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Periodic orbits of two particular classes of planar continuous piecewise linear differential systems with three zones separated by two parallel straight lines

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Abstract

We study the existence and non-existence of periodic orbits in the planar continuous piecewise linear differential systems with three zones separated by two parallel straight lines. First with the additional hypothesis that the three vector fields have no equilibrium points neither real nor virtual we prove that these differential systems have no periodic orbits. After assuming that the three linear vector fields are Hamiltonian we prove that these differential systems have either no periodic orbits or a continuum of periodic orbits. These results study the periodic orbits of the continuous piecewise linear differential systems with three zones separated by two parallel straight lines that until now were only done for the discontinuous piecewise linear differential systems with three zones separated by two parallel straight lines.

Keywords Continuous piecewise linear differential systems · Linear Hamiltonian vector fields · Periodic orbits · Three zones

Mathematics Subject Classification 34C05 · 34A36 · 37J06

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1 Introduction and contextualization of the problem

The study of planar piecewise differential systems yields several interesting results which often differ from those found in the classical theory of differential equations. A key reference is the book by Andronov et al. [1]. While planar linear differential systems have no limit cycles, planar continuous or discontinuous piecewise differential systems consisting of two linear differential systems separated by a straight line can have limit cycles.

Lum and Chua [15] conjectured that continuous piecewise differential systems separated by one straight line have at most one limit cycle. The proof of this conjecture can be found in [3] and [7].

Another well-known result from the classical theory is that in the region bounded by a periodic orbit of a continuous differential system there exists at least one equilibrium point (see for instance [4]), but discontinuous planar differential systems formed by two linear differential systems without equilibria, neither real nor virtual, and separated by a straight line can have at most one limit cycle, see [11].

Without being exhaustive we will cite some results on the limit cycles of the planar piecewise linear differential systems with three zones separated by two parallel straight lines.

In [12], the authors analyzed continuous and discontinuous piecewise linear planar differential systems formed exclusively by linear centers, separated by one or two parallel lines. They proved that continuous systems separated by one or two lines, as well as discontinuous systems separated by a single line do not have limit cycles. On the other hand, it is proved that these discontinuous differential systems separated by two parallel lines can exhibit at most one limit cycle, and an explicit example is provided in which this limit cycle exists.

In [10], the authors investigated piecewise linear planar differential systems defined in three zones separated by two parallel lines, considering boundary focus-center-focus and focus-center-center configurations. Using Hamiltonian functions of the centers and the Poincaré map, the authors demonstrate that such systems can have at most three limit cycles.

In [14], the authors studied piecewise linear planar differential systems, both continuous and discontinuous, formed by combinations of linear centers and linear Hamiltonian saddles, separated by one or two parallel lines. All results are obtained by studying the level curves of the Hamiltonian functions on the separating lines. The authors showed that continuous or discontinuous systems separated by a line and formed by a linear center and a linear Hamiltonian saddle do not have limit cycles. If the system is continuous, separated by two parallel lines and given by two linear centers and a linear Hamiltonian saddle or a linear center and two linear Hamiltonian saddles, it also does not have limit cycles. On the other hand, with the same combinations of centers and saddles, they demonstrated that discontinuous systems separated by two parallel lines can exhibit at most one limit cycle, and explicit examples are provided in which this limit cycle actually exists.

Based on motivation provided by a piecewise linear mechanical model (represented by a non-linear oscillator dynamics) in [8] the authors investigated the global dynamics of piecewise degenerate linear planar differential systems, symmetrical with respect

to the origin and separated by two parallel straight lines. From the qualitative analysis of finite and infinite equilibria, the authors obtained a complete classification of global phase portraits in the Poincaré disk. Necessary and sufficient conditions were established for the existence of crossing limit cycles, crossing heteroclinic loops, and sliding heteroclinic loops, proving that the maximum number of each of these types of closed orbits is equal to one.

In [16], the authors investigated piecewise linear planar Hamiltonian differential systems, defined in two or three zones separated by parallel lines in the second case, in which the linear vector fields exhibit isolated equilibrium points of the center or saddle type. They demonstrated that such continuous or discontinuous systems with two zones have no limit cycles. In the case of three zones, they proved that continuous systems have no limit cycles, while discontinuous systems can have at most one limit cycle.

In [13], the authors studied piecewise planar differential systems defined in three zones separated by two parallel lines, in which the system is continuous on one of the separating lines and discontinuous on the other, considering linear Hamiltonian systems in each zone. Through an algebraic analysis based on Grobner bases, they demonstrated that such systems have no limit cycles. For the discontinuous case, they showed that there is at most one limit cycle.

In [19], the authors studied the global dynamics of a piecewise linear planar differential system with three zones separated by two parallel lines considering the case where the system is degenerate in the left and central regions and continuous along the separation lines. Using global geometric techniques, it was proved that the system admits at most two limit cycles.

In [2], the authors studied piecewise discontinuous linear planar differential systems defined in three zones separated by two parallel lines, in which the central region is given by a linear Hamiltonian saddle and the lateral regions have isolated singularities of the focus or center type located on the separation lines. The authors distinguished limit cycles of two intersection points and four intersection points and established the maximum number of limit cycles for each configuration.

In [17], the authors analyzed piecewise linear planar differential systems defined in three zones separated by two parallel lines with discontinuity in the separation lines. They showed that, under appropriate conditions, the system can be reduced to a canonical Liénard-type form, obtaining necessary and sufficient conditions for the existence of period annuli and the complete classification of global phase portraits in the Poincaré disk. Furthermore, they proved that a period annulus enclosing a singular segment can give rise to at least three limit cycles.

The authors of the paper [9] studied the dynamics of continuous planar piecewise linear differential systems with three zones separated by two parallel straight lines, where the linear differential systems in the one of the exterior linear zones is saddle and the remaining one is anti-saddle. They provide the phase portraits of these piecewise differential systems in the Poincaré disk and its bifurcation diagram.

In [18], the authors studied the limit cycles of continuous piecewise linear differential systems with three zones separated by two parallel straight lines having two degenerate linear differential systems, i.e. the matrix of these linear differential sys-

tems is zero. They study the maximum number of limit cycles that such piecewise differential systems can exhibit.

2 Statement of the main results

We consider in the plane $\mathbb{R}^2$ piecewise smooth vector fields with three zones separated by two parallel straight lines, that without loss of generality we can assume that they are $\Sigma_L = \{x = -1\}$ and $\Sigma_R = \{x = 1\}$. The three vector fields are denoted by $\mathcal{X}_L$, $\mathcal{X}_C$ and $\mathcal{X}_R$ and they are in the regions $x \leq -1$, $|x| \leq 1$ and $x \geq 1$, respectively. So we are working with the piecewise smooth vector fields

$$\mathcal{F}(x, y) = \begin{cases} \mathcal{X}_L(x, y), & x \leq -1, \\ \mathcal{X}_C(x, y), & |x| \leq 1, \\ \mathcal{X}_R(x, y), & x \geq 1. \end{cases} \tag{1}$$

As usual we identify the piecewise smooth vector fields (1) with the piecewise differential system

$$(\dot{x}, \dot{y}) = \mathcal{F}(x, y), \tag{2}$$

where the dot denotes the derivative with respect to the independent variable t, called time.

In this article, we are interested in the piecewise smooth vector fields $\mathcal{F}$ that are continuous in $\Sigma_L \cup \Sigma_R$, i.e. $\mathcal{X}_L(p) = \mathcal{X}_C(p)$ and $\mathcal{X}_C(q) = \mathcal{X}_R(q)$ for all $p \in \Sigma_L$ and for all $q \in \Sigma_R$.

First, we consider planar piecewise linear differential systems (2) without equilibrium points, neither real nor virtual, as follows

$$\mathcal{X}_L(x, y) = \big(a_L x + b_L y + \gamma_L, \lambda_L(a_L x + b_L y) + \delta_L\big), \quad x \leq -1, \tag{3}$$

$$\mathcal{X}_C(x, y) = \big(a_C x + b_C y + \gamma_C, \lambda_C(a_C x + b_C y) + \delta_C\big), \quad |x| \leq 1, \tag{4}$$

$$\mathcal{X}_R(x, y) = \big(a_R x + b_R y + \gamma_R, \lambda_R(a_R x + b_R y) + \delta_R\big), \quad x \geq 1, \tag{5}$$

where $\delta_k \neq \lambda_k \gamma_k$, $k \in \{L, C, R\}$. Note that the vector fields $\mathcal{X}_L(x, y)$, $\mathcal{X}_C(x, y)$ and $\mathcal{X}_R(x, y)$ define the more general linear vector fields without equilibria because the two straight lines defined by the two components of these vector fields are parallel lines.

For a vector field $\mathcal{X}_k$ with $k \in \{L, C, R\}$ an equilibrium point is real if it belongs to the region where the vector field is defined. While an equilibrium point is virtual if it belongs to a region where the vector field is not defined. When the piecewise differential system has an equilibrium point on $\Sigma_L \cup \Sigma_R$ we call such an equilibrium a boundary equilibrium.

We can state the first result of this article.

Theorem 1 *Consider planar piecewise linear differential systems formed by* (3), (4) *and* (5) *that are continuous in* $\Sigma_L \cup \Sigma_R$. *Then these differential systems do not have periodic orbits.*

The proof of Theorem 1 is given in Sect. 3.

The particular case of the result stated in Theorem 1 when the linear differential systems are Hamiltonian was already proved in [6].

Under the assumptions of Theorem 1, but for discontinuous piecewise linear differential systems separated by two parallel lines, these systems can exhibit periodic orbits, in fact they can have at most one limit cycle, and there are differential systems in this class having one limit cycle, see [6, 12, 14]. Recall that a limit cycle of a differential system is an isolated periodic orbit in the set of all periodic orbits of the differential system.

Now, we consider the vector fields

$$\mathcal{X}_k(x, y) = \left(-\frac{\partial \mathcal{H}_k}{\partial y}(x, y), \frac{\partial \mathcal{H}_k}{\partial x}(x, y)\right),$$

for $k \in \{L, C, R\}$. These are linear Hamiltonian vector fields defined by the Hamiltonian functions

$$\mathcal{H}_L(x, y) = c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2, \quad x \leq -1, \tag{6}$$

$$\mathcal{H}_C(x, y) = b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2, \quad |x| \leq 1, \tag{7}$$

$$\mathcal{H}_R(x, y) = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \quad x \geq 1, \tag{8}$$

with $a_i, b_i, c_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, 5\}$. We remark that there are 27 possibilities for such piecewise vector fields $\mathcal{F}$, depending on the choices of coefficients a_i, b_i, c_i, $i \in \{1, 2, \ldots, 5\}$, because each Hamiltonian vector field can have a saddle, a center, or no equilibrium point in each one of the three pieces.

The main result for the continuous piecewise differential systems formed by the linear Hamiltonian systems is the following one.

Theorem 2 *Consider planar piecewise linear Hamiltonian systems defined by* (6), (7) *and* (8) *that are continuous in* $\Sigma_L \cup \Sigma_R$. *Then these differential systems either they do not have periodic orbits, or they have a continuum of periodic orbits.*

The proof of Theorem 2 is given in Sect. 4.

Again under the assumptions of Theorem 2 for the discontinuous piecewise linear differential systems separated by two parallel lines, these systems can exhibit periodic orbits, in fact they can have at most one limit cycle, and there are differential systems in this class having one limit cycle, see [12, 14, 16].

From Theorems 1 and 2 it follows the next corollary.

Corollary 3 *The piecewise differential systems of Theorems* 1 *and* 2 *do not have limit cycles.*

We note that the result of this corollary, in the particular case where each linear Hamiltonian system has only one equilibrium, was also obtained in [16].

3 Proof of Theorem 1

Denote by $\mathcal{F}$ the piecewise smooth vector field with three zones in the plane associated with the planar piecewise linear systems formed by (3), (4) and (5). Assume that $\mathcal{F}$ is continuous in $\Sigma_L \cup \Sigma_R$, then we have

$$b_L = b_C = b_R = b, \quad \lambda_L = \lambda_C = \lambda_R = \lambda, \ a_C = a_L + \gamma_C - \gamma_L,$$
$$\delta_C = \delta_L + \lambda(a_C - a_L), \ a_R = a_C + \gamma_C - \gamma_R, \ \delta_R = \delta_C + \lambda(a_C - a_R).$$

Therefore, the continuous piecewise smooth vector field $\mathcal{F}$ can be written as

$$\mathcal{F}(x,y) = \begin{cases} \mathcal{X}_L(x,y) = (a_L x + by + \gamma_L, \lambda(a_L x + by) + \delta_L), & x \le -1, \\ \mathcal{X}_C(x,y) = (a_C x + by + \gamma_C, \lambda(a_C x + by) + \delta_C), & |x| \le 1, \\ \mathcal{X}_R(x,y) = (a_R x + by + \gamma_R, \lambda(a_R x + by) + \delta_R), & x \ge 1. \end{cases} \tag{9}$$

Let X be a continuous vector field defined on the unit circle centered at the origin of coordinates $\mathbb{S}^1$. Given a point $p = e^{2\pi i t} \in \mathbb{S}^1$ we identify the vector $X(p) = (X_1(p), X_2(p))$ with the point $\sigma(t) \in \mathbb{R}^2$ such that $\overrightarrow{0\sigma(t)} = X(p)$. This define a continuous map $\sigma : [0,1] \to \mathbb{R}^2$ where $t \mapsto \sigma(t)$ with $\sigma(0) = \sigma(1)$. Then σ is a closed path in $\mathbb{R}^2$. If $X(p) \neq 0$ for all $p \in \mathbb{S}^1$, then the origin of coordinates cannot belong to the image of the closed path σ. We can define the *topological index* $i(X)$ as the number of revolutions around the origin of coordinates, taking into account their orientation, that the vector field X makes when moving along $\mathbb{S}^1$.

The next result is proved in Proposition 6.16 of [5].

Proposition 4 *Let X be a continuous vector field defined on the closed unit disk $\mathbb{D}^2$ such that $X(p) \neq 0$ for every p in the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$. If $i(X) \neq 0$ then there is a point $p \in \mathbb{D}^2$ such that $X(p) = 0$.*

Proof of Theorem 1 Assume that γ is a periodic orbit of the continuous piecewise smooth vector field $\mathcal{F}$ given in (9), and we shall arrive to a contradiction. Let R be the closed region bounded by the periodic orbit γ. There exists a homeomorphism $h : R \to \mathbb{D}^2$. Then $X = h(\mathcal{F})$ is a continuous vector field in $\mathbb{D}^2$. The vector fields X and $\mathcal{F}$ are topologically equivalent. Since the vector field $\mathcal{F}$ does not vanish on the boundary of R, the vector field X does not vanish on $\mathbb{S}^1$. Therefore, by Proposition 4 the continuous vector field X has an equilibrium point in $\mathbb{D}^2$, and consequently the continuous vector field $\mathcal{F}$ has an equilibrium point in R. This is a contradiction because by assumption the vector field $\mathcal{F}$ has no equilibria. □

4 Proof of Theorem 2

The Hamiltonian vector fields defined by the Hamiltonian functions in (6), (7) and (8) are

$$\begin{aligned}
\mathcal{X}_L(x, y) &= (-c_2 - c_4 x - 2c_5 y, c_1 + 2c_3 x + c_4 y), \quad x \le -1,\\
\mathcal{X}_C(x, y) &= (-b_2 - b_4 x - 2b_5 y, b_1 + 2b_3 x + b_4 y), \quad |x| \le 1,\\
\mathcal{X}_R(x, y) &= (-a_2 - a_4 x - 2a_5 y, a_1 + 2a_3 x + a_4 y), \quad x \ge 1,
\end{aligned}$$

respectively.

From the hypothesis of continuity of $\mathcal{F} = (\mathcal{X}_L, \mathcal{X}_C, \mathcal{X}_R)$ in $\Sigma_L \cup \Sigma_R$ we have the system of equations

$$\begin{aligned}
b_2 - b_4 - c_2 + c_4 + 2(b_5 - c_5)y = 0,\\
-b_1 + 2b_3 + c_1 - 2c_3 + (c_4 - b_4)y = 0,\\
-a_2 - a_4 + b_2 + b_4 - 2(a_5 - b_5)y = 0,\\
a_1 + 2a_3 - b_1 - 2b_3 + (a_4 - b_4)y = 0,
\end{aligned}$$

which has the solution

$$\begin{aligned}
&b_2 = a_2, \quad b_4 = a_4, \quad b_5 = a_5, \quad b_3 = \frac{a_1 - b_1}{2} + a_3,\\
&c_2 = a_2, \quad c_4 = a_4, \quad c_5 = a_5, \quad c_3 = \frac{a_1 + c_1}{2} + a_3 - b_1.
\end{aligned} \tag{10}$$

Using (10) the Hamiltonian functions (6), (7) and (8) can be written as

$$\mathcal{H}_L(x, y) = c_1 x + a_2 y + \left(\frac{a_1 + c_1}{2} + a_3 - b_1\right) x^2 + a_4 xy + a_5 y^2, \quad x \le -1, \tag{11}$$

$$\mathcal{H}_C(x, y) = b_1 x + a_2 y + \left(\frac{a_1 - b_1}{2} + a_3\right) x^2 + a_4 xy + a_5 y^2, \quad |x| < 1, \tag{12}$$

$$\mathcal{H}_R(x, y) = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2. \quad x \ge 1. \tag{13}$$

Clearly that the continuous piecewise linear Hamiltonian systems separated by two parallel straight lines have no limit cycles, and no periodic orbits contained in one of the three regions, except if in one of these regions there is a linear Hamiltonian system having a center. Then depending if the center is real or virtual, the piecewise linear Hamiltonian system has a continuum of periodic orbits, or no periodic orbits, respectively.

From Theorem 1 and [16] we know that the continuous piecewise linear Hamiltonian systems separated by two parallel straight lines have no limit cycles contained in two of the three regions, but eventually they can have a continuum of periodic orbits when the two consecutive linear Hamiltonian systems have centers or saddles.

To have a periodic orbit that intersect the straight line $x = 1$ at the points $(1, y_1)$ and $(1, y_2)$ with $y_1 < y_2$, and the straight line $x = -1$ at the points $(-1, y_3)$ and

$(-1, y_4)$ with $y_3 > y_4$, it is necessary that

$$\begin{aligned} \mathcal{H}_R(1, y_1) - \mathcal{H}_R(1, y_2) = 0, \quad y_1 \neq y_2, \\ \mathcal{H}_C(1, y_2) - \mathcal{H}_C(-1, y_3) = 0, \\ \mathcal{H}_L(-1, y_3) - \mathcal{H}_L(-1, y_4) = 0, \quad y_3 \neq y_4, \end{aligned}$$

or equivalently

$$a_2 + a_4 + a_5\, y_1 + a_5\, y_2 = 0, \tag{14}$$

$$2b_1 + (a_2 + a_4)\, y_2 + a_5\, y_2^2 + (a_4 - a_2)\, y_3 - a_5\, y_3^2 = 0, \tag{15}$$

$$a_2 - a_4 + a_5\, y_3 + a_5\, y_4 = 0. \tag{16}$$

Equations (14), (15) and (16) have a continuum of solutions when y_1 varies, given by

$$y_2 = -\frac{a_2 + a_4 + a_5\, y_1}{a_5}, \tag{17}$$

$$y_3 = -\frac{a_2 - a_4 + \sqrt{(a_2 - a_4)^2 + 8a_5 b_1 + 4a_5\, y_1\, (a_2 + a_4 + a_5\, y_1)}}{2a_5}, \tag{18}$$

$$y_4 = \frac{-a_2 + a_4 + \sqrt{(a_2 - a_4)^2 + 8a_5 b_1 + 4a_5\, y_1\, (a_2 + a_4 + a_5\, y_1)}}{2a_5}, \tag{19}$$

under the conditions

$$(a_2 - a_4)^2 + 8a_5 b_1 + 4a_5\, y_1\, (a_2 + a_4 + a_5\, y_1) > 0, \quad y_1 < y_2, \quad y_3 > y_4. \tag{20}$$

This implies that the planar continuous piecewise linear Hamiltonian systems defined by the Hamiltonian functions (11), (12) and (13) have a continuum of periodic orbits in the three zones, or no periodic orbits if the conditions (20) do not hold.

We will conclude the proof of Theorem 2 by giving an example of a continuous planar piecewise linear Hamiltonian system defined from the Hamiltonian functions (11), (12) and (13) with a continuum of periodic solutions intersecting the three zones.

With the values

$$a_2 = 0, \quad a_4 = -1, \quad a_5 = -1, \quad b_1 = 0,$$

equations (17), (18) and (19) have the solutions

$$y_2 = -1 - y_1, \quad y_3 = \left(1 + \sqrt{1 + 4(1 + y_1)y_1}\right)/2, \quad y_4 = \left(1 - \sqrt{1 + 4(1 + y_1)y_1}\right)/2.$$

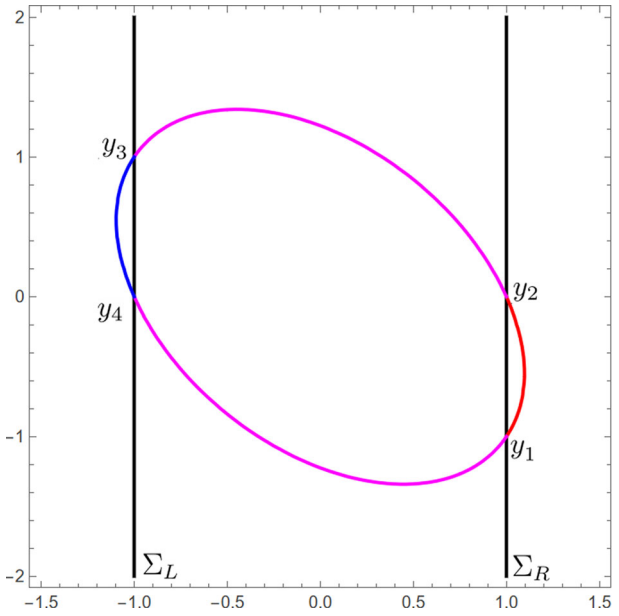

Fig. 1 One of the periodic orbits of the continuous piecewise linear Hamiltonian system (21), (22) and (23)

The Hamiltonian functions (6), (7) and (8) now are

$$\mathcal{H}_L(x, y) = x - x^2 - xy - y^2, \quad x \leq -1, \tag{21}$$

$$\mathcal{H}_C(x, y) = -\frac{3x^2}{2} - xy - y^2, \quad |x| \leq 1, \tag{22}$$

$$\mathcal{H}_R(x, y) = -x - x^2 - xy - y^2, \quad x \geq 1. \tag{23}$$

See an illustration of the periodic orbit with $y_1 = -1$ of this continuous piecewise linear Hamiltonian system in Figure 1. Hence Theorem 2 is proved.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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