

RESEARCH

Period function of planar piecewise reversible quadratic systems



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Abstract

This paper presents an analysis of lower bounds on local criticality within the family of piecewise quadratic reversible centers. Through a study of perturbations within this class, we determine that a minimum of five (in the linear case) or four (in the nonlinear case) local critical periods can bifurcate from the isochronous center. Notably, five represents the highest level of weakness observed in this context.

Keywords: Piecewise planar reversible quadratic systems, Period function, Isochronicity, Critical period

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1 Introduction and statement of the main results

The measurement of time has been an enormous incentive for the development of physics and mathematics. Galileo discovered in 1583 that pendulums of the same length have the same oscillation time, i.e., all periods are the same independently of the initial angle. This phenomenon became known as the Law of Isochrony. Inspired by this fact and realizing the need for accurate time measurement, Huygens created a pendulum clock in 1656, which was the most accurate for a long time. The study of period functions dates back at least to 1673 when Huygens observed that the pendulum clock has a monotone period function and, therefore, oscillates with a shorter period when the energy is decreasing, that is when the clock spring unwinds.

In recent decades, there has been an increasing interest in studying the isochronicity of vector fields in the plane having a continuum of periodic orbits, i.e., a period annulus. We find an overview of the results obtained in the survey by Chavarriga and Sabatini [1]. However, few families of polynomial vector fields have a complete classification. For our current study, where we focus particularly on the quadratic family, we can obtain the isochronicity characterization property from the work of Sibirsky [14], but in the Loud [8] normal form

$$\begin{aligned}\dot{x} &= -y + Bxy, \\ \dot{y} &= x + Dx^2 + Ey^2,\end{aligned}\tag{1}$$

see again [1]. In this context, additional intriguing families for characterizing isochronicity include potential Hamiltonian systems, Kukles systems, and linear centers perturbed by third, fourth, and fifth-degree homogeneous polynomials.

Although we will delve into more precise details shortly, roughly speaking, when considering a differential system with a period annulus, an associated and interesting problem arises: studying the qualitative characteristics of the period function linked to each periodic orbit. Specifically, we examine its monotonicity and the number of oscillations within its defined interval. For the quadratic family, Chicone and Jacobs [3, Theorem 3.1] prove that at most one critical period can bifurcate from an isochronous center of a nonlinear quadratic system and at most two critical periods can bifurcate from the linear center. In both cases, there are perturbations in family (1) with the maximum number of critical periods. As we will explore in the forthcoming sections, this paper aims to expand upon some of these findings concerning piecewise reversible quadratic systems. Additionally, various studies have sought to establish the conditions under which the piecewise system can possess an isochronous center at the origin and ascertain the number of critical periods, as detailed in works such as [2, 4, 15]. Traditionally, this kind of problem is analyzed near equilibrium points of linear center type, where the period function is constant. In the piecewise scenario, this condition seems natural to impose, since there are degenerate cases in which the period function is monotonically increasing from zero. This behavior can be observed, for example, in equilibria of fold-fold type; see more details in [10]. Chen and Zhang [2] studied the isochronicity of a center in a switching quadratic system and provided all the conditions for the origin to be a center and an isochronous center. To demonstrate the conditions identified for the origin to serve as an isochronous center, they employed a modification of two techniques commonly used in analytic systems: linearization and identification of commuting systems (see [9, 12]). Additionally, they used the observation that the switching line fulfills the role of radial isochronicity.

We consider a piecewise system of differential equations given by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + P^+(x, y, \lambda^+), x + Q^+(x, y, \lambda^+)), & \text{if } y \geq 0, \\ (-y + P^-(x, y, \lambda^-), x + Q^-(x, y, \lambda^-)), & \text{if } y < 0, \end{cases} \quad (2)$$

where $\lambda = (\lambda^+, \lambda^-) \in \mathbb{R}^m$ are the parameters, $P^\pm(x, y, \lambda^\pm)$ and $Q^\pm(x, y, \lambda^\pm)$ are analytic functions in x and y starting with at least second-order terms. The system defined in the upper half-plane ($y \geq 0$) is called *upper system*, while the system defined in the lower half-plane ($y < 0$) is referred to as the *lower system*. For a fixed parameter $\lambda = (\lambda^+, \lambda^-)$ in the case that the origin of family (2) is a center, the *period annulus* is the largest neighborhood Ω of such a point where each trajectory in $\Omega \setminus \{(0, 0)\}$ is a periodic function. Consider ρ^* sufficiently small so that the segment $\Sigma = \{(x, y) : 0 < x < \rho^*\}$ of the x -axis is entirely contained in the period annulus. For $\rho \in (0, \rho^*)$, let $T(\rho, \lambda)$ be the least period of the trajectory through $(x, y) = (\rho, 0) \in \Sigma$. The function $T(\rho, \lambda)$ is said to be the *period function*. We say that a center is *isochronous*, if the period function is constant. Each $\rho > 0$ for which $T'(\rho, \lambda) = 0$ is called a *critical period* or an *oscillation* of the period function. Similar to the smooth case, the period function can be analytically extended to $\rho = 0$ and, consequently, for each fixed parameter $\lambda = (\lambda^+, \lambda^-)$, it can be expressed as a power series

$$T(\rho, \lambda) = 2\pi + \sum_{j=1}^{\infty} T_j(\lambda) \rho^j. \quad (3)$$

The coefficients $T_j(\lambda)$ are known as the *period constants* of the center, as outlined in, for instance, [2, 11]. The isochronicity of the system can be characterized in terms of these period constants. Specifically, the system possesses an isochronous center at the origin if and only if $T_j = 0$ for all $j \in \mathbb{N}$. The center of (2) is denoted as a *weak center of finite order* k of the system if $T_1(\lambda) = \dots = T_k(\lambda) = 0$, but $T_{k+1}(\lambda) \neq 0$. In the case of an isochronous center, such a singularity is termed a *weak center of infinite order*, where $T_j(\lambda) = 0$ for all $j \geq 1$. Here, the definition of order is linked to the generic unfolding, implying that a weak center of order k unfolds, in a generic sense, k critical periods. For further details, refer again to [3].

In [2], the authors introduced the concept of an isochronous center, defining it as a *regular isochronous center* if all orbits of the upper system (resp. the lower system) take equal time to transition from the positive (resp. negative) x -axis to the negative (resp. positive) x -axis. Cases outside this definition are referred to as *irregular isochronous centers*.

In this paper, we study the period function associated with a piecewise planar system of the form (2), which is invariant under the transformation $(x, y, t) \mapsto (-x, y, -t)$, rendering the system reversible with respect to the y -axis. Thus, we investigate the *piecewise planar reversible quadratic systems* described by

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + a_1x^2 + a_2y^2, x + a_3xy), & \text{if } y \geq 0, \\ (-y + b_1x^2 + b_2y^2, x + b_3xy), & \text{if } y < 0, \end{cases} \quad (4)$$

where a_i, b_i ($i = 1, 2, 3$) are real numbers. Notably, the origin is a center of system (2) due to its reversibility. It is important to mention that characterizing centers in piecewise systems is a challenging task and can significantly depend on the separation curve. In our study, to ensure the existence of a substantial family of centers, we adopt a natural extension of (1), focusing on reversible centers concerning the separation line, which coincides with the reversibility line. Thus, the center condition is automatically fulfilled.

One of our primary focuses will be to provide an estimate for the maximum number of zeros of T' that can bifurcate near a center for planar polynomial vector fields of degree n , denoted by $\mathcal{C}_0(n)$. The problem of determining $\mathcal{C}_0(n)$ is called *problem of bifurcations of critical periods* or *criticality problem*. As mentioned earlier, the problem was addressed in [3] for the quadratic family, yielding $\mathcal{C}_0(2) = 2$. Specifically, they characterized the isochronous centers, the weak centers of finite order, and their unfoldings to determine the number of critical periods in each center scenario. Our findings closely resemble theirs, given that the piecewise reversible quadratic family (4) encompasses six parameters. We have established that the highest weak center orders are four (for nonlinear centers) and five (for linear centers). However, we have not been able to confirm $\mathcal{C}_0^p(2) = 5$, where $\mathcal{C}_0^p(n)$ extends $\mathcal{C}_0(n)$ to piecewise polynomial systems of degree n defined in two zones separated by a straight line. We will outline the primary (algebraic) obstacles to proving the existence of such an upper bound at the conclusion of Sect. 2. Our main result is as follows.

Theorem 1 *In systems (4), precisely three families exhibit 1-parameter curves of weak centers of order 5. Furthermore, the maximum number of local critical periods bifurcating from the origin, upon perturbing within a neighborhood of each curve but within the class of reversible quadratic systems, is limited to 5.*

In the proof of the aforementioned result, we will observe that one of the families yields 18 distinct 1-parameter curves. While these have been derived explicitly, they are too extensive to be fully detailed here. However, the other two 1-parameter families (depending on w) are considerably simpler and are provided below:

- (i) $\{6a_1 + 5w = 2a_2 - w = 3a_3 + w = 6b_1 + 5w = 2b_2 - w = 3b_3 + w = 0\};$
- (ii) $\{2a_1 - \alpha w = 2a_2 - w = 2a_3 - 5\alpha w - 5 = 2b_1 - \alpha w = 2b_2 - w = 2b_3 - 5\alpha w - 5 = 0\},$
where α is each one of the two roots of the polynomial $p(\alpha) = 2\alpha^2 + 15\alpha + 15$.

Furthermore, within the proof of Theorem 1, we will demonstrate that the three families precisely offer 21 distinct 1-parameter curves.

The problem of isochronicity of analytical quadratic systems (1) was also investigated by Loud [8]. He established that the origin is an isochronous center if and only if, via a change of coordinates, the quadratic system can be expressed in one of the following forms, labeled as S_1, S_2, S_3 , and S_4 in [1]:

$$\begin{aligned} S_1 : \begin{cases} \dot{x} = -y + x^2 - y^2, \\ \dot{y} = x + 2xy, \end{cases} & S_2 : \begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy, \end{cases} \\ S_3 : \begin{cases} \dot{x} = -y - 4x^2/3, \\ \dot{y} = x - 16xy/3, \end{cases} & S_4 : \begin{cases} \dot{x} = -y + 16x^2/3 - 4y^2/3, \\ \dot{y} = x + 8xy/3. \end{cases} \end{aligned} \quad (5)$$

For family (4), we prove the following result that gives us information about isochronous center conditions. Note that, after applying the change of variables $(x, y, t) \mapsto (x, -y, -t)$, the upper system and the lower system in (4) exchange their equations. Note that what actually happens is an interchange of parameters

$$(a_1, a_2, a_3, b_1, b_2, b_3) \leftrightarrow (-b_1, -b_2, -b_3, -a_1, -a_2, -a_3).$$

Theorem 2 *Up to a linear change of coordinates and a time rescaling, including the change $(x, y, t) \mapsto (x, -y, -t)$, the origin is an isochronous center of the piecewise quadratic system (4) if and only if either the system is the isochronous linear center or one of the following conditions holds:*

- (i) $a_1 = b_1 = 1, a_2 = b_2 = -1, a_3 = b_3 = 2;$
- (ii) $a_2 = b_2 = a_3 - a_1 = b_3 - b_1 = 0;$
- (iii) $a_1 = b_1 = -1, a_2 = b_2 = 0, a_3 = b_3 = -4;$
- (iv) $a_1 + 4a_2 = a_3 + 2a_2 = b_1 + 4b_2 = b_3 + 2b_2 = 0;$
- (v) $a_1 = -1, a_2 = 1, a_3 = -2, b_1 = -1, b_2 = 0, b_3 = -4;$
- (vi) $a_2 = a_3 - a_1 = b_1 + 4b_2 = b_3 + 2b_2 = 0.$

We notice that not all piecewise quadratic isochronous centers are the ones listed in (5). The scenarios described in cases (i) and (iii) are equivalent to the quadratic systems S_1 and S_3 , (ii) (resp. (iv)) is equivalent to S_2 (resp. S_4), only if $a_1 = b_1$ (resp. $a_2 = b_2$).

The last main theorem presents our best result on the number of local critical periods bifurcating from the nonlinear isochronous centers. We work with higher-order developments with respect to the perturbation parameters.

Theorem 3 *For the piecewise reversible quadratic family (4), at least four local critical periods can bifurcate from a nonlinear isochronous center when it is perturbed in the class of piecewise reversible quadratic system, using developments up to order 4.*

The paper is organized as follows. Section 2 outlines the procedure for obtaining the period constants of the piecewise system (2). The results used for studying the perturbations of the reversible isochronous centers are given in Sect. 3. The proofs of Theorems 1 and 2 are provided in Sect. 4. Lastly, in Sect. 5, we demonstrate that at least four local critical periods can bifurcate from the piecewise isochronous center presented in Theorem 3. It is worth noting that all computations have been carried out using the computer algebra system Maple.

2 Computation of the period constants

In this section, we present the process of obtaining the period constants, which are the coefficients of the power series of the period map near a center equilibrium point, as defined in (3). But, before introducing the method, we define the *positive half-period function* $T^+(\rho, \lambda^+)$ (resp. the *negative half-period function* $T^-(\rho, \lambda^-)$) as the least (positive) time of the trajectory of (2) passing through $(x, y) = (\rho, 0)$ on the positive x -axis (resp. on the negative x -axis) to reach the negative x -axis (resp. the positive x -axis).

The method used for determining the period constants is essentially an adaptation of the classical technique employed for such computations in the analytical context, as presented in [11]. This method is also outlined in [2], where it was introduced to address isochronicity in a piecewise quadratic system. It is described as follows.

We carry out a change to polar coordinates, $(x, y) = (r \cos \theta, r \sin \theta)$, in system (2), resulting in:

$$(\dot{r}, \dot{\theta}) = \begin{cases} (R^+(r, \theta, \lambda^+), 1 + \Theta^+(r, \theta, \lambda^+)), & \text{for } \theta \in [0, \pi], \\ (R^-(r, \theta, \lambda^-), 1 + \Theta^-(r, \theta, \lambda^-)), & \text{for } \theta \in (\pi, 2\pi], \end{cases} \quad (6)$$

where $R^\pm$ (resp. $\Theta^\pm$) are analytic functions in $r > 0, \sin \theta$ and $\cos \theta$. They start with at least first (resp. second) order terms in $r > 0$ and with third-order terms in $\sin \theta$ and $\cos \theta$. These functions are described by:

$$\begin{aligned} R^\pm(r, \theta, \lambda^\pm) &= P^\pm(r \cos \theta, r \sin \theta, \lambda^\pm) \cos \theta + Q^\pm(r \cos \theta, r \sin \theta, \lambda^\pm) \sin \theta, \\ \Theta^\pm(r, \theta, \lambda^\pm) &= (Q^\pm(r \cos \theta, r \sin \theta, \lambda^\pm) \cos \theta - P^\pm(r \cos \theta, r \sin \theta, \lambda^\pm) \sin \theta) / r. \end{aligned}$$

By eliminating time from (6), we obtain:

$$\frac{dr}{d\theta} = \begin{cases} R^+(r, \theta, \lambda^+) / (1 + \Theta^+(r, \theta, \lambda^+)), & \text{for } \theta \in [0, \pi], \\ R^-(r, \theta, \lambda^-) / (1 + \Theta^-(r, \theta, \lambda^-)), & \text{for } \theta \in (\pi, 2\pi], \end{cases}$$

whose expansion around $r = 0$ can be expressed in the form of

$$\frac{dr}{d\theta} = \begin{cases} \sum_{j \geq 1} R_j^+(\theta, \lambda^+) r^j, & \text{for } \theta \in [0, \pi], \\ \sum_{j \geq 1} R_j^-(\theta, \lambda^-) r^j, & \text{for } \theta \in (\pi, 2\pi], \end{cases} \quad (7)$$

where $R_j^\pm(\theta, \lambda^\pm)$ are 2π -periodic functions of θ and the series is convergent for all θ and for all sufficiently small r , because of the analyticity of the functions $P^\pm, Q^\pm$ in (2). The initial value problem for the upper system of (7) (resp. lower system), with the initial condition $(r, \theta) = (\rho, 0)$ (resp. (ρ, π)), has unique solution. Hence, we have convergent power series

$$\begin{cases} r^+(\rho, \theta, \lambda^+) = \rho + \sum_{j \geq 2} u_j^+(\theta, \lambda^+) \rho^j, & \text{for } \theta \in [0, \pi], \\ r^-(\rho, \theta, \lambda^-) = \rho + \sum_{j \geq 2} u_j^-(\theta, \lambda^-) \rho^j, & \text{for } \theta \in (\pi, 2\pi], \end{cases} \quad (8)$$

for all θ and $\rho < r^*$. Of course, all is valid for some sufficiently small $r^* > 0$. Then, the *positive (resp. negative) half-return map* is $\Pi^+(\rho, \lambda^+) = r^+(\rho, \pi, \lambda^+)$ (resp. $\Pi^-(\rho, \lambda^-) = r^-(\rho, 2\pi, \lambda^-)$), and the *complete return map* Π can be defined as $(\rho, 0) \rightarrow \Pi(\rho, \lambda) = \Pi^-(\Pi^+(\rho, \lambda^+), \lambda^-)$.

Upon substituting (8) into the right-hand side of the upper and lower systems of (6), we obtain two equations (one denoted with the superscript $+$ and the other with the superscript $-$):

$$\dot{\theta} = 1 + \sum_{j \geq 1} \xi_j^\pm(\theta, \lambda^\pm) \rho^j.$$

Rewriting these equations as

$$dt = \frac{d\theta}{1 + \sum_{j \geq 1} \xi_j^\pm(\theta, \lambda^\pm) \rho^j} = \left(1 + \sum_{j \geq 1} \phi_j^\pm(\theta, \lambda^\pm) \rho^j \right) d\theta,$$

and integrating them, we get

$$t - \theta = \sum_{j \geq 1} \psi_j^\pm(\theta) \rho^j, \quad (9)$$

where

$$\psi_j^+(\theta) = \int_0^\theta \phi_j^+(s) ds, \quad \psi_j^-(\theta) = \int_\pi^\theta \phi_j^-(s) ds,$$

and the power series in (9) converges, for all θ and sufficiently small $\rho \geq 0$.

From (9), it follows that the positive half period of (2) passing through $(x, y) = (\rho, 0)$, in the positive x -axis for $\rho \neq 0$, to reach the negative x -axis is given by

$$T^+(\rho, \lambda^+) = \pi + \sum_{j \geq 1} T_j^+(\lambda^+) \rho^j,$$

where $T_j^+(\lambda^+) = \frac{1}{\pi} \psi_j^+(\pi) = \frac{1}{\pi} \int_0^\pi \phi_j^+(s) ds$.

The *negative half period* from the point $(x, y) = (\Pi^+(\rho, \lambda^+), 0)$, along the negative x -axis for $\rho \neq 0$, is given by

$$\begin{aligned} T^-(\Pi^+(\rho, \lambda^+), \lambda^-) &= \pi + \sum_{j \geq 1} T_j^-(\lambda^-) (\Pi^+(\rho, \lambda^+))^j \\ &= \pi + \sum_{j \geq 1} T_j^-(\lambda^-) (r^+(\rho, \pi, \lambda^+))^j \\ &= \pi + \sum_{j \geq 1} T_j^-(\lambda^-) \left(\rho + \sum_{i \geq 2} u_i^+(\pi, \lambda^+) \rho^i \right)^j \\ &= \pi + \sum_{j \geq 1} \hat{T}_j^-(\lambda^+, \lambda^-) \rho^j, \end{aligned}$$

where $T_j^-(\lambda^-) = \psi_j^-(2\pi) = \int_\pi^{2\pi} \phi_j^-(s) ds$; in the second equality we have used the expression obtained in (8), and each $\hat{T}_j^-(\lambda^+, \lambda^-)$ represent the Taylor coefficients of the last expression.

Thus, the least period of the trajectory of (2) passing through $(x, y) = (\rho, 0)$, for $\rho \neq 0$, is given by (3), where $T_j(\lambda) = T_j^+(\lambda^+) + \hat{T}_j^-(\lambda^+, \lambda^-)$ is the j -th *period constant*. Obviously, the center O of (2) is isochronous if and only if all period constants vanish.

However, determining the coefficients $\hat{T}_j^-(\lambda^+, \lambda^-)$ is generally challenging, as it needs knowledge of the expression for $\Pi^+(\rho, \lambda^+)$. To circumvent this obstacle, we can employ the transformation $(x, y, t) \rightarrow (x, -y, -t)$ in the lower system of (2). Consequently, the negative half-period function becomes the positive half-period function of

$$(\dot{x}, \dot{y}) = \begin{cases} (-y - P^-(x, -y, \lambda^-), x + Q^-(x, -y, \lambda^-)), & \text{if } y \geq 0, \\ (-y - P^+(x, -y, \lambda^+), x + Q^+(x, -y, \lambda^+)), & \text{if } y < 0, \end{cases}$$

which is denoted by $\tilde{T}^-(\rho, \lambda^-)$ and it is found in an analogous way as $T^+(\rho, \lambda^+)$. Hence, its power series writes as

$$\tilde{T}^-(\rho, \lambda^-) = \pi + \sum_{j \geq 1} \tilde{T}_j^-(\lambda^-) \rho^j.$$

Hence, the expansion of the period function becomes

$$\begin{aligned} T(\rho, \lambda^+, \lambda^-) &= T^+(\rho, \lambda^+) + \tilde{T}^-(\rho, \lambda^-) \\ &= 2\pi + \sum_{j \geq 1} \left(T_j^+(\lambda^+) + \tilde{T}_j^-(\lambda^-) \right) \rho^j, \end{aligned}$$

and the period constants are $T_j(\lambda) = T_j^+(\lambda^+) + \tilde{T}_j^-(\lambda^-)$. In fact, $\tilde{T}_j^-(\lambda^-) = \hat{T}_j^-(\lambda^+, \lambda^-)$.

The integrals yielding $T_j^+(\lambda^+)$ and $\tilde{T}_j^-(\lambda^-)$ become increasingly challenging to compute explicitly for vector fields with higher-degree polynomials, primarily due to computational complexities. Consequently, to extend the analysis to higher degrees, alternative techniques must be employed.

Remark 4 For $j \geq 1$, the period constant T_j lies within the ideal $\langle T_1, T_2, T_3, \dots, T_{j-1} \rangle$ within the ring $\mathbb{R}[\lambda]$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$. Thus, the expression of a period constant $T_j(\lambda)$ holds significance only if $T_1(\lambda) = \dots = T_{j-1}(\lambda) = 0$.

The subsequent lemma furnishes the initial six period constants derived through the method described above for family (4), with the remaining statements following straightforwardly. Due to the extensive size of their expressions, we avoid providing further details. It is worth noting that, in the case of system (4), the corresponding period constants can be regarded as $T_j(\mathbf{a}, \mathbf{b}) = T_j^+(\mathbf{a}) + \tilde{T}_j^-(\mathbf{b})$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

Lemma 5 *The period constants T_j for family (4) are homogeneous polynomials with rational coefficients in the parameters and they are given by $T_j = \bar{T}_j(a_1, a_2, a_3) - \bar{T}_j(b_1, b_2, b_3)$, for $j = 1, 3, 5$, and $T_j = \bar{T}_j(a_1, a_2, a_3) + \bar{T}_j(b_1, b_2, b_3)$, for $j = 2, 4, 6$, where:*

$$\begin{aligned}\bar{T}_1(z_1, z_2, z_3) &= \frac{1}{3}(2z_1 + 4z_2 - 2z_3), \\ \bar{T}_2(z_1, z_2, z_3) &= \frac{\pi}{24}(4z_1^2 + 10z_1z_2 - 5z_1z_3 + 10z_2^2 - z_2z_3 + z_3^2), \\ \bar{T}_3(z_1, z_2, z_3) &= \frac{1}{405}(206z_1^3 + 624z_1^2z_2 - 312z_1^2z_3 + 960z_1z_2^2 - 204z_1z_2z_3 + 114z_1z_3^2 \\ &\quad + 640z_2^3 + 120z_2^2z_3 + 12z_2z_3^2 - 8z_3^3), \\ \bar{T}_4(z_1, z_2, z_3) &= \frac{\pi}{2304}(400z_1^4 + 1424z_1^3z_2 - 712z_1^3z_3 + 2772z_1^2z_2^2 - 828z_1^2z_2z_3 + 369z_1^2z_3^2 \\ &\quad + 3080z_1z_2^3 + 168z_1z_2^2z_3 + 126z_1z_2z_3^2 - 58z_1z_3^3 + 1540z_2^4 + 700z_2^3z_3 \\ &\quad + 21z_2^2z_3^2 - 2z_2z_3^3 + z_3^4), \\ \bar{T}_5(z_1, z_2, z_3) &= \frac{1}{42525}(26342z_1^5 + 107708z_1^4z_2 - 53854z_1^4z_3 + 247520z_1^3z_2^2 - 90956z_1^3z_2z_3 \\ &\quad + 35786z_1^3z_3^2 + 360640z_1^2z_2^3 - 20832z_1^2z_2^2z_3 + 23772z_1^2z_2z_3^2 - 8882z_1^2z_3^3 \\ &\quad + 313600z_1z_2^4 + 98560z_1z_2^3z_3 + 3192z_1z_2^2z_3^2 - 1604z_1z_2z_3^3 + 592z_1z_3^4 \\ &\quad + 125440z_2^5 + 89600z_2^4z_3 + 9520z_2^3z_3^2 - 56z_2^2z_3^3 - 40z_2z_3^4 + 16z_3^5), \\ \bar{T}_6(z_1, z_2, z_3) &= \frac{\pi}{2488320}(578624z_1^6 + 2670432z_1^5z_2 - 1335216z_1^5z_3 + 6994704z_1^4z_2^2 \\ &\quad - 2964576z_1^4z_2z_3 + 1076988z_1^4z_3^2 + 12236840z_1^3z_2^3 - 1891308z_1^3z_2^2z_3 \\ &\quad + 1121838z_1^3z_2z_3^2 - 366193z_1^3z_3^3 + 14294280z_1^2z_2^4 + 2721180z_1^2z_2^3z_3 \\ &\quad + 262458z_1^2z_2^2z_3^2 - 149091z_1^2z_2z_3^3 + 46227z_1^2z_3^4 + 10210200z_1z_2^5 \\ &\quad + 5825820z_1z_2^4z_3 + 404250z_1z_2^3z_3^2 - 11379z_1z_2^2z_3^3 + 1380z_1z_2z_3^4 \\ &\quad - 291z_1z_3^5 + 3403400z_2^6 + 3303300z_2^5z_3 + 690690z_2^4z_3^2 + 11935z_2^3z_3^3 \\ &\quad - 699z_2^2z_3^4 + 417z_2z_3^5 - 139z_3^6).\end{aligned}$$

Additionally, the ideal $\langle T_1, \dots, T_6 \rangle$ is not radical, and $T_7^2, T_8^2, T_9^2 \in \langle T_1, \dots, T_6 \rangle$, but $T_7, T_8, T_9 \notin \langle T_1, \dots, T_6 \rangle$.

We decided to present the expressions of the period constants obtained directly, without the simplification commented in Remark 4, as they are short.

In the context of a piecewise scenario, typical for the center-focus and cyclicity problems, as seen in works such as [11], the analysis of criticality requires additional effort due to the doubling of the number of functions to be analyzed. In [3], it is shown that the expressions of the period constants with odd subscripts vanish, and only three of them generate the ideal formed by all the coefficients of the Taylor series of the period function. This is the key point for proving the exact upper bound $C_0(2) = 2$; see again [3]. In the non-smooth scenario, the non-radicality as indicated in the aforementioned result does not directly

establish, as we suspect, that $C_0^p(2) = 5$, despite having solved the isochronicity problem. It is important to note that while we have identified the maximum order of weakness, this does not automatically translate into being the upper limit of local criticality.

3 Preliminary results

In this section, we present the main technique that can be used to increase the number of critical periods with respect to the bounds obtained by linear developments. These results were introduced in [7] and further developed in [5,6] for studying cyclicity in families of centers. These ideas were extended in [13] for studying the number of critical periods. In this latter reference, the authors stated and proved the following results, which we include here for completeness and to aid the reader in avoiding the need to search through multiple references. They describe the structure of the first-order terms of the period constants after perturbing a family of isochronous centers and how it can be used to increase the previously established lower bounds for the number of critical periods in a class of planar smooth systems of polynomial differential equations of fixed degree with a center.

The following proposition illustrates the structure of the first-order terms of the period constants for a perturbed family of isochronous centers within a fixed class of centers.

Proposition 6 *Consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$, for some $P \in \mathbb{N}$, and add a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$, for some $N \in \mathbb{N}$, which does not break the center property.*

- (i) *The j -th period constant T_j of the perturbed system is a polynomial on the perturbative parameters of the form*

$$T_j = \sum_{i=1}^N g_j^{(i)}(A) \lambda_i + O_2(\lambda_1, \lambda_2, \dots, \lambda_N),$$

where $g_j^{(i)}(A)$ are polynomials in A which are the coefficients of the linear part of T_j with respect to λ , and $O_2(\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes a sum of monomials of degree at least 2 on the parameters.

- (ii) *The matrix of coefficients of linear parts of the first m period constants is the $m \times m$ matrix $G_m(A)$, whose element in position (i, j) is $g_i^{(j)}(A)$ in (6). If $\det G_N(A) = 0$ and $\det G_{N-1}(A) \neq 0$, there exists a linear change of variables such that the first $N - 1$ period constants take the form*

$$T_j = u_j + O_2(u_1, u_2, \dots, u_N),$$

for $j = 1, \dots, N - 1$, where the linear part of T_j is u_j , $u_N := \lambda_N$, and we denote the higher-order terms by $O_2(u_1, u_2, \dots, u_N)$.

- (iii) *Under the assumptions of (ii), the first $N + M$ period constants for some $M \in \mathbb{N}$ can be written as*

$$T_j = \begin{cases} v_j, & \text{if } j = 1, \dots, N - 1, \\ \sum_{i=1}^{N-1} \tilde{g}_j^{(i)}(A) v_i + f_{j-N}(A) u_N + O_2(v, u_n), & \text{if } k = N, \dots, N + M, \end{cases} \quad (10)$$

where $v = (v_1, \dots, v_{N-1}, u_N)$ are new variables, $f_{j-N}(A)$ and $\tilde{g}_j^{(i)}(A)$ are the corresponding coefficients of $v_1, v_2, \dots, v_{N-1}, u_N$, which are rational functions in $A \in \mathbb{R}^P$, and $O_2(v, u_N)$ are analytical functions of order two in $v_1, \dots, v_{N-1}, u_N$.

Now we can present the aforementioned results which are useful to obtain a better lower bound for the local criticality on a parameterized family of isochronous centers.

Theorem 7 *We consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$, for some $P \in \mathbb{N}$, and a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$, for some $N \in \mathbb{N}$, which does not break the center property. We denote by $G_m(A)$ the $m \times m$ matrix as defined in Proposition 6.*

- (i) *If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) \neq 0$, then the linear parts of the first period constants have rank N and at least $N - 1$ simple critical periods can bifurcate.*
- (ii) *If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) = 0$, $\det G_{N-1}(A^*) \neq 0$, $f_i(A^*) = 0$, for $i = 0, \dots, M - 1$, $f_M(A^*) \neq 0$ (where $f_0, \dots, f_M$ are those defined in (10)) and the Jacobian determinant satisfies $J(A^*) := \det \text{Jac}_{(f_0, \dots, f_{M-1})}(A^*) \neq 0$, then M extra critical periods can bifurcate, which leads to a total of at least $N + M - 1$ critical periods.*

The subsequent corollaries affirm that the same conclusion holds true when the number of parameters is greater than or equal to N .

Corollary 8 *We consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$, for some $P \in \mathbb{N}$, and a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^m$, for some $m \in \mathbb{N}$, where $m \geq N$ and which does not break the center property. For each $j \leq m$, we denote by $G_j(A)$ the $j \times j$ matrix as defined in Proposition 6. If there exists $A^* \in \mathbb{R}^P$ such that $\det G_j(A^*) \neq 0$, then generically at least $j - 1$ simple critical periods bifurcate from the origin.*

Corollary 9 *Under the same conditions of Corollary 8, if there exists $A^* \in \mathbb{R}^P$ such that $\det G_j(A^*) = 0$, $\det G_{j-1}(A^*) \neq 0$, $f_i(A^*) = 0$, for $i = 0, \dots, l - 1$, $f_l(A^*) \neq 0$ (where $f_0, \dots, f_l$ are defined in an analogous way for those ones presented in (ii) of Proposition 6) and the Jacobian determinant satisfies $J(A^*) := \det \text{Jac}_{(f_0, \dots, f_{l-1})}(A^*) \neq 0$, then l extra critical periods can bifurcate, which leads to a total of $j + l - 1$ critical periods.*

We emphasize that we can restrict the analysis to a representative system from family (4).

4 Weak centers of order five and its unfoldings

In this section, we shall prove Theorems 1 and 2. Henceforth, we will denote by $T_j^{[i]}$ the i -th order terms of the period constant T_j . Firstly, we notice that it is very convenient to make a change in the parameter space given by

$$a_j = (w_j + z_j)/2 \text{ and } b_j = (w_j - z_j)/2 \text{ for } j = 1, 2, 3.$$

This change is natural because the upper system is equal to the lower one. After the change $(a_1, a_2, a_3) \rightarrow (b_1, b_2, b_3)$, it is reasonable to consider the case in which both systems coincide, making the difference equal to zero. Also, note that in Lemma 5, the

period constants are either differences or sums of the same expression in three variables depending on whether they correspond to odd or even subscripts.

Again, we use the period constants found in Lemma 5 and consider p_1 as T_1 after the previous change. Subsequently, for even $j \geq 2$, p_j represents T_j/π after the change of variables reduced with respect to the ideal $\langle p_1, \dots, p_{j-1} \rangle$, and for odd $j > 2$, p_j represents T_j after the change of variables reduced with respect to the ideal $\langle p_1, \dots, p_{j-1} \rangle$. We notice that the expressions of p_j can be written up to multiplicative non-vanishing constants, as we are only interested in common solutions or whether they are zero or not. In the upcoming section, we will observe that the isochronicity conditions are established by solving $\{p_1 = p_2 = \dots = p_6 = 0\}$. Hence, the maximum order of weakness arises when $p_6 \neq 0$. The forthcoming proof outlines all families that satisfies such a condition.

Proof of Theorem 1 Initially, we observe that the expressions p_j are homogeneous of degree j with respect to the parameters $(z_1, z_2, z_3, w_1, w_2, w_3)$. Hence, to simplify computations, we can consider two cases: $w_2 = 0$ and $w_2 = 1$. We will check that only the second case has some interest for us; the first case is rather straightforward and does not provide our desired result. Straightforward computations show that the solution of $S = \{p_1 = p_2 = \dots = p_5 = 0\}$ gives us three families:

$$\begin{aligned} \{w_2 = w_3 - w_1 = z_3 - z_2 = z_1 = 0\}, \\ \{w_1 = w_2 = w_3 = z_1 + 4z_2 = z_3 + 2z_2 = 0\}, \\ \{w_2 = w_3 - 4w_1 = z_1 = z_2 = z_3 = 0\}, \end{aligned}$$

where, for each family, $p_6 = p_7 = p_8 = p_9 = 0$. Henceforth, for the remainder of the proof, we will assume $w_2 = 1$. We will now present step-by-step how we can obtain the solutions of S .

Starting with $p_1 = 0$, we can obtain

$$z_1 = -2z_2 + z_3,$$

and by substituting it, we get

$$p_2 = 4w_1^2 - 5w_1w_3 + w_3^2 + 6z_2^2 + 3z_2z_3 + 10w_1 - w_3 + 10.$$

When $z_2 = 0$, we get four families, but only the following two satisfy the condition $p_6 \neq 0$:

$$\begin{aligned} \{3w_1 + 5 = 3w_3 + 2 = z_1 = z_2 = z_3 = 0\}, \\ \{w_1 - \alpha = w_3 - 5 - 5\alpha = z_1 = z_2 = z_3 = 0\}, \end{aligned} \tag{11}$$

being α each one of the two roots, which are simple, of $p(\alpha) = 2\alpha^2 + 15\alpha + 15$. Moreover, after recovering the rescaling by w_2 and the original parameters, we obtain the families (i) and (ii) indicated in the introduction. Both families are indeed smooth.

When $z_2 \neq 0$, we get, equating the above expression of p_2 to zero,

$$z_3 = -\frac{4w_1^2 - 5w_1w_3 + w_3^2 + 6z_2^2 + 10w_1 - w_3 + 10}{3z_2}.$$

Then, looking the expression of p_3 we can consider only $10w_1^2 - 21w_1w_3 + 5w_3^2 + 14w_1 - 22w_3 \neq 0$, because, when this last expression vanishes, we can again check that $p_6 = p_7 = p_8 = p_9 = 0$. Hence, we have

$$z_2^2 = -\frac{(w_1^2 - 2w_1w_3 + w_3^2 + 18w_1 - 6w_3 + 20)(4w_1^2 - 5w_1w_3 + w_3^2 + 10w_1 - w_3 + 10)}{6(10w_1^2 - 21w_1w_3 + 5w_3^2 + 14w_1 - 22w_3)} \quad (12)$$

and, consequently,

$$\begin{aligned} z_1 &= -\frac{(4w_1^2 - 5w_1w_3 + w_3^2 + 10w_1 - w_3 + 10)(8w_1^2 - 17w_1w_3 + 3w_3^2 - 22w_1 - 10w_3 - 40)}{3(10w_1^2 - 21w_1w_3 + 5w_3^2 + 14w_1 - 22w_3)z_2}, \\ z_3 &= -\frac{(9w_1^2 - 19w_1w_3 + 4w_3^2 - 4w_1 - 16w_3 - 20)(4w_1^2 - 5w_1w_3 + w_3^2 + 10w_1 - w_3 + 10)}{3(10w_1^2 - 21w_1w_3 + 5w_3^2 + 14w_1 - 22w_3)z_2}. \end{aligned} \quad (13)$$

From the above expressions for z_1, z_2, z_3 , depending on (w_1, w_3) , we can check that $p_6 \neq 0$ only when we encounter a solution to the next algebraic system of two equations and two variables:

$$\begin{aligned} &2560w_1^7 + 2624w_1^6w_3 - 46308w_1^5w_3^2 + 96733w_1^4w_3^3 - 82996w_1^3w_3^4 + 32934w_1^2w_3^5 - 5936w_1w_3^6 \\ &+ 389w_3^7 + 215424w_1^6 - 637424w_1^5w_3 + 512466w_1^4w_3^2 - 94327w_1^3w_3^3 - 63007w_1^2w_3^4 \\ &+ 28635w_1w_3^5 - 3239w_3^6 + 1185392w_1^5 - 3201252w_1^4w_3 + 2281022w_1^3w_3^2 - 615572w_1^2w_3^3 \\ &+ 45294w_1w_3^4 + 1852w_3^5 + 2715544w_1^4 - 6569316w_1^3w_3 + 3596736w_1^2w_3^2 - 652072w_1w_3^3 \\ &+ 28692w_3^4 + 3104200w_1^3 - 6717712w_1^2w_3 + 2580144w_1w_3^2 - 232464w_3^3 + 1715440w_1^2 \\ &- 3340560w_1w_3 + 700720w_3^2 + 316000w_1 - 632000w_3 - 40000 = 0, \\ &569584w_1^9 - 2365502w_1^8w_3 + 398507w_1^7w_3^2 + 12308694w_1^6w_3^3 - 25464623w_1^5w_3^4 \\ &+ 22828196w_1^4w_3^5 - 10575579w_1^3w_3^6 + 2608102w_1^2w_3^7 - 323089w_1w_3^8 + 15710w_3^9 \\ &+ 28667500w_1^8 - 152373736w_1^7w_3 + 290658808w_1^6w_3^2 - 242710514w_1^5w_3^3 + 83522270w_1^4w_3^4 \\ &+ 4185748w_1^3w_3^5 - 10560284w_1^2w_3^6 + 2531942w_1w_3^7 - 189254w_3^8 + 108696132w_1^7 \\ &- 666739632w_1^6w_3 + 1330184388w_1^5w_3^2 - 1075741404w_1^4w_3^3 + 417042612w_1^3w_3^4 \\ &- 72781536w_1^2w_3^5 + 3240420w_1w_3^6 + 297804w_3^7 - 212720w_1^6 - 866497448w_1^5w_3 \\ &+ 2374884848w_1^4w_3^2 - 1770975488w_1^3w_3^3 + 555461360w_1^2w_3^4 - 74933816w_1w_3^5 \\ &+ 3469360w_3^6 - 722839232w_1^5 + 617633072w_1^4w_3 + 1831552624w_1^3w_3^2 - 1418359936w_1^2w_3^3 \\ &+ 320517136w_1w_3^4 - 21401968w_3^5 - 1679605920w_1^4 + 2922519936w_1^3w_3 + 204490752w_1^2w_3^2 \\ &- 531055776w_1w_3^3 + 72280128w_3^4 - 1738765760w_1^3 + 3252123520w_1^2w_3 - 489135040w_1w_3^2 \\ &- 62799040w_3^3 - 861094400w_1^2 + 1588764800w_1w_3 - 196716800w_3^2 - 145728000w_1 \\ &+ 291744000w_3 + 14400000 = 0. \end{aligned}$$

This system is equivalent to solving $p_4 = p_5 = 0$, with the condition $p_6 \neq 0$. Consequently, this indicates that we are not dealing with isochronous cases. Applying the Groebner basis simplification algorithm enables us to express all solutions of the above

2×2 system as $(w_1, w_3) = (\alpha, q(\alpha))$, where $q(\alpha)$ is a polynomial with integer coefficients of degree 31. Using this polynomial, we can obtain polynomials $q_1(\alpha), \dots, q_6(\alpha)$ of degree 30 with integer coefficients such that the expressions in (12) and (13) provide our last family

$$\left\{ z_1 = \frac{q_1(\alpha)}{q_2(\alpha)z_2}, z_2^2 = \frac{q_3(\alpha)}{q_4(\alpha)}, z_3 = \frac{q_5(\alpha)}{q_6(\alpha)z_2}, w_1 = \alpha, w_3 = q(\alpha) \right\}. \quad (14)$$

We avoided providing the polynomials q_j and q due to their extensive sizes. Straightforward computations show that p_6 and the Jacobian of $(p_1, \dots, p_5)$ on $(z_1, z_2, z_3, w_1, w_3)$ in (14) remain non-vanishing. More concretely, they also write as rational polynomials, as z_i , of degree 30 in α . Additionally, regarding q_j , the resultant with q with respect to α results in a non-zero integer. Consequently, (14) is well-defined, and we have a versal unfolding of exactly 5 critical periods near each curve in the parameter space defined by (14). However, there is a last aspect to verify. Although the polynomial $q(\alpha)$ yields eleven real solutions, only those close to the values in

$$S^+ = \{-30.4809, -5.0964, -3.4076, -1.6457, -1.5645, -1.4294, \\ -1.1524, -1.1225, -1.1068\}$$

can be selected. This is because the solutions in $S^- = \{-1.3228, -0.81783\}$ result in complex values for z_2 in (12).

We conclude the proof recovering the original parameters for the first family (11), as denoted by (i) in the introduction, and proving the existence of a versal unfolding in a more explicit manner. Family (2) is established in a similar fashion.

System (4), when considering this particular (smooth) family and applying a rescaling on (x, y) , can be expressed as:

$$(\dot{x}, \dot{y}) = (-y - 5x^2 + 3y^2, x - 2xy). \quad (15)$$

By using the period constants discovered in Lemma 5, we find that $T_1 = \dots = T_5 = 0$ and $T_6 = 63\pi/5$. We consider the time-reversible perturbation in terms of the parameters $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ in system (15):

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + (-5 + e_1)x^2 + (3 + e_2)y^2, x + (-2 + e_3)xy), & \text{if } y \geq 0, \\ (-y + (-5 + e_4)x^2 + (3 + e_5)y^2, x + (-2 + e_6)xy), & \text{if } y < 0. \end{cases}$$

Then, the linear developments in power series with respect to the perturbative parameters of the first six period constants are given by

$$\begin{aligned} T_1^{[1]} &= \frac{2}{3}(e_1 + 2e_2 - e_3 - e_4 - 2e_5 + e_6), \\ T_2^{[1]} &= \frac{\pi}{4}(2e_2 + 3e_3 + 2e_5 + 3e_6), \\ T_3^{[1]} &= \frac{2}{5}(5e_1 + 4e_2 - 10e_3 - 5e_4 - 4e_5 + 10e_6), \\ T_4^{[1]} &= \pi(-2e_1 + 5e_3 - 2e_4 + 5e_6), \\ T_5^{[1]} &= \frac{2}{525}(7415e_1 + 1834e_2 - 13949e_3 - 7415e_4 - 1834e_5 + 13949e_6). \end{aligned}$$

Table 1 Isochronous center conditions

Type	Conditions
I	$w_1 + w_2 = 2w_2 + w_3 = z_i = 0, i = 1, 2, 3$
II	$w_1 - w_3 = w_2 = z_1 - z_3 = z_2 = 0$
III	$w_2 = w_3 - 4w_1 = z_i = 0, i = 1, 2, 3$
IV	$w_1 + 4w_2 = 2w_2 + w_3 = z_1 + 4z_2 = 2z_2 + z_3 = 0$
V	(1) $w_1 + 2w_2 = 6w_2 + w_3 = w_2 - z_2 = 2w_2 - z_3 = z_1 = 0$ (2) $w_1 + 2w_2 = 6w_2 + w_3 = w_2 + z_2 = 2w_2 + z_3 = z_1 = 0$
VI	(1) $w_1 + 2w_2 - w_3 = w_1 + 8w_2 + z_1 = w_2 - z_2 = w_1 + 6w_2 + z_3 = 0$ (2) $w_1 + 2w_2 - w_3 = w_1 + 8w_2 - z_1 = w_2 + z_2 = w_1 + 6w_2 - z_3 = 0$

We can check easily that the rank of the above linear expressions with respect to the parameters $\{e_1, e_2, e_3, e_4, e_5\}$ is 5. Then, by using the Implicit Function Theorem, there exists a linear change of variables in the parameter space, well defined in a neighborhood of the origin, such that $T_j = v_j$, for $j = 1, \dots, 5$. Furthermore, as the sixth period constant is not equal to zero, we obtain that at most 5 local critical periods bifurcate from this center.

In summary, we have three families, as defined in (11) and (14), which altogether yield $1 + 2 + 18 = 21$ curves of weak centers unfolding exactly 5 critical periods.

A final remark is that, as all the equations involved are polynomial with rational coefficients, all the computations are made in an exact way using a computer algebra system and arithmetic interval analysis. $\square$

Proof of Theorem 2 By solving $p_1 = p_2 = \dots = p_6 = 0$, we find 8 necessary conditions to obtain isochronous center which are expressed and separated into their respective types in the Table 1.

The conditions in Type I, Type II, Type IV, Type V, and Type VI in Table 1 are respectively equivalent to conditions (i), (ii), (iii), (iv), (v), and (vi) of Theorem 2.

Recalling the families in (5), systems (i) and (iii) correspond to the isochronous quadratic systems S_1 and S_3 , respectively. For case (ii), it is equivalent to S_2 if $a_1 = b_1$, and for case (iv), it is equivalent to S_4 if $a_2 = b_2$. In [2], the conditions for systems (4) to have isochronous centers at the origin in each case (ii) with $a_1 \neq b_1$, (iv) with $a_2 \neq b_2$, and cases (v) and (vi) are proven.

Thus, the condition $p_1 = \dots = p_6 = 0$ implies that the origin is an isochronous center, as for each case, the isochronous center condition does not require the period constant p_7 . $\square$

5 Perturbing piecewise isochronous quadratic systems

In this section, we delve into determining lower bounds for the criticality of the isochronous center derived in Theorem 2. We will demonstrate that at least 4 local critical periods can unfold within the class of reversible piecewise quadratic systems only by employing first-order developments. Across all the cases examined, our analysis reveals that no additional critical periods can be found by employing higher-order developments up to order 4.

Proposition 10 Consider the isochronous quadratic systems given by condition (i) of Theorem 2. The number of critical periods bifurcating from the origin when perturbing into the class of piecewise reversible quadratic system is at most 3.

Proof We consider the time-reversible quadratic perturbation

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + (1 + e_1)x^2 + (-1 + e_2)y^2, x + (2 + e_3)xy), & \text{if } y \geq 0, \\ (-y + (1 + e_4)x^2 + (-1 + e_5)y^2, x + (2 + e_6)xy), & \text{if } y < 0. \end{cases}$$

The matrix of $T_1^{[1]}$, $T_2^{[1]}$, $T_3^{[1]}$, and $T_5^{[1]}$ with respect to the parameters $\{e_1, e_2, e_3, e_4\}$ is given by

$$\begin{pmatrix} \frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{\pi}{2} & -\frac{\pi}{2} & 0 & -\frac{\pi}{2} \\ -\frac{2}{15} & -\frac{8}{15} & \frac{4}{5} & \frac{2}{15} \\ \frac{2}{35} & \frac{12}{35} & -\frac{6}{7} & -\frac{2}{35} \end{pmatrix}, \quad (16)$$

and its rank is four, since its determinant is equal to $-64\pi/1575$. Then, there exists a linear change of variables given by

$$\begin{aligned} e_1 &= (99u_1 - 32u_2 + 570\pi u_3 + 455\pi u_4 - 32\pi u_5)/(32\pi), \\ e_2 &= (-27u_1 - 210u_3 - 175u_4 + 16u_5)/16, \\ e_3 &= (-3u_1 - 30u_3 - 35u_4 + 8u_6)/8, \\ e_4 &= -(45\pi u_1 + 32u_2 + 150\pi u_3 + 105\pi u_4 + 32\pi u_5)/(32\pi), \end{aligned} \quad (17)$$

where $u_5 = e_5$ and $u_6 = e_6$, such that $T_1^{[1]} = u_1$, $T_2^{[1]} = u_2$, $T_3^{[1]} = u_3$, $T_5^{[1]} = u_4$. Using these expressions, we can express $T_1, \dots, T_5$ in terms of these new variables. By the properties presented in Proposition 6, we can simplify the period constant T_4 and, then, we get $T_4^{[1]} = 0$. Using the simplification $u_1 = u_2 = u_3 = 0$, we obtain the second-order terms of T_j , for $j = 4, 5$, which depend on the remaining parameters (u_4, u_5, u_6) :

$$\begin{aligned} T_4^{[2]} &= \pi(385875u_4^2 - 53760u_4u_5 - 26880u_4u_6 + 4096u_5^2 + 4096u_5u_6 + 1024u_6^2)/24576, \\ T_5^{[2]} &= u_4(480 + 82215u_4 - 12848u_5 - 5464u_6)/480. \end{aligned}$$

As the solutions of $T_4^{[2]} = 0$ are complex, the only way the fourth period constant could be zero is if $u_4 = u_5 = u_6 = 0$, causing all subsequent period constants to vanish. Thus, we cannot obtain more than 3 oscillations. $\square$

Proposition 11 Consider the isochronous quadratic systems given by conditions (iii) and (v) presented in Theorem 2. The number of critical periods bifurcating from the origin when perturbing in the class of piecewise reversible quadratic system is at least 4, using developments up to order 4.

Table 2 Regions in the plane of the parameters

Case	Condition
II*	$a_1 b_1 (a_1 + b_1) \neq 0$
II**	$a_1 (a_1 + b_1)(a_1 + 2b_1) \neq 0$
IV*	$a_2 b_2 (a_2 + b_2) \neq 0$
IV**	$a_2 (a_2 + b_2)(a_2 + 2b_2) \neq 0$
IV***	$a_2 (a_2 + b_2)(4a_2 + 5b_2) \neq 0$
VI*	$a_1 b_2 (a_1 - 4b_2) \neq 0$
VI**	$a_1 \neq 0$

Proof We present the proof only for condition (iii), and for the other, the analysis is carried out in a similar way.

We consider the time-reversible quadratic perturbation

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + (-1 + e_1)x^2 + e_2y^2, x + (-4 + e_3)xy), & \text{if } y \geq 0, \\ (-y + (-1 + e_4)x^2 + e_5y^2, x + (-4 + e_6)xy), & \text{if } y < 0. \end{cases}$$

Initially, we compute the first eight period constants and we obtain that the rank of the linear developments of the first five period constants, with respect to the parameters $\{e_1, e_2, e_3, e_4, e_5\}$, is 5. Then, up to a linear change of variables in the parameter space as in (17), we can express $T_j^{[1]} = u_j$, for $j = 1, \dots, 5$. Using Proposition 6, we can simplify the next period constants to get $T_j^{[1]} = 0$, for $j = 6, 7, 8$. Applying item (i) of Theorem 7 up to order 1, we get that only four critical periods bifurcate from the origin. Computing the higher developments up to order 4, and using the simplification mechanism described in item (ii) of Theorem 7, we get that $T_6^{[j]} = T_7^{[j]} = T_8^{[j]} = 0$, for $j = 2, 3, 4$. Then, the result follows up to order 4. $\square$

For the conditions (ii), (iv), and (vi) of Theorem 2, we have a polynomial family of isochronous centers parametrized by two parameters. In the plane of the parameters, there exist some curves for which the rank of the linear parts of the period constants is not maximal along them. In these more degenerate cases, it would be possible to obtain a greater number of oscillations by using Theorem 7 (ii). Thus, it is possible to split into cases, and some of them are presented in Table 2.

Proposition 12 *Consider the isochronous quadratic systems given by conditions II*, IV*, and VI* of Table 2. The number of critical periods bifurcating from the origin when perturbing in the class of piecewise reversible quadratic system is at least 3, using developments up to order 4.*

Proof We shall see why we consider the restriction presented in the Table 2 for the isochronous center of condition (ii) of Theorem 2; analogous analysis applies to the other cases.

Consider the time-reversible quadratic perturbation for the isochronous center presented in condition (ii) of Theorem 2:

$$(\dot{x}, \dot{y}) = \begin{cases} (-y + (a_1 + e_1)x^2 + e_2y^2, x + (a_1 + e_3)xy), & \text{if } y \geq 0, \\ (-y + (b_1 + e_4)x^2 + e_5y^2, x + (b_1 + e_6)xy), & \text{if } y < 0. \end{cases}$$

Performing a first-order analysis, we observe that the matrix, as the one presented in (16), associated with the linear terms of the initial four period constants, concerning parameters $\{e_1, e_2, e_4, e_5\}$, yields a determinant of $-\pi^2 a_1 b_1 (a_1 + b_1)^4 / 720$. Thus, provided $a_1 b_1 (a_1 + b_1) \neq 0$, a linear transformation, as the one in (17), exists. Consequently, we obtain the region of Type II*.

The remaining part of the proof is carried out in a similar way to those of Propositions 10 and 11. $\square$

Remark 13 Under the remaining conditions in Table 2, we were unable to obtain a greater number of oscillations than those ones obtained in the last results and, therefore, we shall not detail them here.

Finally, we prove our last main result.

Proof of Theorem 3 The proof follows by using Propositions 10, 11, and 12. Moreover, by Proposition 11, it is deduced that the greater number of critical periods bifurcating from the origin, upon perturbation within the class of piecewise reversible quadratic systems up to order 4, amounts to 4. This is achieved under the conditions outlined in Theorem 2. $\square$

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Data availability

No datasets were generated or analyzed during the current study.

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