



A Primer on Resultants and Their Applications

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Abstract

We briefly introduce the resultant of two polynomials and prove its main properties. We illustrate how to use this tool in many mathematical problems that involve polynomials or rational functions. Finally, we show its utility to face several questions appearing in the study of some dynamical systems.

Keywords Resultant · Discriminant · Polynomial system · Dynamical systems · Periodic points

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1 Introduction

This is a survey paper and it is divided in three parts. In the first one, corresponding to Sect. 2, we briefly introduce the resultant of two polynomials and prove its main properties. In few words, recall that the resultant of two one-variable polynomials is a polynomial expression of their coefficients that is equal to zero if and only if both have a common root, real or complex.

In the second one, given in Sect. 3, we illustrate its utility in many different mathematical questions. For instance, we introduce the discriminant, we give a proof of the Casas-Alvero conjecture for polynomials of degrees 4 and 5, we study problems related with algebraic curves that are rationally parametrizable, we prove some prop-

Dedicated to my friend and colleague Marco Antonio Teixeira in occasion of his eighty birthday.

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erties of the algebraic numbers, we compute primitives of rational functions, and study many other subjects.

Most of the results explained in these two chapters appear disseminated in several books [8, 38–40, 50, 66], and some papers that are quoted along this work. We have collected them trying to show the ubiquity of the resultant in many situations where polynomials appear. We do not pretend to be exhaustive.

Finally, Sect. 4 is devoted to present applications of the use of resultants into dynamical systems given by polynomials vector fields or polynomial maps. It includes several situations where the dynamical question is reduced to know the solution of a polynomial system of equations. Some of them are the center-focus or the isochronicity problems, the study of invariant algebraic curves, or the global periodicity problem for discrete dynamical systems. In other situations, the sign of a function gives the desired information about a dynamical property of the dynamical system. This is the situation, for instance, of the Lyapunov or Bendixson–Dulac functions, in the study of some Wronskians to prove the Chebyshev property of an Abelian integral, or other questions. We also study some classes of centers for Liénard systems and Abel differential equations.

In this paper, computations of resultants and discriminants are performed with Maple [54].

2 The Resultant

As usual we denote by $\mathbb{K}[x]$ the ring of one-variable polynomials with coefficients in $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. Given two polynomials $p, q \in \mathbb{K}[x]$ there is a simple and classical criterion to know whether they have or not a common root (real or complex). It consists on computing a number in $\mathbb{K}$, called the resultant of both polynomials and checking whether it is zero or not. This resultant can be traced back to Newton (for special equations) and to Cayley, Bézout, and Euler, see for instance [38, 49, 68].

For completeness in this section we prove some basic and well-known properties of the resultant and for simplicity we fix $\mathbb{K} = \mathbb{C}$.

Theorem 1 *Set $p, q \in \mathbb{C}[x]$, given by*

$$p = p_n x^n + \cdots + p_2 x^2 + p_1 x + p_0, \quad q = q_m x^m + \cdots + q_2 x^2 + q_1 x + q_0, \quad p_n q_m \neq 0,$$

and define its resultant $\text{Res}_x(p, q) \in \mathbb{C}$, as the $(m+n) \times (m+n)$ determinant¹,

¹ In the formula, without loss of generality, we have assumed that $n > m$. In (2) you can see its correspondent matrix for the values $n = 3$ and $m = 2$.

$$\begin{array}{cccccccccccc}
 p_n & 0 & \cdots & 0 & q_m & 0 & \cdots & \cdots & \cdots & 0 \\
 p_{n-1} & p_n & \ddots & \vdots & \vdots & q_m & 0 & \ddots & \ddots & \vdots \\
 \vdots & p_{n-1} & \ddots & 0 & \vdots & \vdots & q_m & \ddots & \ddots & \vdots \\
 \vdots & \vdots & \ddots & p_n & q_1 & \vdots & \vdots & \ddots & \ddots & \vdots \\
 \vdots & \vdots & \vdots & p_{n-1} & q_0 & q_1 & \vdots & \vdots & q_m & 0 \\
 \vdots & \vdots & \vdots & \vdots & 0 & q_0 & q_1 & \vdots & \vdots & q_m \\
 p_0 & \vdots & \vdots & \vdots & \vdots & 0 & q_0 & \ddots & \vdots & \vdots \\
 0 & p_0 & \ddots & \vdots & \vdots & \ddots & 0 & \ddots & q_1 & \vdots \\
 \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & q_0 & q_1 \\
 0 & \cdots & 0 & p_0 & 0 & \cdots & \cdots & \cdots & 0 & q_0
 \end{array} \quad (1)$$

$\underbrace{\hspace{10em}}_m \quad \underbrace{\hspace{10em}}_n$

Then p and q share a root (real or complex) if and only if $\text{Res}_x(p, q) = 0$.

The squared matrix associated to the above determinant is usually called Sylvester matrix.

An intermediate step in the proof of the above theorem is the following characterization. Given $p \in \mathbb{C}[x]$ we denote by ∂p its degree.

Proposition 2 *Given $p, q \in \mathbb{C}[x]$, they have a common root (real or complex) if and only if there exist non trivial polynomials $r, s \in \mathbb{C}[x]$, such that*

$$rp + sq = 0 \quad \text{with} \quad \partial r < \partial q \quad \text{and} \quad \partial s < \partial p.$$

Proof We prove first that if p and q have a common root then such r and s do exist.

Set $u = \gcd(p, q)$. By hypothesis $\partial u \geq 1$. Then $p = uP$ and $q = uQ$ with $P, Q \in \mathbb{C}[x]$ and $\partial P < \partial p$ and $\partial Q < \partial q$. Hence, taking $r = -Q$ and $s = P$,

$$0 = -qp + qp = -uQp + uPq = u(-Qp + Pq) = u(rp + sq) \implies rp + sq = 0,$$

with $\partial r = \partial Q < \partial q$ and $\partial s = \partial P < \partial p$.

To have the converse implication we will prove the equivalent assertion: If p and q do not share roots there are no r and s satisfying the above properties. We will proceed by contradiction.

Since p and q have no common roots, by Bézout's identity there exist polynomials $v, w \in \mathbb{C}[x]$ such that $vp + wq = 1$. Hence, if we assume that p and q satisfying $rp + sq = 0$ do exist, we get

$$r = r \times 1 = r(vp + wq) = rpv + rqw = -sqv + rqw = q(rw - sv).$$

In particular q divides r and $\partial r \geq \partial q$, proving that a polynomial r , satisfying $\partial r < \partial q$ and $rp + sq = 0$, does not exist. $\square$

Notice that in the proof of Proposition 2 the condition on the degrees of r and s is relevant. This is so, because it is trivial to find r, s with higher degrees satisfying $rp + sq = 0$. For instance, by taking $r = -q$ and $s = p$ (then $\partial r = \partial q$ and $\partial s = \partial p$),

$$rp + sq = -qp + pq = 0.$$

Proof of Theorem 1 Let us prove that the existence of $r, s \in \mathbb{C}[x]$ such that $rp + sq = 0$ with $\partial r < \partial q$ and $\partial s < \partial p$ given in the above proposition, is equivalent to the fact that the determinant given in (1) involving the coefficients of p and q is 0.

To avoid cumbersome notations we continue the proof when $\partial p = 3$ and $\partial q = 2$. It will become clear that the proof for general degrees is essentially the same.

We write

$$p = p_3x^3 + p_2x^2 + p_1x + p_0, \quad q = q_2x^2 + q_1x + q_0,$$

with $p_3q_2 \neq 0$, and assume that there exist $r, s \in \mathbb{Q}[x]$ with $\partial r < 2$ and $\partial s < 3$, that is,

$$r = r_1x + r_0, \quad s = s_2x^2 + s_1x + s_0,$$

such that $rp + sq = 0$. Then, $rp + sq = 0$ writes as

$$(r_1x + r_0)(p_3x^3 + p_2x^2 + p_1x + p_0) + (s_2x^2 + s_1x + s_0)(q_2x^2 + q_1x + q_0) \equiv 0,$$

or, equivalently,

$$r_1(p_3x^4 + p_2x^3 + p_1x^2 + p_0x) + r_0(p_3x^3 + p_2x^2 + p_1x + p_0) + s_2(q_2x^4 + q_1x^3 + q_0x^2) + s_1(q_2x^3 + q_1x^2 + q_0x) + s_0(q_2x^2 + q_1x + q_0) \equiv 0.$$

This identity writes in matrix form as

$$S \begin{pmatrix} r_1 \\ r_0 \\ s_2 \\ s_1 \\ s_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad S = \begin{pmatrix} p_3 & 0 & q_2 & 0 & 0 \\ p_2 & p_3 & q_1 & q_2 & 0 \\ p_1 & p_2 & q_0 & q_1 & q_2 \\ p_0 & p_1 & 0 & q_0 & q_1 \\ 0 & p_0 & 0 & 0 & q_0 \end{pmatrix}. \quad (2)$$

Notice that S is the Sylvester $k \times k$ matrix, with $k = 2 + 3 = \partial q + \partial p$.

It is well known that the this system has a non trivial solution $(r_1, r_0, s_2, s_1, s_0) \neq (0, 0, 0, 0, 0)$ if and only if $\det(S) = \text{Res}_x(p, q) = 0$, as we wanted to prove.

□

Simple manipulations in the determinant that provides the resultant allows us to prove the following nice and useful result.

Proposition 3 Set $p, q \in \mathbb{C}[x]$, polynomials of degree n and m respectively. Then there exist polynomials r and s , with respective degrees at most $m - 1$ and $n - 1$, such that $rp + sq = \text{Res}_x(p, q)$.

Proof Consider the expression of the resultant as the determinant given in (1). This determinant remains invariant if we pursue the following row operations: we add in sequence to the last row $n + m - 1$ rows as follows, the $(n + m - 1)$ -th row multiplied by t , the $(n + m - 2)$ -th row multiplied by t^2 , and in general the $(n + m - k)$ -th row multiplied by t^k , until $k = n + m - 1$. We get

$$\text{Res}_x(p, q) = \begin{vmatrix} p_n & 0 & \cdots & 0 & q_m & 0 & \cdots & \cdots & \cdots & 0 \\ p_{n-1} & p_n & \ddots & \vdots & \vdots & q_m & 0 & \ddots & \ddots & \vdots \\ \vdots & p_{n-1} & \ddots & 0 & \vdots & \vdots & q_m & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & p_n & q_1 & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & p_{n-1} & q_0 & q_1 & \vdots & \vdots & q_m & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & q_0 & q_1 & \vdots & \vdots & q_m \\ p_0 & \vdots & \vdots & \vdots & \vdots & 0 & q_0 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & 0 & \ddots & q_1 & \vdots \\ \vdots & \ddots & p_0 & \vdots & 0 & 0 & \ddots & \ddots & q_0 & q_1 \\ t^{m-1}p & t^{m-2}p & \cdots & p & t^{n-1}q & t^{n-2}q & \cdots & \cdots & tq & q \end{vmatrix},$$

where here $p = p(t)$ and $q = q(t)$. By developing this determinant with respect the last row we obtain that $\text{Res}_x(p, q) \equiv r(t)p(t) + s(t)q(t)$ for some polynomials r and s , with respective degrees at most $m - 1$ and $n - 1$, as we wanted to prove. $\square$

Notice that if p and q are coprime, since they have no common roots, $\text{Res}_x(p, q) \neq 0$. Hence we have a constructive proof of Bézout's identity because

$$Rp + Sq = 1, \quad \text{with} \quad R = \frac{r}{\text{Res}_x(p, q)}, \quad S = \frac{s}{\text{Res}_x(p, q)},$$

where r and s are given in Proposition 3.

Next proposition gives the resultant in terms of the roots of p and q . Although it is not useful for computing it, this result illustrates quite well the reason why the resultant vanishes if and only if the two polynomials share a root. We will not use it in this paper. Its proof can be found in any of the books listed in the introduction.

Proposition 4 Consider $p, q \in \mathbb{C}[x]$ and write them as

$$p(x) = p_n \prod_{i=1}^n (x - a_i), \quad q(x) = q_m \prod_{j=1}^m (x - b_j).$$

Then

$$\text{Res}_x(p, q) = p_n^m q_m^n \prod_{i=1}^n \prod_{j=1}^m (a_i - b_j) = p_n^m \prod_{i=1}^n q(a_i) = (-1)^{nm} q_m^n \prod_{j=1}^m p(b_j).$$

The resultant can be also computed for polynomials $p \in \mathbb{C}[x]$ in several variables, where $x = (x_1, x_2, \dots, x_n)$. Simply select one of the variables, say x_1 , and consider that the polynomials are in this variable, with coefficients in the ring of polynomials in the remainder variables. This resultant is in this context a polynomial in these variables $x_2, x_3, \dots, x_n$. This process can be repeatedly applied giving rise to the so called *elimination method*, very useful to treat polynomial systems of equations [66]. We will see several examples of this approach in next sections.

By using similar tools as those in the proof of Proposition 2 it is not difficult to prove the following extension:

Proposition 5 *Given $p, q \in \mathbb{C}[x]$, they have at least $k \geq 1$ common roots (real or complex) if and only if there exist non trivial polynomials $r, s \in \mathbb{C}[x]$, such that*

$$rp + sq = 0 \quad \text{with} \quad \partial r \leq \partial q - k \quad \text{and} \quad \partial s \leq \partial p - k.$$

By using this proposition and similar computations as in the proof of Theorem 1 we can see that some minors of the Sylvester matrix, called *subresultants*, control whether two polynomials share at least $k \geq 1$ roots. These subresultants are also related with Euclides' algorithm. For more details, see for instance [7, 38, 39, 49].

There is also another generalization of the resultant, namely the resultant for homogeneous forms in several variables. It is known as *multivariate resultant*, or *Macaulay's resultant*, or either *multipolynomial resultant*. A nice reference for this topic is the book [40, Chp. 13]. We will not use either of these extensions in this work.

2.1 An Elementary Computation of the Resultant

In this section we include a very simple approach that also gives the resultant of two polynomials. This is a point of view that can be introduced even to students in high school without knowledge of determinants. It simply consists on the treatment of the non linear system of two equations like a linear one with a variation of the reduction method. The main idea is that at each step we can reduce by one the degree of both polynomials.

To simplify the explanation we will assume that the involved polynomials start with the same degree (this can be achieved by combining linearly both initial equations) and that at each step the degree reduces exactly by one. If it reduces more it is even easier to continue the procedure.

We start with a system with two polynomial equations of degree n and a single unknown x ,

$$p_n(x) = a_n x^n + c_{n-1} x^{n-1} + \cdots = 0, \quad q_n(x) = b_n x^n + d_{n-1} x^{n-1} + \cdots = 0, \quad (3)$$

where $p_n, q_n \in \mathbb{C}[x]$, and we are interested to know if both equations are compatible (then both polynomials will have a common root). To do this we construct a similar system, but of degree $n - 1$, with the property that if (3) has some solution, then the same holds for this new one and vice versa. It can be constructed as

$$\begin{aligned} p_{n-1}(x) &= b_n p_n(x) - a_n q_n(x) =: a_{n-1} x^{n-1} + \cdots, \\ q_{n-1}(x) &= a_{n-1} p_n(x) - a_n x p_{n-1}(x) =: b_{n-1} x^{n-1} + \cdots. \end{aligned}$$

We can continue this procedure until arriving to constant values:

$$[p_n, q_n] \longrightarrow [p_{n-1}, q_{n-1}] \longrightarrow [p_{n-2}, q_{n-2}] \longrightarrow \cdots \longrightarrow [p_2, q_2] \longrightarrow [p_1, q_1] \longrightarrow [p_0, q_0].$$

Then the compatibility condition is that $p_0 = q_0 = 0$. These expressions give the resultant.

Let us illustrate it with an example. Consider

$$p_3(x) = x^3 + ax + 3, \quad q_3(x) = 2x^3 + 7x^2 + 3x + 1,$$

Then

$$\text{Res}_x(p_3, q_3) = \begin{vmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 7 & 2 & 0 \\ a & 0 & 1 & 3 & 7 & 2 \\ 3 & a & 0 & 1 & 3 & 7 \\ 0 & 3 & a & 0 & 1 & 3 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{vmatrix} = 4a^3 + a^2 - 44a + 1936.$$

By applying the reduction algorithm we obtain:

$$p_2(x) = -7x^2 + (2a - 3)x + 5, \quad q_2(x) = (-2a + 3)x^2 - (7a + 5)x - 21,$$

then

$$\begin{aligned} p_1(x) &= -(4a^2 + 37a + 44)x - 10a - 132, \\ q_1(x) &= -(8a^3 + 62a^2 + 47a + 792)x - 20a^2 - 185a - 220, \end{aligned}$$

and, finally,

$$\begin{aligned} p_0(x) &= 49(4a^3 + a^2 - 44a + 1936), \\ q_0(x) &= -98(4a^3 + a^2 - 44a + 1936)(5a + 66). \end{aligned}$$

Hence the initial system can have some solution only if $4a^3 + a^2 - 44a + 1936 = 0$. Moreover, for these values of a , with this approach we also get that the solution is

$$x = -\frac{2(5a + 66)}{4a^2 + 37a + 44},$$

value obtained from the equations $p_1(x) = 0$ or $q_1(x) = 0$.

3 Applications

In this section we collect several applications of the resultant in mathematical problems involving polynomials. The subjects that we cover are discriminants, Heron's formula, cubic equations, Casas-Alvero conjecture, weak Bézout's theorem, Kouchirenko's conjecture, distance between two real algebraic curves, algebraic numbers, computation of primitives, and rational parametrizations.

3.1 The Discriminant

One of the most common uses of the resultant is to characterize when a polynomial p has a multiple root. As usual, p' denotes the derivative of p .

Proposition 6 *Define the discriminant of $p \in \mathbb{C}[x]$ as*

$$\Delta_x(p) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{p_n} \text{Res}_x(p, p').$$

Then p has a multiple root if and only if $\Delta_x(p) = 0$.

The proof is a simple consequence of the properties of the resultant and the fact that for a polynomial p , to have a multiple root is equivalent to say that p and p' share a root.

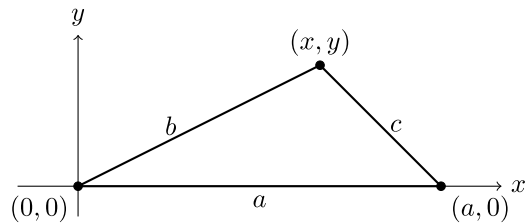
The factor $(-1)^{\frac{n(n-1)}{2}} \frac{1}{p_n}$ was introduced historically. In particular, people wanted that for $p_2x^2 + p_1x + p_0$ the classical discriminant of the quadratic equation was recovered. Notice that

$$\Delta_x(p) = \frac{(-1)^{2/2}}{p_2} \begin{vmatrix} p_2 & 2p_2 & 0 \\ p_1 & p_1 & 2p_2 \\ p_0 & 0 & p_1 \end{vmatrix} = p_1^2 - 4p_2p_0.$$

Consider now the reduced cubic equation $x^3 + px + q = 0$. It is well known that a translation reduces the general cubic to this one. In this case

$$\Delta_x(x^3 + px + q) = -4p^3 - 27p^2.$$

Fig. 1 A general triangle after some rotations and symmetries



3.2 A Simple Proof of Heron's Formula

Let $a > b \geq c > 0$ be the lengths of the sides of a triangle. By means of rotations and symmetries, it is not restrictive to locate its vertices at the points $(0, 0)$, $(a, 0)$ and (x, y) , with $0 < a/2 \leq x < a$ and $y > 0$, see Fig. 1. Then, its area is $A = ay/2$. Moreover, by applying twice Pythagoras' theorem we get that $x^2 + y^2 = b^2$ and $(a - x)^2 + y^2 = c^2$. By replacing $y = 2A/a$, in both equalities we obtain

$$p(x, a, b, c, A) = a^2x^2 + 4A^2 - a^2b^2 = 0, \quad q(x, a, b, c, A) = a^2(x - a)^2 + 4A^2 - a^2c^2 = 0.$$

To arrive to a relation involving only a , b , c , and A , we compute the resultant in x , and impose that it vanishes,

$$\text{Res}_x(p, q) = 16A^2 - (a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 0.$$

Hence, if we introduce the semiperimeter of the triangle $s = (a + b + c)/2$, we get that

$$A^2 = \frac{1}{16}(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = s(s - a)(s - b)(s - c),$$

and finally we obtain the classical formula:

$$A = \sqrt{s(s - a)(s - b)(s - c)}.$$

3.3 Solving Cubic Equations Via Tschirnhaus' Transformations

Tschirnhaus in 1683 had the idea that polynomial equations of degree n could be simplified by using the today called *Tschirnhaus' transformations*. In particular, he succeed in removing always the terms of degree $n - 1$ and $n - 2$, when $n > 2$. He also commented that this idea was already suggested by Descartes. Let us explain, following [2], how his idea allows to solve the cubic in a totally different way to the one followed by Cardano.

We will give the detailed computations to solve the reduced cubic $x^3 + px + q = 0$. Moreover, we also give without details the final solution for the general cubic $x^3 + rx^2 + px + q = 0$ because it provides an explicit formula of the roots in terms of its coefficients.

The key idea to solve $P(x) = x^3 + px + q = 0$, is to find a quadratic equation that shares some root with the cubic one. If we succeed, we are done. Consider $Q(x) = x^2 + yx + w + z$, where y, z and w are unknowns to be determined. After some computations we get that the condition of sharing a root is equivalent to the equation

$$H = \text{Res}_x(P, Q) = z^3 + (3w - 2p)z^2 + (py^2 + p^2 - 4pw + 3qy + 3w^2)z + pwy^2 - qy^3 + p^2w - pqy - 2pw^2 + 3qwy + w^3 + q^2 = 0.$$

If we impose that the coefficient of z^2 vanishes, that is $w = 2p/3$, we obtain that

$$H = z^3 + \left(py^2 + 3qy - \frac{p^2}{3}\right)z - qy^3 + \frac{2p^2}{3}y^2 + pqy + \frac{2p^3}{27} + q^2 = 0.$$

Finally, if we impose that y is such that $py^2 + 3qy - p^2/3 = 0$, the coefficient of z in H vanishes and the degree of its independent term in the variable y can be reduced to one, arriving to

$$H = z^3 + \frac{(4p^3 + 27q^2)(2p^2 - 9qy)}{27p^2} = 0.$$

In short, for solving $x^3 + px + q = 0$, it suffices to take y as one of the roots of $py^2 + 3qy - p^2/3 = 0$,

$$z = \frac{1}{3} \sqrt[3]{\frac{(4p^3 + 27q^2)(9qy - 2p^2)}{p^2}} \quad \text{and} \quad w = \frac{2p}{3},$$

and, finally, solve the corresponding quadratic equation

$$Q(x) = x^2 + yx + w + z = 0.$$

The transformation $z = -x^2 - yx - w$ is the so called Tschirnhaus' transformation. Of course, with this approach we obtain more than three solutions and we must discard some of them.

If we start from the general cubic equation $x^3 + rx^2 + px + q = 0$, we can mimic the process and we arrive to the next procedure to solve it. Consider y as any of the roots of the quadratic equation

$$\left(p - \frac{r^2}{3}\right)y^2 + \left(3q - \frac{7pr}{3} + \frac{2r^3}{3}\right)y - \frac{p^2}{3} - 2qr + \frac{4pr^2}{3} - \frac{r^4}{3} = 0,$$

$$z = \sqrt[3]{\frac{AB}{9(3p - r^2)^2}} \quad \text{and} \quad w = \frac{2p + ry - r^2}{3},$$

where

$$A = 4p^3 + 27q^2 - 18pqr - p^2r^2 + 4qr^3,$$

$$B = 27qy - 6p^2 - 9(py + 2q)r + 10pr^2 + 2yr^3 - 2r^4.$$

The values of w and z are obtained, like in the study of the reduced cubic case, by imposing that the coefficients of z^2 and z , respectively, vanish.

Notice that for each y there are three possible values of z . Finally, the solutions of the initial cubic are contained in the set of solutions of the six corresponding quadratic equations $x^2 + yx + w + z = 0$.

Tschirnhaus also succeed in solving quartic equations.

3.4 Casas-Alvero Conjecture

The Casas-Alvero conjecture claims that if a complex polynomial P of degree $n > 1$ shares roots with all its derivatives, $p^{(k)}$, $k = 1, 2, \dots, n - 1$, then there exist two complex numbers, a and $b \neq 0$, such that $p(z) = b(z - a)^n$.

Notice that, in principle, the common root between p and each $p^{(k)}$ might depend on k . Casas-Alvero arrived to this problem at the turn of this century, when he was working trying to obtain an irreducibility criterion for two variable power series with complex coefficients.

Although several authors have got partial answers, to the best of our knowledge the conjecture remains open, being $n = 24$, the lowest degree open case. A good reference on the subject is [19].

In this section, we present a simple computational proof when $n \in \{4, 5\}$, by using resultants. This approach works up to degree seven, see [17].

Proposition 7 *Casas-Alvero conjecture holds for $n \in \{4, 5\}$,*

Proof It is clear that we can restrict our attention to monic polynomials of degree n such that the coefficient of degree $n - 1$ is zero. This second assumption is simply achieved by a translation of the independent variable. Then the conjecture reduces to prove that under its hypotheses the polynomial is x^n .

Case $n = 4$. Write $p(x) = x^4 + ax^2 + bx + c$. The conditions of the conjecture can be algebraically written as

$$C_1 = \text{Res}_x(p, p') = 0, \quad C_2 = \text{Res}_x(p, p'') = 0, \quad C_3 = \text{Res}_x(p, p''') = 0. \quad (4)$$

Notice that $C_3 = 2^{12}3^4c = 0$. Thus we get that $c = 0$. By fixing $c = 0$, the remaining two equations write as

$$C_1 = -b^2(4a^3 + 27b^2) = 0 \quad \text{and} \quad C_2 = 16a(25a^3 + 216b^2) = 0.$$

The only solution is $a = b = 0$, proving that $p(x) = x^4$, as desired.

Case n = 5. Similarly to the previous case, write $p(x) = x^5 + ax^3 + bx^2 + cx + d$ and impose the same set of conditions (4), together with $C_4 = \text{Res}_x(p, p''') = 0$. In this case, $C_4 = -2^{15}15^5d = 0$, and thus $d = 0$. Then the other three conditions write as

$$\begin{aligned} C_1(a, b, c) &= c^2(16a^4c - 4a^3b^2 - 128a^2c^2 + 144ab^2c - 27b^4 + 256c^3), \\ C_2(a, b, c) &= 32b(441a^4c - 98a^3b^2 - 4200a^2c^2 + 4320ab^2c - 729b^4 + 10000c^3), \\ C_3(a, b, c) &= 7776a(81a^4 - 1800a^2c + 1000ab^2 + 10000c^2). \end{aligned}$$

We continue by using the so called *elimination theory* for solving systems of polynomial equations. In few words, it essentially consists on changing the equations by simpler ones, in such a way that the new equations involve less variables. We use resultants to obtain these new equations. Other methods, like for instance Gröbner basis, could be used to obtain these simplifications.

More concretely, we perform successive resultants, eliminating at each step, one variable more. Consider

$$\begin{aligned} D_1(a, c) &= \text{Res}_b(C_1, C_3) = 6^{20}c^4a^4u^2(a, c), \\ D_2(a, c) &= \text{Res}_b(C_2, C_3) = 2^{35}3^{25}a^5(81a^4 - 1800a^2c + 10000c^2)v^2(a, c), \end{aligned}$$

where

$$\begin{aligned} u(a, c) &= 146853a^8 + 5009200a^6c + 3998 \cdot 10^4a^4c^2 - 212 \cdot 10^6a^2c^3 - 27 \cdot 10^8c^4, \\ v(a, c) &= 3155031a^8 + 127256400a^6c + 101306 \cdot 10^4a^4c^2 - 6956 \cdot 10^6a^2c^3 - 3^6 \cdot 10^8c^4. \end{aligned}$$

Then the solutions of our initial system must satisfy $D_1 = D_2 = 0$. Finally, $E_1(a) = \text{Res}_c(D_1, D_2) = ka^{340}$, with $0 \neq k \in \mathbb{Z}$, and the solutions also must satisfy $E_1 = 0$.

Hence the only solution of the four equations $C_j = 0, j = 1, 2, 3, 4$ is $a = b = c = d = 0$ giving the polynomial $p(x) = x^5$, as we wanted to see. $\square$

3.5 Weak Bézout's Theorem

Consider a system of two polynomial equations, $p(x, y) = 0, q(x, y) = 0$, with $p, q \in \mathbb{C}[x]$. Bézout's theorem asserts that either it has a continuum of solutions, or it has exactly $\partial p \partial q$ solutions when the system is considered in the complex projective space (taking into account solutions at infinity) and, moreover, each solution is counted with its multiplicity, see for instance [23]. Let us see that by using resultants it is not difficult to prove the following weak version, see also [57, Sec. 5.1].

Proposition 8 Consider the system

$$p(x, y) = 0, \quad q(x, y) = 0, \quad (5)$$

with $p, q \in \mathbb{C}[x]$ and assume that it has finitely many solutions. Then it has at most $\partial p \partial q$ solutions in $\mathbb{C}^2$.

Proof By the elimination theory, and by using the properties of the resultants, it is clear that all the solutions (in $\mathbb{C}^2$) of the initial system are contained in the set of solutions of the new system

$$u(x) = \text{Res}_y(p(x, y), q(x, y)) = 0, \quad v(y) = \text{Res}_x(p(x, y), q(x, y)) = 0.$$

Moreover, the hypothesis of the existence of only finitely many solutions for the initial system implies that none of the polynomials $u, v \in \mathbb{C}[x]$ is identically zero. For convenience, we denote $N = \partial p$ and $M = \partial q$.

Let us prove that the degree of $v(y)$ is at most NM . Then, clearly, the same property will hold for $u(x)$. Write

$$\begin{aligned} p(x, y) &= p_n(y)x^n + p_{n-1}(y)x^{n-1} + \cdots + p_0(y), \quad \text{with } n \leq N, \\ q(x, y) &= q_m(y)x^m + p_{m-1}(y)x^{m-1} + \cdots + q_0(y), \quad \text{with } m \leq M, \end{aligned}$$

where $\partial p_j(y) \leq N - j$ and $\partial q_j(y) \leq M - j$. Let $C(y)$ denote the $(m+n) \times (m+n)$ Sylvester determinant (1), but with $p_j = p_j(y)$ and $q_j = q_j(y)$. Indeed, the elements $c_{i,j}(y)$ of this matrix are: for $1 \leq j \leq m$ and all i ,

$$c_{i,j}(y) = \begin{cases} p_{n+j-i}(y), & 0 \leq i-j \leq n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{with } \partial c_{i,j}(y) \leq N - n + i - j.$$

Similarly, for $m+1 \leq j \leq m+n$ and all i ,

$$c_{i,j}(y) = \begin{cases} q_{j-i}(y), & 0 \leq j-i \leq m, \\ 0, & \text{otherwise,} \end{cases} \quad \text{with } \partial c_{i,j}(y) \leq M + i - j.$$

We remark that the above bounds for $\partial c_{i,j}(y)$ work for all i and j if we take as usual the degree of the polynomial identically 0 as minus infinity. Let us compute an upper bound for the degree of $v(y)$. By definition

$$v(y) = \det(C(y)) = \sum_{\sigma \in G_{m+n}} \text{sgn}(\sigma) \prod_{j=1}^{m+n} c_{\sigma(j),j}(y),$$

where G_{m+n} is the group of permutations of $m+n$ elements. Notice that

$$\begin{aligned}
 \partial \left(\prod_{j=1}^{m+n} c_{\sigma(j),j}(y) \right) &= \sum_{j=1}^{m+n} \partial(c_{\sigma(j),j}(y)) = \sum_{j=1}^m \partial(c_{\sigma(j),j}(y)) + \sum_{j=m+1}^{m+n} \partial(c_{\sigma(j),j}(y)) \\
 &\leq \sum_{j=1}^m (N - n + \sigma(j) - j) + \sum_{j=m+1}^{m+n} (M + \sigma(j) - j) \\
 &\stackrel{(*)}{=} \sum_{j=1}^m (N - n) + \sum_{j=m+1}^{m+n} M \\
 &= m(N - n) + nM = NM + (N - n)(m - M) \leq NM,
 \end{aligned}$$

where in $(*)$ we have used that $\sum_{j=1}^{m+n} (\sigma(j) - j) = 0$. Hence $\partial v(y) \leq NM$.

Finally, since the system $u(x) = 0$, $v(y) = 0$ is of separated variables it is clear that its maximum number of solutions is $\partial u \partial v \leq (NM)^2 = (\partial p \partial q)^2$. Notice that the candidates to be solutions of our original system form a rectangular grid and this upper bound is precisely the square of the desired bound.

To obtain the sharp bound we will use a well-known and simple trick. Let us denote by $(x_j, y_k) \in \mathbb{C}^2$, varying j and k , all the points of the grid introduced above. It is clear that there exist infinitely many $b \in \mathbb{C}$ such that all quantities $x_j - by_k$ are different. Fix any of these values of b . By means of the change of variables $X = x - by$, $Y = y$, we obtain a new system of polynomial equations

$$P(X, Y) = p(X + bY, Y) = 0, \quad Q(X, Y) = q(X + bY, Y) = 0, \quad (6)$$

with $\partial P = \partial p$ and $\partial Q = \partial q$. By construction, system (6) is equivalent to system (5). Arguing as above, but with this new system, we arrive to another system whose solutions contain all solutions of system (6):

$$U(X) = \text{Res}_Y (P(X, Y), Q(X, Y)) = 0, \quad V(Y) = \text{Res}_X (P(X, Y), Q(X, Y)) = 0.$$

In particular observe that if $(x, y) = (x_j, y_k)$ is a solution of system (5) then $(X, Y) = (x_j - by_k, y_k)$ is a solution of system (6) and, as a consequence, $X = x_j - by_k$ is a zero of U . Since there are at most $\partial P \partial Q = \partial p \partial q$ roots of U , and by our choice of b , for different couples j, k the values $x_j - by_k$ never coincide, we know that there are at most $\partial p \partial q$ points of the initial $(\partial p \partial q) \times (\partial p \partial q)$ grid that can be solutions of system (5). Hence the proposition follows. $\square$

3.6 An Application Related with Kouchnirenko's Conjecture

Firstly, we study the number of real solutions of the system

$$p(x, y) = x^6 + ay^3 - y = 0, \quad q(x, y) = p(y, x) = y^6 + ax^3 - x = 0, \quad (7)$$

with $a = 61/43$, by using the ideas introduced in Sect. 3.5. Then, at the end of this section we will explain the relation between this question and the Kouchnirenko's

conjecture. We follow a similar approach to the one developed in [36], but with small modifications.

By using the notation introduced in the previous section, for the above system,

$$u(x) = cx(43x^5 + 61x^2 - 43)r(x) = 0, \quad v(y) = u(y) = cy(43y^5 + 61y^2 - 43)r(y) = 0,$$

where $0 \neq c \in \mathbb{Q}$ and r is a fixed polynomial of degree 30 with integer coefficients that we omit for the sake of simplicity.

By computing the Sturm's sequences [64] of u we can prove that it has exactly 6 real solutions, all simple, and with approximate values:

$$x_0 = 0 < x_1 \approx 0.597 < x_2 \approx 0.689 < x_3 \approx 0.7403 < x_4 \approx 0.780 < x_5 \approx 0.816.$$

Indeed, they can be known with any desired precision, by obtaining sharp enough rational upper and lower bounds. By symmetry the same six values are the possible real values of y . As a consequence, we obtain that the initial system has at most 6×6 real solutions and that the candidates to be actual solutions are $(x, y) = (x_j, x_k)$ varying j and k between 0 and 5. The solutions in $\mathbb{C}^2$ (at most 36 by Bézout's theorem) are among the 36×36 points of a bigger grid formed by the 36 complex roots of u .

Let us use the trick explained in previous section to reduce the possible 36 real solutions until 6. Take $b = -2$ and define

$$P(X, Y) = p(X - 2Y, Y) = 0, \quad Q(X, Y) = q(X - 2Y, Y) = 0.$$

Then

$$U(X) = CX(43X^5 + 1647X^2 - 10449)R(X),$$

where $0 \neq C \in \mathbb{Q}$ and R is another polynomial of degree 30 that we also omit. In any case, by using again the Sturm's approach, we obtain that U has exactly 6 real solutions, all simple, and with approximate values:

$$X_0 = 0 < X_1 \approx 2.0097 < X_2 \approx 2.158 < X_3 \approx 2.221 < X_4 \approx 2.229 < X_5 \approx 2.249.$$

That the value $b = -2$ is well chosen can be simply checked by proving that $\Delta_X(U) \neq 0$, because this fact implies that U has not double roots, implying that all values $x_j + 2x_k$ are different.

Next notice that it must happen for each $\ell = 0, 1, \dots, 5$, that $X_\ell = x_j + 2x_k$ for some j and k . Since we know all these numbers with the desired precision we obtain that the only possible relations are:

$$\begin{aligned} X_0 &= x_0 + 2x_0, & X_1 &= x_5 + 2x_1, & X_2 &= x_4 + 2x_2, \\ X_3 &= x_3 + 2x_3, & X_4 &= x_1 + 2x_5, & X_5 &= x_2 + 2x_4. \end{aligned}$$

Hence there are only 6 candidates to be real solutions of system (7). They are:

$$(x_0, x_0), (x_1, x_5), (x_2, x_4), (x_3, x_3), (x_4, x_2), (x_5, x_1).$$

It is clear that $(x_0, x_0) = (0, 0)$ is actually one of these solutions. Another one is (x_3, x_3) . This can be easily proved because the system reduces to a one-variable polynomial when we study the solutions of the form (x, x) and, for instance by Bolzano's theorem, x_3 is an actual solution of $x^6 + 61x^3/43 - x = 0$.

We need a different tool to ensure that the other four couples are also solutions of system (5). This tool can be Poincaré–Miranda's theorem, that indeed is an extension of Bolzano's theorem in the space. This is done with detail in [36] and here we skip the details.

In short, we have proved that the planar system (7) has exactly 6 real solutions and that 5 of them are in the first quadrant. Next we explain the interest of this result under the light of Kouchnirenko's conjecture. This conjecture was stated by Kouchnirenko in the late 70's, and published in 1980, in [46]. His intention was to extend the celebrated Descartes' rule to more than one variable.

In the two variables case this conjecture said that: A real polynomial system $p_1(x, y) = 0$, $p_2(x, y) = 0$ would have at most $(m_1 - 1)(m_2 - 1)$ simple solutions with positive coordinates, where m_i is the number of monomials of each p_i .

In 2000, B. Haas [43] constructed a family of counterexamples given by two trinomials. In them, 106 was the lowest degree counterexample. In 2007 a much simpler family of counterexamples was presented by Dickenstein, Rojas, Rusek, and Shih in [18]. The simplest one, again was formed by two trinomials, but it had degree 6. Both counterexamples have exactly 5 simple solutions with positive coordinates instead of the 4 predicted by the conjecture.

Indeed, the degree 6 counterexample in [18] corresponds to the system (7) but with $a = 44/31$. Here we simply have presented a new counterexample, clearly inspired in the previous one, although our study of the number of solutions is different to that of [18].

In fact, in [36] it is explained that for any $a \in (\underline{a}, \bar{a})$, where $\underline{a} \approx 1.4176595$ and $\bar{a} = 7 \times 12^{4/5}/36 \approx 1.4195168$, the corresponding system gives a counterexample to the conjecture. The extrema of this interval are the zeroes of the polynomial equation $\Delta_x(\text{Res}_y(p, q)) = 0$, see more details in Sect. 4.4.

3.7 Algebraic Numbers

Let us prove the following classical result.

Proposition 9 *Let α and β be complex numbers such that $p(\alpha) = q(\beta) = 0$, for some non trivial polynomials $p, q \in \mathbb{Z}[x]$. If we define the two new polynomials r and s , also in $\mathbb{Z}[x]$,*

$$r(x) = \text{Res}_y(p(y), q(x - y)), \quad s(x) = \text{Res}_y(p(y), y^n q(x/y)),$$

where n is the degree of q , it holds that

$$r(\alpha + \beta) = 0 \quad \text{and} \quad s(\alpha\beta) = 0.$$

Proof Notice that when $x = \alpha + \beta$, the polynomials $p(y)$ and $q(\alpha + \beta - y)$ have both the root $y = \alpha$ and hence $r(x)$ at $x = \alpha + \beta$ must vanish.

Similarly, when $x = \alpha\beta$ the polynomials $p(y)$ and $y^n q(\alpha\beta/y)$ share the root $y = \alpha$, and so $s(\alpha\beta) = 0$. □

Let us illustrate this proposition with a simple example. Consider $\alpha = \varphi = (1 + \sqrt{5})/2$, the golden ratio, and $\beta = \cos(\pi/7)$, and $p(x) = x^2 - x - 1$ and $q(x) = 8x^3 - 4x^2 - 4x + 1$. After some computations we get that

$$\begin{aligned} r(x) &= 64x^6 - 256x^5 + 112x^4 + 464x^3 - 336x^2 - 88x + 41, \\ s(x) &= 64x^6 - 32x^5 - 112x^4 + 16x^3 + 28x^2 - 4x - 1, \end{aligned}$$

are such that $r(\varphi + \cos(\pi/7)) = 0$ and $s(\varphi \cos(\pi/7)) = 0$.

Recall that the complex numbers α for which there exists a non trivial polynomial $p_\alpha \in \mathbb{Z}[x]$ such that $p_\alpha(\alpha) = 0$ are called *algebraic* numbers. The above proposition, together with the simple observation that when $\alpha \neq 0$, $p_{1/\alpha}(x) = x^m p_\alpha(1/x)$, where m is the degree of p_α , prove that the set of algebraic numbers is a field. The proof that we have presented, that is not at all original, has the advantage of providing explicit vanishing polynomials for $1/\alpha$, $\alpha + \beta$, and $\alpha\beta$, however there are also more abstract proofs of this fact.

We finish this subsection by proving that the roots of one-variable polynomials whose coefficients are algebraic numbers are algebraic numbers as well. We will argue with a concrete polynomial but by the reasoning used it is transparent how to prove the result in the general case.

Consider the polynomial equation

$$u(x) = x^5 + \varphi x + \cos(\pi/7) = 0$$

and let us prove that all their solutions are algebraic numbers. By following the above notation we can reformulate the problem as follows: prove that all the roots x of the polynomial

$$u(x, y, z) = x^5 + yx + z = 0, \quad \text{where} \quad p(y) = 0 \quad \text{and} \quad q(z) = 0,$$

are algebraic numbers. Notice that

$$v(x, z) = \text{Res}_y(u(x, y, z), p(y)) = x^{10} + x^6 + 2x^5z - x^2 + xz + z^2,$$

$$\begin{aligned} w(x) = \text{Res}_z(v(x, z), q(z)) &= 64x^{30} + 192x^{26} + 64x^{25} + 160x^{21} - 48x^{20} - 320x^{18} \\ &\quad - 96x^{16} - 48x^{15} - 160x^{13} - 112x^{12} - 72x^{11} + 200x^{10} \\ &\quad - 64x^8 - 56x^7 - 56x^6 + 40x^5 + 112x^4 - 16x^3 - 28x^2 \\ &\quad + 4x + 1 \end{aligned}$$

and for the solutions of our problem $v(x, z) = 0$ and $w(x) = 0$. Hence all zeroes of the initial polynomial u are also roots of $w \in \mathbb{Z}[x]$ and, as a consequence, are algebraic numbers.

3.8 Distance Between Two Algebraic Curves

Given two real algebraic curves $p(x, y) = 0$ and $P(x, y) = 0$, we will construct a polynomial D such that the distance between them in $\mathbb{R}^2$ is one of the roots of D . Moreover, if the maximum of the distances between the points of both curves is reached, this value is also a root of the same polynomial D .

We start proving a similar result for a simpler case. We study the distance between a real algebraic curve $p(x, y) = 0$ and a point $(u, v) \in \mathbb{R}^2$. Recall that by using the method of Lagrange multipliers the minimum (and maximum if it exists) of our problem must be located in one of the local extrema of the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = (x - u)^2 + (y - v)^2 + \lambda p(x, y).$$

Notice that, equivalently, instead of considering the distance function we have taken the square of the distance.

Hence, by imposing that $\nabla \mathcal{L} = 0$, we obtain the system with three equations

$$2(x - u) + \lambda p_x(x, y) = 0, \quad 2(y - v) + \lambda p_y(x, y) = 0, \quad p(x, y) = 0.$$

Simple computations give that the necessary conditions for x, y to have a local extremum are

$$q(x, y) = (x - u)p_y(x, y) - (y - v)p_x(x, y) = 0, \quad p(x, y) = 0.$$

Actually, the first equation $q(x, y) = 0$ has a clear geometrical meaning: the points candidate to be extrema are such that ∇p at the point (x, y) is parallel to the vector joining the points (x, y) and (u, v) .

In short, the searched distance is one of the solutions of the following system of three equations

$$p(x, y) = 0, \quad q(x, y) = 0, \quad r(x, y, d) = (x - u)^2 + (y - v)^2 - d^2 = 0.$$

To study its solutions we proceed as in the previous sections, by doing successive resultants. More concretely

$$s(x) = \text{Res}_y(p, q), \quad t(x, d) = \text{Res}_y(p, r), \quad \text{and} \quad D(d) = \text{Res}_x(s, t).$$

This final polynomial equation $D(d) = 0$ is the searched one. Notice that the following result is also a consequence of our proof:

Corollary 10 *If $p \in \mathbb{Q}[x, y]$ and $(u, v) \in \mathbb{Q}^2$ the distance between $p(x, y) = 0$ and (u, v) is an algebraic number.*

For completeness, let us develop a couple of examples. The first one is the well-known formula of the distance of a point to a straight line.

If $p(x, y) = ax + by + c$, then some computations give that

$$\begin{aligned}s(x) &= (a^2 + b^2)x + abv - b^2u + ac, \\ t(x, y) &= (a^2 + b^2)x^2 + (2abv - 2b^2u + 2ac)x - b^2d^2 + b^2u^2 + b^2v^2 + 2bcv + c^2, \\ \text{and finally,}\end{aligned}$$

$$D(d) = -b^2(a^2 + b^2)((a^2 + b^2)d^2 - (au + bv + c)^2).$$

Hence

$$d = \frac{|au + bv + c|}{\sqrt{a^2 + b^2}}.$$

Consider now $p(x, y) = x^2 - 2y^2 + x + 3$ and $(u, v) = (7, 3)$. Then $q(x, y) = 6(1 - y)x + 27y + 3$, $r(x, y, d) = (x - 7)^2 + (y - 3)^2 - d^2$ and

$$\begin{aligned}s(x) &= 36x^4 - 288x^3 + 441x^2 - 315x + 2169, \\ t(x, y) &= 9x^4 - 162x^3 + (-12d^2 + 1371)x^2 + (108d^2 - 6498)x \\ &\quad + 4d^4 - 476d^2 + 13945, \\ D(d) &= 104976D_1(d)D_2(d),\end{aligned}$$

where

$$\begin{aligned}D_1(d) &= 64d^8 - 11344d^6 + 717729d^4 - 16593151d^2 + 55757089, \\ D_2(d) &= 64d^8 - 12880d^6 + 965409d^4 - 29778871d^2 + 264227101.\end{aligned}$$

It can be seen that $d^* \approx 2.003482$, the smallest positive root of D_1 , is the searched distance between the hyperbola $p(x, y) = 0$ and the point.

Similarly, for studying the distance between two real algebraic curves $p(x, y) = 0$ and $P(x, y) = 0$, we consider the Lagrangian function

$$\mathcal{L}(x, y, X, Y, \lambda, \mu) = (x - X)^2 + (y - Y)^2 + \lambda p(x, y) + \mu P(X, Y).$$

By doing the corresponding computations we arrive to the conditions of extremum

$$\begin{aligned}q(x, y, X, Y) &= (x - X)p_y(x, y) - (y - Y)p_x(x, y) = 0, \\ Q(x, y, X, Y) &= (x - X)P_Y(X, Y) - (y - Y)P_X(X, Y) = 0,\end{aligned}$$

and we introduce the function

$$r(x, y, X, Y, d) = (x - X)^2 + (y - Y)^2 - d^2.$$

Hence, we arrive to the following five equations

$$\begin{aligned} p(x, y) &= 0, & q(x, y, X, Y) &= 0, \\ q(x, y) &= 0, & Q(x, y, X, Y) &= 0, & r(x, y, X, Y, d) &= 0. \end{aligned}$$

To study the above system we use elimination theory by performing many successive resultants. In fact we need $4 + 3 + 2 + 1 = 10$ resultants to arrive to a single polynomial $D(d)$ that contains the desired information. We skip the details and do not include any specific example. We only comment that usually D is a huge polynomial. Like in the previous case we have proved:

Corollary 11 *If $p, P \in \mathbb{Q}[x, y]$, the distance between the two algebraic curves $p(x, y) = 0$ and $P(x, y) = 0$ is an algebraic number.*

Clearly the approach developed in this section can be also extended to higher dimensions taking hypersurfaces instead of curves.

3.9 A Careless Assertion

Next example was introduced in [22] and illustrates quite well that a property can hold for all natural numbers $n < N$, being N huge, but the property fails for this value. The first fact could lead to the careless assertion that the property is always true.

Concretely, we wonder if $n^{17} + 9$ and $(n + 1)^{17} + 9$ are coprime for all $n \in \mathbb{N}$. It turns out that they are for all $n < N$, with

$$N = 8424432925592889329288197322308900672459420460792433,$$

but for $n = N$ they have the common divisor

$$M = 8936582237915716659950962253358945635793453256935559,$$

which is a prime number.

Without entering in full details we give some comments about the computations involved to find these numbers. One of the key points is that

$$\text{Res}_n(n^{17} + 9, (n + 1)^{17} + 9) = M.$$

In particular, this implies that both polynomials have a common root in $\mathbb{Z}/M\mathbb{Z}$. Finally, by computing the greatest common divisor of these polynomials modulus M one gets the polynomial $n - N$.

It is not difficult to find $k, m \in \mathbb{N}$ for which $n^k + m$ and $(n + 1)^k + m$, present the same phenomenon for bigger N .

3.10 Computing Residues and Primitives

For any given rational function $P(x)/Q(x)$, with $P, Q \in \mathbb{C}[x]$, there are several ways to compute its primitive. The first step, attributed to Hermite and Ostrovsii, consists in writing

$$\int \frac{P(x)}{Q(x)} dx = \frac{p_0(x)}{q_0(x)} + \int \frac{p(x)}{q(x)} dx.$$

In the above formula we can take $p_0, q_0, p, q \in \mathbb{C}[x]$, with $q_0 = \gcd(Q, Q')$, $q = Q/q_0$, $\partial p < \partial q$ and q is monic and square free, that is such that all its roots in $\mathbb{C}$ are simple. Notice that this last property is equivalent to $\gcd(q, q') = 1$.

Therefore it suffices to concentrate on the computation of the primitive of p/q . By using the decomposition in partial fractions we know that if $q(x) = \prod_{j=1}^n (x - x_j)$ then

$$\frac{p(x)}{q(x)} = \sum_{j=1}^n \frac{r_j}{x - x_j}, \quad \text{where} \quad r_j = \frac{p(x_j)}{q'(x_j)},$$

and the values r_j are called the residues of the poles $x = x_j$. Notice that then

$$\int \frac{p(x)}{q(x)} dx = \sum_{j=1}^n r_j \log(x - x_j). \quad (8)$$

There is an indirect form of obtaining a polynomial $r \in \mathbb{C}[y]$ whose zeroes are all the residues. Define

$$r(y) = \text{Res}_x (q(x), p(x) - yq'(x)).$$

Then $r(x_j) = 0$ for all residues because r vanishes only for $y = p(x_j)/q'(x_j) = r_j$. The reason is that for these values of y both polynomials share the root $x = x_j$ and there are no other values for which both polynomials share roots. Notice that two different x_j could give rise to the same residue r_j .

By joining the terms in (8) that have the same residue we arrive to the so called method of Rothstein–Trager for computing primitives. More concretely, we obtain

$$\int \frac{p(x)}{q(x)} dx = \sum_{j=1}^k r_j \log \left(\gcd(q(x), p(x) - r_j q'(x)) \right), \quad (9)$$

where k is the number of different zeroes of r . See [39, Thm. 11.7] for a detailed proof of the explicit nice expression of the log term that accompanies each residue.

Let us apply this method when $p(x) = 11x^5 + 62x^4 + 34x^3 - 96x^2 + 49x + 14$ and $q(x) = x^6 + 9x^5 + 10x^4 - 26x^3 + 15x^2 + 7x$. The polynomials p and q are coprime because $\text{Res}_x(p, q) \neq 0$ and q is square free because $\Delta_x(q) \neq 0$. We get that

$$r(y) = \text{Res}_x(q(x), p(x) - yq'(x)) = k(y - 2)(y - 5)(y - 1)^4,$$

where $0 \neq k \in \mathbb{Q}$. Moreover,

$$\begin{aligned} \gcd(q, p - 2q') &= x, & \gcd(q, p - 5q') &= x + 7, \\ \gcd(q, p - q') &= x^4 + 2x^3 - 4x^2 + 2x + 1. \end{aligned}$$

Hence, by applying (9), we obtain that

$$\int \frac{p(x)}{q(x)} dx = 2 \log(x) + 5 \log(x + 7) + \log(x^4 + 2x^3 - 4x^2 + 2x + 1).$$

Notice that one of the advantages of this approach is that we do not need to find the roots of q . In any case, for this example

$$\begin{aligned} q(x) &= x(x + 7)(x^4 + 2x^3 - 4x^2 + 2x + 1) \\ &= x(x + 7)(x^2 + (1 + \sqrt{7})x + 1)(x^2 + (1 - \sqrt{7})x + 1). \end{aligned}$$

3.11 Rational Parametrizable Planar Algebraic Curves

It is said that an algebraic curve $F(x, y) = 0$, can be rationally parametrized when there exist non-trivial rational functions $p(t)/r(t)$ and $q(t)/s(t)$ such that $F(p(t)/r(t), q(t)/s(t)) \equiv 0$.

Thanks to Cayley–Riemann’s theorem it is known that an algebraic curve can be rationally parametrized if and only if its genus is 0. See for instance [1] for more details and other references.

The genus of an irreducible algebraic curve $F(x, y) = 0$ of degree n is a non-negative integer. It is equal to $(n - 1)(n - 2)/2$ minus some non-negative quantities which depend on the singularities on $F(x, y) = 0$. In particular, when this curve has no singularities at all its genus is exactly $(n - 1)(n - 2)/2$.

Notice also that algebraic curves can also be seen as surfaces when we look them in $\mathbb{C}^2$. We remark that there are effective methods to find the genus of a surface and to obtain rational parametrizations for genus 0 surfaces, see for instance [62]. Moreover, many programs of symbolic calculus like Maple or Mathematica have implemented some of these methods. For instance, in Maple this can be done by using the instruction `parametrization(F, x, y, t)` that is included in the package `with(algcurves)`.

In some cases, resultants also give a simple way for obtaining rational parametrizations. Consider for instance the conic $F(x, y) = x^2 + y^2 - 3x + y - 4 = 0$. Take the point $(x, y) = (4, 0)$ on the curve, and consider the pencil of lines passing by

this point $L(x, y, t) = x - 4 + ty = 0$. Then, the intersection points between both algebraic curves are given by

$$\begin{aligned}\operatorname{Res}_y(F(x, y), L(x, y, t)) &= (x - 4)((t^2 + 1)x + t^2 - t - 4) = 0, \\ \operatorname{Res}_x(F(x, y), L(x, y, t)) &= y((t^2 + 1)y + 1 - 5t) = 0.\end{aligned}$$

Hence a rational parametrization of $F(x, y) = 0$ is

$$(x, y) = \left(\frac{4 + t - t^2}{1 + t^2}, \frac{5t - 1}{1 + t^2} \right).$$

We finish this short introduction by some comments about which are the simpler parametrizations, see [18]. The maximum degree in t of the four polynomials involved in a rational parametrization is called its degree. It is clear that given a parametrization we can obtain many others, simply by changing t by any rational function $u(s)/v(s)$. A parametrization with minimal degree is called *proper* and it can be seen that, modulus Möbius transformations, it is unique. It is known [18, Thm. 4.21] that a parametrization is proper if and only if its degree is the maximum of the degrees of F with respect to x and y .

Next we develop several applications of rational parametrizable curves, some of them involving also resultants.

3.11.1 An Inverse Problem

As we have already explained, Cayley–Riemann’s theorem characterizes the existence of a rational parameterization in terms of the genus. We can wonder about the following inverse problem: given a couple of non-trivial rational functions $p(t)/r(t)$ and $q(t)/s(t)$, does there exist a planar algebraic curve $G(x, y) = 0$ (of course, of genus 0) such $x = p(t)/q(t)$ and $y = r(t)/s(t)$ is one of its rational parametrizations?

The answer to this question is yes, and it is clear from the properties of the resultant, that one of these curves can be obtained as

$$G(x, y) = \operatorname{Res}_t(p(t) - xq(t), r(t) - ys(t)). \quad (10)$$

This is so because if for some $(x, y) = (x^*, y^*) \in \mathbb{C}^2$, it holds that $G(x^*, y^*) = 0$, then there exists at least one $t = t^*$ such that the above two polynomials in t have this common root, or in other words, such that

$$(x^*, y^*) = (p(t^*)/q(t^*), r(t^*)/s(t^*)).$$

Let us give a simple example. Consider

$$x = \frac{t}{1 + t^2}, \quad y = \frac{t^3}{1 - t^2}.$$

By computing the resultant given in (10) it holds that

$$G(x, y) = -4x^3y^2 - x^3 - 4x^2y + xy^2 + y = 0.$$

Similarly, rationally parametrized curves in $\mathbb{C}^n$ can also be obtained implicitly, by doing resultant in couples. For instance, if we consider the curve

$$x = \frac{t}{1+t^2}, \quad y = 5+t^2, \quad z = \frac{7+t^2}{t},$$

we get that

$$\begin{aligned} G(x, y) &= \text{Res}_t(t - x(1+t^2), 5+t^2 - y) = x^2y^2 - 8x^2y + 16x^2 - y + 5, \\ H(x, z) &= \text{Res}_t(t - x(1+t^2), 7+t^2 - zt) = x^2z^2 + 36x^2 - 8xz + 7, \end{aligned}$$

and, hence, the curve is included in $\{G(x, y) = 0, H(x, z) = 0\}$.

3.11.2 Fermat's Equation for Polynomials

The celebrated Fermat's theorem asserts that the equation

$$x^n + y^n = z^n, \quad 1 \leq n \in \mathbb{N}, \quad (11)$$

has non trivial integer solutions if and only if $n \in \{1, 2\}$. It is common knowledge that the proof of this result (achieved in 1994 by Andrew Wiles) is really profound and difficult. On the other hand, let us see that the same result for polynomials is relatively easy to be proved. Before stating it properly we need to define what is a *trivial* polynomial solution of (11). For any polynomial $p(t)$, any $n \in \mathbb{N}$, and any $a, b \in \mathbb{C}$, observe that

$$x = ap(t), \quad y = bp(t) \quad \text{and} \quad z = (a^n + b^n)^{1/n}p(t),$$

are solutions of (11) but not interesting at all. We will say that a solution is trivial if the corresponding polynomials are not coprime.

We will need the following lemma.

Lemma 12 *The algebraic curve $x^n + y^n - 1 = 0$ is irreducible and has genus $(n-1)(n-2)/2$.*

Its proof is a direct consequence of the discussion made in the beginning of this section because the surface $x^n + y^n - 1 = 0$ has no singularities.

The main result of this subsection is the following proposition, which is already known. The proof that we present is the one of [16, Thm. 1.2].

Proposition 13 *The equation $x^n + y^n = z^n$, $1 \leq n \in \mathbb{N}$ has non trivial one-variable polynomial solutions if and only if $n \in \{1, 2\}$.*

Proof Assume that

$$x = p(t), \quad y = q(t), \quad \text{and} \quad z = r(t)$$

are non trivial solutions of $x^n + y^n = z^n$. Then

$$p^n(t) + q^n(t) = r^n(t) \implies \left(\frac{p(t)}{r(t)}\right)^n + \left(\frac{q(t)}{r(t)}\right)^n = 1,$$

being the involved rational functions not identically constant. Hence the algebraic curve $x^n + y^n - 1 = 0$ can be rationally parametrized and due to Cayley–Riemann’s theorem its genus must be 0. By Lemma 12 its genus is $(n-1)(n-2)/2$. Hence, it is 0 only when $n \in \{1, 2\}$. As a consequence, non trivial solutions only could exist for these two cases. To finish the proof it suffices to see that indeed they do exist.

When $n = 1$, obviously the equation $x + y = z$ has many non trivial solutions. For instance, we can take

$$x = p(t), \quad y = q(t) \quad \text{and} \quad z = p(t) + q(t).$$

For $n = 2$, the equation $x^2 + y^2 = z^2$ has for instance the non trivial solution

$$x = 1 - t^2, \quad y = 2t \quad \text{and} \quad z = 1 + t^2.$$

□

In fact, it is also known from the classical works of Montel [55, 56] that the equation $x^n + y^n = z^n$, for $n > 2$, has no non trivial solutions given by entire functions. See also the popularizing paper [42].

Notice that for $n = 2$, apart of the polynomial non trivial solution given in the above proof, the triplet $(x, y, z) = (\sin t, \cos t, 1)$ is another non trivial entire solution.

3.11.3 Local Determination of an Invariant Manifold

Although we will not use resultants in this subsection we include an example, extracted from [31], of the advantages of rational parameterizations to simplify the involved expressions when we compute the local expansion of an invariant manifold. Many more examples of this method and the use of resultants in dynamical systems are developed in [34]. For instance, in that paper we study some Abelian integrals or some central configurations.

To study the travelling wave solutions of Fisher–Kolmogorov equation it is interesting to know the local and global expression of the invariant manifolds $y = h(x)$ of the saddle point $(0, 0)$ of the planar system

$$\begin{cases} x' = -y, \\ y' = cy + x(x-1), \end{cases}$$

where $c > 0$ is a real parameter, see for instance [31].

The condition for $y - h(x) = 0$ to be invariant is

$$y' - h'(x)x'|_{y=h(x)} \equiv 0.$$

Equivalently,

$$cy + x(x-1) + h'(x)y|_{y=h(x)} = (c + h'(x))h(x) - x + x^2 \equiv 0.$$

From the above equation it is easy to obtain the local expressions of the invariant manifolds of the saddle. From $(c + h'(x))h(x) - x + x^2 \equiv 0$, writing $h(x) = h_1x + h_2x^2 + O(x^3)$, we get that

$$(c + h_1 + 2h_2x + O(x^2))(h_1x + h_2x^2 + O(x^3)) - x + x^2 \equiv 0.$$

The condition on the degree 1 terms on x gives $h_1^2 + ch_1 - 1 = 0$. Hence

$$h_1 = \frac{-c \pm \sqrt{c^2 + 4}}{2}.$$

Similarly, by equating the degree 2 terms on x

$$(c + h_1)h_2 + 2h_2h_1 + 1 = (c + 3h_1)h_2 + 1 = 0 \implies h_2 = \frac{-1}{c + 3h_1},$$

and hence we obtain h_2 from h_1 . In general, it can be readily seen that for $n \geq 2$, h_{n+1} can be obtained from h_n .

To simplify all the expressions of the invariant manifolds of the saddle point we will rename the parameter c .

To eliminate the square root in h_1 we impose that $c^2 + 4 = d^2$, arriving to the genus zero curve $d^2 - c^2 = 4$. Writing $d^2 - c^2 = (d + c)(d - c) = 4$ and by introducing r as $d + c = 2r$ and $d - c = 2/r$, we get the rational parametrization

$$c = r - 1/r, \quad d = r + 1/r.$$

Hence, by writing c as $c = r - 1/r$ it holds that

$$h_1^2 + ch_1 - 1 = h_1^2 + (r - 1/r)h_1 - 1 = (h_1 + r)(h_1 - 1/r) = 0.$$

As consequence the unstable (resp. stable) invariant manifold of the saddle is $y = -rx + O(x^2)$ (resp. $y = -x/r + O(x^2)$).

In terms of r , the expression of the manifolds is much more compact and easy to be obtained with a computer algebra system. For instance

$$h_2 = \frac{-1}{c + 3h_1} = \frac{-1}{(r - 1/r) + 3(-r)} = \frac{r}{2r^2 + 1},$$

and the first five terms of the unstable manifold of the saddle are:

$$\begin{aligned} y = h^r(x) = & -rx + \frac{r}{2r^2 + 1}x^2 + \frac{2r^3}{(2r^2 + 1)^2(3r^2 + 1)}x^3 \\ & + \frac{10r^5}{(2r^2 + 1)^3(3r^2 + 1)(4r^2 + 1)}x^4 \\ & + \frac{12r^7(19r^2 + 6)}{(2r^2 + 1)^4(3r^2 + 1)^2(4r^2 + 1)(5r^2 + 1)}x^5 + O(x^6). \end{aligned}$$

It is interesting to remark that in the celebrated paper of Fisher [21] the change $c = r - 1/r$ was already introduced.

4 Applications in Dynamical Systems

This section collects several applications of resultants to continuous and discrete dynamical systems given by polynomial or rational functions.

4.1 Genericity of Hyperbolic Equilibrium Points

I have decided to start this section with a nice result of Jorge Sotomayor that in fact was the first work where I saw resultants, see [63, Chp. II.4]. The result of that section entitled “Genericity” that I have chosen is Proposition 4.1. It reads as

Proposition 14 *Let $\mathcal{X}_n$ be the set of planar polynomial vector fields of degree n . Let $\mathcal{Y}_n$ be the subfamily of them such that all their equilibrium points in the Poincaré compactification are hyperbolic. Set $\mathcal{Z}_n = \mathcal{X}_n \setminus \mathcal{Y}_n$. Then $\mathcal{Z}_n$ is contained into an affine algebraic hypersurface of $\mathcal{X}_n$. More precisely, there exists a polynomial $P : \mathcal{X}_n \rightarrow \mathbb{R}$, $P \neq 0$, such that $\mathcal{Z}_n \subset P^{-1}(0)$.*

To prove the above result, consider the planar polynomial vector field $X = X(x, y) = (p(x, y), q(x, y))$. To impose that a (finite) equilibrium point is hyperbolic we consider the new function

$$r(x, y) = \det(DX)(x, y) = p_x(x, y)q_y(x, y) - p_y(x, y)q_x(x, y).$$

Then the planar systems such that some finite equilibrium point is non hyperbolic are such that the following system with three equations has some (real) solution

$$p(x, y) = 0, \quad q(x, y) = 0, \quad r(x, y) = 0.$$

From successive elimination we can obtain the polynomial

$$P_1(\mathbf{c}) = \text{Res}_y (\text{Res}_x(p, r), \text{Res}_x(q, r)),$$

where $\mathbf{c}$ denotes all the coefficients of p and q . Then, clearly if all finite equilibrium points are hyperbolic then the system has no solution and $P_1(\mathbf{c}) \neq 0$. In the same way, we introduce a new function by changing r by the divergence, $s(x, y) = p_x(x, y) + q_y(x, y)$, giving rise to a new hyperbolicity condition $P_2(\mathbf{c}) \neq 0$. Recall that a finite equilibrium point is hyperbolic if and only if $r(x, y)s(x, y) \neq 0$. Then, when for a planar vector field $P_1(\mathbf{c})P_2(\mathbf{c}) \neq 0$ all its finite equilibrium points are hyperbolic.

Two similar polynomials $P_3(\mathbf{c})$ and $P_4(\mathbf{c})$ can be constructed to control the hyperbolicity of the equilibrium points at infinite of the Poincaré compactification. Finally, the polynomial P appearing in the statement is $P_1P_2P_3P_4$.

This is a key result in the path chosen by him to prove the result of Poincaré that asserts that, generically, polynomial vector fields have finitely many limit cycles, see again that Sotomayor's chapter for more details.

4.2 Period 3 for the Logistic Map

Consider the celebrated logistic map $f_r(x) = rx(1 - x)$, $r > 0$. Its dynamics are well understood. In particular, thanks for instance to Sharkovskii's theorem, it is known that the starting of the complicated dynamics coincides with the born of a 3-periodic orbit. According [65, Prob. 10.4.10] the search for this bifurcation value is a hard exercise, see also also [60]. Let us see how by using resultants it is a routine task.

The main point is to realize that this 3-periodic orbit must appear for the values of r such that the degree seven polynomial equation

$$\begin{aligned} P_r(x) = \frac{f_r(f_r(f_r(x))) - x}{x} = & -r^7x^7 + 4r^7x^6 - 2r^6(3r+1)x^5 + 2r^6(2r+3)x^4 \\ & - r^4(r^3 + 6r^2 + r + 1)x^3 + 2r^4(r^2 + r + 1)x^2 \\ & - r^3(r^2 + r + 1)x + (r-1)(r^2 + r + 1) = 0 \end{aligned}$$

has a multiple root. These values are detected by vanishing the discriminant. Some computations give that

$$\Delta_x(P_r(x)) = r^{49} (r^2 + r + 1)^2 (r^2 - 2r - 7)^3 (r^2 - 5r + 7)^4 = 0.$$

The only positive root of the above equation is $1 + 2\sqrt{2} \approx 3.8284$, which is the searched value.

4.3 Problems that Can Be Reduced to Solve Polynomial Systems

There are many situations when studying polynomial differential equations that can be reduced to study the solutions of a system of polynomial equations. For them, it is

clear that the use of resultants is a very convenient tool. Without aiming to be exhaustive let us list some of them:

4.3.1 Number of Equilibrium Points of a Differential System

This is the most obvious application and can be used for polynomial systems in any dimension, real or complex. In particular, for the bidimensional case, Bézout's theorem gives only two alternative for the number of equilibrium points: either the system has a continuum of equilibrium points, or it has finitely many, see Proposition 8. Moreover, in this last case the maximum number is given by the product of the degrees of the components of the vector field.

Furthermore, more characteristics of the points can be taken into account, like in Sect. 4.1. For instance, points that are on the divergence zero curve are called *weak* equilibrium points. If one considers a planar polynomial differential equation $\dot{x} = p(x, y)$, $\dot{y} = q(x, y)$, its weak equilibrium points are solutions of the polynomial system

$$p(x, y) = 0, \quad q(x, y) = 0, \quad r(x, y) = p_x(x, y) + q_y(x, y) = 0.$$

In particular notice that all the centers of the system are included in this set of solutions. Of course, among these solutions some of them can be non centers, like for instance all weak saddles.

Clearly, similar questions can be addressed to study fixed or periodic points of rational discrete dynamical systems, see for instance [35].

4.3.2 Center-Focus Problem for Nondegenerated Equilibrium Points

Any planar polynomial differential system with a nondegenerated monodromic equilibrium point can be written by means of affine changes as

$$\dot{x} = -y + \lambda x + p(x, y), \quad \dot{y} = x + \lambda y + q(x, y), \quad (12)$$

where $p, q \in \mathbb{R}[x, y]$ start with terms of degree at least 2 and have some given degree n . The center-focus problem consists on distinguishing between the center or focus cases. A first necessary condition to have a center is that $\lambda = 0$. In this case, from the works of Poincaré and Lyapunov [6], together with the Hilbert's basis theorem, it is known that there are finitely many polynomial equations $L_j(\mathbf{c}) = 0$, $j = 2, 3, \dots, k$, whose real solutions characterize the centers. Here $\mathbf{c}$ denotes the set of all coefficients of p and q , the value $k = k(n)$ is known to exist but in general its precise value is totally unknown, and these expressions are called *Lyapunov constants*, see for instance [20, Chp. 4]. This problem remains open even for $n = 3$, and also for many natural families of systems due to the huge complexity of solving the associated system of equations, see [58].

4.3.3 Isochronicity for Nondegenerated Equilibrium Points

Recall that a center of a planar differential equation is called *isochronous* if all its periodic solutions have the same period. Given a family of planar polynomial system of the form (12), but with $\lambda = 0$, the characterization of their isochronous centers is given by a system of some polynomial equations

$$L_j(\mathbf{c}) = 0, j = 2, 3, \dots, k, \quad \text{and} \quad T_j(\mathbf{c}) = 0, j = 1, 2, \dots, \ell,$$

where $L_j(\mathbf{c})$ are as in the previous subsection, $T_j(\mathbf{c})$ are also polynomials in $\mathbf{c}$ and ℓ is finite but also usually unknown. These new polynomials are called *period constants*, see again [58]. For a survey on isochronous centers, see [9]. This problem has similar difficulties than the center-focus problem.

4.3.4 Invariant Algebraic Curves

Given a planar polynomial differential system $\dot{x} = p(x, y)$, $\dot{y} = q(x, y)$ of degree n , recall that it is said that it has an *invariant algebraic curve* $f(x, y) = 0$, with $f \in \mathbb{R}[x, y]$, if there exists a polynomial of degree at most $n - 1$, $k(x, y)$, called its *cofactor*, such that

$$f_x(x, y)p(x, y) + f_y(x, y)q(x, y) = k(x, y)f(x, y).$$

It is clear that the set $\{f(x, y) = 0\} \cap \mathbb{R}^2$ is invariant by the flow of the differential equation.

If in the above equation we fix the degree of f , say by m , then k has at most degree $n - 1$. Therefore, to impose the above condition provides a polynomial system being the unknowns the coefficients of f and k . Usually, it is a very difficult system to be treated directly. See for instance [20, Chp. 8] for more details and for some illustration of the use of these invariant algebraic curves to Darboux's theory of integrability.

From Jouanolou's results [44] it follows that for a given fixed polynomial system the maximum degree of its irreducible invariant algebraic curves is bounded. Thus for each polynomial system there is a natural number m which bounds the degree of all its irreducible invariant algebraic curves. A first natural open question, that goes back to Poincaré, is to give an effective procedure to obtain this value m .

Moreover, it is also well known that given a family of planar polynomial systems of a fixed degree n there is no upper bound for the possible m . For instance, by taking the linear systems $\dot{x} = x$, $\dot{y} = my$, $m \in \mathbb{N}$, (thus $n = 1$), they have the invariant algebraic curves of degree m , $y = cx^m$ for all $c \in \mathbb{R}$. Another question proposed by Poincaré in 1891 was to know if it is true that when a planar system of degree n has an invariant algebraic curve of degree higher than some $M(n)$ then this system is rationally integrable. At the beginning of this century several authors gave a negative answer to Poincaré's question. Nowadays people are trying to know if his question has an affirmative answer changing rationally integrable by having what is called a *Darboux invariant*. To see more details and references on this subject see [13, 51, 52].

4.3.5 Global Periodicity and Involutions

Given a rational map $f : \mathbb{C}^m \rightarrow \mathbb{C}^m$ we will say that it is *globally periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) \equiv x$, for all x where the composition is well defined. In this section f^n denotes the composition of f , n times. The minimal n with this property is called the *period* of f . When $n = 2$, then f is called an *involution*. Some examples of simple globally periodic maps with their corresponding m and n are:

- $f(x) = a - x$, with $m = 1$ and $n = 2$,
- $f(x, y) = (y/(1 + x + y), -x/(1 + x + y))$, with $m = 2$ and $n = 4$,
- $f(x, y) = (y, (1 + y)/x)$, with $m = 2$ and $n = 5$,
- $f(x, y, z) = (y, z, (1 + y + z)/x)$, with $m = 3$ and $n = 8$.

Given a value n and a map f of a given degree, it is clear that the condition $f^n(x) \equiv x$, provides a finite set of polynomial equations with unknowns the coefficients of the numerator and denominator of the map. The most difficult part is to determine a priori which are the possible value of n . For more details and explicit examples of application, see [15] and its references.

4.3.6 Central Configurations

The central configurations are special and important solutions studied in Celestial Mechanics. Their existence is reduced to the existence of real solutions of an associated system of equations, that in general is not polynomial, but involves polynomials and square roots of polynomials (essentially due to the distances among the bodies). In several cases, these systems can be transformed into other equivalent polynomial systems, by using the rational parametrizations, introduced in Sect. 3. To see some concrete examples look at [3, 34] and their references.

4.4 Problems that Can Be Reduced to Control the Sign of a Polynomial

That a given polynomial in one variable, without parameters, does not change sign on a given interval can be studied for instance by computing its Sturm's sequence and by checking the signs of this sequence evaluated at the extrema of the interval. For polynomials depending on one parameter the situation is more complicated but results like next lemma help to study the problem.

Lemma 15 *Let $P_a(x) = p_n(a)x^n + p_{n-1}(a)x^{n-1} + \dots + p_1(a)x + p_0(a)$, be a family of real polynomials depending also polynomially on a real parameter a and set $x \in \Omega = \mathbb{R}$. Suppose that there exists an open interval $I \subset \mathbb{R}$ such that:*

- (i) *There is some $a_0 \in I$, such that $P_{a_0}(x) > 0$ on Ω .*
- (ii) *For all $p \in I$, $\Delta_x(P_a) \neq 0$.*
- (iii) *For all $a \in I$, $p_n(a) \neq 0$.*

Then for all $a \in I$, $P_a(x) > 0$ on Ω . Moreover, if $\Omega = \Omega_a = (c(a), \infty)$ for some polynomial function $c(a)$, the same result holds changing Ω by this new Ω_a if we add the additional hypothesis:

(iv) For all $a \in I$, $P_a(c(a)) \neq 0$.

The key point of its proof is that the roots (real and complex) of P_a depend continuously of a , because $p_n(a) \neq 0$. Moreover, hypotheses (iii) and (iv) prevent that moving a some root enters in Ω either from infinity or from the boundary of Ω , respectively.

On the other hand if moving a some real root appears from $\mathbb{C}$, it does appear trough a multiple real root that is detected by the vanishing of $\Delta_x(P_a)$. Since by item (ii), $\Delta_x(P_a) \neq 0$ no real root does appear either in this way. Hence, for all $a \in I$, the number of real roots of any P_a is the same. Since by item (i) for $a = a_0$, $P_{a_0} > 0$ on Ω , the same holds for all $a \in I$.

An equivalent result can be proved when $\Omega = \Omega_a = (c(a), d(a))$. Also a similar result holds for proving that $P_a(x)$ has exactly K simple real roots when $a \in I$ and $x \in \Omega$ if, in item (i), $P_{a_0}(x)$ has K simple real roots in Ω .

As an example of application of the lemma we detail the computations of the result stated at the end of Sect. 3.6. Recall that there we were interested to study the number of positive solutions of

$$p(x, y, a) = x^6 + ay^3 - y = 0, \quad q(x, y) = p(y, x, a) = y^6 + ax^3 - x = 0,$$

with a fixed value of a . Here we consider a as a parameter. First we compute $\text{Res}_y(p, q) = x(x^5 + ax^2 - 1)P_a(x)$, where P_a is a polynomial of degree 30.

By Lemma 15, the values of a where the number of solutions can change are given by the zeroes of

$$Q(a) = \Delta_x(\text{Res}_y(p, q)) = \Delta_x(x(x^5 + ax^2 - 1)P_a(x)).$$

After many computations, we obtain that

$$Q(a) = -(108a^5 - 3125)(2916a^5 - 16807)^3(24a^5 - 1)^4 R_{35}^2(a) S_{35}^2(a),$$

where R_{35} and S_{35} are two polynomials of degree 35 that we skip for the sake of shortness. The value $\underline{a}$ given in Sect. 3.6 is a root of one of these two polynomials and the value $\bar{a}$ is a root of the second factor of Q .

Polynomials depending on more variables and parameters are much more difficult to be controlled. See for instance some results for an one parameter family of polynomials in two variables in [24, App. II].

We continue this section we give list of several situations where the sign of a function gives dynamical properties. Most of them refer to planar differential equations

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y). \quad (13)$$

Results similar to the above lemma and its extensions are applied to study the described problems.

4.4.1 Curves Without Contact

If a curve $C(x, y) = 0$, with $C : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that

$$\dot{C}|_{C=0} = (C_x p + C_y q)|_{C=0}$$

does not vanish, then it is called a *curve without contact* for the flow given by (13).

Some examples of application of their existence can be seen in [30, 41]. In particular, in these papers the authors construct positively (or negatively) Poincaré annular regions, and as a consequence, they deduce the existence and location of some limit cycles.

4.4.2 Quasi Lyapunov Functions

If there exists a smooth function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\dot{V} = V_x p + V_y q$$

does not change sign, then the differential equation (13) does not have periodic orbits, see for instance [53]. The proof of this assertion is straightforward, because the hypothesis implies that the function $t \rightarrow V(x(t), y(t))$, where $(x(t), y(t))$ is any solution of the system, is strictly monotonous.

Notice that this result is not exactly the same that saying that V is a Lyapunov function. We have removed some of the hypotheses on the classical result and obtain less information, because only the non existence of periodic orbits is asserted.

For instance, this idea is applied to some families of planar systems having an algebraic invariant curve, by using this curve and its cofactor to construct a suitable function V , see [29, Thms. 1.2, 2.1]. To find conditions on $V_x p + V_y q$ to ensure that it does not change sign is not easy at all. If in some situations we succeed in obtaining that $V_x p + V_y q$ is a function of a single variable, the condition can be achieved by forcing that this function has double roots, and this is done by imposing that some associated discriminants vanish. When the function keeps the two variables similar ideas can also be used, see again [29, Thm. 1.2].

4.4.3 Dulac Functions

In few words, Bendixson–Dulac’s theorem asserts that if there exist a smooth function $B : \mathbb{R}^2 \rightarrow \mathbb{R}$, usually called Dulac function, and $s \in \mathbb{R}^+$ such that

$$\operatorname{div}(Bp, Bq) = B_x p + B_y q - s \operatorname{div}(p, q)B$$

does not change sign, then the shape of the set $\mathbb{R}^2 \setminus \{(x, y) : B(x, y) = 0\}$ controls the maximum number of limit cycles of the differential equation (13). For more details and several examples of application, see [27, 28].

4.4.4 Properties of the Period Function

Given a nondegenerated center, a problem of current interest is the study of the monotonicity or number of oscillations of the so called *period function*. Recall that this function, $T(s)$, is the one that assigns to each periodic orbit its period, where s is a real parameter that determines the orbit.

As an example of result for proving the monotonicity of $T(s)$ we state a criterion of Schaaf [61]: Consider a smooth potential differential equation $\dot{x} = -y$, $\dot{y} = v'(x)$, with a center at the origin and let I be the intersection of the set of all its period annuli with the Ox -axis. Then the center has a monotonous increasing period function in the case that

$$5v'''(x)^2 - 3v''(x)v^{(4)}(x) > 0,$$

for all $x \in I$ with $v''(x) > 0$, and $v'(x)v'''(x) < 0$ for any $x \in I$ with $v''(x) = 0$. On the other hand, in the case that $5v'''(x)^2 - 3v''(x)v^{(4)}(x) < 0$ for any $x \in I$ with $v''(x) \geq 0$ then the period function is monotonous decreasing. For a detailed application of this criterion to reversible quadratic centers, see [67].

4.4.5 Chebyshev Systems

Given $m + 1$ real, analytic and linearly independent functions $\mathcal{F} = \{f_0(x), f_1(x), \dots, f_m(x)\}$, defined on some open interval I , a natural problem is to study the number of real zeroes of any non-zero function $F \in \text{Span } \mathcal{F}$,

$$F(x) = \sum_{j=0}^m \lambda_j f_j(x), \quad \lambda_j \in \mathbb{R}.$$

Let us denote by $\mathcal{Z}(F)$ the number of zeroes in I , counted with multiplicities, of a function F and by

$$\mathcal{Z}(\mathcal{F}) = \max_{F \in \mathcal{F} \setminus \{0\}} \mathcal{Z}(F),$$

whenever they exist. It is not difficult to see that $\mathcal{Z}(\mathcal{F}) \geq m$.

A set of functions $\mathcal{F}$ for which $\mathcal{Z}(\mathcal{F}) = m$ is usually called an *Extended complete Chebyshev system* on I and denoted in short as an ECT-system. The “T” stands for Tchebycheff, which is one of the transcriptions of the name of the Russian mathematician.

The set of polynomials of degree m , $\{1, x, x^2, \dots, x^m\}$, on any open interval is a well-known example. Other nice examples are

$$\begin{aligned} &\{1, \log x, x, x \log x, x^2, x^2 \log x, \dots, x^n, x^n \log x\} \text{ on } (0, \infty); \\ &\{1, \cos x, \cos(2x), \dots, \cos(mx)\} \text{ on } (0, \pi). \end{aligned}$$

For general properties of ECT-systems, see for instance [45]. A very useful and well-known characterization of ECT-systems is the following:

Theorem 16 *The set $\{f_0, \dots, f_m\}$ is an ECT-system on I if and only if for all $k = 0, 1, \dots, m$, $W[f_0, \dots, f_k](x) \neq 0$ for all $x \in I$, where*

$$W[f_0, \dots, f_k](x) \doteq \det \left(f_j^{(i)}(x) \right)_{0 \leq i, j \leq k}$$

is the Wronskian of $(f_0, \dots, f_k)$ at $x \in I$.

A common situation where this problem appears is when people study the number of zeroes of the Melnikov (or Abelian) integrals. They control the number of limit cycles that bifurcate by perturbing Hamiltonian or integrable planar systems. See [12, 33] for some examples of application.

4.5 Centers in Two Families of Differential Equations

In this section we will study the centers in two different families of differential equations: the planar Liénard systems and the Abel differential equations. As we will see, the two situations present big similarities.

4.5.1 Polynomial Liénard Systems

We characterize the centers of the celebrated family of Liénard systems in terms of resultants. These systems can be written as

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \quad (14)$$

where $g(0) = 0, g'(0) = 0, F(0) = 0$ and F and g are analytic functions. We also introduce the function

$$G(x) = \int_0^x g(s) \, ds.$$

The following result due to Cherkas, and also proved by Christopher, characterizes when the origin is a center, see [10, 11].

Theorem 17 *The system (14) has a center at the origin if and only if there exist analytic functions F_1, G_1 and z satisfying*

$$F(x) = F_1(z(x)), \quad G(x) = G_1(z(x)), \quad (z(0) = 0, z'(0) < 0). \quad (15)$$

When F and G are polynomials, the function $z(x)$ must correspond to a common factor between the functions $F(x) - F(y)$ and $G(x) - G(y)$ other than $x - y$. We introduce the polynomial map

$$\Phi(x, y) = \left(\frac{F(x) - F(y)}{x - y}, \frac{G(x) - G(y)}{x - y} \right).$$

Then, the following nice corollary is also given in [11]:

Corollary 18 *If system (14) with F and g polynomials has a center at the origin then $\text{Res}_x(\Phi(x, y)) \equiv \text{Res}_y(\Phi(x, y)) \equiv 0$. This condition is also sufficient if the common factor of both polynomials vanishes at $x = z = 0$.*

It is remarkable that (14) is one of the few families where centers can be characterized in an elegant and compact way.

In [32] there is a version of the above results when the functions involved in the Liénard system are trigonometric polynomials.

In fact, the function Φ has much more dynamical information. It is proved in [14, 37] that its multiplicity at the origin, divided by two, gives the order of the weak focus at the origin of the Liénard system. In consequence, it provides information of the number of limit cycles that can bifurcate from it.

4.5.2 Abel Differential Equations

Consider the following family of periodic Abel differential equations

$$y' = a(x)y^3 + b(x)y^2, \quad (16)$$

where $y = y(x)$, and a and b are smooth and T -periodic. Clearly, $y = 0$ is one of its solutions, and by continuity with respect to initial conditions, the solutions $y(x, y_0)$ starting at $y = y_0$, when $x = 0$, are defined at $x = T$, when $|y_0|$ is small enough.

The Eq. (16) is said to have a center (at $y = 0$) if $y(x, y_0) \equiv y_0$, for $|y_0|$ small enough. These equations are said to have a center because they have a continuum of periodic orbits when the differential equation is thought on the cylinder. To see more properties of Abel equations and some relations with planar polynomial vector fields, see [26] and its references.

For Eq. (16) the center problem is not solved. Nevertheless there is a nice family of centers, introduced in [5], the *composition centers*. To define them, we will use the following functions:

$$A(x) = \int_0^x a(s) \, ds, \quad B(x) = \int_0^x b(s) \, ds.$$

We will say that the couple A, B satisfies the composition conditions if there exist smooth functions A_1 and B_1 , and a T -periodic smooth map $z(x)$ such that

$$A(x) = A_1(z(x)), \quad B(x) = B_1(z(x)). \quad (17)$$

Notice the similarities between (17) and (15).

Lemma 19 Consider differential equations (16) and assume that their associated functions A and B satisfy the composition conditions (17). Then $y = 0$ is a center for (16).

Proof Consider the new Abel differential equation

$$Y' = A_1'(u)Y^3 + B_1'(u)Y^2.$$

Let $Y = Y(u)$ be any of its solutions in a neighborhood of $Y = 0$. Then $y(x) = Y(z(x))$ is a solution of the initial Abel equation because

$$\begin{aligned} y'(x) &= Y'(z(x))z'(x) = A_1'(z(x))z'(x)Y^3(z(x)) + B_1'(z(x))z'(x)Y^2(z(x)) \\ &= (A_1(z(x)))'y^3(x) + (B_1(z(x)))'y^2(x) \\ &= A'(x)y^3(x) + B'(x)y^2(x) = a(x)y^3(x) + b(x)y^2(x). \end{aligned}$$

Clearly all these solutions are T -periodic, because $y(T) = Y(z(T)) = Y(z(0)) = y(0)$ and the result follows. $\square$

The centers described in this lemma are the so called composition centers. At the very beginning people thought that maybe all centers of Abel equations would be of composition type, especially when the two periodic functions appearing in (16) were trigonometric polynomials. Quickly, it was proved that this is not the case. In [4] there is the first counterexample.

Without entering into computations the main points in the construction of that counterexample were the following:

- (i) Take any quadratic system with a center at the origin. By the so called Cherkas' transformation it gives rise to an Abel equation (16), with coefficients a and b being trigonometrical polynomial and a center, see [4, 26].
- (ii) Notice also that if the Abel equation was of composition type, then

$$a(x) = A'(x) = A_1'(z(x))z'(x), \quad b(x) = B'(x) = B_1'(z(x))z'(x).$$

Hence, in particular, both function $a(x)$ and $b(x)$ have all zeroes of $z'(x)$ as common zeroes. Moreover, since $z(x)$ is T -periodic, it has at least one real zero. In short, the system $a(x) = 0$, $b(x) = 0$, should have a real solution.

- (iii) Observe that both trigonometrical polynomials a and b are homogeneous. Indeed their respective degrees are 6 and 3. Therefore, they can be seen as polynomials in one variable, say $t = \tan(x)$. Then, from the system $a(x) = 0$, $b(x) = 0$, we obtain an equivalent one, $\alpha(t) = 0$, $\beta(t) = 0$, which is polynomial, with polynomials of respective degrees 6 and 3.
- (iv) Finally, if it holds that $\text{Res}_t(\alpha, \beta) \neq 0$, we have proved that the initial center for the Abel equation was not a composition center. This is what happens in the example presented in [4].

4.6 Common Solutions of Two Polynomial Differential Equations

In this section we describe a method to find conditions for two families of polynomial differential systems

$$(a) \dot{x} = p_1(x, y), \dot{y} = q_1(x, y), \quad (b) \dot{x} = p_2(x, y), \dot{y} = q_2(x, y), \quad (18)$$

to share a non trivial solution. This method is already presented in [25].

It is clear that if the systems share a non trivial solution, this curve has to be contained in the region where both vector fields are parallel. This region is given by the algebraic set $H(x, y) = 0$ where $H(x, y) = q_1(x, y)p_2(x, y) - p_1(x, y)q_2(x, y)$. Notice that, in particular, this already implies that common solutions must be algebraic.

Next condition is that the set $H(x, y) = 0$ contains actual solutions of both systems. To impose this we introduce two new polynomials

$$J_i(x, y) = H_x(x, y)p_i(x, y) + H_y(x, y)q_i(x, y), \quad i = 1, 2,$$

and force that $J_i(x, y) = 0, i = 1, 2$, on this set.

In short, we have to find conditions on the parameters of systems (18) that force that the next three equations are compatible and have a continuum of solutions.

$$H(x, y) = 0, \quad J_1(x, y) = 0, \quad J_2(x, y) = 0.$$

These conditions are obtained by imposing that $G_i(y) = \text{Res}_x(H, J_i) \equiv 0$ for $i = 1, 2$. For each i , this leads to an algebraic system constructed by imposing that all the coefficients of G_i vanish.

Let us illustrate our approach with an example. Consider

$$\begin{aligned} p_1(x, y) &= axy^2 + 5y^3 - x^2 - 5xy + 5, & q_1(x, y) &= 5y^3 - 4xy + b, \\ p_2(x, y) &= cx^2y^2 + xy^4 + xy + 3y^2 - x, & q_2(x, y) &= xy^4 - x^2y^2 + xy + y. \end{aligned}$$

We want to find conditions on the parameters a, b , and c in order that the correspondent differential systems (18) have a common solution.

The computations are huge and we skip the details. We only comment that $G_2(y) = y^9 P_9(y, c) P_{41}(y, a, b, c)$, where the subscript indicates the degree of the polynomial with respect to y . The polynomial P_9 can not be identically zero. By forcing that $P_{41} = 0$ we obtain a single solution $a = 3, b = 5$, and $c = -1$. For instance, the equation that forces the coefficient of degree 41 in y of P_{41} to be zero is

$$(a - 3)(a - 1)^2 a^4 = 0.$$

Once we have fixed these 3 values of the parameters we obtain that

$$H(x, y) = (y^3 - xy + 1)(-3y^2 + x + y)(x^2y - 5).$$

Each of the above three factors, equated to zero, gives a candidate to be a common solution of both differential systems. Direct computations show that the only common solution is the algebraic invariant curve $y^3 - xy + 1 = 0$, given by the first factor.

We finish with some comments addressed to contextualize the question that we have considered in this section. They are based on the results of Khovanskii [47, 48]. See also the preprint [59]. Consider the following two types of planar real curves:

- Type I: algebraic curves $r(x, y) = 0$, with $r \in \mathbb{R}[x, y]$, and degree ∂r .
- Type II: non constant solutions of a real planar polynomial differential system $\dot{x} = p(x, y)$, $\dot{y} = q(x, y)$, where $p, q \in \mathbb{R}[x, y]$. The degree of this polynomial vector field, $X = (p, q)$, is $\partial X = \max(\partial p, \partial q)$.

A first natural question is how many intersection points have two elements of class I. As we have already explained in Sect. 3.5 the answer is well known and it is a consequence of Bézout's theorem. Recall that if the curves are $r_1 = 0$ and $r_2 = 0$ then either they coincide on a continuum of points or this number is bounded by $\partial r_1 \partial r_2$.

A second question is what happens when one takes one element of each class. It is clear that one possibility is that they have a continuum of intersections. It suffices for instance to take $r = r_1 r_2 = 0$, where $r_1 = 0$ is an invariant algebraic curve of a planar polynomial system that gives the curve of type II.

It is also easy to construct an example for which both elements cut at infinitely many points and these intersection points form a numerable set. Consider for instance $r(x, y) = x = 0$, and $(p(x, y), q(x, y)) = (-y - x, x - y)$. This vector field has an attracting focus at the origin and our assertion is clear.

Nevertheless, there is a Bézout's type result when the solutions of type II are much restricted. Instead of considering all solutions of the polynomial differential system, we will consider only the so called *separating solutions*. In few words, separating solutions are a particular finite set of solutions of type II such that, if we consider the solutions parametrized by the time of the differential system, they form the boundary of an oriented domain in $\mathbb{R}^2$. For instance, a solution tending to a focus or to a limit cycle is not a separating solution, but the limit cycle itself is a separating solution. These especial solutions of the differential system are also called *Pfaff functions*. For more details, see again [47, 48, 59].

The first result is that if we consider a function of type I and a separating solution then either they have a continuum of common points or that the maximum number of intersection points among them is $(\partial X + \partial r)\partial r$. For more details, see [47, 48].

Similarly, if we consider two separating solutions, associated to vector fields X and Y , with $\partial X \leq \partial Y$, and they cut at isolated points, then the maximum number of points is $(\partial X + \partial Y)(2\partial X + \partial Y) + \partial X + 1$. For more details see again [47, 48].

From this point of view, the problem considered in this section can be thought as the characterization of the case where two functions of type II coincide on a continuum of points.

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