



On the Invariant Sphere Theorem

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Abstract. For a system in $\mathbb{R}^n$ consisting of an unstable star node and homogeneous non-linearity, i.e. $\dot{x} = \mu x + Q(x)$ with $\mu > 0$ and $Q(x)$ a homogeneous polynomial satisfying $xQ(x) < 0$ for all $x \neq 0$, Field in 1989 proved the existence of a unique topological invariant sphere. Here restricted to $n = 2$, we prove that the topological invariant circle is of star shape, and obtain the classification of all their phase portraits in the Poincaré disc when Q is of degree 5 and the invariant circle is the circle $\mathbb{S}^1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Moreover, without the restriction $xQ(x) < 0$ we characterize the relation between equilibria (both finite and infinite) and the invariant straight lines passing through the origin. These results extend and improve the previous results of other authors.

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1. Introduction and statement of the main result

As an application of his results, Field [4, Theorem 5.1] in 1989 studied the n -dimensional differential system

$$\dot{x} = \frac{dx}{dt} = \mu x + R(x), \quad (1)$$

and obtained the next result, called *Invariant Sphere Theorem*.

Invariant Sphere Theorem. Assume that $\mu > 0$, and $R(x)$ is an ℓ -dimensional homogeneous polynomial vector field whose components all have the same degree $2d + 1$ satisfying $\langle R(x), x \rangle < 0$ for all $x \in \mathbb{R}^\ell \setminus \{0\}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^\ell$. Then there exists a unique $(\ell - 1)$ -dimensional invariant topological sphere $S(\mu) \subset \mathbb{R}^\ell \setminus \{0\}$ whose complement has the bounded component containing the origin. $S(\mu)$ is globally attracting in the sense that every orbit of system (1) through $x \neq 0$ is forward asymptotic to $S(\mu)$.

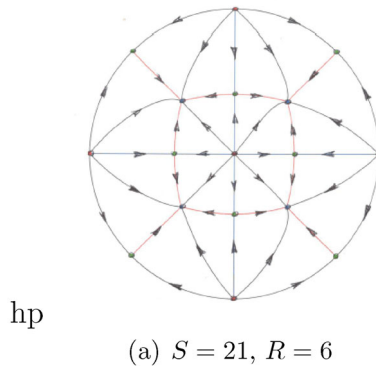


FIGURE 1. The topological phase portrait of the differential system (3) in the Poincaré disc with a topological invariant circle that is a square

Note that in the Invariant Sphere Theorem, we can take the time rescaling $t \rightarrow \mu t$ without changing the orientation and topological dynamics. So, restricted system (1) to 2-dimensional case, we have the class of polynomial differential systems of the form

$$\begin{aligned}\dot{x} &= x + P_n(x, y), \\ \dot{y} &= y + Q_n(x, y),\end{aligned}\tag{2}$$

with P_n and Q_n being homogeneous polynomials of degree n odd. Then, the Invariant Sphere Theorem becomes the following Invariant Circle Theorem.

Theorem 1. (Invariant Circle Theorem) *Consider the polynomial differential systems (2) of degree $n > 1$ odd. If $xP_n(x, y) + yQ_n(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, then there exist a topological circle S invariant by the flow of the differential system (2). Moreover, S is globally attracting in the sense that every orbit of the differential system (2) distinct to $(0, 0)$ is forward asymptotic to S .*

The Invariant Circle Theorem provides the existence of an invariant topological circle surrounding the origin under the flow of system (2). In general the invariant topological circle in the statement of Theorem 1 is not the circle $\mathbb{S}^1$. Also in general we do not know its geometric shape, as well if it is algebraic or not. For instance, the differential system

$$\dot{x} = x - x^5, \quad \dot{y} = y - y^5,\tag{3}$$

satisfies all the assumptions of the Invariant Circle Theorem and its topological invariant circle is a square, see Figure 1.

Related with the Invariant Circle Theorem, Alarcón et al. [1] characterized topological structures of system (2) with an invariant topological circle when $n = 3$.

In this paper we first provide some properties on the equilibria and the invariant straight lines of system (2). Then we characterize the global dynamics of system (2) with invariant unit circle $\mathbb{S}^1$ when $n = 3$ and $n = 5$.

To study the global dynamics one needs the classification of the system at infinity. Roughly speaking, the Poincaré disc is the closed disc of radius one centered at the origin of coordinates of the plane $\mathbb{R}^2$ whose interior has been identified with the whole $\mathbb{R}^2$, and the circle of its boundary, denoted by $\mathbb{S}_\infty$, is identified with the infinity of the plane $\mathbb{R}^2$. Note that in the plane $\mathbb{R}^2$ we can go or come from the infinity in as many directions as points has the circle $\mathbb{S}_\infty$. Since a polynomial differential system can be extended analytically to the whole Poincaré disc, i.e. including its infinity $\mathbb{S}_\infty$, this fact allows to describe the orbits that go to infinity or that come from the infinity. For more details on the Poincaré disc and the so called Poincaré compactification, see Section 2.

Our first main result establishes relations among the invariant straight lines and the equilibria of the differential system (2).

Theorem 2. *For $n = 2m + 1$ the following statements hold.*

- (a) *The straight line passing through the origin and any of the infinite equilibrium points is invariant under the flow of system (2).*
- (b) *Any finite equilibrium is located on an invariant straight line of the ones stated in statement (a).*
- (c) *System (2) has either infinitely many finite equilibria, or at most $4m + 5$ finite equilibria.*
 - (c₁) *There exist systems (2) which have exactly $4k + 1$, $k \in \{0, 1, \dots, m + 1\}$, finite equilibria, or infinitely many finite equilibria.*
 - (c₂) *If system (2) has infinitely many equilibria, then it is of the form $\dot{x} = x(1 + M(x, y))$, $\dot{y} = y(1 + M(x, y))$ with $M(x, y)$ a homogeneous polynomial of degree $n - 1$. So the invariant topological circle filled with equilibria is given by the curve $1 + M(x, y) = 0$.*
- (d) *Any straight line passing through the origin and another finite equilibrium distinct to the origin is invariant, and contains exactly three finite equilibria with two symmetric with respect the origin.*

Theorem 2 is proved in Section 3.

As mentioned previously we do not know the geometric shape of the invariant topological cycle for system (2), any way our next result shows that it is of star shape, that is, each straight line passing through the origin has exactly two intersection points with the topological circle.

Proposition 3. *The invariant topological circle in Theorem 1 is of star shape.*

Proposition 3 is proved in Section 3.

Next we turn to system (2) with the invariant unit cycle for $n = 3$ and $n = 5$. As we will show, system (2) having the invariant circle $\mathbb{S}^1$ when $n = 3$ is

$$\begin{aligned}\dot{x} &= x - x^3 + x^2y =: P_{3c}(x, y), \\ \dot{y} &= y - x^3 - 2x^2y - y^3 =: Q_{3c}(x, y),\end{aligned}\tag{4}$$

and when $n = 5$ is

$$\begin{aligned}\dot{x} &= x - x^5 + ax^4y + bx^3y^2 + cx^2y^3 + dxy^4 + ey^5 =: P_{5c}(x, y), \\ \dot{y} &= y - ax^5 - (3+b)x^4y - cx^3y^2 - (3+d)x^2y^3 - exy^4 - y^5 =: Q_{5c}(x, y).\end{aligned}\tag{5}$$

One can check that $xP_{3c}(x, y) + yQ_{3c}(x, y) < 0$ and $xP_{5c}(x, y) + yQ_{5c}(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. This means that for system (2) with $n = 3$ and $n = 5$ the existence of the invariant unit circle $\mathbb{S}^1$ implies the assumption $xP_n(x, y) + yQ_n(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ of Invariant Circle Theorem. But what occurs for the unit circle $\mathbb{S}^1$ that when it is invariant for system (5) automatically this system satisfies the assumption of the Invariant Circle Theorem, this does not occur if the system (5) has an invariant ellipse. Indeed, consider the system

$$\begin{aligned}\dot{x} &= x - \frac{x^5}{16} + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 =: x + P_{5c}(x, y), \\ \dot{y} &= y - \frac{1}{4}x^5 - \frac{7}{16}x^4y - \frac{1}{4}x^3y^2 - x^2y^3 - \frac{1}{4}xy^4 - y^5 =: y + Q_{5c}(x, y),\end{aligned}$$

having the invariant ellipse

$$\frac{x^2}{4} + y^2 = 1,$$

and satisfying $xP_{5c}(x, y) + yQ_{5c}(x, y)|_{x=-y} = -\frac{11}{4}y^6 < 0$ for $y \neq 0$, while $xP_{5c}(x, y) + yQ_{5c}(x, y)|_{x=y} = \frac{7}{4}y^6 > 0$ for $y \neq 0$.

Let U be an open and dense subset of $\mathbb{R}^2$ and let $H : U \rightarrow \mathbb{R}$ an analytic map. We say that the function $H = H(x, y)$ is a *first integral* of the differential system (4) if H is constant on every solution $(x(t), y(t))$ of system (4) contained in U , i.e. if

$$\left. \frac{dH}{dt} \right|_{(x,y) \in U} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} \Big|_{(x,y) \in U} = 0.$$

Our second and third main results are the following.

Proposition 4. *The phase portrait in the Poincaré disc of the polynomial differential system (4) is the one of Figure 2. Moreover, this differential system has the first integral $H = x^2(x^2 + y^2 - 1)/(x + y)^2$.*

Proposition 4 is proved in Section 4.

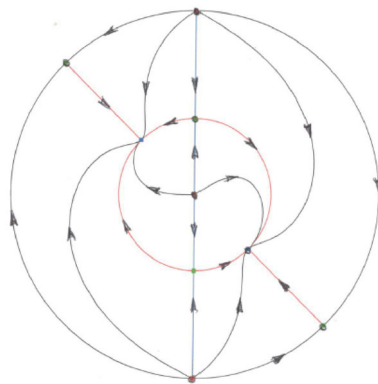
(a) $S = 21, R = 6$

FIGURE 2. The topological phase portrait of the differential system (4)

Theorem 5. *The phase portrait in the Poincaré disc of a polynomial differential system (5) is topologically equivalent to one of Figure 3.*

Theorem 5 is proved in Section 5.

Of course, providing the phase portraits of the polynomial differential systems (4) and (5), we are providing all the qualitative dynamics of these differential systems.

The phase portraits in the Poincaré disc of the polynomial differential systems (2) with $n = 3$ were classified by Alarcón, Castro and Labouriau in [1]. In fact these differential systems only have three topological distinct phase portraits in the Poincaré disc, see Figure 1 of [1]. Of course, the phase portrait of Proposition 4 is topologically one of these phase portraits, but it is the unique phase portrait of Figure 1 of [1] that can be realized with the invariant circle $\mathbb{S}^1$ provided that the system has no a common factor. Otherwise, the invariant circle $\mathbb{S}^1$ is filled with equilibria.

The class of polynomial differential systems whose nonlinearities are homogeneous polynomials of the same degree has been studied by several authors, see for instance the papers [2, 5–8]. The classification of global dynamics of system (2) satisfying the conditions in Theorem 1 is in general still an open problem to be solved.

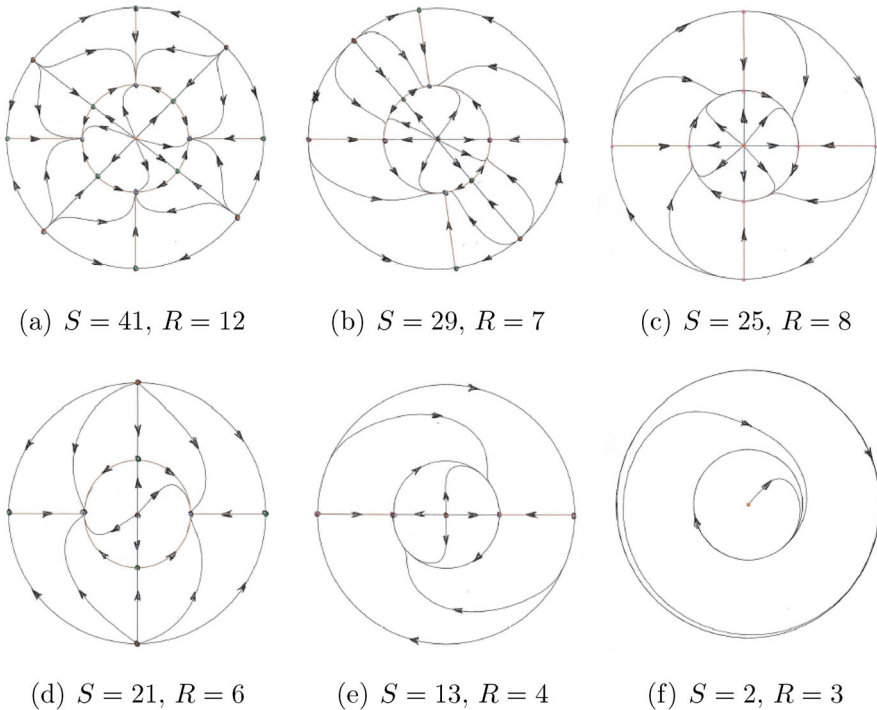


FIGURE 3. The distinct topological phase portraits of the differential system (5) in the Poincaré disc, where S and R denote the number of separatrices and canonical regions of each phase portrait, respectively

2. Preliminaries

2.1. Poincaré compactification

Here we summarize the equations of the Poincaré compactification that we need for proving our results. Poincaré [12] shown us how to study the dynamics of a polynomial differential system in the plane $\mathbb{R}^2$ near the infinity.

Let $\mathbb{S}^2$ be the sphere of radius one centered at the origin of coordinates of $\mathbb{R}^3$. It is known that the polynomial vector field X induces an analytic vector field $p(X)$ on $\mathbb{S}^2$, in such a way that there are in $p(X)$ two copies of X , one in the open northern hemisphere and the other in the open southern hemisphere, and the equator of the sphere has been identified with the infinity $\mathbb{S}_\infty$ of $\mathbb{R}^2$, for details see [3, Chapter 5].

To obtain the analytical expression for $p(X)$ we consider the sphere as a smooth manifold. We choose the six local charts given by $U_i = \{\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{S}^2 : z_i > 0\}$ and $V_i = \{\mathbf{z} \in \mathbb{S}^2 : z_i < 0\}$, for $i = 1, 2, 3$, with the corresponding coordinate maps $\varphi_i : U_i \rightarrow \mathbb{R}^2$ and $\psi_i : V_i \rightarrow \mathbb{R}^2$, defined by $\varphi_k(\mathbf{z}) = \psi_k(\mathbf{z}) = (z_m/z_k, z_n/z_k)$ for $m < n$ and $m, n \neq k$. The local coordinates on U_i and V_i for $i = 1, 2, 3$ are denoted by (u, v) . From [3, Chapter 5] the vector field $p(X)$ in these local charts writes

$$\begin{aligned}(\dot{u}, \dot{v}) &= \left(v^n \left(-uP \left(\frac{1}{v}, \frac{u}{v} \right) + Q \left(\frac{1}{v}, \frac{u}{v} \right) \right), -v^{n+1} P \left(\frac{1}{v}, \frac{u}{v} \right) \right) \text{ in } U_1; \\(\dot{u}, \dot{v}) &= \left(v^n \left(-uQ \left(\frac{u}{v}, \frac{1}{v} \right) + P \left(\frac{u}{v}, \frac{1}{v} \right) \right), -v^{n+1} Q \left(\frac{u}{v}, \frac{1}{v} \right) \right) \text{ in } U_2; \quad (6) \\(\dot{u}, \dot{v}) &= (P(u, v), Q(u, v)) \text{ in } U_3,\end{aligned}$$

where n is the *degree* of the polynomial vector field X , i.e. the maximum of the two degrees of the two polynomial components of the vector field X . Recall that the expressions of the vector field $p(X)$ in the local chart (V_i, ψ_i) is the same as that in the local chart (U_i, φ_i) multiplied by $(-1)^{n-1}$ for $i = 1, 2, 3$.

The Poincaré disc $\mathbb{D}$ is the projection by $(z_1, z_2, z_3) \rightarrow (z_1, z_2)$ of the closed northern hemisphere.

Note that the coordinates of the points on the circle $\mathbb{S}_\infty$ (identified with the infinity of $\mathbb{R}^2$, i.e. the boundary of the Poincaré disc $\mathbb{D}$) are of the form $(u, 0)$ in the local charts U_i and V_i for $i = 1, 2$.

From the expressions of the vector field $p(X)$ it follows that the circle $\mathbb{S}_\infty$ of the infinity is invariant by the flow of $p(X)$. The equilibrium points of $p(X)$ which lie in $\mathbb{S}_\infty$ are the *infinite equilibrium points* of X .

Note that if $p \in \mathbb{S}_\infty$ is an infinite equilibrium point, then $-p$ is also an infinite equilibrium point and these two points have the same local phase portrait on the sphere if the degree of vector field is odd. Such phase portraits change the orientation of their orbits if the degree of the vector field is even. We also note that the pair of diametrically opposite infinite equilibria of the differential system (2) are given by the real linear factors $rx + sy$ of the homogeneous polynomial $yP_n(x, y) - xQ_n(x, y)$.

From the expressions of X in the charts U_i and V_i for $i = 1, 2$ it follows that for determining the infinite equilibrium points it is sufficient to study the ones in the chart U_1 , and to check if the origin of the local chart U_2 is or not an equilibrium.

For additional details on the results mentioned in this summary on the Poincaré compactification, see [3, Chapter 5].

2.2. Topological equivalence of two polynomial vector fields

Let X_i for $i = 1, 2$ be two polynomial vector fields on $\mathbb{R}^2$. These two vector fields are *topologically equivalent* if there exists a homeomorphism in the Poincaré disc $\mathbb{D}$ sending the infinity $\mathbb{S}^1$ to the infinity and sending the orbits of $p(X_1)$ into orbits of $p(X_2)$, preserving or reversing the orientation of all the orbits.

A *separatrix* of the Poincaré compactification $p(X)$ is one of following orbits: the (finite) equilibrium points of X , the limit cycles, all the orbits at the infinity $\mathbb{S}_\infty$, and the two orbits of the boundary of the hyperbolic sectors from a finite or an infinite equilibrium point, for more information on the separatrices, see [9, 10].

We denote the set of all separatrices of $p(X)$ by Σ_X , this set is closed, see [10].

An open connected component of $\mathbb{D} \setminus \Sigma_X$ is called a *canonical region* of $p(X)$. The set Σ'_X formed by the union of Σ_X with an orbit from each canonical region is called the *separatrix configuration* of $p(X)$. The number of separatrices of a phase portrait in the Poincaré disc is denoted S , and by R we denote the number of canonical regions. We recall that the separatrices in a Poincaré disc are all the orbits on the infinity circle $\mathbb{S}_\infty$, the finite equilibria, the limit cycles if they exist, and the two orbits in the boundary of the hyperbolic sectors either from infinite or finite equilibria.

The separatrix configurations Σ'_{X_1} and Σ'_{X_2} are *topologically equivalent* if there is a homeomorphism h of the Poincaré disc $\mathbb{D}$ verifying that $h(\Sigma'_{X_1}) = \Sigma'_{X_2}$.

The next theorem, due to Markus [9], Neumann [10] and Peixoto [11], they proved it independently, allows to determine if two phase portraits in the Poincaré disc are topologically equivalent looking at the topological equivalence of their separatrix configurations.

Theorem 6. *Two Poincaré compactified polynomial vector fields $p(X_1)$ and $p(X_2)$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations Σ'_{X_1} and Σ'_{X_2} are topologically equivalent.*

3. Proof of Theorem 2 and Proposition 3

Proof of Theorem 2. (a) From Subsection 2.1 an infinite equilibrium must be located at an endpoint of the straight line $rx + sy = 0$ with $rx + sy$ being a real linear factor of the homogeneous polynomial $yP_n(x, y) - xQ_n(x, y)$. Since

$$\left. \frac{d}{dt} \left(\frac{y}{x} \right) \right|_{rx+sy=0} = \left. \frac{\dot{y}x - y\dot{x}}{x^2} \right|_{rx+sy=0} = \left. \frac{xQ_n(x, y) - yP_n(x, y)}{y^2} \right|_{rx+sy=0} = 0,$$

the straight line $rx + sy = 0$ is invariant. The statement follows.

(b) Let $(\bar{x}, \bar{y})$ be an equilibrium point of the differential system (2). We shall prove that the straight line $f = f(x, y) = \bar{x}y - \bar{y}x = 0$ passing through the origin of coordinates and the equilibrium point $(\bar{x}, \bar{y})$ is invariant. Indeed, the invariance of the straight line $f = 0$ follows from the next computation

$$\begin{aligned} \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \Big|_{y=\bar{y}x/\bar{x}} &= -\bar{y}(x + P_n(x, y)) + \bar{x}(y + Q_n(x, y)) \Big|_{y=\bar{y}x/\bar{x}} \\ &= (-x\bar{y} + \bar{x}y) - \bar{y}P_n(x, y) + \bar{x}Q_n(x, y) \Big|_{y=\bar{y}x/\bar{x}} \\ &= -\bar{y}P_n\left(x, \frac{\bar{y}}{\bar{x}}x\right) + \bar{x}Q_n\left(x, \frac{\bar{y}}{\bar{x}}x\right) \\ &= x^n \left(-\bar{y}P_n\left(1, \frac{\bar{y}}{\bar{x}}\right) + \bar{x}Q_n\left(1, \frac{\bar{y}}{\bar{x}}\right)\right) \\ &= \frac{x^n}{\bar{x}^n}(-\bar{y}P_n(\bar{x}, \bar{y}) + \bar{x}Q_n(\bar{x}, \bar{y})) = 0. \end{aligned}$$

Therefore, since the invariant straight line $f = 0$ intersects the infinity in an equilibrium point, it must be one of the invariant straight lines $rx + sy = 0$. The proof of the statement is done.

(c) Note that a rotation does not change the form of system (2) and the number of equilibria, both finite and infinite. So we can assume without loss of generality that any finite equilibrium is not located on the axes, except the origin.

Let $(\bar{x}, \bar{y})$ be a finite equilibrium, then $\bar{x} \neq 0, \bar{y} \neq 0$ by the above assumption. From the equations that a finite equilibrium satisfies, $(\bar{x}, \bar{y})$ must satisfy

$$\bar{y}P_n(\bar{x}, \bar{y}) - \bar{x}Q_n(\bar{x}, \bar{y}) = 0, \quad (7)$$

and

$$\bar{x} + P_n(\bar{x}, \bar{y}) = 0. \quad (8)$$

Note that $R_{n+1}(x, y) := yP_n(x, y) - xQ_n(x, y)$ is a homogeneous polynomial of degree $2m + 2$, and the solution (x_j, y_j) of $R_{n+1}(x, y) = 0$ exactly corresponds to $\lambda_j = y_j/x_j$ which satisfies $R_{n+1}(1, \lambda_j) = 0$. Here we have used the assumption that any equilibrium $(\bar{x}, \bar{y})$ satisfies $\bar{x} \neq 0, \bar{y} \neq 0$. Observe that for any given $P_n(x, y) \not\equiv 0$, we can choose suitable $Q_n(x, y)$ such that $\lambda P_n(1, \lambda) - Q_n(1, \lambda) = 0$ has exactly $k \in \{0, 1, 2, \dots, 2m + 2\}$ solutions. Denote these solutions by $\lambda_1, \dots, \lambda_k$ when $k > 0$.

We now claim that each straight line $y = \lambda_j x$ contains either exactly one equilibrium (i.e. the origin), or exactly three equilibria: the origin and the other two ones symmetric to the origin. Indeed, if in the former case, we are done. If the line $y = \lambda_j x$ contains an equilibrium (x_j, y_j) , different from the origin. Then by symmetry of system (2) with respect to the origin, it follows that $(-x_j, -y_j)$ is also an equilibrium. By contrary, if there is other equilibrium saying $(\bar{x}, \bar{y})$ on $y = \lambda_j x$, different from $(0, 0)$, (x_j, y_j) and $(-x_j, -y_j)$, then

$(\bar{x}, \bar{y}) = \rho(x_j, y_j)$ with $\rho \notin \{0, 1, -1\}$. By the equation

$$\bar{x} + P_n(\bar{x}, \bar{y}) = 0$$

that $(\bar{x}, \bar{y})$ satisfies, it gives that

$$0 = \rho x_j + \rho^n P_n(x_j, y_j) = \rho x_j + \rho^n(-x_j) = \rho(1 - \rho^{n-1})x_j,$$

a contradiction. Hence, system (2) has at most $2(n+1)+1 = 4m+5$ equilibria provided that the number is finite.

(c₁) We first claim that the maximum number of equilibria can be realized. Indeed, for any $P_n(x, y)$, we can choose $Q_n(x, y)$ such that $yP_n(x, y) - xQ_n(x, y) = 0$ has exactly $2m+2$ nonzero and different solutions $(x, y) = x(1, \lambda_j)$, $j = 1, \dots, 2m+2$ with 'exact' in the sense $\lambda_i \neq \lambda_j$ for $i \neq j$ and $i, j \in \{1, \dots, 2m+2\}$. For any chosen (x_j, y_j) with $y_j = \lambda_j x_j$ and $x_j \neq 0$, $j = 1, \dots, 2m+2$, one has

$$x_j + x_j^n P_n(1, \lambda_j) = 0, \quad j = 1, \dots, 2m+2. \quad (9)$$

To ensure that all (x_j, y_j) 's are finite equilibria of system (2), we need to find $P_n(x, y)$ such that all the equations in (9) hold simultaneously. For that aim, set

$$P_n(x, y) = \sum_{j=0}^n a_{n-j,j} x^{n-j} y^j.$$

Then the equations in (9) can be written in

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{2m} & \lambda_1^{2m+1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{2m} & \lambda_2^{2m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \lambda_{2m+2} & \lambda_{2m+2}^2 & \cdots & \lambda_{2m+2}^{2m} & \lambda_{2m+2}^{2m+1} \end{pmatrix} \begin{pmatrix} a_{2m+1,0} \\ a_{2m,1} \\ \vdots \\ a_{0,2m+1} \end{pmatrix} = - \begin{pmatrix} x_1^{-2m} \\ x_2^{-2m} \\ \vdots \\ x_{2m+2}^{-2m} \end{pmatrix}. \quad (10)$$

The coefficient matrix is a Vandermonde matrix due to $\lambda_i \neq \lambda_j$ for $i \neq j$. So the linear equation on $(a_{2m+1,0}, a_{2m,1}, \dots, a_{0,2m+1})$ has always a unique nontrivial solution. This proves that such a chosen $P_n(x, y)$ with coefficients $a_{2m+1,0}, a_{2m,1}, \dots, a_{0,2m+1}$ ensures that the equation $x + P_n(x, y) = 0$ has $(x_j, \lambda_j x_j)$, $j = 1, \dots, 2m+2$ as its solutions. In addition, by symmetry $(-x_j, -\lambda_j x_j)$, $j = 1, \dots, 2m+2$, are also its solutions. This proves the claim. So, including the equilibrium at the origin system (2) can have the maximum number $4m+5$ of finite equilibria.

For any given $k \in \{0, 1, \dots, 2m+2\}$, the claim that the exact $2k+1$ number of finite equilibria can be realized by systems of form (2) follows from the facts that there exist $P_n(x, y)$ and $Q_n(x, y)$ such that the equations (7) and (8) have exactly $2k+1$ solutions. Indeed, for any $P_n(x, y)$, one can choose $Q_n(x, y)$ such that $\lambda P_n(1, \lambda) - Q_n(1, \lambda) = 0$ has exactly k different solutions, denoted by $\lambda_1, \dots, \lambda_k$. Then the system of linear equations (10) with λ_{2m} replaced by λ_k and x_{2m+2} by x_k has always solutions in $(a_{2m+1,0}, a_{2m,1}, \dots, a_{0,2m+1})$. In fact,

when $k < 2m + 2$, the solutions form a linear space of dimension $2m + 2 - k$ in the coefficients space of $P_n(x, y)$.

(c_2) If the number of finite equilibria is infinite, then $R_{n+1}(x, y) \equiv 0$. Set

$$P_n(x, y) = \sum_{j=0}^n a_{n-j,j} x^{n-j} y^j, \quad Q_n(x, y) = \sum_{j=0}^n b_{n-j,j} x^{n-j} y^j.$$

Then

$$R_{n+1}(x, y) = \sum_{j=0}^{n-1} (a_{n-j,j} - b_{n-1-j,j+1}) x^{n-j} y^{j+1} + a_{0,n} y^{n+1} - b_{n,0} x^{n+1}.$$

It follows

$$a_{0,n} = 0, \quad b_{n,0} = 0, \quad a_{n-j,j} = b_{n-1-j,j+1}, \quad j = 0, 1, \dots, n-1.$$

Consequently $P_n(x, y) = xM_{n-1}(x, y)$, $Q_n(x, y) = yM_{n-1}(x, y)$ where M_{n-1} is a homogeneous polynomial of degree $n-1$. In order that system (2) has infinitely many finite equilibria is equivalent to the fact that $1 + M_{n-1}(x, y) = 0$ is a real curve in the (x, y) plane. In fact, in this case system (2) has the common factor $1 + M_{n-1}(x, y)$, and so it is essentially a linear differential system after a convenient rescaling of the time.

(d) The proof follows from those of the previous three ones. $\square$

Proof of Proposition 3. Let Γ be the invariant topological circle obtained from Theorem 1, which is in counterclockwise sense. By contrary, if Γ is not of the star shape, then there exists a ray starting from the origin, denoted by $y = \sigma x$, which intersects Γ transversally in at least three points. Let I_1 and I_2 be the two consecutive intersection points, see Figure 4.

Without loss of generality (if necessary, one can take a rotation), we can take the ray with $\sigma > 0$, and so the points (x, y) on the ray satisfy $x > 0$ and $y > 0$. Denote by $\mathcal{X}$ the vector field associated to system (2), and by $\mathcal{L}$ and $\mathcal{H}$ the linear and homogeneous parts of $\mathcal{X}$. Then $\mathcal{X} = \mathcal{L} + \mathcal{H}$. Denote by $\mathcal{A}_{I_j}$ the vector of the vector field $\mathcal{A}$ at I_j with $\mathcal{A} \in \{\mathcal{X}, \mathcal{L}, \mathcal{H}\}$ and $j = 1, 2$. Then the three vectors at I_1 have the structure as those shown in Figure 4. Note that $\mathcal{L}_{I_1}$ is upwards and $\mathcal{H}_{I_1}$ is leftwards. While the three vectors at I_2 must have $\mathcal{L}_{I_2}$ upwards and $\mathcal{H}_{I_2}$ rightwards.

Set $I_1 = (x_1, y_1)$ and $I_2 = (x_2, y_2)$. Then $y_1 = \sigma x_1$ and $y_2 = \sigma x_2$. Hence, one has

$$\begin{aligned} \mathcal{X}_{I_1} &= (x_1, \sigma x_1) + (P_n(x_1, \sigma x_1), Q_n(x_1, \sigma x_1)) = x_1(1, \sigma) + x_1^n(P_n(1, \sigma), Q_n(1, \sigma)), \\ \mathcal{X}_{I_2} &= (x_2, \sigma x_2) + (P_n(x_2, \sigma x_2), Q_n(x_2, \sigma x_2)) = x_2(1, \sigma) + x_2^n(P_n(1, \sigma), Q_n(1, \sigma)). \end{aligned}$$

This shows that $\mathcal{H}_{I_1}$ and $\mathcal{H}_{I_2}$ are parallel and point the same direction. We have a contradiction, so the invariant topological circle Γ must be of star shape. $\square$

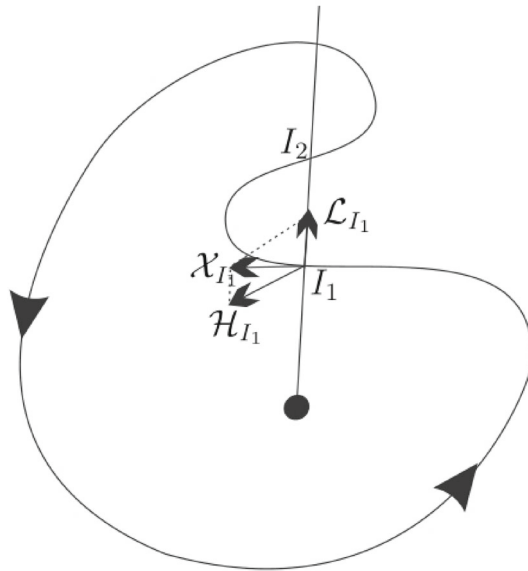


FIGURE 4. Illustration of the invariant topological circle which is not star shape

4. Proof of Proposition 4

First we claim that the unique polynomial differential system (2) with $n = 3$ satisfying the condition $xP_3(x, y) + yQ_3(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and having the invariant circle $\mathbb{S}^1$ is the differential system (4).

Now we shall prove the claim. The polynomial differential systems (2) with $n = 3$ are

$$\begin{aligned}\dot{x} &= x + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3, \\ \dot{y} &= y + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3.\end{aligned}\tag{11}$$

We recall that the circle $\mathbb{S}^1$ is *invariant* by the flow of the differential system (11) if $f = f(x, y) = x^2 + y^2 - 1 = 0$ satisfies

$$\left. \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \right|_{f=0} = 0,$$

for more details see [3, Chapter 8]. Since

$$\begin{aligned}\left. \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \right|_{x=\pm\sqrt{1-y^2}} &= 1 + b_1 - (2b_1 - b_2)y^2 + a_3y^3 \pm (b_1 - b_2 + b_4)y^4 - a_3y^5 \\ &\quad \pm (a_2 + b_1)y\sqrt{1-y^2} \mp (a_2 - a_4 + b_1)y^3\sqrt{1-y^2} \\ &\quad \pm b_3y^4\sqrt{1-y^2} = 0.\end{aligned}$$

It follows that

$$b_1 = -1, \quad b_2 = -2, \quad a_3 = 0, \quad b_3 = 0, \quad a_2 = 1, \quad a_4 = 0, \quad b_4 = -1.$$

Hence the differential system (11) becomes the next one

$$\begin{aligned}\dot{x} &= x - x^3 + x^2y = x + P_3(x, y), \\ \dot{y} &= y - x^3 - 2x^2y - y^3 = y + Q_3(x, y),\end{aligned}\tag{12}$$

i.e. we obtain the differential system (4). Now we shall see that forcing that the differential system (4) has the invariant circle $\mathbb{S}^1$, then it automatically satisfies the condition $xP_3(x, y) + yQ_3(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Indeed,

$$xP_3(x, y) + yQ_3(x, y) = -(x^2 + y^2)^2 < 0, \quad \text{for all } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

So the claim is proved.

Since the differential system (4) satisfies all the assumptions of Theorem 1 we have that the circle $\mathbb{S}^1$ is invariant and globally attracting. Hence all the equilibria distinct to the origin of coordinates live on the circle $\mathbb{S}^1$.

The algebraic curve $g = g(x, y) = 0$ is *invariant* by the flow of the polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, if there exists a polynomial $K_g = K_g(x, y)$ such that

$$\frac{\partial g}{\partial x}P + \frac{\partial g}{\partial y}Q = K_g g.$$

The polynomial K_g is called the *cofactor* of the invariant algebraic curve $g = 0$.

Additional to the invariant circle $f_1 = f = 0$, the differential system (4) has the invariant equilibrium point $f_2 = f_2(x, y) = x^2 + y^2 = 0$, the invariant straight line $f_3 = f_3(x, y) = x = 0$ (this follows directly from the expression of $\dot{x}$), and the invariant straight line $f_4 = f_4(x, y) = x + y = 0$. It is easy to verify that the cofactors of these four invariant algebraic curves are

$$\begin{aligned}K_{f_1} &= -2(x^2 + y^2), \quad K_{f_2} = -2(x^2 + y^2 - 1), \quad K_{f_3} = 1 - x^2 + xy, \\ K_{f_4} &= 2x^2 - xy + y^2 - 1.\end{aligned}$$

It is known that if there are constants ρ_i , for $i = 1, 2, 3, 4$, not all zero such that $\sum_{i=1}^4 \rho_i K_{f_i} = 0$, then the function $H = f_1^{\rho_1} f_2^{\rho_2} f_3^{\rho_3} f_4^{\rho_4}$ is a first integral of the differential system (4). This result can be found in [3, Chapter 8]. From

$$\begin{aligned}\sum_{i=1}^4 \rho_i K_{f_i} &= -2\rho_1(x^2 + y^2) - 2\rho_2(x^2 + y^2 - 1) \\ &\quad + \rho_3(1 - x^2 + xy) + \rho_4(2x^2 - xy + y^2 - 1) = 0,\end{aligned}$$

we obtain that $\rho_2 = 0$, $\rho_3 = 2\rho_1$ and $\rho_4 = -2\rho_1$. Therefore the differential system (4) has the first integral

$$H = f_1^{\rho_1} f_2^{\rho_2} f_3^{\rho_3} f_4^{\rho_4} = \frac{x^2(x^2 + y^2 - 1)}{(x + y)^2},$$

after taking $\rho_1 = 1$.

Easy computations show that the finite equilibria of the differential system (4) are the $(0, 0)$ a hyperbolic unstable node, and the following four equilibria on the invariant circle $\mathbb{S}_\infty$ ordered in counterclockwise:

-) $(0, 1)$ a hyperbolic saddle,
-) $(-1/\sqrt{2}, 1/\sqrt{2})$ a hyperbolic stable node,
-) $(0, -1)$ a hyperbolic saddle,
-) $(1/\sqrt{2}, -1/\sqrt{2})$ a hyperbolic stable node.

Recall that a *hyperbolic equilibrium* satisfies that the eigenvalues of the Jacobian matrix of the differential system evaluated at the equilibrium have non-zero real part. The Hartman-Grobman Theorem classifies the local phase portraits at the hyperbolic equilibria, a version of this theorem for 2-dimensional differential systems can be found in [3, Theorem 2.15].

From (6) we get that the differential system (4) in the local chart U_1 writes

$$\dot{u} = -(1+u)(1+u^2), \quad \dot{v} = v(1-u-v^2).$$

This differential system has only the infinite equilibrium $(-1, 0)$, that is a hyperbolic saddle. While the differential system (4) in the local chart U_2 becomes

$$\dot{u} = u(1+u)(1+u^2), \quad \dot{v} = v(1+2u^2+u^3-v^2).$$

So the origin $(0, 0)$ is a hyperbolic unstable node. Of course, we have the additional two infinite equilibria symmetric to the previous two ones with respect the center of the Poincaré disc, for more details see Subsection 2.1.

Using the invariant circle $\mathbb{S}^1$ and the two invariant straight lines $x = 0$ and $x + y = 0$, together with the local phase portraits at the finite and infinite equilibria, it follows easily the phase portrait of this differential system in the Poincaré disc given in Figure 1. This completes the proof of Proposition 4.

5. Proof of Theorem 5

Again we claim that the unique polynomial differential system (2) with $n = 5$ satisfying the condition $xP_5(x, y) + yQ_5(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and having the invariant circle $\mathbb{S}^1$ is the differential system (5).

Now we shall prove the claim. The polynomial differential systems (2) with $n = 5$ are

$$\begin{aligned} \dot{x} &= x + a_1x^5 + a_2x^4y + a_3x^3y^2 + a_4x^2y^3 + a_5xy^4 + a_6y^5, \\ \dot{y} &= y + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5. \end{aligned} \quad (13)$$

Since

$$\begin{aligned} \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} \Big|_{x=\pm\sqrt{1-y^2}} &= 1 + a_1 - (3a_1 - a_3 - b_2)y^2 + (3a_1 - 2a_3 + a_5 - 2b_2 + b_4)y^4 \\ &\quad - (a_1 - a_3 + a_5 - b_2 + b_4 - b_6)y^6 \pm (a_2 + b_1)y\sqrt{1-y^2} \\ &\quad \mp (2a_2 - a_4 + 2b_1 - b_3)y^3\sqrt{1-y^2} \pm (a_2 - a_4 + a_6 + b_1 \\ &\quad - b_3 + b_5)y^5\sqrt{1-y^2} = 0, \end{aligned}$$

It follows easily that

$$\begin{aligned} a_1 &= -1, \quad b_1 = -a_2, \quad b_2 = -3 - a_3, \quad b_3 = -a_4, \quad b_4 = -3 - a_5, \\ b_5 &= -a_6, \quad b_6 = -1. \end{aligned}$$

Therefore the differential system (13) writes as

$$\begin{aligned} \dot{x} &= x - x^5 + ax^4y + bx^3y^2 + cx^2y^3 + dxy^4 + ey^5 = x + P_5(x, y), \\ \dot{y} &= y - ax^5 - (3+b)x^4y - cx^3y^2 - (3+d)x^2y^3 - exy^4 - y^5 = y + Q_5(x, y), \end{aligned} \quad (14)$$

where $a = -a_2$, $b = -a_3$, $c = -a_4$, $d = -a_5$ and $e = -a_6$. Note that the differential system (14) is the differential system (5). As in the case $n = 3$ also for $n = 5$ when we force that the differential system (5) has the invariant circle $\mathbb{S}^1$ automatically it satisfies the condition $xP_5(x, y) + yQ_5(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Indeed,

$$xP_5(x, y) + yQ_5(x, y) = -(x^2 + y^2)^3 < 0, \quad \text{for all } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Hence the claim is proved.

Since the differential system (5) satisfies all the assumptions of Theorem 1, it follows that the circle $\mathbb{S}^1$ is invariant and globally attracting. Therefore all the equilibria of the differential system (5) distinct to the origin of coordinates live on the circle $\mathbb{S}^1$.

Let $rx + sy$ be a real linear factor of the homogeneous polynomial $yP_5(x, y) - xQ_5(x, y)$. By Theorem 2, the invariant straight line $rx + sy = 0$ intersects the invariant circles $\mathbb{S}^1$ and $\mathbb{S}_\infty$ in equilibrium points, and since the invariant circle $\mathbb{S}^1$ is globally attracting, all the finite equilibria of the differential system (5) are the ones in the intersection of the invariant straight lines $rx + sy = 0$ with the invariant circle $\mathbb{S}^1$ plus the origin of coordinates. Moreover, we know that the real linear factors $rx + sy$ of the homogeneous polynomial $yP_n(x, y) - xQ_n(x, y)$ provide all the pairs of diametrically opposite infinite equilibria.

For the differential system (5) we have that

$$yP_5(x, y) - xQ_5(x, y) = (x^2 + y^2)(ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4).$$

Hence the differential system (5) can have the following eight distinct possibilities for its invariant straight lines $rx + sy = 0$.

Case 1111: four real invariant straight lines $r_i x + s_i y = 0$ for $i = 1, 2, 3, 4$, i.e.

$$\begin{aligned} &ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4 \\ &= (r_1x + s_1y)(r_2x + s_2y)(r_3x + s_3y)(r_4x + s_4y). \end{aligned}$$

Then the differential system (5) having these invariant straight lines has coefficients: $a = r_1 r_2 r_3 r_4$, $b = -2 + r_2 r_3 r_4 s_1 + r_1 r_3 r_4 s_2 + r_1 r_2 r_4 s_3 + r_1 r_2 r_3 s_4$, $c = r_3 r_4 s_1 s_2 + r_2 r_4 s_1 s_3 + r_1 r_4 s_2 s_3 + r_2 r_3 s_1 s_4 + r_1 r_3 s_2 s_4 + r_1 r_2 s_3 s_4$, $d = -1 + r_4 s_1 s_2 s_3 + r_3 s_1 s_2 s_4 + r_2 s_1 s_3 s_4 + r_1 s_2 s_3 s_4$, $e = s_1 s_2 s_3 s_4$.

Case 211: one real double invariant straight line and two simple real straight lines, i.e.

$$ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4 = (r_1x + s_1y)^2(r_2x + s_2y)(r_3x + s_3y).$$

Then the differential system (5) having these invariant straight lines has coefficients: $a = r_1^2 r_2 r_3$, $b = -2 + 2r_1 r_2 r_3 s_1 + r_1^2 r_3 s_2 + r_1^2 r_2 s_3$, $c = r_2 r_3 s_1^2 + 2r_1 r_3 s_1 s_2 + 2r_1 r_2 s_1 s_3 + r_1^2 s_2 s_3$, $d = -1 + r_3 s_1^2 s_2 + r_2 s_1^2 s_3 + 2r_1 s_1 s_2 s_3$, $e = s_1^2 s_2 s_3$.

Case 22: two real double invariant straight lines, i.e.

$$ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4 = (r_1x + s_1y)^2(r_2x + s_2y)^2.$$

Then the differential system (5) having these invariant straight lines has coefficients: $a = r_1^2 r_2^2$, $b = 2(-1 + r_1 r_2^2 s_1 + r_1^2 r_2 s_2)$, $c = r_2^2 s_1^2 + 4r_1 r_2 s_1 s_2 + r_1^2 s_2^2$, $d = -1 + 2r_2 s_1^2 s_2 + 2r_1 s_1 s_2^2$, $e = s_1^2 s_2^2$.

Case 31: one real triple invariant straight line and another real simple straight line, i.e.

$$ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4 = (r_1x + s_1y)^3(r_2x + s_2y).$$

Then the differential system (5) having these invariant straight lines has coefficients: $a = r_1^3 r_2$, $b = -2 + 3r_1^2 r_2 s_1 + r_1^3 s_2$, $c = 3(r_1 r_2 s_1^2 + r_1^2 s_1 s_2)$, $d = -1 + r_2 s_1^3 + 3r_1 s_1^2 s_2$, $e = s_1^3 s_2$.

Case 4: one real quadruple invariant straight line, i.e.

$$ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4 = (r_1x + s_1y)^4.$$

Then the differential system (5) having this invariant straight line has coefficients: $a = r_1^4$, $b = 2(-1 + 2r_1^3 s_1)$, $c = 6r_1^2 s_1^2$, $d = -1 + 4r_1 s_1^3$, $e = s_1^4$.

Case 11a: two real simple invariant straight lines and two parallel complex straight lines $(ux + vy + w)^2 + k^2 = 0$ with $k(u^2 + v^2) \neq 0$. Then, since $(r_1x + s_1y)(r_2x + s_2y)((ux + vy + w)^2 + k^2)$ cannot be a homogeneous polynomial, this case cannot occur.

Case 11b: two real simple invariant straight lines and two complex straight lines intersecting in a real point $(ux + vy + w)^2 + (rx + sy + t)^2 = 0$ with $(u^2 + v^2)(r^2 + s^2) \neq 0$. In order that $(r_1x + s_1y)(r_2x + s_2y)(ux + vy + w)^2 + (rx + sy + t)^2$ be homogeneous we must take $w = t = 0$. Then, one has

$$\begin{aligned} ax^4 + (2+b)x^3y + cx^2y^2 + (1+d)xy^3 + ey^4 \\ = (r_1x + s_1y)(r_2x + s_2y)(ux + vy)^2 + (rx + sy)^2. \end{aligned}$$

Hence the differential system (5) having these invariant straight lines has coefficients: $a = r_1 r_2 (r^2 + u^2)$, $b = -2 + 2rr_1 r_2 s + r^2(r_2 s_1 + r_1 s_2) + u(r_2 s_1 u + r_1 s_2 u +$

$2r_1r_2v$), $c = 2rr_2ss_1 + r^2s_1s_2 + s_1u(s_2u + 2r_2v) + 2r_1s_2(rs + uv) + r_1r_2(s^2 + v^2)$, $d = -1 + 2s_1s_2(rs + uv) + r_2s_1(s^2 + v^2) + r_1s_2(s^2 + v^2)$, $e = s_1s_2(s^2 + v^2)$.

Case 2a: one real double invariant straight line and two parallel complex straight lines $(ux + vy + w)^2 + k^2 = 0$ with $k(u^2 + v^2) \neq 0$. Then, since $(r_1x + s_1y)^2((ux + vy + w)^2 + k^2)$ cannot be a homogeneous polynomial, this case cannot occur.

Case 2b: one real double invariant straight line and two complex straight lines intersecting in a real point $(ux + vy + w)^2 + (rx + sy + t)^2 = 0$ with $(u^2 + v^2)(r^2 + s^2) \neq 0$. In order that $(r_1x + s_1y)(r_2x + s_2y)(ux + vy + w)^2 + (rx + sy + t)^2$ be homogeneous we must take $w = t = 0$. Then, it follows

$$\begin{aligned} ax^4 + (2 + b)x^3y + cx^2y^2 + (1 + d)xy^3 + ey^4 \\ = (r_1x + s_1y)^2((ux + vy)^2 + (rx + sy)^2). \end{aligned}$$

Hence, the differential system (5) having these invariant straight lines has coefficients: $a = r^2 + u^2$, $b = 2(-1 + rs + uv)$, $c = r^2 + s^2 + u^2 + v^2$, $d = -1 + 2rs + 2uv$, $e = s^2 + v^2$.

Case 0a: two pairs of complex parallel straight lines, $(ux + vy + w)^2 + k^2 = 0$ with $k(u^2 + v^2) \neq 0$ and $(rx + sy + t)^2 + \ell^2 = 0$ with $\ell(r^2 + s^2) \neq 0$. Since $((ux + vy + w)^2 + k^2)((rx + sy + t)^2 + \ell^2)$ cannot be a homogeneous polynomial, this case cannot occur.

Case 0b: two complex parallel straight lines and two complex straight lines intersecting in a real point. As in the previous case when two complex parallel straight lines are present, the corresponding case cannot occur.

Case 0c: two pairs of complex straight lines intersecting in a real point, one can be $(ux + vy + w)^2 + (rx + sy + t)^2 = 0$ and the other $(\bar{u}x + \bar{v}y + \bar{w})^2 + (\bar{r}x + \bar{s}y + \bar{t})^2 = 0$ with $(\bar{u}^2 + \bar{v}^2)(\bar{r}^2 + \bar{s}^2) \neq 0$, and in order to have a homogeneous polynomial we take $w = \bar{w} = t = \bar{t} = 0$. Then, it follows

$$\begin{aligned} ax^4 + (2 + b)x^3y + cx^2y^2 + (1 + d)xy^3 + ey^4 \\ = ((ux + vy)^2 + (rx + sy)^2)((\bar{u}x + \bar{v}y)^2 + (\bar{r}x + \bar{s}y)^2). \end{aligned}$$

Hence, the differential system (5) having these invariant straight lines has coefficients: $a = (r^2 + u^2)(\bar{r}^2 + \bar{u}^2)$, $b = 2(-1 + r\bar{r}s + r^2\bar{r}\bar{s} + \bar{r}\bar{s}u^2 + rs\bar{u}^2 + \bar{r}^2uv + u\bar{u}^2v + r^2\bar{u}\bar{v} + u^2\bar{u}\bar{v})$, $c = \bar{r}^2s^2 + 4r\bar{r}s\bar{s} + r^2\bar{s}^2 + \bar{s}^2u^2 + s^2\bar{u}^2 + 4\bar{r}\bar{s}uv + \bar{r}^2v^2 + \bar{u}^2v^2 + 4rs\bar{u}\bar{v} + 4u\bar{u}\bar{v}\bar{v} + r^2\bar{v}^2 + u^2\bar{v}^2$, $d = -1 + 2\bar{r}s^2\bar{s} + 2rs\bar{s}^2 + 2\bar{s}^2uv + 2\bar{r}\bar{s}v^2 + 2s^2\bar{u}\bar{v} + 2\bar{u}v^2\bar{v} + 2rs\bar{v}^2 + 2uv\bar{v}^2$, $e = (s^2 + v^2)(\bar{s}^2 + \bar{v}^2)$.

Case 0d: a double pair of complex parallel straight lines. Again by homogeneity this case cannot occur.

Case 0e: a double pair of complex straight lines that intersect in a real point, and by homogeneity we have $((ux + vy)^2 + (rx + sy)^2)^2 = 0$, i.e.

$$ax^4 + (2 + b)x^3y + cx^2y^2 + (1 + d)xy^3 + ey^4 = ((ux + vy)^2 + (rx + sy)^2)^2.$$

Then the differential system (5) having these invariant straight lines has coefficients: $a = (r^2 + u^2)^2$, $b = 2(-1 + 2r^3s + 2rsu^2 + 2r^2uv + 2u^3v)$, $c =$

$$2(3r^2s^2 + s^2u^2 + 4rsuv + r^2v^2 + 3u^2v^2), \quad d = -1 + 4rs^3 + 4s^2uv + 4rsv^2 + 4uv^3, \quad e = (s^2 + v^2)^2.$$

Now we must study the phase portraits in the Poincaré disc of the nine cases that are non-empty.

Of course, the origin of coordinates of the differential system (5) always is a hyperbolic unstable node.

Case 1111. Then the differential system (5) has on the invariant circle $\mathbb{S}^1$ the following eight finite equilibria

$$p_i = \left(-\frac{s_i}{\sqrt{r_i^2 + s_i^2}}, \frac{r_i}{\sqrt{r_i^2 + s_i^2}} \right)$$

and $-p_i$ for $i = 1, 2, 3, 4$. The determinant of the Jacobian matrix of the differential system (5) evaluated at p_i is

$$4 \frac{\prod_{k=1, k \neq i}^4 (r_k s_i - r_i s_k)}{r_i^2 + s_i^2} \neq 0, \quad (15)$$

because if some $r_k s_i - r_i s_k = 0$ then the two straight lines $r_i x + s_i y = 0$ and $r_k x + s_k y = 0$ would coincide. From (15) it follows that the equilibrium p_i must be either a hyperbolic node, or a hyperbolic saddle, or a focus or a center. But these last two possibilities can occur because the equilibrium p_i is on the invariant circle $\mathbb{S}^1$. So the eight equilibria on the invariant circle $\mathbb{S}^1$ are either hyperbolic nodes or saddles. Moreover, by Theorem 1 the invariant circle $\mathbb{S}^1$ is globally attracting, so all the nodes are stable, consequently we cannot have two consecutive nodes on the invariant circle. In summary, we have four stable nodes and four saddles on the circle $\mathbb{S}^1$ located alternatively.

Now we study the infinite equilibria. The differential system (5) in the local chart U_1 writes

$$\begin{aligned} \dot{u} &= -(r_1 + s_1 u)(r_2 + s_2 u)(r_3 + s_3 u)(r_4 + s_4 u)(1 + u^2), \\ \dot{v} &= v(1 - r_1 r_2 r_3 r_4 u - (r_1 r_3 r_4 s_2 + r_2(r_3 r_4 s_1 + r_1 r_4 s_3 + r_1 r_3 s_4) - 2)u^2 - \\ &\quad (s_3(r_2 r_4 s_1 + r_1 r_4 s_2 + r_1 r_2 s_4) + r_3(r_4 s_1 s_2 + r_2 s_1 s_4 + r_1 s_2 s_4))u^3 - \\ &\quad (r_4 s_1 s_2 s_3 + r_3 s_1 s_2 s_4 + r_2 s_1 s_3 s_4 + r_1 s_2 s_3 s_4 - 1)u^4 - s_1 s_2 s_3 s_4 u^5 - v^4). \end{aligned} \quad (16)$$

With a rotation of the coordinates (if necessary) in the Poincaré disc we can assume without loss of generality that all the infinite equilibria are in the local charts U_1 and V_1 . The differential system (16) has eight infinite equilibria $P_i = (-r_i/s_i, 0)$ and $-P_i$ for $i = 1, 2, 3, 4$. Note that since we are assuming that all infinite equilibria are contained in $U_1 \cup V_1$ all the $s_i \neq 0$. The determinant of the Jacobian matrix of the differential system (16) evaluated at P_i is

$$\frac{(r_i^2 + s_i^2)^3 \prod_{k=1, k \neq i}^4 (r_i s_k - r_k s_i)}{s_i^8} \neq 0.$$

The same arguments used for studying the local phase portraits at the equilibria on the circle $\mathbb{S}^1$ work here for studying the local phase portraits at the equilibria on the circle $\mathbb{S}_\infty$, and we obtain that at infinity there are four hyperbolic unstable nodes and four hyperbolic saddles, and they are located alternatively on the circle $\mathbb{S}_\infty$. Moreover, on every straight line $r_i x + s_i y = 0$ there are five equilibria, two at infinity saddles (resp. unstable nodes), two at $\mathbb{S}^1$ stable nodes (resp. saddles) and the unstable node at the origin of coordinates.

Taking into account all this information we obtain that the phase portrait in the Poincaré disc of the differential system (5) under the assumptions 1111 is topologically equivalent to the phase portrait of Figure 2(a).

Case 211. Then the differential system (5) has on the invariant circle $\mathbb{S}^1$ the following six finite equilibria

$$p_i = \left(-\frac{s_i}{\sqrt{r_i^2 + s_i^2}}, \frac{r_i}{\sqrt{r_i^2 + s_i^2}} \right)$$

and $-p_i$ for $i = 1, 2, 3$. The determinant of the Jacobian matrix of the differential system (5) evaluated at p_i is

$$4 \frac{(r_i s_1 - r_1 s_i)^2 (r_k s_i - r_i s_k)}{r_i^2 + s_i^2} \neq 0, \quad \text{for } i = 2, 3 \text{ and } k \neq i.$$

So, by the arguments used in the case 1111 one of the two equilibria p_2 and p_3 is a hyperbolic stable node and the other is a hyperbolic saddle. On the other hand, the eigenvalues of the Jacobian matrix of the differential system (5) evaluated at p_1 are 0 and -4 , so p_1 is a semihyperbolic equilibrium. By Theorem 2.19 of [3] p_1 is a saddle-node.

Now we study the infinite equilibria. The differential system (5) in the local chart U_1 writes

$$\begin{aligned} \dot{u} &= -(r_1 + s_1 u)^2 (r_2 + s_2 u) (r_3 + s_3 u) (1 + u^2), \\ \dot{v} &= v (1 - r_1^2 r_2 r_3 u - (-2 + 2r_1 r_2 r_3 s_1 + r_1^2 (r_3 s_2 + r_2 s_3)) u^2 - \\ &\quad (r_1 s_2 (2r_3 s_1 + r_1 s_3) + r_2 s_1 (r_3 s_1 + 2r_1 s_3)) u^3 - \\ &\quad (r_3 s_1^2 s_2 + r_2 s_1^2 s_3 + 2r_1 s_1 s_2 s_3 - 1) u^4 - s_1^2 s_2 s_3 u^5 - v^4). \end{aligned} \quad (17)$$

Again we can assume without loss of generality that all the infinite equilibria are in the local charts U_1 and V_1 . The differential system (17) has six infinite equilibria $P_i = (-r_i/s_i, 0)$ and $-P_i$ for $i = 1, 2, 3$. As in the case 1111 we have that $s_i \neq 0$. The determinant of the Jacobian matrix of the differential system (17) evaluated at P_i is

$$\frac{(r_i^2 + s_i^2)^3 (r_i s_1 - r_1 s_i)^2 (r_k s_i - r_i s_k)}{s_i^8} \neq 0, \quad \text{for } i = 2, 3 \text{ and } k \neq i.$$

The same arguments than in the case 1111 work here for studying the local phase portraits at the equilibria on the circle $\mathbb{S}_\infty$, and we obtain that one of the

equilibria P_2 and P_2 is a hyperbolic unstable node and the other a hyperbolic saddle. The eigenvalues of the Jacobian matrix of the differential system (17) evaluated at P_1 are 0 and $(r_1^2 + s_1^2)^2/s_1^4$, so P_1 is a semihyperbolic equilibrium, and by Theorem 2.19 of [3] P_1 is a saddle-node. Furthermore, on the straight line $r_ix + s_iy = 0$ for $i = 2, 3$ there are five equilibria, two at infinity saddles (resp. unstable nodes), two at $\mathbb{S}^1$ stable nodes (resp. saddles) and the unstable node at the origin of coordinates. While on the straight line $r_1x + s_1y = 0$ there are five equilibria, two saddle-nodes at infinity, two saddle-nodes on $\mathbb{S}^1$ and the unstable node at the origin of coordinates.

Taking into account all this information we obtain that the phase portrait in the Poincaré disc of the differential system (5) under the assumptions 211 is topologically equivalent to the phase portrait as in Figure 2(b).

The remainder seven cases are studied using the same arguments than in the analysis of the cases 1111 and 211. Thus the phase portrait of the case 22 is shown in Figure 2(c). This phase portrait has two invariant straight lines with five equilibria, the four that are on the two invariant circles are semihyperbolic saddle-nodes.

The cases 31 and 11 have topologically equivalent phase portraits in the Poincaré disc shown in Figure 2(d). This phase portrait has two invariant straight lines with five equilibria. In one invariant straight line the two equilibria on the circle $\mathbb{S}_\infty$ are saddles and on the circle $\mathbb{S}^1$ are stable nodes, and in other on the circle $\mathbb{S}_\infty$ are unstable nodes and on the circle $\mathbb{S}^1$ are saddles.

Also the phase portraits of the cases 4 and 2 are topologically equivalent to the one of Figure 2(e). This phase portrait has only one invariant straight line with five equilibria. The four equilibria that are on the circles $\mathbb{S}_\infty$ and $\mathbb{S}^1$ are all semihyperbolic saddle-nodes.

Finally, the cases 0c and 0e have their phase portraits topologically equivalent to the one of Figure 2(f). In this phase portrait the invariant circle $\mathbb{S}^1$ is a stable limit cycle, and the unique equilibria in the Poincaré disc is the unstable node at the origin of coordinates. This completes the proof of Theorem 5.

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Declarations

Competing interests The authors declare no competing interests.

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