + c_5^+))v^3 - 4((2c_6^+ + c_7^+)u - 4(2c_8^+ - c_1^+))v^2 \\
&+ (2c_3^+ u^3 - 4(c_2^+ a^+ - c_4 + c_5^+)u^2 + 8(2c_9^+ - c_3^+)u - 16(c_2^+ a^+ - c_5^+ - c_4))v \\
&+ ((c_2^+ - c_1^+)u^2 + 4(c_7^+ u + c_2^+ + c_1^+))(u^2+4))A^+ \\
&+ 2v((a^+ c_3^+ + c_6^+)u - 2(a^+)^2 c_1^+ - 2(c_5^+ + c_4)a^+ - 2c_8^+)(u^2+v^2+4)]/A^+, \\
\mathcal{P}_2 =& [(c_3^+ u^4 + (+2(c_1^+ - c_2^+)v - 4(c_2^+ a^+ + c_5^+))u^3 + (-4(c_6^+ + 2c_7^+)v + 8(2c_9^+ - c_3^+))u^2 \\
&+ (2(c_1^+ - c_2^+)v^3 - 4(c_2^+ a^+ + 2c_4 + c_5^+)v^2 + 8(2c_8^+ - c_2^+ - c_1^+)v - 16(c_2^+ a^+ - c_5^+))u \\
&- c_3^+ v^4 + 4c_6^+ v^3 + 16c_6^+ v + 16c_3^+)A^+ + ((a^+ c_3^+ + c_6^+)u^2 + (-4(a^+)^2 c_1^+ \\
&- 4(c_5^+ + c_4)a^+ - 4c_8^+)u - (v^2+4)(a^+ c_3^+ + c_6^+))(u^2+v^2+4)]/A^+, \\
\mathcal{P}_3 =& [((c_2^- - c_1^-)v^4 + (-2c_3^- u - 4c_2^- a^- + 4c_4^- + 4c_5^-)v^3 + (4(2c_6^- + c_7^-)u - 8(2c_8^- - c_1^-))v^2 \\
&+ (-2c_3^- u^3 - 4(c_2^- a^- + c_4^- - c_5^-)u^2 - 8(2c_9^- - c_3^-)u - 16(c_2^- a^- + c_5^- + c_4^-))v \\
&- ((c_2^- - c_1^-)u^2 + 4c_7^- u + 4c_2^- + 4c_1^-)(u^2+4))A^- \\
&+ 2v(v^2+u^2+4)((-a^- c_3^- + c_6^-)u - 2c_1^-(a^-)^2 + 2(c_5^- + c_4^-)a^- - 2c_8^-)]/A^-, \\
\mathcal{P}_4 =& [(-c_3^- v^4 + 4c_6^- v^3 + 16c_6^- v + c_3^- u^4 + 2((c_1^- - c_2^-)v + 2(c_2^- a^- - c_5^-))u^3 - 4((c_6^- + 2c_7^-)v \\
&- 2(2c_9^- + c_3^-))u^2 + (2(c_1^- - c_2^-)v^3 + 4(c_2^- a^- - 2c_4^- - c_5^-)v^2 + 8(2c_8^- - c_2^- - c_1^-)v \\
&+ 16(c_2^- a^- + c_5^-))u + 16c_3^-)A^- - (v^2+u^2+4)((c_6^- - a^- c_3^-)u^2 \\
&+ (-4c_1^-(a^-)^2 + (4c_5^- + 4c_4^-)a^- - 4c_8^-)u - (v^2+4)(c_6^- - a^- c_3^-))]/A^-,
\end{aligned}$$

where $A^\pm = \sqrt{(a^\pm)^2+1}$. We note that the expressions of $\mathcal{P}_i$ are quite intricate, they are special degree four polynomials because they come from the quadratic polynomial perturbation (7).

Let $J_{\mathcal{P}_{\widetilde{Y}^\pm}}$ be the Jacobian matrix of $\mathcal{P}_{\widetilde{Y}^\pm}$. By setting

$$c_9^\pm = \mp(c_7^\pm - c_6^\pm)a^\pm - c_3^\pm - \frac{\alpha^\pm}{32}(a^\pm + \sqrt{(a^\pm)^2+1}), \tag{8}$$

we obtain $\mathrm{tr}(J_{\mathcal{P}_{\widetilde{Y}^+}}(p_1)) = \varepsilon\alpha^+$ and $\mathrm{tr}(J_{\mathcal{P}_{\widetilde{Y}^-}}(p_2)) = \varepsilon\alpha^-$, where tr denotes the trace of the Jacobian matrix of the projected vector field at the origin. To simplify the calculations, we assume (8) from now on.

3 Local Cyclicity in a Neighborhood of a Monodromic Equilibrium Point

First, we briefly recall how to obtain the Lyapunov quantities for planar polynomial vector fields. Consider

$$\begin{aligned} \dot{x} &= \alpha x - \beta y + \sum_{k=2}^{n} P_k(x, y), \\ \dot{y} &= \beta x + \alpha y + \sum_{k=2}^{n} Q_k(x, y), \end{aligned} \tag{9}$$

where $P_k(x, y)$ and $Q_k(x, y)$ are homogeneous polynomials of degree k. As a non-degeneracy condition, we impose $\beta \neq 0$. Recall that the Grobman–Hartman Theorem guarantees that if $\alpha \neq 0$, the origin $(0, 0)$ is a hyperbolic focus and the sign of α determines the stability of $(0, 0)$.

If $\alpha = 0$, the origin is a non-hyperbolic equilibrium point and we can apply the Poincaré–Lyapunov Theorem, searching for a function $H(x, y) = x^2 + y^2 + \mathcal{O}_3(x, y)$ which satisfies

$$\dot{H} = \frac{dH}{dt} = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = \sum_{k=2}^{\infty} h_{2k} r^{2k}, \tag{10}$$

where $r^2 = x^2 + y^2$, to study the local behavior of (9) in a neighborhood of the origin. It was proved that the first non-vanishing coefficient of (10) always has an even subscript. So we can write it as h_{2K}. It is called the *focal value* of (9) and it determines the stability of the origin of this system. Moreover, if there exists $K \geq 1$ such that $h_{2K} \neq 0$, we say that the origin of system (9) is a weak focus of order K. Otherwise, we say that the origin is a center. In this case, (9) admits an analytical first integral. It is important to note that the function H plays the role of a Lyapunov function for (9) and that as the differential system (9) is analytic, the Taylor series of $H(x, y) = x^2 + y^2 + \mathcal{O}_3(x, y)$ converges to an analytic first integral of it.

Still considering $\alpha = 0$, a second approach can be used to investigate the stability of the origin. In this case, writing (9) in polar coordinates, we calculate the *Lyapunov quantities* L_k (see [1] for more details). It is a well-known result that the first non-vanishing Lyapunov quantity always has an odd subscript, $k = 2K + 1$, for $K \geq 0$. Moreover, the coefficients h_{2K+2} and L_{2K+1} differ only by a multiplicative nonzero constant. Because of that, we do not distinguish Lyapunov quantities and focal values in what follow and we use the term Lyapunov quantity hereinafter.

Considering the trace parameter being $\alpha = 0$, the classical *Hopf bifurcation* takes place when the first focal value h_2 is non-vanishing. In this case, if $h_2 > 0$ ($h_2 < 0$), the origin is an unstable (stable, respectively) weak focus of order 1 and a stable (unstable) limit cycle bifurcates from the origin when α becomes negative (positive, respectively) and sufficiently small. When $\alpha = 0, h_i = 0$ for $i = 1, \ldots, k-1$ and $h_k \neq 0$, the degenerate Hopf bifurcation occurs. In this case, at most k limit cycles bifurcate from the weak focus of order k. For more details about the results presented in this section, see [1, 9, 27, 28].

Proposition 5 *Consider system* (6)*. Using the second-order of Lyapunov quantities, there exists at least one smooth small amplitude limit cycle bifurcating from the admissible center of* $X^{\pm}_{|_{\Sigma^{\pm}}}$*, on the sphere* $\mathbb{S}^2_1$*, with a quadratic perturbation.*

Proof Following the method explained above, when $\alpha^+ = \alpha^- = 0$, we compute the Taylor series of the first Lyapunov quantities $L_1^{\pm}$ up to second-order in ε and we obtain:

$$\begin{aligned} L_1^{\pm} = \mp \frac{512\varepsilon^2}{\left(3\sqrt{(a^{\pm})^2+1}\right)}\big[&c_1^{\pm}(c_7^{\pm}-c_6^{\pm})(a^{\pm})^2 + c_7^{\pm}c_8^{\pm} + c_1^{\pm}c_6^{\pm} \\ &\pm ((c_4^{\pm}+c_5^{\pm})c_7^{\pm} + c_1^{\pm}c_3^{\pm} - (c_4^{\pm}+c_5^{\pm})c_6^{\pm} + c_3^{\pm}c_8^{\pm})a^{\pm}\big] + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{11}$$

So, there exist free perturbation parameters such that $L_1^+ \neq 0$ and $L_1^- \neq 0$, when $\alpha^+ = \alpha^- = 0$. By the classic Hopf bifurcation, there exists at least one hyperbolic limit cycle of small amplitude bifurcating from the admissible center point of $X^{\pm}_{|_{\Sigma^{\pm}}}$. □

Note that Theorem 2(a) is an immediate consequence of Proposition 5. We point out that we calculate three Lyapunov quantities in the last Proposition and when the first one vanishes, the others also do. So, using this technique, we only obtain one smooth small amplitude limit cycle bifurcating from the admissible center of $X^{\pm}_{|_{\Sigma^{\pm}}}$, on the sphere $\mathbb{S}^2_1$, with a quadratic perturbation. We also observe that the first-order of Lyapunov quantities are identically zero.

4 Melnikov Method for Planar Smooth and Piecewise Smooth Vector Fields

The Melnikov method is a classic and important tool in the study of bifurcations of limit cycles. It is used to analyze the behavior of dynamical systems under small perturbations and to predict, under autonomous perturbations, the occurrence of isolated periodic orbits This method can be applied to both smooth and piecewise smooth vector fields. For more details, see [15, 22].

4.1 Smooth Method

We start by recalling how to obtain the Melnikov functions for planar polynomial vector fields. Let

$$X(\mathrm{x}) = (H_{x_2}(\mathrm{x}), -H_{x_1}(\mathrm{x})), \tag{12}$$

be a planar smooth Hamiltonian system with Hamiltonian function H, where $\mathrm{x} = (x_1, x_2)$. Consider the ε *perturbation* of (12) given by

$$X_\varepsilon(\mathrm{x}) = (H_{x_2}(\mathrm{x}) + \varepsilon f_1(\mathrm{x}), -H_{x_1}(\mathrm{x}) + \varepsilon f_2(\mathrm{x})),$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions and $\varepsilon \in \mathbb{R}$ is the small perturbative parameter. If system (12) has a continuum of periodic orbits $L_h = \{\mathrm{x} \in \mathbb{R}^2 : H(\mathrm{x}) = h > 0\}$ when $\varepsilon = 0$, then

$$\mathrm{M}(h) = \oint_{L_h} (f_2\, dx - f_1\, dy) \tag{13}$$

is called the *first-order Melnikov function* of (12). Moreover, an isolated zero h_0 of odd multiplicity of (13) generates a limit cycle tending to the level H_{h_0} when ε goes to zero. For more details, we refer the reader to [15].

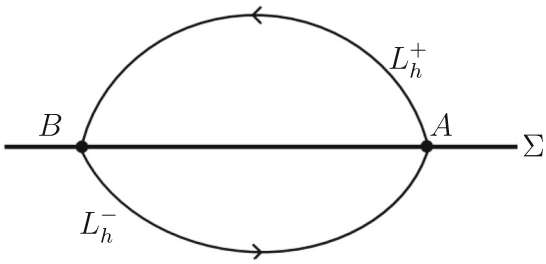


Fig. 3 A closed crossing trajectory of (14).

4.2 Piecewise Method

Now, consider the piecewise planar smooth system

$$Y(\mathrm{x}) = \begin{cases} X^+(\mathrm{x}), & x_2 > 0, \\ X^-(\mathrm{x}), & x_2 < 0, \end{cases} \tag{14}$$

with separation set $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Suppose that $X^{\pm}$ are Hamiltonian systems of the form

$$X^{\pm}(\mathrm{x}) = (H^{\pm}_{x_2}(\mathrm{x}), -H^{\pm}_{x_1}(\mathrm{x})).$$

In this case, (14) is called a piecewise Hamiltonian system. Following the same idea presented above, we consider the perturbative systems

$$X^+_\varepsilon(\mathrm{x}) = (H^+_{x_2}(\mathrm{x}) + \varepsilon f_1^+(\mathrm{x}), -H^+_{x_1}(\mathrm{x}) + \varepsilon f_2^+(\mathrm{x})),$$

and

$$X^-_\varepsilon(\mathrm{x}) = (H^-_{x_2}(\mathrm{x}) + \varepsilon g_1^-(\mathrm{x}), -H^-_{x_1}(\mathrm{x}) + \varepsilon g_2^-(\mathrm{x})),$$

where $f_1^+, f_2^+, g_1^-, g_2^- : \mathbb{R}^2 \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. The piecewise system

$$Y_\varepsilon(\mathrm{x}) = \begin{cases} X^+_\varepsilon(\mathrm{x}), & x_2 > 0, \\ X^-_\varepsilon(\mathrm{x}), & x_2 < 0, \end{cases} \tag{15}$$

is a perturbation of the piecewise Hamiltonian system (14). Now, we suppose that system (15) satisfies the following assumptions when $\varepsilon = 0$:

(I) There is an open interval $J = (\alpha, \beta)$, and two points in the separation set, $A(h) = (a(h), 0)$ and $B(h) = (a_1(h), 0)$, $A(h) \neq B(h)$, such that for $h \in J$,

$$H^+(A(h)) = H^+(B(h)) = h, \qquad H^-(A(h)) = H^-(B(h)).$$

(II) The non-perturbed system $X^+(\mathrm{x}) = (H^+_{x_2}(\mathrm{x}), -H^+_{x_1}(\mathrm{x}))$ has an orbital arc L_h^+ starting from $A(h)$ and ending at $B(h)$ defined by $H^+(x_1, x_2) = h$, $x_2 > 0$. Moreover, $X^-(\mathrm{x}) = (H^-_{x_2}(\mathrm{x}), -H^-_{x_1}(\mathrm{x}))$ has an orbital arc L_h^- starting from $B(h)$ and ending at $A(h)$ defined by $H^-(x_1, x_2) = H^-(B)$, $x_2 \leq 0$.

This means that system (15) has a continuum of crossing closed periodic orbits $L_h = L_h^+ \cup L_h^-$ around the origin when $\varepsilon = 0$, for $h \in J$. For more details, see [22]. We will suppose that L_h has an anticlockwise orientation, as in Figure 3.

The following theorem was proved in [22], where the authors consider a planar piecewise smooth vector field with separation set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$. With a simple change of coordinates, this result could be adapted for our case of interest, as follows.

Theorem 6 *Under the conditions (I) and (II), the first-order Melnikov function of the piecewise smooth vector field (15) is*

$$\mathcal{M}(h) = \frac{H^+_{x_1}(A)}{H^-_{x_1}(A)}\left[\frac{H^-_{x_1}(B)}{H^+_{x_1}(B)}\int_{\widehat{AB}} f_2^+\,dx_1 - f_1^+\,dx_2 + \int_{\widehat{BA}} g_2^-\,dx_1 - g_1^-\,dx_2\right].$$

Moreover, if h_0 is a simple zero (resp. odd multiplicity zero) of $\mathcal{M}$, then for $|\varepsilon|$ small enough, the piecewise smooth vector field (15) has a unique (resp. at least one) limit cycle near L_{h_0}.

4.3 Technical Lemmas: The Melnikov Functions

Firstly, we summarize the results of this section. By Lemma 7, the first-order of the (smooth) Melnikov function $\mathrm{M}^{\pm}$ vanishes identically when the traces of $X^{\pm}_{|_{\Sigma^{\pm}}}$ at their admissible centers, called respectively $\alpha^{\pm}$, are identically null, and they are non-vanishing otherwise. Therefore, to obtain small amplitude limit cycles bifurcating from the admissible centers of $X^+_{|_{\Sigma^+}}$ or $X^-_{|_{\Sigma^-}}$ using the Melnikov method, we need to go further and study the second-order Melnikov function.

As the first-order Melnikov functions and Lyapunov quantities are equivalent (see [13], for instance), this result was expected after Proposition 5. Moreover, since the calculations of higher-order Lyapunov quantities are simpler, as shown in Proposition 5, we do not proceed further with the calculations of Melnikov functions in a neighborhood of the admissible centers of $X^{\pm}_{|_{\Sigma^{\pm}}}$. On the other hand, in Lemma 8, we exhibit the first-order Melnikov function $\mathcal{M}$, which is defined in the separation set Σ of the piecewise smooth vector field (7).

Lemma 7 *Consider system (6). Then, either the first-order Melnikov function of $X^{\pm}_{|_{\Sigma^{\pm}}}$ is identically null or it does not vanish. The first possibility occurs if, and only if, the trace of $X^{\pm}_{|_{\Sigma^{\pm}}}$ at its admissible center point is zero.*

Proof Consider

$$\mathcal{P}(u,v) = \begin{cases} ((-4a^+v + u^2 - v^2 + 4), 2u(2a^+ + v))/(v + 2a^+)^2, & v \geq 0, \\ ((-4a^-v - u^2 + v^2 - 4), 2u(2a^- - v))/(v - 2a^-)^2, & v \leq 0. \end{cases} \tag{16}$$

So, system $\mathcal{P}^+(u,v)$ is given by

$$\begin{aligned} \dot{u}^+ &= \frac{-4a^+v + u^2 - v^2 + 4}{(v + 2a^+)^2}, \\ \dot{v}^+ &= \frac{2u(2a^+ + v)}{(v + 2a^+)^2}. \end{aligned} \tag{17}$$

It is obtained from $\mathcal{P}_{X^+}$ of (6), after the time reparametrization $\tau = (v + 2a^+)^2 t$, which is well defined in Σ^+.

Similarly, system $\mathcal{P}^-(u,v)$, given by

$$\begin{aligned} \dot{u}^- &= \frac{-4a^-v - u^2 + v^2 - 4}{(v - 2a^-)^2}, \\ \dot{v}^- &= \frac{2u(2a^- - v)}{(v - 2a^-)^2}. \end{aligned} \tag{18}$$

is obtained from $\mathcal{P}_{X^-}$ of (6), after the time reparametrization $\tau^- = (v - 2a^-)^2 t$, which is well defined in Σ^-. Moreover, (17) and (18) are Hamiltonian with Hamiltonian functions

H^+ and H^-, respectively. With a straightforward computation, we can see that

$$\widetilde{H}^{\pm}(w) = \frac{4((a^{\pm})^2+1) \pm 2a^{\pm}v + u^2 + v^2}{2a^{\pm} \pm v} - \frac{4((a^{\pm})^2+1) \pm 2a^{\pm}w + w^2}{2a^{\pm} \pm w}, \tag{19}$$

determines the level curves of H^+ (H^-, respectively) passing by the points $(0, w)$, where we fix $w > 0$ ($w < 0$, respectively).

Observe that $\alpha(t) = (U^+(t), V^+(t))$ and $\beta(t) = (U^-(t), V^-(t))$, with

$$U^{\pm} = \frac{t(w^2 \pm 4a^{\pm}w - 4)^2}{(w^4 \pm 8a^{\pm}w^3 + 8(2(a^{\pm})^2 - 1)w^2 \mp 32a^{\pm}w + 16)t^2 + (w^2 \pm 2a^{\pm})^2},$$

$$V^{\pm} = \frac{(w^4 \pm 8a^{\pm}w^3 + 8(2(a^{\pm})^2 - 1)w^2 \mp 32a^{\pm}w + 16)wt^2 \mp 2a^{\pm}w^2 + 4(1 - (a^{\pm})^2)w \pm 8a^{\pm}}{(w^4 \pm 8a^{\pm}w^3 + 8(2(a^{\pm})^2 - 1)w^2 \mp 32a^{\pm}w + 16)t^2 + (w^2 \pm 2a^{\pm})^2},$$

are parametrizations of (19) such that $\lim_{t\to\pm\infty} \alpha(t) = (0, w)$ and $\lim_{t\to\pm\infty} \beta(t) = (0, w)$.

As time reparametrizations do not change the level curves, we can work with the piecewise Hamiltonian system (16) instead of (6). This time reparametrization is necessary because we cannot apply the Melnikov method for non-Hamiltonian systems. After the above analysis, we can calculate the first-order Melnikov functions M^+ and M^- of the quadratic perturbation of the projection of X^+ restricted to Σ^+ and X^- restricted to Σ^-, respectively, following the smooth method explained in Section 4.1. Doing that, we obtain

$$\mathrm{M}^{\pm} = \frac{\varepsilon\alpha^{\pm}\pi \operatorname{sgn}(4a^{\pm}w \pm w^2 \mp 4)(4a^{\pm}w \pm w^2 \mp 4)^2}{128((a^{\pm})^2+1)^{\frac{3}{2}}\left(a^{\pm} - \sqrt{(a^{\pm})^2+1}\right)(2a^{\pm} \pm w)^2}.$$

Since $4a^+w + w^2 - 4 \neq 0$ (resp. $4a^-w - w^2 + 4 \neq 0$) in the integration region of $X^+|_{\Sigma^+}$ (resp. $X^-|_{\Sigma^-}$), we conclude the proof. □

Lemma 8 *Consider system (6). Then the first-order Melnikov function of (7) is given by*

$$\mathcal{M}(w) = f_1 w + f_2 w^3 + \alpha^+ f_3(w) + \alpha^- f_4(w), \tag{20}$$

where

$$f_1 = \frac{16s_1((c_3^- s_2 + c_6^-)(s_1^4 - 1)s_2^2 - (c_3^+ s_1 - c_6^+)(s_2^4 - 1)s_1^2)}{(s_1^2+1)(s_1^2-1)^2(s_2^4-1)},$$

$$f_2 = \frac{4s_1((c_6^- s_2 - c_3^-)(s_1^4 - 1)s_2 + (c_6^+ s_1 + c_3^+)(s_2^4 - 1)s_1)}{(s_1^2+1)(s_1^2-1)^2(s_2^4-1)},$$

$$f_3 = \frac{s_1^2\left(2s_1 w(s_1^2+1)(w^2-4) - (4s_1^2+w^2)(s_1^2w^2+4)\operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right)\right)}{16(s_1^2+1)^3(s_1^2-1)^2},$$

$$f_4 = \frac{s_1 s_2\left(2s_2 w(s_2^2+1)(w^2-4) - (4s_2^2+w^2)(s_2^2w^2+4)\operatorname{arccot}\left(\frac{s_2(w^2-4)}{2(s_2^2+1)w}\right)\right)}{16(s_1^2-1)(s_2^2+1)^2(s_2^4-1)},$$

and s_1, s_2 satisfy $a^+ = (1-s_1^2)/(2s_1)$, $a^- = (1-s_2^2)/(2s_2)$, with $0 < s_1, s_2 < 1$. Moreover, if $s_1 \neq s_2$, (20) is generated by the following four linearly independent functions:

$$\begin{aligned} \mathcal{M}_1 &= w, \quad & \mathcal{M}_3 &= (4s_1^2+w^2)(s_1^2w^2+4)\operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right), \\ \mathcal{M}_2 &= w^3, \quad & \mathcal{M}_4 &= (4s_2^2+w^2)(s_2^2w^2+4)\operatorname{arccot}\left(\frac{s_2(w^2-4)}{2(s_2^2+1)w}\right). \end{aligned} \tag{21}$$

On the other hand, if $s_1 = s_2$, $\mathcal{M}_3 = \mathcal{M}_4$ and (20) is generated by $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$.

Proof At first, we indicate that we use the notation established in the proof of Lemma 7 in the sequence. Following the same approach as Lemma 7, we can see that

$$H_w^{\pm} = \frac{u^2 + v^2 + 4}{2a^{\pm} \pm v} - \frac{w^2 + 4}{2a^{\pm}}, \tag{22}$$

determines the level curve of $H^{\pm}$ passing by the points $(\pm w, 0)$, where we can assume $w > 0$, without loss of generality. Observe that $\alpha(t) = (U^+(t), V^+(t))$ and $\beta(t) = (U^-(t), V^-(t))$ with

$$U^{\pm}(t) = \frac{\pm t w^2 - 2a^{\pm}t^2 w + 2a^{\pm} w \pm 4t}{2a^{\pm}(t^2 + 1)},$$
$$V^{\pm}(t) = \frac{\pm w^2 - 4a^{\pm} t w \pm 4}{2a^{\pm}(t^2 + 1)},$$

are parametrizations of (22) such that $\lim_{t \to -\infty} \alpha(t) = (-w, 0)$, $\alpha(t_0) = (w, 0)$, with $t_0 = (w^2+4)/(4a^+w)$, and $V^+(t) > 0$ for $t \in (-\infty, t_0)$. Besides, $\lim_{t \to \infty} \beta(t) = (-w, 0)$, $\beta(t_0^-) = (w, 0)$, with $t_0^- = -(w^2 + 4)/(4a^-w)$, and $V^-(t) < 0$ for $t \in (t_0^-, \infty)$.

We calculate the first-order Melnikov function $\mathcal{M}$ of (16) following the piecewise method explained in Section 4.2.

Since $\mathcal{M}$ depends on $\sqrt{(a^+)^2 + 1}$ and $\sqrt{(a^-)^2 + 1}$, we use the parametrizations

$$a^+ = \frac{1 - s_1^2}{2s_1} \quad \text{and} \quad a^- = \frac{1 - s_2^2}{2s_2}, \tag{23}$$

with $0 < s_1, s_2 < 1$ to simplify the calculations. Clearly, $s_1 = s_2$ if, and only if, $a^+ = a^-$. After that, we obtain (20). With a straightforward calculation, we can prove that (20) is generated by (21), finishing the proof. □

5 Lower Bounds of Crossing Limit Cycles

In this section, we apply Lemmas 7 and 8 to our study. In Propositions 9 and 10, we use Lemma 8 to provide lower bounds for the number of crossing limit cycles bifurcating from (6) on the sphere $\mathbb{S}_1^2$, with a quadratic perturbation when $a^+ = a^-$ and $a^+ \neq a^-$, respectively. Finally, in Proposition 11, we increase this lower bound, proving Theorem 2(b).

Proposition 9 *Consider system (6) with $a^+ \neq a^-$. Using the first-order Melnikov's method, the lower bound for the number of small amplitude crossing limit cycles that bifurcate from (6), on the sphere $\mathbb{S}_1^2$, under a quadratic polynomial perturbation, is* 3.

Proof Consider $a^+ \neq a^-$. Let $\widetilde{\mathcal{M}}$ be the expansion of (20) in a neighborhood of $w = 0$. Our main goal is to find a sequence of changes of coordinates in the parameter space for which the Melnikov function is described by independent functions.

Using the reparametrization (23) and changing coordinates in the parameter space, introducing b_1 and b_3 instead of c_3^- and c_6^-,

$$\begin{aligned} c_6^- =& \frac{\left(s_2^4-1\right)\left(s_1^2+1\right)\left(s_1^2-1\right)^2 b_1 - 16\left(c_3^- s_1^5 s_2^3 + \left(s_2^4-1\right)\left(c_6^+ s_1^3 - c_6^+ s_1^4\right) - c_3^- s_1 s_2^3\right)}{16(s_1^4-1)s_1 s_2^2}, \\ c_3^- =& \frac{\left((s_1^2-1)^2(b_1-4b_3)+16c_3^+ s_1^2\right)\left(s_2^2-1\right)}{16(s_1^2-1)s_1 s_2}, \end{aligned} \tag{24}$$

we obtain

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}((s_1, s_2, \alpha^+, \alpha^-, b_1, b_3); w) + \mathcal{O}(w^4),$$

such that

$$\widetilde{\mathcal{M}}((s_1, s_2, 0, 0, b_1, b_3) = b_1 w + b_3 w^3.$$

Moreover, by making a new change of coordinates in the parameter space described by:

$$\begin{aligned} \alpha^- =& -\frac{(s_2^2+1)^3(s_2^2-1)(4(s_1^2-1)^2(s_1^2+1)^3 b_2 + \pi\alpha^+(s_1^4+1)s_1^2)}{\pi s_1 s_2 (s_2^4+1)(s_1^4-1)(s_1^2+1)^2}, \\ \alpha^+ =& \frac{(s_1^2-1)^2(s_1^2+1)^3((s_2^4+1)b_0 - 4s_2^2 b_2)}{\pi s_1^2(s_1^2 s_2^2-1)(s_1^2-s_2^2)}, \end{aligned} \tag{25}$$

we obtain

$$\widetilde{\mathcal{M}}((s_1, s_2, b_0, b_1, b_2, b_3); w) = \varepsilon(b_0 + b_1 w + b_2 w^2 + (g_0 b_0 + g_2 b_2 + g_3 b_3) w^3),$$

where

$$\begin{aligned} g_0 =& (s_2^4+1)(s_1^4+1) - 2s_2(s_1^2+1)(s_2^2+1)s_1 - 4s_2^2 s_1^2, \\ g_2 =& -4(s_2^2 s_1^4 + s_2(s_2^2+1)s_1^3 + (s_2^4+4s_2^2+1)s_1^2 + s_2(s_2^2+1)s_1 + s_2^2), \\ g_3 =& 12\pi s_1 s_2((s_1^2+1)s_2 + (s_2^2+1)s_1). \end{aligned}$$

By the derivation division algorithm, it follows that there exist at least 3 small amplitude crossing limit cycles bifurcating from (6) on the sphere $\mathbb{S}_1^2$ with a quadratic perturbation when $s_1 \neq s_2$ or, equivalently, when $a^+ \neq a^-$. □

Proposition 10 *Consider system (6) with $a^+ = a^-$. Using the first-order Melnikov's method, the lower bound for the number of small amplitude crossing limit cycles that bifurcate from (6), on the sphere $\mathbb{S}_1^2$, under a quadratic polynomial perturbation, is* 2.

Proof We follow the same approach as in the last result. Let $a^+ = a^-$ and let $\widetilde{\mathcal{M}}$ be the Taylor series expansion of (20) in a neighborhood of $w = 0$. Using the reparametrization (23) and changing coordinates in the parameter space:

$$\begin{aligned} c_6^- =& \frac{1}{16 s_1^3}(b_1 s_1^6 - (b_1 - 16c_3^+ + 16c_3^-)s_1^4 - 16c_6^+ s_1^3 - b_1 s_1^2 + b_1), \\ c_3^- =& \frac{1}{16 s_1^2}((b_1 - 4b_3)s_1^4 - 2(b_1 - 4b_3 - 8c_3^+)s_1^2 + b_1 - 4b_3), \end{aligned}$$

we obtain

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}((s_1, s_2, \alpha^+, \alpha^-, b_1, b_3); w) + \mathcal{O}(w^4),$$

such that $\widetilde{\mathcal{M}}((s_1, s_2, 0, 0, b_1, b_3) = b_1 w + b_3 w^3$.

Moreover, by changing coordinates introducing b_0 instead of α^-,

$$\alpha^- = -\frac{1}{s_1^4 \pi}(\pi \alpha s_1^4 + b_0(s_1^{10} + s_1^8 - 2s_1^6 - 2s_1^4 + s_1^2 + 1)),$$

we obtain

$$\widetilde{\mathcal{M}}((s_1, s_2, b_0, b_1, b_3); w) = \varepsilon(b_0 + b_1 w + \tilde{g}_1 b_0 w^2 + (\tilde{g}_2 b_0 + b_3) w^3),$$

where $\tilde{g}_1 = (s_1^4 + 1)/(4s_1^2)$ and $\tilde{g}_2 = -(s_1^2 + 1)^3/(12\pi s_1^3)$.

By the derivation division algorithm, it follows that there exist at least 2 small amplitude crossing limit cycles bifurcating from (6), on the sphere $\mathbb{S}_1^2$, under quadratic polynomial perturbation, when $s_1 = s_2$ or, equivalently, when $a^+ = a^-$. □

As we saw in Theorem 6, when $\mathcal{M}$ has an odd multiplicity zero $h_0 \in J$, there exists at least one crossing limit cycle near L_{h_0}. We can use the results presented in [25] to get upper bounds for the number of limit cycles that we can obtain in these cases. To recall these results, we first fix some notations.

Let $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ be an ordered set of $\mathcal{C}^r$-class functions, with $r \geq n$, defined on a closed interval $[a, b]$, let Span$(\mathcal{F})$ be the set of all possible functions generated by linear combinations of elements of $\mathcal{F}$ and let $Z(\mathcal{F})$ be the maximum number of zeros, counting multiplicity, of a function $F \in$ Span$(\mathcal{F})$. We also recall that the Wronskian of $\mathcal{F}$ is defined by $W_k(t) = W_k(u_0, \ldots, u_k)(t) = \det(A(u_0, \ldots, u_k))(t)$, where

$$A(u_0, \ldots, u_k)(t) = \begin{pmatrix} u_0(t) & \cdots & u_k(t) \\ u_0'(t) & \cdots & u_k'(t) \\ \vdots & \ddots & \vdots \\ u_0^{(k)}(t) & \cdots & u_k^{(k)}(t) \end{pmatrix}.$$

When all the Wronskians of the ordered polynomial set $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ are nonvanishing except W_{n-1} and W_n, which have $\mu_{n-1} = k$ and $\mu_n = l$ simple zeros on the open interval (a, b), respectively, we obtain the upper bound $Z(\mathcal{F}) = n + k + l$. Moreover, if the functions u_i, with $i = 0, \ldots, n$ are $\mathcal{C}^\infty$ and all the Wronskians are nonvanishing, except W_n, which has exactly one simple zero on (a, b), then $Z(\mathcal{F}) = n + 1$, and for any configuration of $m \leq n + 1$ zeros there exists an element in Span$(\mathcal{F})$ realizing it. For more details about these results, see Theorem 1.2 and Proposition 4.1 of [25].

By Propositions 9 and 10, the best lower bound for the number of crossing limit cycles is obtained when $a^+ \neq a^-$. It occurs because, in this case, the Melnikov function $\mathcal{M}$ is generated by four linear independent functions. So we fix this assumption in the following analysis.

Proposition 11 *Consider system* (6) *with* $a^+ \neq a^-$. *The lower bound of the number of crossing limit cycles bifurcating from* (6), *on the sphere* $\mathbb{S}_1^2$, *under quadratic polynomial perturbation is* 4.

Proof Let $a^+ \neq a^-$. Following the notation established in the proof of Lemma 8, we know that $\mathcal{M}$ is generated by $\mathcal{M}_1 = w$ and $\mathcal{M}_2 = w^3$, when $\alpha^+ = \alpha^- = 0$. Moreover, when $\alpha^+ \alpha^- \neq 0$, $\mathcal{M}$ is generated by $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, and $\mathcal{M}_4$ given by (21). So, there exist at least 3 crossing limit cycles bifurcating from (6). To obtain an upper bound for the number of limit cycles that bifurcate from (6), applying the results of [25], we need to determine the number of zeros of all the Wronskians of $\mathcal{F} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$. We have, $W_1(w) =$

$W_1(\mathcal{M}_1)(w) = w$, $W_2(w) = W(\mathcal{M}_1, \mathcal{M}_2)(w) = 2w^3$, which never vanish. Unfortunately, we are not able to analyze the sign of $W_3(w) = W(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 - \mathcal{M}_4)(w)$ and $W_4(w) = W(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4)(w)$ for all w, s_1, s_2, analytically.

So we propose the following approach to prove our result. We will suppose that s_1 and s_2 are sufficiently closed. With this idea we will be able to obtain the fourth crossing limit cycle, finishing the proof.

We recall that we are using the parametrizations $a^+ = (1 - s_1^2)/(2s_1)$ and $a^- = (1 - s_2^2)/(2s_2)$, with $0 < s_1, s_2 < 1$, given in (23), and that $\mathcal{M}$ is generated by (21). Let $s_2 = s_1 + \delta$, with δ sufficiently small. The expansion of $\mathcal{M}_4(w)$ in a neighborhood of $\delta = 0$, is given by:

$$\mathcal{M}_4 = \mathcal{M}_3 + \frac{\delta}{2}\left((w^4 + 8s_1^2w^2 + 16)s_1 \operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right) + w(s_1^2-1)(w^2-4)\right) + \mathcal{O}(\delta^2).$$

With a straightforward calculation, it follows that

$$\mathcal{M}_1 = w, \quad \widehat{\mathcal{M}_3} = w^2 \operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right),$$
$$\mathcal{M}_2 = w^3, \quad \widehat{\mathcal{M}_4} = (w^4+16) \operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right),$$

generates $\mathcal{M}$, up to first-order in δ.

In this case, the sequence or ordered Wronskians of $\mathcal{F} = \{\mathcal{M}_1, \mathcal{M}_2, \widehat{\mathcal{M}_3}, \widehat{\mathcal{M}_4}\}$ are:

$$W_1(w) = W(\mathcal{M}_1)(w) = w,$$
$$W_2(w) = W(\mathcal{M}_1, \mathcal{M}_2)(w) = 2w^3,$$
$$W_3(w) = W(\mathcal{M}_1, \mathcal{M}_2, \widehat{\mathcal{M}_3})(w) = -2w^3 \operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right) + \frac{4s_1w^4(s_1^2+1)(s_1^2w^4 - 4(s_1^4 - 4s_1^2 + 1)w^2 + 16s_1^2)(w^2-4)}{(4s_1^2+w^2)^2(s_1^2w^2+4)^2},$$
$$W_4(w) = W(\mathcal{M}_1, \mathcal{M}_2, \widehat{\mathcal{M}_3}, \widehat{\mathcal{M}_4})(w) = \frac{\mathcal{W}_0(s_1, w)}{[\mathcal{W}_3(s_1, w)]^4} + \frac{\mathcal{W}_1(s_1, w)}{[\mathcal{W}_3(s_1, w)]^3} \operatorname{arccot}\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right) - \mathcal{W}_2(s_1, w) \arctan^2\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right) + \pi\mathcal{W}_2(s_1, w) \arctan\left(\frac{s_1(w^2-4)}{2(s_1^2+1)w}\right).$$

Being the expressions of $\mathcal{W}_i$, for $i = 1, \ldots, 4$ as follows:

$$\begin{aligned}
\mathcal{W}_0 =& (16 - w^4)\Big(3(\mathcal{W}_3)^4\pi^2 + 16s_1^2w^2(s_1^2+1)^2(3s_1^4(w^{12}+12288) - 8s_1^2w^2(5s_1^4 - 11s_1^2 + 5)\times \\
& (w^8+256) - 16w^4(9s_1^8 - 20s_1^6 + 41s_1^4 - 20s_1^2 + 9)(w^4+16) + 384w^6\times \\
& (s_1^8 - 6s_1^6 + 4s_1^4 - 6s_1^2 + 1)) - 256s_1^2(s_1^2+1)^2w^2\Big((9s_1^8 - 20s_1^6 + 41s_1^4 - 20s_1^2 + 9)w^8 \\
& -(9s_1^8 - 20s_1^6 + 41s_1^4 - 20s_1^2 + 9)w^6 + 1536s_1^4\Big)\Big), \\
\mathcal{W}_1 =& 16ws_1(w^4+16)(s_1^2+1)(w^2+4)(3s_1^4(w^8+256) - 8s_1^2w^2(s_1^4 - 4s_1^2 + 1)(w^4+16) \\
& + 16(3s_1^8 + 8s_1^6 - 20s_1^4 + 8s_1^2 + 3)w^4), \\
\mathcal{W}_2 =& 12(w^4 - 16), \\
\mathcal{W}_3 =& (4s_1^2 + w^2)(s_1^2w^2 + 4).
\end{aligned}$$

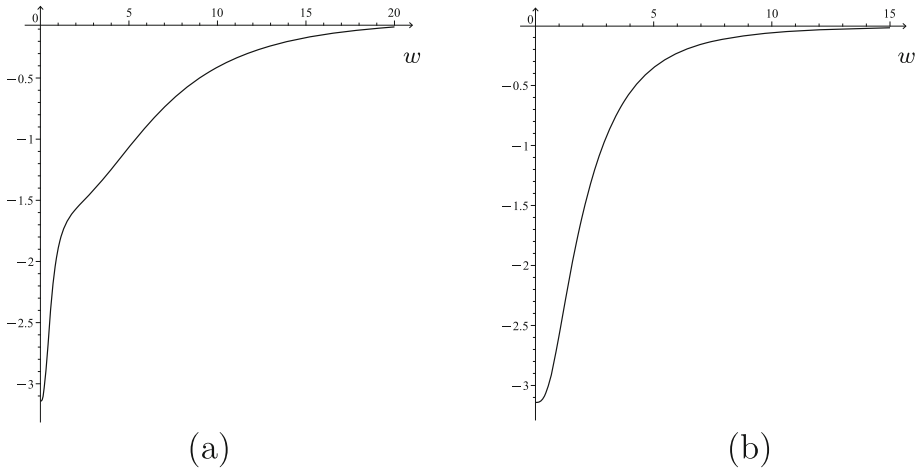


Fig. 4 (a) Graph of $\widetilde{W}_3$ with $s_1 = 3/10$; (b) graph of $\widetilde{W}_3$, with $s_1 = 7/10$.

The property $W_i(w) \neq 0$, for $i = 1, 2$, for all $w > 0$, is immediate.

We assert that $W_3(w) \neq 0$, for all $w > 0$. In fact, for $w > 0$ we can analyze the number of roots of $\widetilde{W}_3 = W_3/(2w^3)$ instead of the number of roots of W_3.

Note that

$$\frac{\partial \widetilde{W}_3}{\partial w} = 64w^2 s_1^3 \frac{(w^6 + 64)(s_1^6 + 1) + 12w^2 s_1^2 (w^2 + 4)(s_1^2 + 1)}{(s_1^2 w^2 + 4)^3 (4s_1^2 + w^2)^3},$$

is a rational function. As $\partial \widetilde{W}_3/\partial w > 0$ is positive if $w > 0$, then it is a increasing function. Moreover,

$$\lim_{w \to 0^+} \widetilde{W}_3 = -\pi, \quad \lim_{w \to \infty} \widetilde{W}_3 = 0.$$

Consequently, the w-axes is an horizontal asymptote of $\widetilde{W}_3(w)$. Thus, $\widetilde{W}_3 < 0$, which implies that $W_3 < 0$ for all $w > 0$. See Figure 4.

Despite not being able to analyze the sign of $W_4(w)$ analytically for all $0 < s_1 < 1$, $w > 0$, we can show that $W_4(w)$ has one zero if $0 < s_1 < \tilde{s}_1$, where $\tilde{s}_1$ is the unique root of $3s_1^8 - 18s_1^6 + 28s_1^4 - 18s_1^2 + 3$ in $(0, 1)$. This is because the asymptotic expansion of $W_4(w)$ near $w = \infty$ is the following

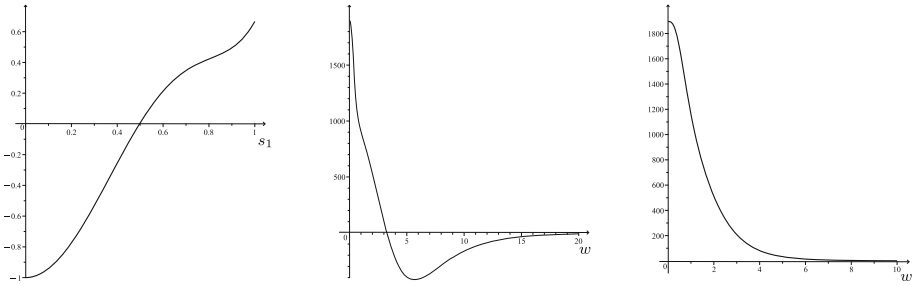
$$W_4^{\infty}(w) = -\frac{1048576}{525} \frac{(s_1^2 + 1)^6}{s_1^{10}} (3s_1^8 - 18s_1^6 + 28s_1^4 - 18s_1^2 + 3) w^6 + \mathcal{O}(w^8).$$

Doing the same analysis near $w = 0$, we obtain

$$W_4^0(w) = 192\pi^2 + \mathcal{O}(w).$$

Thus, W_4 does not vanish in a sufficiently small neighborhood of $w = 0$.

Combining this information, we conclude that $\mathcal{F} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$ is such that $W_i \neq 0$, $i = 1, \ldots, 3$ and W_4 has at least one zero. So, by Proposition 4.1 of [25], the lower bound for the number of crossing limit cycles bifurcating from (6), on the sphere $\mathbb{S}_1^2$, under quadratic polynomial perturbation is 4, when $s_1 \neq s_2$ are sufficiently close and $0 < s_1 < \tilde{s}_1$. □



(a) (b) (c)

Fig. 5 (a) Graph of $\mathcal{C} = \{s_1 \in (0, 1) : 3s_1^8 - 18s_1^6 + 28s_1^4 - 18s_1^2 + 3 = 0\}$; (b) Graph of $\widetilde{W}_4$, with $s_1 = 3/10$; (c) Graph of $\widetilde{W}_4$, with $s_1 = 6/10$.

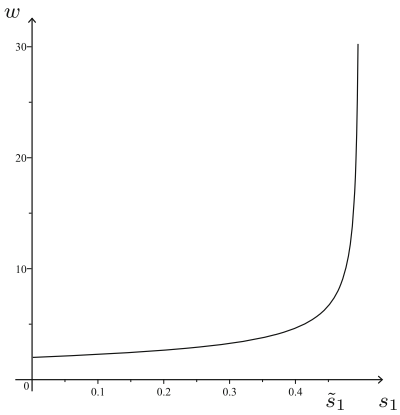


Fig. 6 Numerical roots of $W_4(w)$, for $s_1 \in (0, \tilde{s}_1)$.

The idea of the following analysis is to show numerical evidence that corroborates that 4 can be, in fact, an upper bound for the number of limit cycles bifurcating from (6), on the sphere $\mathbb{S}_1^2$ under quadratic polynomial perturbation when $s_1 \neq s_2$ are sufficiently close. By performing a numerical analysis, we can see that for fixed values of $s_1 \in (0, \tilde{s}_1)$, $W_4(w)$ has only one simple zero. See Figure 6. Thus, applying Proposition 4.1 of [25], it follows that the lower bound for the number of crossing limit cycles bifurcating from (6), on the sphere $\mathbb{S}_1^2$, under quadratic polynomial perturbation, is 4 when $s_1 \in (0, \tilde{s}_1)$. Moreover, we cannot found any root of $W_4(w)$ in $s_1 \in (\tilde{s}_1, 1)$ using the same numerical approach. So, we believe that $W_4(w)$ is sign-defined for these values of the parameter.

As we mentioned in the proof of Proposition 11, we are not able to analyze the sign of $W_3(w) = W(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 - \mathcal{M}_4)(w)$ and $W_4(w) = W(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4)(w)$ for all w, s_1, s_2, analytically. Despite that, we would like to note that we conducted several graphical analyses of the functions $W_3(w)$ and $W_4(w)$, for fixed values of s_1 and s_2 and in all cases, we obtained the same result as explained before, for s_1 sufficiently closed to s_2.

Remark 12 When $a^+ = a^-$, $\mathcal{M}$ is generated by $\mathcal{M}_i$, $i = 1, \ldots, 3$, as given in (21). In this case, using the same analysis that we present before, we can see that $W_i \neq 0$, for $i = 1, 2$ and that W_3 can have at least one zero for small values of s_1.

6 Mixed Simultaneous Cyclicity

The main goal of this section is to prove Theorem 2(c). We investigate a piecewise linear vector field with three periodic annuli from which we obtain a mixed simultaneous configuration with at least five limit cycles bifurcating at the same time with perturbations in the family $\mathcal{X}_2$. We divide the study of this problem into two steps. Firstly, we consider the functions L^+ and L^-, which depends only of $X^+_{|_{\Sigma^+}}$ or $X^-_{|_{\Sigma^-}}$ and that are defined in a neighborhood of the admissible center points of (7), as given in Proposition 5. Secondly, we consider a crossing Melnikov function, which involves both $X^\pm$ and is defined in the crossing region Σ, as given in Lemma 8.

Theorem 13 *Consider system (6) with $a^+ \neq a^-$. Using the first-order of Melnikov function and the second-order of Lyapunov quantities, there exists a mixed simultaneous configuration with at least 5 limit cycles bifurcating concurrently from (6) on the sphere $\mathbb{S}^2_1$ under quadratic polynomial perturbation. Moreover, all the mixed configurations (i, j, k), with $i = 0, \ldots, 3$, $j, k = 0, 1$, are also possible.*

Proof The proof follows applying Proposition 9 and Proposition 5, and observing that there exist a sufficient number of free parameters in the Lyapunov quantities (11) such that after (24) and (25), we can obtain $L^\pm_1 \neq 0$, when $\alpha^+ = \alpha^- = 0$. Thus, these two facts occur simultaneously on the sphere $\mathbb{S}^2_1$ under a quadratic polynomial perturbation: (i) there exists at least 1 small amplitude limit cycle bifurcating from the admissible center point of $X^\pm_{|_{\Sigma^\pm}}$; (ii) there exist at least 3 small amplitude crossing limit cycles bifurcating from (6). Therefore, all the mixed configurations (i, j, k), with $i = 0, \ldots, 3$, $j, k = 0, 1$ are realizable. □

Theorem 14 *Consider system (6) with $a^+ = a^-$. Using the first-order of Melnikov method and the second-order of Lyapunov quantities, there exists a mixed simultaneous configuration with at least 4 of limit cycles bifurcating concurrently from (6) on the sphere $\mathbb{S}^2_1$ under quadratic polynomial perturbation. Moreover, all the mixed configurations (i, j, k), with $i = 0, \ldots, 2$, $j, k = 0, 1$, are also possible.*

Proof The proof is analogous to the proof of Theorem 13, applying Proposition 10 and Proposition 5. □

We remark that Theorem 2 is an immediate consequence of Proposition 5, Theorem 13, and Proposition 11.

Remark 15 Note that to prove Proposition 11, obtaining 4 crossing limit cycles on the sphere $\mathbb{S}^2_1$, we need to assume that $\alpha^+\alpha^- \neq 0$, where $\alpha^\pm$ denotes the trace parameter of $X^\pm_{|_{\Sigma^\pm}}$ at its admissible center points. Therefore, a Hopf-bifurcation cannot occur joint with the bifurcation of these 4-crossing limit cycles. Thus, using the first-order of Melnikov function and the second-order of Lyapunov quantities, the configuration (4, 1, 1) cannot be obtained under a quadratic perturbation.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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