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Pitt's Inequalities and Bounds for Dirichlet Kernels

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Abstract
We establish an analogue of Pitt's inequality for Fourier series on $\mathbb{T}^d$, extending the range of parameters by employing averages of Fourier coefficients. We show that this inequality is closely connected to sharp estimates for the L_p norms of the Dirichlet kernels over various subsets of $\mathbb{Z}^d$.

Keywords Fourier series · Pitt's inequality · Dirichlet kernel

Mathematics Subject Classification Primary 42B10 · Secondary 46E30 · 26D10

1 Introduction

Classical Pitt's inequality states that, for an integrable function $f \sim \sum_{m\in\mathbb{Z}^d} a_m e^{\imath(m,x)}$, there holds

$$\left(\sum_{k\in\mathbb{Z}^d} (1+|k|)^{\beta dq}\ |a_k|^q\right)^{1/q} \lesssim \left(\int_{\mathbb{T}^d} |x|^{\alpha dr}\,|f(x)|^r\ dx\right)^{1/r} \tag{1.1}$$

provided that

$$1 < r \leqslant q < \infty \qquad \text{and} \qquad \max\left(0, \frac{1}{r'} - \frac{1}{q}\right) \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}, \tag{1.2}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{T}^d$.

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In the one-dimensional case, this is the classical result of Pitt and Stein [25, 28]. In the multivariate case, the result follows from the well-known Fourier inequality in the Lorentz spaces, namely, $\|a\|_{p',q} \lesssim \|f\|_{p,q}$, $1 < p < 2$, $0 < q \leqslant \infty$. Alternatively, (1.1) can be obtained from the corresponding result for the Fourier transform using the technique based on Plancherel–Polya–type inequalities for entire functions of exponential type, see [13, (5.3)]. For the Fourier transform this inequality is well known [3, 4]. See also [5, 12, 14–16, 21, 26] for the related results on $\mathbb{T}^d$ and $\mathbb{R}^d$. It is worth mentioning that a complete characterization of the weight pairs (u, v) for which the weighted Pitt inequality $\|u_k a_k\|_{l_q(\mathbb{Z}^d)} \lesssim \|v(\cdot) f(\cdot)\|_{L_r(\mathbb{R}^d)}$, $1 \leqslant r, q \leqslant \infty$, holds has been recently obtained in [27].

The inequality (1.1) with $1 < r = q \leqslant 2$, in particular, recovers the Hardy-Littlewood estimate

$$\left(\sum_{k\in\mathbb{Z}^d} (1+|k|)^{d(r-2)} \ |a_k|^r \right)^{1/r} \lesssim \|f\|_{L_r(\mathbb{T}^d)} . \tag{1.3}$$

For $2 < r < \infty$, Nursultanov proved the following estimate [20]:

$$\left(\sum_{k=1}^{\infty} k^{r-2} \left(\overline{a}_k(M) \right)^r \, dt \right)^{1/r} \lesssim \|f\|_{L_r(\mathbb{T}^d)} , \tag{1.4}$$

where M is the set of all parallelepipeds in $\mathbb{Z}^d$ and

$$\overline{a}_k(M) = \sup_{\substack{U\in M \\ |U|\geqslant k}} \frac{1}{|U|} \left| \sum_{m\in U} a_m \right| .$$

Here $|U|$ denotes the number of the points $n \in U \cap \mathbb{Z}^d$.

Our goal in this paper is to obtain analogues of Pitt-type (in particular, Hardy-Littlewood-type) which hold for the extended range of the parameter α namely, $0 \leqslant \alpha < \frac{1}{r'}$. In this case the classical inequality (1.1) does not always hold. We will show that, for some $C(d) > 1$, the right analogue of Pitt's estimate is

$$\left(\sum_{k=1}^{\infty} k^{\beta dq+d-1} \left| \frac{1}{|U_k|} \sum_{m\in U_k} a_m \right|^q \right)^{1/q} \lesssim \left(\int_{\mathbb{T}^d} |x|^{\alpha dr} \, |f(x)|^r \, dx \right)^{1/r}, \quad r \leqslant q, \tag{1.5}$$

provided that $1 < r < C(d)$ and $0 \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}$, where the sets U_k satisfy certain properties. Note that for U_k, the ball (or cube) centered at 0 with radius k, equation (1.1) implies (1.5). More precisely, the left-hand side of (1.5) can be bounded by the left-hand side of (1.1) (see [6]).

It turns out that the key restriction $1 < r < C(d)$ is closely related to the suitable bound of the L_p norm of the Dirichlet kernel with respect to U_k; see Section 2. Recall

that the Dirichlet kernel with respect to $U \subset \mathbb{R}^d$ is given by

$$D_U(x) = \sum_{m \in U \cap \mathbb{Z}^d} e^{\imath (m,x)}.$$

Our main examples of U_k for which (1.5) holds include:

- balls with radii k, in this case $C(d) = 2d/(d-1)$;
- cubs with edges of the length k, $C(d) = \infty$;
- more generally, convex polyhedrons such that $|U_k| \asymp k^d$, $C(d) = \infty$.

Another interesting example is when U_k are hyperbolic crosses (see Section 4). In this case $C(d) = \infty$ but (1.5) has a different form, namely,

$$\left(\sum_{k=1}^{\infty} k^{\beta q} \ln(k+1)^{(d-1)(q-\beta q-1)} \left| \frac{1}{|G_k|} \sum_{m \in G_k} a_m \right|^q \right)^{1/q} \lesssim \left(\int_{\mathbb{T}^d} |x|^{\alpha d r} \, |f(x)|^r \, dx \right)^{1/r}$$

provided that $0 \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}$ holds. Note that this estimate and (1.5) are not comparable; see Section 4.2. This paper develops the approach recently introduced in [10].

2 Pitt's Inequalities via Estimates of the Norm of Dirichlet Kernels

Let M be a fixed family of bounded subsets of $\mathbb{R}^d$.

Definition 2.1 Let $0 < p < \infty$. For $0 < q < \infty$, the net space $n_{p,q}(M)$ contains all sequences $\{a_m\}_{m \in \mathbb{Z}^d}$ for which the quasi-norm

$$\|a\|_{n_{p,q}(M)} := \left(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1} (\overline{a}_k(M))^q \right)^{\frac{1}{q}} < \infty,$$

where

$$\overline{a}_k(M) = \sup_{\substack{U \in M \\ |U| \geqslant k}} \frac{1}{|U|} \left| \sum_{m \in U} a_m \right|.$$

For $q = \infty$, we define the space with the usual modification:

$$\|a\|_{n_{p,\infty}(M)} = \sup_{1 \leqslant k < \infty} k^{\frac{1}{p}} \overline{a}_k(M) < \infty.$$

For the main properties of these spaces, see [20, 24]. Now we introduce the Lorentz spaces. The non-increasing rearrangement of a measurable function f is given by

$$f^*(t) := \inf \{\rho : |\{|f| > \rho\}| \leqslant t\}.$$

For $0 < p < \infty, 0 < q < \infty$, the Lorentz space $L_{p,q}(\mathbb{T}^d)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,q}(\mathbb{T}^d)} := \left(\int_0^1 f^*(t)^q t^{q/p} \frac{dt}{t}\right)^{1/q} < \infty.$$

It is easy to see that $L_{p,p}(\mathbb{T}^d) = L_p(\mathbb{T}^d)$ and $L_{p,r}(\mathbb{T}^d) \subset L_{p,q}(\mathbb{T}^d)$ for $0 < r \leqslant q < \infty$.

The next result provides necessary and sufficient conditions for the Fourier inequality between Lorentz and net spaces, expressed in terms of the corresponding bounds for the norms of the Dirichlet kernels.

Theorem 2.2 *Let $1 \leqslant q, r \leqslant \infty$ and $1 < C(d) \leqslant \infty$. The following conditions are equivalent:*

(i) For any $1 < p < C(d)$ and for any $U \in M$, we have

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{1/p}. \tag{2.1}$$

(ii) For any $1 < p < C(d)$ and for any $U \in M$, we have

$$\|D_U\|_{L_{p',r}(\mathbb{T}^d)} \lesssim |U|^{1/p}.$$

(iii) For any $1 < p < C(d)$ and for any $f \in L_{p,q}(\mathbb{T}^d)$ such that $f \sim \sum_{m\in\mathbb{Z}^d} a_m e^{\imath(m,x)}$, we have

$$\|a\|_{n_{p',q}(M)} \lesssim \|f\|_{L_{p,q}(\mathbb{T}^d)}. \tag{2.2}$$

In particular, if (i) holds, then, for $1 < p < C(d)$,

$$\left(\sum_{k=1}^{\infty} k^{p-2}(\overline{a}_k(M))^p\right)^{\frac{1}{p}} \lesssim \|f\|_{L_p(\mathbb{T}^d)}.$$

We note that the first-named author studied estimates of type (2.2) in [20].

Proof First, we show that *(i)* implies *(iii)*. For $U \in M$, by Hölder's inequality, we obtain

$$\left|\sum_{m\in U} a_m\right| = \left|\int_{\mathbb{T}^d} f(x)\overline{D_U(x)}\right| \leqslant \|f\|_{L_p(\mathbb{T}^d)}\|D_U\|_{L_{p'}(\mathbb{T}^d)},$$

that is,

$$\sup_{U\in M} \frac{1}{|U|^{1/p}}\left|\sum_{m\in U} a_m\right| \lesssim \|f\|_{L_p(\mathbb{T}^d)}.$$

Using this, we write

$$\|a\|_{n_{p',\infty}(M)} = \sup_{k\in\mathbb{N}} k^{\frac{1}{p'}} \sup_{\substack{U\in M\\ |U|\geqslant k}} \frac{1}{|U|}\left|\sum_{m\in U} a_m\right|$$
$$\leqslant \sup_{U\in M} \frac{1}{|U|^{1/p}}\left|\sum_{m\in U} a_m\right| \lesssim \|f\|_{L_p(\mathbb{T}^d)} \tag{2.3}$$

for all $p \in (1, C(d))$. Interpolating this estimate and noting that (see [20, 24])

$$\Big(n_{p_1,q_1}(M), n_{p_2,q_2}(M)\Big)_{\theta,q} \hookrightarrow n_{p,q}(M), \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad 0<\theta<1, \tag{2.4}$$

where $1 \leqslant p_1 < p_2 < C(d)$, $1 \leqslant q, q_1, q_2 \leqslant \infty$ and M is any set in $\mathbb{R}^d$, we arrive at (2.2).

Second, we show that *(iii)* implies *(ii)*. Let $U \in M$. By duality and an inequality similar to (2.3), we conclude

$$\|D_U\|_{L_{p',r}(\mathbb{T}^d)} = \sup_{\|g\|_{L_{p,r'}(\mathbb{T}^d)}=1} \int_{\mathbb{T}^d} D_U(x)\overline{g(x)}\,dx$$
$$= \sup_{\|g\|_{L_{p,r'}(\mathbb{T}^d)}=1} \left|\sum_{m\in U} \widehat{g}(m)\right|$$
$$\lesssim |U|^{1/p} \sup_{\|g\|_{L_{p,r'}(\mathbb{T}^d)}=1} \|\widehat{g}\|_{n_{p',\infty}(M)} \leqslant |U|^{1/p} \sup_{\|g\|_{L_{p,r'}(\mathbb{T}^d)}=1} \|\widehat{g}\|_{n_{p',r'}(M)}.$$

In light of (2.2), we have $\sup_{\|g\|_{L_{p,r'}(\mathbb{T}^d)}=1} \|\widehat{g}\|_{n_{p',r'}(M)} \lesssim 1$. Thus,

$$\|D_U\|_{L_{p',r}(\mathbb{T}^d)} \leqslant C|U|^{1/p},$$

where C is a constant that depends only on p and the dimension d.

Third, *(ii)* always implies *(i)*. □

Note that, for $C(d) \leqslant 2$, (2.1) always holds by the Hausdorff-Young inequality. Thus, the only interesting case is $C(d) > 2$.

Theorem 2.3 *Suppose that $1 < r \leqslant q < \infty$ and that there exists $1 < C(d) \leqslant \infty$ such that for any $1 < p < C(d)$ and for any $U \in M$, we have*

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{1/p}. \tag{2.5}$$

Let $U_k \in M$ such that $|U_k| \gtrsim k^d$ for each $k \in \mathbb{N}_+$ and let $f \sim \sum_{m\in\mathbb{Z}^d} a_k e^{\imath(m,x)}$ such that, for some $0 \leqslant \alpha < \infty$,

$$\left(\int_{\mathbb{T}^d} |x|^{\alpha d r}\, |f(x)|^r\, dx\right)^{1/r} < \infty. \tag{2.6}$$

(i) If

$$1 < r < C(d) \quad \text{and} \quad 0 \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'},$$

then

$$\left(\sum_{k=1}^{\infty} k^{\beta d q + d - 1} \left| \frac{1}{|U_k|} \sum_{m\in U_k} a_m \right|^q \right)^{1/q} \lesssim \left(\int_{\mathbb{T}^d} |x|^{\alpha d r}\, |f(x)|^r\, dx\right)^{1/r}. \tag{2.7}$$

In particular, taking $1 < r = q < C(d)$, $\alpha = 0$ and $\beta = 1 - 2/r$, we have

$$\left(\sum_{k=1}^{\infty} k^{d(r-1)-1} \left| \frac{1}{|U_k|} \sum_{m\in U_k} a_m \right|^r \right)^{1/r} \lesssim \|f\|_{L_r(\mathbb{T}^d)}. \tag{2.8}$$

(ii) If

$$C(d) \leqslant r < \infty \quad \text{and} \quad \frac{1}{C(d)} - \frac{1}{r} < \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'},$$

then (2.7) holds.

Proof We choose p such that $\alpha = \frac{1}{p} - \frac{1}{r}$. We claim that then $p \in (1, C(d))$. Indeed, since $\alpha < 1/r'$, we have that $p > 1$. In the first case, if $1 < r < C(d)$ and $\alpha \geqslant 0$, then $1 < p \leqslant r < C(d)$. In the second case, when $C(d) \leqslant r < \infty$ and $\frac{1}{C(d)} - \frac{1}{r} < \alpha = \frac{1}{p} - \frac{1}{r}$, then $1 < p < C(d)$.

We observe that

$$\beta = \frac{1}{r'} - \frac{1}{q} - \alpha = \frac{1}{r'} - \frac{1}{q} - \frac{1}{p} + \frac{1}{r} = \frac{1}{p'} - \frac{1}{q}$$

and

$$\beta d q + d - 1 = \frac{dq}{p'} - 1.$$

Further, we have

$$\begin{aligned}\|a\|^q_{n_{p',q}(M)} &= \sum_{k=1}^{\infty} k^{\frac{q}{p'}-1}(\overline{a}_k(M))^q = \sum_{k=1}^{\infty}\sum_{m=k^d}^{(k+1)^d-1} m^{\frac{q}{p'}-1}(\overline{a}_m(M))^q \\ &\gtrsim \sum_{k=1}^{\infty}(\overline{a}_{(k+1)^d}(M))^q (k+1)^{\frac{dq}{p'}-1}.\end{aligned}$$

Together with $\|a\|^q_{n_{p',q}(M)} \geqslant (\bar{a}_1(M))^q$ and Theorem 2.2, this yields

$$\begin{aligned}\left(\sum_{k=1}^{\infty} k^{\beta dq+d-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k} a_m\right|^q\right)^{1/q} &= \left(\sum_{k=1}^{\infty} k^{\frac{dq}{p'}-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k} a_m\right|^q\right)^{1/q} \\ &\leqslant \left(\sum_{k=1}^{\infty}(\overline{a}_{k^d}(M))^q k^{\frac{dq}{p'}-1}\right)^{1/q} \\ &\lesssim \|a\|_{n_{p',q}(M)} \lesssim \|f\|_{L_{p,q}(\mathbb{T}^d)}.\end{aligned}$$

Due to the the facts that $L_{p,r}(\mathbb{T}^d) \subset L_{p,q}(\mathbb{T}^d)$ for $r \leqslant q$ and $\frac{1}{p} - \frac{1}{r} \geqslant 0$, condition (2.6) implies

$$\begin{aligned}\|f\|_{L_{p,q}(\mathbb{T}^d)} \lesssim \|f\|_{L_{p,r}(\mathbb{T}^d)} &= \left(\int_0^{\infty}\left(t^{\frac{1}{p}-\frac{1}{r}} f^*(t)\right)^r dt\right)^{1/r} \\ &= \left(\int_0^{\infty}\left(t^{\alpha} f^*(t)\right)^r dt\right)^{1/r} \\ &\lesssim \left(\int_{\mathbb{T}^d}\left(|x|^{\alpha d}|f(x)|\right)^r dx\right)^{1/r}.\end{aligned}$$

Here in the last estimate we have used the Hardy–Littlewood–Pólya inequality for rearrangements (see, e.g., [3, p. 7])

$$\int_0^{\infty} g^*(t)\frac{1}{(1/\varphi)^*(t)}\,dt \leqslant \int_{\mathbb{R}^d} g(x)\varphi(x)\,dx.$$

This completes the proof. □

Note that we do not assume in this result that $0 \in U_k \subset U_{k+1}$. Under certain structural conditions on U_k, the converse of Theorem 2.3 also holds.

Remark 2.4 (i) Let $U_k = U_{2^s}$ for $2^{s-1} < k \leqslant 2^s$ and $|U_k| \asymp k^d$. Then (2.8) with $1 < r < \infty$ implies $\|D_{U_k}\|_{L_{r'}(\mathbb{T}^d)} \lesssim |U_k|^{1/r}$ for all k; cf. (2.5).

(ii) If the parameter $C(d)$ is sharp in (2.5), that is, if

$$\sup_{U\subset M}\frac{\|D_U\|_{L_{r'}(\mathbb{T}^d)}}{|U|^{1/r}}=\begin{cases}\infty, & r\geqslant C(d),\\ c_{r,d}<\infty, & 1<r<C(d),\end{cases}$$

then it is sharp in (2.7) and (2.8).
(iii) Estimate (2.8) is also sharp in the following sense. Let $C(d)>2$. Take $U_k\subset U_{k+1}$ and $|U_k|\asymp k^d$ satisfying condition (2.5). Then there exists a function $f\in L_r(\mathbb{T}^d)$ such that

$$\left(\sum_{k=1}^{\infty}k^{d(r-1)-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k}a_m\right|^r\right)^{1/r}\asymp\|f\|_{L_r(\mathbb{T}^d)},\qquad C(d)'<r<C(d).\quad(2.9)$$

Proof *(i)* Applying (2.8) to $f\sim\sum_{s\in\mathbb{Z}^d}a_s e^{\imath(x,s)}$, $f\in L_r(\mathbb{T}^d)$, and taking into account conditions on U_k, we have

$$\|f\|_{L_r(\mathbb{T}^d)}\gtrsim\left(\sum_{m=0}^{\infty}2^{d(r-1)m}\left|\frac{1}{|U_{2^m}|}\sum_{s\in U_{2^m}}a_s\right|^r\right)^{1/r}\geqslant\sup_m 2^{\frac{d}{r'}m}\left|\frac{1}{|U_{2^m}|}\sum_{s\in U_{2^m}}a_s\right|.$$

Hence,

$$\left|\sum_{s\in U_{2^m}}a_s\right|\lesssim 2^{\frac{dm}{r}}\|f\|_{L_r(\mathbb{T}^d)}=|U_{2^m}|^{1/r}\|f\|_{L_r(\mathbb{T}^d)}.$$

In light of this estimate, we conclude that

$$\|D_{U_k}\|_{L_{r'}(\mathbb{T}^d)}=\sup_{\|f\|_{L_r(\mathbb{T}^d)}=1}\left|\sum_{m\in U_k}a_m\right|\lesssim|U_k|^{1/r}.$$

(ii) This follows from item *(i)*.
(iii) To see the part $\gtrsim$ in (2.9), consider $f=D_{U_N}$. Then, by (2.5), for $C(d)'<r<C(d)$ we have $\|f\|_{L_r(\mathbb{T}^d)}\lesssim|U_N|^{1/r'}$. Thus,

$$\begin{aligned}\|f\|_{L_r(\mathbb{T}^d)}&\lesssim\left(\sum_{k=1}^{N}k^{d(r-1)-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k}1\right|^r\right)^{1/r}\\&\lesssim\left(\sum_{k=1}^{\infty}k^{d(r-1)-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k}a_m\right|^r\right)^{1/r}.\end{aligned}$$

□

3 Estimates of L_p Norms of Dirichlet Kernels and Examples

In this section, we first introduce several classes of sets in $\mathbb{Z}^d$ and then estimate the corresponding Dirichlet kernels in the $L_p(\mathbb{T}^d)$-norms.

3.1 L_p Norms of Dirichlet Kernels

A bounded set $U \subset \mathbb{R}^d$ is

- *solid* (written $U \in M_S$) if $x \in U$ implies that

$$\prod_{j=1}^{d} \big[\min\{x_j, 0\}, \max\{x_j, 0\}\big] \subset U.$$

  (Note that for $x_j \geqslant 0$, $j = 1, \cdots, d$, this means $[0, x_1] \times \cdots \times [0, x_d] \subset U$.)
- *dense* (written $U \in M_D$) if

$$|R_U| \leqslant C|U|,$$

  where $R_U = \prod_{j=1}^{d}[-n_j, n_j]$ denotes the smallest rectangle for which $U \subset R_U$.
- *in the class* M_{SD} if

$$U \in M_{SD} = \{l + V : l \in \mathbb{Z}^d, V \in M_S \cap M_D\}.$$

  (Note that M_{SD} contains the sets of balls and rectangles in $\mathbb{R}^d$.)
- *in the class* M_{PD} if U is a convex polyhedron such that $U \in M_D$ (written $U \in M_{PD}$).
  (Note that $\big\{l + (M_P \cap M_S) : l \in \mathbb{Z}^d\big\} \subset M_{PD}$, where M_P denotes the collection of all convex polyhedra.)

First, observe that if $1 < p \leqslant 2$, for any bounded set U, by Hausdorff-Young inequality,

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{1/p}.$$

The following result was proved by Dyachenko [8].

Lemma 3.1 *Let $U \in M_S$. If $\frac{2d}{d-1} < p \leqslant \infty$, then*

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{\frac{d-1}{2d}}. \tag{3.1}$$

If $1 < p < \frac{2d}{d-1}$, then

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \asymp \left(\sum_{n\in U}(1 + |n_1|)^{p'-2} \cdots (1 + |n_d|)^{p'-2}\right)^{1/p'}. \tag{3.2}$$

Here, the constants depend only on p and the dimension d.

For the class $U \in M_{SD}$, this result can be formulated in the same terms.

Lemma 3.2 *Let $U \in M_{SD}$ and $0 < r \leqslant \infty$. If $\frac{2d}{d-1} < p < \infty$, then*

$$\|D_U\|_{L_{p',r}(\mathbb{T}^d)} \lesssim |U|^{\frac{d-1}{2d}}. \tag{3.3}$$

If $1 < p < \frac{2d}{d-1}$, then

$$\|D_U\|_{L_{p',r}(\mathbb{T}^d)} \lesssim |U|^{1/p}. \tag{3.4}$$

Proof Let us prove (3.3). Let $\frac{2d}{d-1} < p_1 < p < \infty$. Then from (3.1), we have

$$\|D_U\|_{L_{p',r}(\mathbb{T}^d)} \lesssim \|D_U\|_{L_{p_1'}(\mathbb{T}^d)} \lesssim |U|^{\frac{d-1}{2d}}.$$

We now show (3.4). Since

$$\|D_{k+U}\|_{L_p(\mathbb{T}^d)} = \|D_U\|_{L_p(\mathbb{T}^d)},$$

it is enough to prove the lemma for $U \in M_S \cap M_D$. For each $j = 1, \dots, d$, denote by N_j the largest $|n_j|$ which occurs in the sum (3.2). Then $R_U = \prod_{j=1}^d [-N_j, N_j]$ and we obtain that

$$\begin{aligned}\sum_{n\in U}(1+|n_1|)^{p'-2}\cdots(1+|n_d|)^{p'-2} &\lesssim (1+|N_1|)^{p'-1}\cdots(1+|N_d|)^{p'-1}\\ &\lesssim |R_U|^{p'-1} \lesssim |U|^{p'-1},\end{aligned}$$

which means that

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{1/p}.$$

Finally, we use Theorem 2.2 to obtain the desired result. □

Remark 3.3 If $2 < p < \frac{2d}{d-1}$, then (3.4) is not true without the condition $U \in M_D$. Indeed, take the hyperbolic cross $U_N \in M_S \setminus M_D$ (see Section 4 for more details). Then $|U_N| \asymp N(\ln N)^{d-1}$ (see e.g. [7, p. 22]) and, by Theorem 4.2, $\|D_U\|_{L_{p'}(\mathbb{T}^d)} \asymp N^{\frac{1}{p}}(\ln N)^{\frac{d-1}{p'}}$. This yields a contradiction to $\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{1/p}$ for $p' < 2$.

For M_{PD}-sets, Kolomoitsev and Lomako [17] generalized Lemma 3.2 (see the previous results in [1, 2]) as follows:

Lemma 3.4 *If $U \in M_{PD}$ and $1 < p < \infty$, then*

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim |U|^{1/p}.$$

3.2 Examples

In this section, we summarize our results for the sets introduced in the previous section. Using Lemma 3.4, we can apply Theorems 2.2 and 2.3 for $M = M_{SD}$ with $C(d) = \frac{2d}{d-1}$ and for $M = M_{PD}$ with $C(d) = \infty$.

Corollary 3.5 *Let $1 < r \leqslant q < \infty$. Let also $f \sim \sum_{m\in\mathbb{Z}^d} a_m e^{\imath(m,x)}$ and $f \in L_{p,q}(\mathbb{T}^d)$ such that (2.6) holds. Assume that $M = M_{SD}$ or $M = M_{PD}$ and define*

$$C(d) = \begin{cases} \frac{2d}{d-1}, & \text{if } M = M_{SD}, \\ \infty, & \text{if } M = M_{PD}. \end{cases}$$

(i) If $1 < p < C(d)$, then

$$\|a\|_{n_{p',q}(M)} \lesssim \|f\|_{L_{p,q}(\mathbb{T}^d)}, \tag{3.5}$$

and, in particular,

$$\left(\sum_{k=1}^{\infty} k^{p-2}\overline{a}_k(M)^p\right)^{1/p} \lesssim \|f\|_{L_p(\mathbb{T}^d)}. \tag{3.6}$$

(ii) Assume that

$$1 < r < C(d) \quad \text{and} \quad 0 \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}.$$

Then, for any $U_k \in M$ with $|U_k| \gtrsim k^d$ for each $k \in \mathbb{N}_+$,

$$\left(\sum_{k=1}^{\infty} k^{\beta dq+d-1} \left|\frac{1}{|U_k|}\sum_{m\in U_k} a_m\right|^q\right)^{1/q} \lesssim \left(\int_{\mathbb{T}^d} |x|^{\alpha dr}\,|f(x)|^r\,dx\right)^{1/r} \tag{3.7}$$

and, in particular, (taking $r = q$, $\alpha = 0$ and $\beta = 1 - 2/r$),

$$\left(\sum_{k=1}^{\infty} k^{d(r-1)-1} \left|\frac{1}{|U_k|}\sum_{m\in U_k} a_m\right|^r\right)^{1/r} \lesssim \|f\|_{L_r(\mathbb{T}^d)}. \tag{3.8}$$

(iii) If

$$C(d) \leqslant r < \infty \quad \text{and} \quad \frac{1}{C(d)} - \frac{1}{r} < \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'},$$

then, for any $U_k \in M$ with $|U_k| \gtrsim k^d$ for each $k \in \mathbb{N}_+$, (3.7) holds.

Typical examples of acceptable sets M in Corollary 3.5 are s-balls. For $0 < s \leqslant \infty$ and $r > 0$, we set

$$B_r^s := \{x \in \mathbb{R}^d : \|x\|_s \leqslant r\}.$$

Note that if $0 < s < \infty$ and $s \neq 1$, then $B_r^s \in M_{SD}$ and thus, in this case, $C(d) = \frac{2d}{d-1}$. In the case $s = 1, \infty$, B_r^s are convex polyhedrons and $B_r^s \in M_{PD}$, hence $C(d) = \infty$ (note that in this case part (iii) is empty).

Remark 3.6 (i) Both (3.5) and (3.6) hold for $M = M_B$, the class of all bounded sets, with $C(d) = 2$.
(ii) The condition M_D is essential in Pitt-type estimates. For example, consider (3.8). There is a function from $L_r(\mathbb{T}^d)$ and there is a sequence $U_k \in M_S \setminus M_D$ such that (3.8) does not hold for $2 < r < \infty$. (For $1 < r \leqslant 2$ it is true for all bounded sets.) Indeed, let $s \in \mathbb{N}$ be fixed and define the hyperbolic cross

$$G_s = \Big\{n \in \mathbb{Z}^d : \prod_{i=1}^d \max\{|n_i|, 1\} \leqslant s\Big\}.$$

Taking into account that $|G_s| \asymp s(\ln s)^{d-1}$ and using Theorem 4.2, for $f(x) = D_{G_s}(x) = \sum_{m \in G_s} e^{i(m,x)}$, one has $\|f\|_{L_r(\mathbb{T}^d)} \asymp s^{1/r'}(\ln s)^{(d-1)/r}$.

Let $s_0 \in \mathbb{N}$ such that $s_0^d \asymp s(\ln s)^{d-1} \asymp |G_s|$ and G_k be a hyperbolic cross such that $G_k \subset G_s$ and $|G_k| \gtrsim k^d$. Then

$$\left(\sum_{k=1}^{\infty} k^{d(r-1)-1}\Big(\frac{1}{|G_k|}\Big|\sum_{m \in G_k} a_m\Big|\Big)^r\right)^{1/r} \geqslant \left(\sum_{k=1}^{s_0} k^{d(r-1)-1}\right)^{1/r} \asymp s_0^{d/r'}.$$

Therefore, assuming that (3.8) holds,

$$s^{1/r'}(\ln s)^{(d-1)/r'} \asymp |G_s|^{1/r'} \lesssim \|f\|_{L_r(\mathbb{T}^d)} \asymp s^{1/r'}(\ln s)^{(d-1)/r},$$

which leads to a contradiction, since $r > 2$.

4 Hyperbolic Crosses

In Subsections 3.1 and 3.2, we considered dense sets U, that is, those satisfying $U \in M_D$. This condition clearly does not hold for such an important class as hyperbolic crosses. In this section, we study this class in detail and obtain analogues of Pitt's inequalities.

For $N \in \mathbb{N}$, the hyperbolic cross G_N is defined by

$$G_N := \Big\{n \in \mathbb{Z}^d : \prod_{i=1}^d \bar{n}_i \leqslant N\Big\}, \qquad \bar{n}_i = \max\{|n_i|, 1\}.$$

Let $M = \{G_N\}_{N\in\mathbb{N}}$ be the set of all hyperbolic crosses and

$$M_r = \left\{G_N : 2^{r-1} \leqslant N < 2^r\right\}, \qquad r \in \mathbb{N}_+.$$

Let $h = \{h_k\}_{k=0}^{\infty}$ be a sequence such that, for some $\delta > 1$, there holds

$$h_0 = 1, \qquad 1 < \frac{h_{k+1}}{h_k} < \delta.$$

Definition 4.1 Let $0 < p, q < \infty$. The net space $n_{p,q}(M, h)$ contains all sequences $\{a_m\}_{m\in\mathbb{Z}^d}$ for which

$$\|a\|_{n_{p,q}(M,h)} := \left(\sum_{k=1}^{\infty} h_k^{\frac{q}{p}} \left(\tilde{a}_k(M)\right)^q\right)^{\frac{1}{q}} < \infty,$$

where

$$\tilde{a}_k(M) = \sup_{r\geqslant k} \sup_{U\in M_r} \frac{1}{2^r} \left|\sum_{m\in U} a_m\right| \qquad (k \geqslant 1).$$

For $q = \infty$, we define the space with the usual modification:

$$\|a\|_{n_{p,\infty}(M,h)} = \sup_{1\leqslant k<\infty} h_k^{\frac{1}{p}} \tilde{a}_k(M) < \infty.$$

Note that if $U \in M_r$, then $|U| \asymp 2^r r^{d-1}$ (see [7]).

4.1 Estimates for Dirichlet Kernels and Pitt's Inequalities

The proof of the Pitt-type inequality for averages over hyperbolic crosses follows the argument of Theorem 2.3 and relies on the following L_p-estimate for the Dirichlet kernels corresponding to hyperbolic crosses. It is worth noting that these estimates are of independent interest in several areas of harmonic analysis and have been investigated mainly in the L_1-setting; see the recent paper [18] and references therein.

Theorem 4.2 *Let $1 < p < \infty$ and G_N be a hyperbolic cross, then*

$$\left\|D_{G_N}\right\|_{L_{p'}(\mathbb{T}^d)} \asymp N^{1/p} (\ln N)^{\frac{d-1}{p'}}.$$

We would like to stress that for step hyperbolic crosses a similar result is well known (see e.g. [7, p. 22]) and it can be proved using embedding theorems for Besov classes. The result for non-step crosses is new. In contrast to the L_1 case [19], the L_p-norms of the Dirichlet kernels associated with step and non-step hyperbolic crosses turn out to be equivalent.

Proof First, the lower bound

$$\left\| D_{G_N} \right\|_{L_{p'}(\mathbb{T}^d)} \gtrsim \left(\sum_{n \in G_N} \left(\prod_{i=1}^{d} \bar{n}_i \right)^{p'-2} \right)^{1/p'}, \quad \bar{n} := \max(|n|, 1),$$

always holds for all $1 < p < \infty$ and for any $G_N \in M_S$, see [9, Theorem 1].

Find $m \in \mathbb{N}$ such that $2^{m-1} < N \leqslant 2^m$. Define the step hyperbolic cross as in Remark 3.6, i.e.,

$$\mathcal{U}_m = \bigcup_{k_1+\ldots+k_d \leqslant m} \{x \in \mathbb{R}^d : \lfloor 2^{k_i-1} \rfloor \leqslant |x_i| \leqslant 2^{k_i}\},$$

where $k_i \in \mathbb{N}$ and $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$. Then $\mathcal{U}_{m-2} \subset G_N \subset \mathcal{U}_{m+1}$ and therefore we have

$$\left(\sum_{n \in \mathcal{U}_{m-2}} \left(\prod_{i=1}^{d} \bar{n}_i \right)^{p'-2} \right)^{1/p'} \leqslant \left(\sum_{n \in G_N} \left(\prod_{i=1}^{d} \bar{n}_i \right)^{p'-2} \right)^{1/p'} \leqslant \left(\sum_{n \in \mathcal{U}_{m+1}} \left(\prod_{i=1}^{d} \bar{n}_i \right)^{p'-2} \right)^{1/p'}.$$

Noting that

$$\begin{aligned}
&\left(\sum_{n \in \mathcal{U}_s} (1+n_1)^{p'-2} \ldots (1+n_d)^{p'-2} \right)^{1/p'} \\
&= \left(\sum_{l=0}^{s} \sum_{k_1+\ldots+k_d=l} \sum_{n_1=\lfloor 2^{k_1-1} \rfloor}^{2^{k_1}-1} \cdots \sum_{n_d=\lfloor 2^{k_d-1} \rfloor}^{2^{k_d}-1} (1+n_1)^{p'-2} \ldots (1+n_d)^{p'-2} \right)^{1/p'} \\
&\asymp \left(\sum_{l=0}^{s} 2^{l(p'-1)} \sum_{k_1+\ldots+k_d=l} 1 \right)^{1/p'} \asymp \left(\sum_{l=0}^{s} 2^{l(p'-1)} l^{d-1} \right)^{1/p'} \asymp 2^{s/p} s^{(d-1)/p'},
\end{aligned}$$

we complete the lower bound. It is worth mentioning that this proof implies the required result for $1 < p < \frac{2d}{d-1}$, since $G_N \in M_S$ and Lemma 3.1 can be applied.

To prove the upper bound, we use Theorem 3 in [11], which claims that for $f \sim \sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} a_m e^{i(m,x)}$ and $1 < p < \infty$, we have

$$\|f\|_p^p \lesssim \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \beta_n^p \left(\prod_{j=1}^{n} n_j \right)^{p-2},$$

where

$$\beta_n := \sum_{m_1=n_1}^{\infty} \cdots \sum_{m_d=n_d}^{\infty} |\Delta a_m|$$

and the operator Δ is defined by $\Delta := \prod_{j=1}^{n} \triangle^j$ and $\Delta^j a_m = a_m - a_{m_1,\cdots,m_{j-1},m_j+1,m_{j+1},\cdots,m_n}$. The rest of the proof is as follows:

$$\begin{aligned}\left\| D_{G_N} \right\|_{L_{p'}(\mathbb{T}^d)}^{p'} &\lesssim \sum_{n\in G_N} \beta_n \prod_{i=1}^{d} \overline{n}_i^{p'-2} = \sum_{r=0}^{N} \sum_{\prod_{i=1}^{d} \overline{n}_i = r} \beta_n \prod_{i=1}^{d} \overline{n}_i^{p'-2} \\ &= \sum_{r=0}^{N} r^{p'-2} \sum_{\prod_{i=1}^{d} \overline{n}_i = r} \beta_n \leqslant 2^d \sum_{r=0}^{N} r^{p'-2} \sum_{\prod_{i=1}^{d} \overline{n}_i = r} 1 \\ &\asymp \sum_{r=0}^{N} r^{p'-2} (\ln r)^{d-1} \asymp N^{p'-1} (\ln N)^{d-1}.\end{aligned}$$

□

Theorem 4.3 *Let M be the set of all hyperbolic crosses and $h = \{2^k/k^{d-1}\}_{k\in\mathbb{N}}$. If $1 < p < \infty$, $0 < q \leqslant \infty$ and $f \sim \sum_{m\in\mathbb{Z}^d} a_m e^{i(m,x)}$, then*

$$\|a\|_{n_{p',q}(M,h)} \lesssim \|f\|_{L_{p,q}(\mathbb{T}^d)}.$$

In particular,

$$\left(\sum_{k=1}^{\infty} h_k^{p-1} \left(\tilde{a}_k(M)\right)^p \right)^{\frac{1}{p}} \lesssim \|f\|_{L_p(\mathbb{T}^d)}.$$

Proof Let $1 < p < \infty$. First, we prove the weak inequality

$$\|a\|_{n_{p',\infty}(M,h)} \lesssim \|f\|_{L_p(\mathbb{T}^d)}. \tag{4.1}$$

Let $r \in \mathbb{N}$ and $U \subset M_r$, then in light of Theorem 4.2, we have

$$\|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim 2^{\frac{r}{p}} r^{\frac{d-1}{p'}}.$$

Therefore,

$$\left| \sum_{m\in U} a_m \right| = \left| \int_{\mathbb{T}^d} f(x) \overline{D_U(x)} dx \right| \leqslant \|f\|_{L_p(\mathbb{T}^d)} \|D_U\|_{L_{p'}(\mathbb{T}^d)} \lesssim 2^{r/p} r^{\frac{d-1}{p'}} \|f\|_{L_p(\mathbb{T}^d)},$$

or, equivalently,

$$\sup_{u\in M_r} \frac{1}{2^{\frac{r}{p}} r^{\frac{d-1}{p'}}} \left| \sum_{m\in U} a_m \right| \lesssim \|f\|_{L_p(\mathbb{T}^d)}.$$

Thus, we derive

$$\begin{aligned}\|a\|_{n_{p',\infty}(M,h)} &= \sup_k h_k^{\frac{1}{p'}} \sup_{r\geqslant k}\sup_{U\in M_r} \frac{1}{2r}\left|\sum_{m\in U} a_m\right| \\ &\leqslant \sup_r \sup_{U\in M_r} \frac{1}{2^{\frac{r}{p}} r^{\frac{d-1}{p'}}}\left|\sum_{m\in U} a_m\right| \lesssim \|f\|_{L_p(\mathbb{T}^d)}.\end{aligned}$$

Therefore, (4.1) is proved.

Second, using Lemma 4.4, we obtain the required result. □

Lemma 4.4 *If* $0 < p_0 < p_1 < \infty$, $0 < q \leqslant \infty$ *and* $\theta \in (0,1)$, *then*

$$\left(n_{p_0,\infty}(M,h), n_{p_1,\infty}(M,h)\right)_{\theta,q} \subset n_{p,q}(M,h), \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}.$$

Proof Let $a = a_0 + a_1$ such that $a_0 \in n_{p_0,\infty}(M,h)$ and $a_1 \in n_{p_1,\infty}(M,h)$. Then

$$\begin{aligned}h_k^{\frac{1}{p}}\tilde{a}_k(M) &\leqslant h_k^{\frac{1}{p}-\frac{1}{p_0}}\left(h_k^{\frac{1}{p_0}}\tilde{a}_k^0(M) + h_k^{\frac{1}{p_0}}\tilde{a}_k^1(M)\right) \\ &\leqslant h_k^{\frac{1}{p}-\frac{1}{p_0}}\left(\|a_0\|_{n_{p_0,\infty}(M,h)} + h_k^{\frac{1}{p_0}-\frac{1}{p_1}}\|a_1\|_{n_{p_0,\infty}(M,h)}\right).\end{aligned}$$

Since the decomposition $a = a_0 + a_1$ is arbitrary, we obtain

$$h_k^{\frac{1}{p}}\tilde{a}_k(M) \leqslant h_k^{\frac{1}{p}-\frac{1}{p_0}} K(h_k^{\frac{1}{p_0}-\frac{1}{p_1}}, a; n_{p_0,\infty}(M,h), n_{p_1,\infty}(M,h)).$$

Then

$$\begin{aligned}\|a\|_{n_{p,q}(M,h)} &\leqslant \left(\sum_{k=0}^{\infty}\left(h_k^{-\theta(\frac{1}{p_0}-\frac{1}{p_1})} K(h_k^{\frac{1}{p_0}-\frac{1}{p_1}}, a; n_{p_0,\infty}(M,h), n_{p_1,\infty}(M,h))\right)^q\right)^{1/q} \\ &= \left(\sum_{k=0}^{\infty}\left(\eta_k^{-\theta} K(\eta_k, a)\right)^q\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\eta_k = h_k^{\frac{1}{p_0}-\frac{1}{p_1}}, \qquad \eta_0 = h_0 = 1, \qquad 1 < \frac{\eta_{k+1}}{\eta_k} = \left(\frac{h_{k+1}}{h_k}\right)^{\frac{1}{p_0}-\frac{1}{p_1}} < \delta^{\frac{1}{p_0}-\frac{1}{p_1}}.$$

On the other hand, we have

$$\begin{aligned}\|a\|_{(n_{p0,\infty}(M,h),n_{p1,\infty}(M,h))_{\theta,q}} &= \left(\int_0^\infty \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t}\right)^{1/q}\\ &\geqslant \left(\int_1^\infty \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t}\right)^{1/q}\\ &\asymp \left(\sum_{k=0}^\infty \left(\eta_k^{-\theta}K(\eta_k,a)\right)^q\right)^{1/q},\end{aligned}$$

completing the proof. □

Remark 4.5 We note that, for $d = 2$, a result analogous to Theorem 4.3, concerning step hyperbolic crosses, was obtained in [22, Theorem 3]. Moreover, when M is the set of step hyperbolic crosses, inequality (4.1) was obtained for Fourier series and transforms in [22] and [23], respectively.

Now we can prove a similar result to Corollary 3.5 for hyperbolic crosses.

Corollary 4.6 *Suppose that* $1 < r \leqslant q < \infty$, $0 \leqslant \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}$, *and* (2.6) *holds. Then*

$$\left(\sum_{k=1}^\infty k^{\beta q}\ln(k+1)^{(d-1)(q-\beta q-1)}\left|\frac{1}{|G_k|}\sum_{m\in G_k}a_m\right|^q\right)^{1/q} \lesssim \left(\int_{\mathbb{T}^d}|x|^{\alpha d r}\,|f(x)|^r\,dx\right)^{1/r}. \tag{4.2}$$

In particular, taking $r = q$, $\alpha = 0$ *and* $\beta = 1 - 2/r$, *we have*

$$\left(\sum_{k=1}^\infty k^{r-2}\ln(k+1)^{d-1}\left|\frac{1}{|G_k|}\sum_{m\in G_k}a_m\right|^r\right)^{1/r} \lesssim \|f\|_{L_r(\mathbb{T}^d)}. \tag{4.3}$$

Proof The proof repeats the one of Theorem 2.3 with the same p, α, β. Setting M to be the set of all hyperbolic crosses and

$$M_r := \{G_k : 2^{r-1} \leqslant k < 2^r\}, \qquad r \in \mathbb{N}_+,$$

we only note that

$$\begin{aligned}\|a\|^q_{n_{p',q}(M,h)} &= \sum_{k=1}^{\infty}\left(\frac{2^k}{k^{d-1}}\right)^{\frac{q}{p'}}(\tilde{a}_k(M))^q\\ &\gtrsim \sum_{k=1}^{\infty}\frac{k^{q/p'-1}}{\ln(k+1)^{(d-1)q/p'}}\left(\sup_{r\geqslant\lceil\ln(k+1)\rceil}\sup_{U\in M_r}\frac{1}{2^r}\left|\sum_{m\in U}a_m\right|\right)^q\\ &\gtrsim \sum_{k=1}^{\infty}k^{q/p'-1}\ln(k+1)^{(d-1)q/p}\left(\frac{1}{|G_k|}\left|\sum_{m\in G_k}a_m\right|\right)^q,\end{aligned}$$

where $\lceil t\rceil$ denotes the upper integer part of t. □

4.2 Comparison of Corollaries and 3.5 and 4.6

For simplicity, we consider Pitt's inequalities for $d=2$ and $\alpha=0$. Let us recall (see (3.8)) that, for any cube $U_k=[1,k]^d$ and $1<r<\infty$,

$$\left(\sum_{k=1}^{\infty}k^{d(r-1)-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k}a_m\right|^r\right)^{1/r}\lesssim\|f\|_{L_r(\mathbb{T}^d)}. \tag{4.4}$$

For any hyperbolic crosses $G_k\in M$ (see (4.3)), and $1<r<\infty$,

$$\left(\sum_{k=1}^{\infty}k^{r-2}\ln(k+1)^{d-1}\left|\frac{1}{|G_k|}\sum_{m\in G_k}a_m\right|^r\right)^{1/r}\lesssim\|f\|_{L_r(\mathbb{T}^d)}. \tag{4.5}$$

We will show that (4.4) and (4.5) are not comparable. Indeed, for any sufficiently large n, take $f(x)=\sum_{m\in G_n}e^{\imath(m,x)}=D_{G_n}(x)$. First, we know that (4.5) becomes an equivalence for this function. Indeed, from Theorem 4.2,

$$\|f\|_{L_r(\mathbb{T}^d)}\asymp n^{1/r'}(\ln n)^{\frac{d-1}{r}},\quad 1<r<\infty.$$

On the one hand, (4.5) is sharp in this case:

$$\|f\|_{L_r(\mathbb{T}^d)}\gtrsim\left(\sum_{k=1}^{n}k^{r-2}\ln(k+1)^{d-1}\right)^{1/r}\gtrsim n^{1/r'}(\ln n)^{\frac{d-1}{r}}.$$

On the other hand, (4.4) is not optimal for f since

$$J:=\left(\sum_{k=1}^{\infty}k^{d(r-1)-1}\left|\frac{1}{|U_k|}\sum_{m\in U_k}a_m\right|^r\right)^{1/r}\lesssim n^{1/r'}. \tag{4.6}$$

To see this inequality for $d = 2$,

$$J^r = \Big(\sum_{k=1}^{n^{1/2}} + \sum_{k=n^{1/2}}^{n} + \sum_{k=n}^{\infty}\Big) k^{d(r-1)-1} \Big|\frac{1}{|U_k|}\sum_{m\in U_k} a_m\Big|^r =: J_1 + J_2 + J_3.$$

For the first term,

$$J_1 = \sum_{k=1}^{n^{1/2}} k^{2(r-1)-1} \lesssim n^{r-1}.$$

Rewriting the sum as an integral, we have

$$\Big|\sum_{m\in G_n\cap[1,k]^2} 1\Big| \lesssim n \ln \frac{k^2}{n} \tag{4.7}$$

if $n^{1/2} \leqslant k \leqslant n$. Then

$$\begin{aligned} J_2 &= \sum_{k=n^{1/2}}^{n} k^{2(r-1)-1}\frac{1}{k^{2r}}\Big|\sum_{m\in G_n\cap[1,k]^2} 1\Big|^r \\ &\lesssim \sum_{k=n^{1/2}}^{n} k^{-3}\left(n\ln\frac{k^2}{n}\right)^r \lesssim \int_{n^{1/2}}^{n} x^{-3}\left(n\ln\frac{x^2}{n}\right)^r dx \lesssim n^{r-1}\int_1^n x^{-2}(\ln x)^r\, dx \lesssim n^{r-1}. \end{aligned}$$

For the third term, we have

$$J_3 \lesssim \sum_{k=n}^{\infty} k^{2(r-1)-1}\frac{1}{k^{2r}}(n\ln n)^r \lesssim n^{r-1},$$

which shows (4.6).

Let us now consider, for any sufficiently large n, $f(x) = \sum_{m\in[1,n]^2\cap\mathbb{Z}^2} e^{\imath(m,x)}$. For this function, (4.4) is sharp: in light of Lemma 3.4,

$$n^{d/r'} \lesssim \left(\sum_{k=1}^{n} k^{d(r-1)-1}\Big|\frac{1}{k^d}\sum_{m\in[1,k]^d} 1\Big|^r\right)^{1/r} \lesssim \|f\|_{L_r(\mathbb{T}^d)} \lesssim n^{d/r'}.$$

On the contrary, estimate (4.5) is not optimal, since, for $d = 2$,

$$J := \left(\sum_{k=1}^{\infty} k^{r-2}\ln(k+1)\Big|\frac{1}{|G_k|}\sum_{m\in G_k} a_m\Big|^r\right)^{1/r} \lesssim n^{2/r'}(\ln n)^{-1/r'}. \tag{4.8}$$

Indeed, let

$$J^r = \left(\sum_{k=1}^{n} + \sum_{k=n}^{n^2} + \sum_{k=n^2}^{\infty}\right) k^{r-2}\ln(k+1)\Big|\frac{1}{|G_k|}\sum_{m\in G_k} a_m\Big|^r =: J_1 + J_2 + J_3.$$

Then

$$J_1 := \sum_{k=1}^{n} k^{r-2}\ln(k+1) \lesssim n^{r-1}\ln n \lesssim n^{2r-2}(\ln n)^{1-r}.$$

Using (4.7), we conclude

$$\begin{aligned} J_2 &= \sum_{k=n}^{n^2} k^{r-2}\ln(k+1)\Big|\frac{1}{k\ln k}\sum_{m\in G_k\cap[1,n]^2} 1\Big|^r \\ &\lesssim \sum_{k=n}^{n^2} k^{r-2}\ln(k+1)\left|\frac{k\ln\frac{n^2}{k}}{k\ln k}\right|^r \lesssim \int_n^{n^2} x^{r-2}(\ln x)^{1-r}\left(\ln\frac{n^2}{x}\right)^r dx \\ &\lesssim (\ln n)^{1-r}\int_n^{n^2} x^{r-2}\left(\ln\frac{n^2}{x}\right)^r dx \lesssim n^{2r-2}(\ln n)^{1-r}\int_1^n x^{-r}\ln^r x\,dx \lesssim n^{2r-2}(\ln n)^{1-r}. \end{aligned}$$

Finally,

$$J_3 = \sum_{k=n^2}^{\infty} k^{r-2}\ln(k+1)\left|\frac{n^2}{k\ln k}\right|^r \lesssim n^{2r-2}(\ln n)^{1-r},$$

which completes the proof of (4.8).

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