



The Global Injectivity of C^2 Maps in $\mathbb{R}^2$

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Abstract

Let F be a C^2 map from the real plane into itself whose Jacobian determinant is nonzero at every point. By the Inverse Function Theorem, such a map is locally injective. A classical and challenging problem is to determine conditions under which F is globally injective. In this work, using results from the qualitative theory of differential equations, we provide both a necessary and sufficient condition and a sufficient condition for the global injectivity of F .

Keywords Global injectivity · Markus-Yamabe conjecture · Parallel vector fields

Mathematics Subject Classification Primary: 34A26 · Secondary: 26B10 · 14R15

1 Introduction and Statement of the Main Results

In this article we study C^2 maps $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that its Jacobian, defined by

$$JF(z) = \det DF(z) = f_x(z)g_y(z) - f_y(z)g_x(z), \quad z = (x, y) \in \mathbb{R}^2,$$

never vanishes. By the Inverse Function Theorem, this hypothesis is sufficient to guarantee a local conclusion: F is locally injective. As is well known, however, local injectivity does not, in general, imply global injectivity. Therefore, it is natural to seek additional conditions that guarantee the global injectivity of F , thereby making it a diffeomorphism onto its image.

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In the polynomial setting, under the assumption that $JF(z) \neq 0$ for all $z \in \mathbb{R}^2$, the statement that F is injective –and therefore surjective, see (Bialynicki-Birula and Rosenlicht 1962)– is known as the *real Jacobian conjecture* (RJC), or sometimes the *strong real Jacobian conjecture*. According to Pinchuk (1994), this conjecture is false, implying that such a polynomial map F is not, in general, a global diffeomorphism.

Without loss of generality we will assume from now on that

$$JF(z) < 0, \quad \forall z = (x, y) \in \mathbb{R}^2, \quad (1)$$

that is F is a C^2 local orientation reversing diffeomorphism.

A vector field in $\mathbb{R}^n$ is said to be *bounded* if the forward orbit of every point enters and remains in a compact set. Such vector fields are very important in the qualitative theory of differential equations because the ω -limit set of every point is nonempty and compact. In Cima and Llibre (1990), a necessary condition for a polynomial differential system in $\mathbb{R}^n$ to be bounded is established.

In Theorem 1 below, we provide a necessary and sufficient condition for the injectivity of a C^2 map in $\mathbb{R}^2$.

Theorem 1 *Let $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^2 local orientation reversing diffeomorphism, and assume $F(0, 0) = (0, 0)$. Define the C^1 vector field*

$$G : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad G = (P, Q) = (fg_y - gf_y, -fg_x + gf_x). \quad (2)$$

The map F is injective if and only if the vector field G is bounded.

In Theorem 2 below, we provide a sufficient condition for the injectivity of a C^2 map in the plane.

Theorem 2 *Let $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^2 local orientation reversing diffeomorphism, and assume $F(0, 0) = (0, 0)$. Let G be the vector field defined in (2). If*

$$JG(z) = \det DG(z) > 0, \quad \forall z = (x, y) \in \mathbb{R}^2, \quad (3)$$

then F is injective.

Remark 3 In Theorems 1 and 2, the conclusion that F is a global diffeomorphism follows immediately if F is also surjective.

The article is organized as follows. In Sect. 2, we summarize the results and definitions that will be used in the proofs of Theorems 1 and 2. In Sect. 3, we present the proofs of Theorems 1 and 2, along with some examples, using techniques from the qualitative theory of differential equations.

2 Preliminary Results

Let $U \subset \mathbb{R}^2$ be an open set. We say that a vector field $\mathcal{X} : U \rightarrow \mathbb{R}^2$ is *radial* surrounding an equilibrium point if it is topologically equivalent to the vector field (x, y) in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

The following result is part of a theorem due to Braun and Llibre (2024).

Theorem 4 (Braun, Llibre) *Let $U \subset \mathbb{R}^2$ be a simply connected open set. Let $F : U \rightarrow \mathbb{R}^2$ be a C^2 local diffeomorphism, and let $\mathcal{X} : F(U) \rightarrow \mathbb{R}^2$ be a C^1 radial vector field surrounding an equilibrium point. Then the following statements are equivalent:*

- (i) *The pullback of $\mathcal{X}$ by F , i.e. $\mathcal{Y}_{\mathcal{X}}(z) = DF(z)^{-1}\mathcal{X}(F(z))$, with $z \in U$, is topologically equivalent to the vector field $\mathcal{X}$.*
- (ii) *The map F is a global diffeomorphism onto $F(U)$.*

Theorem 4 states that a C^2 local diffeomorphism F is injective if and only if the vector fields $\mathcal{Y}_{\mathcal{X}}$ and $\mathcal{X}$ are topologically equivalent. Moreover, $\mathcal{Y}_{\mathcal{X}}$ is topologically equivalent to $\mathcal{X}$ if and only if $JF \cdot \mathcal{Y}_{\mathcal{X}}$ is also topologically equivalent to $\mathcal{X}$.

If F is a C^2 local diffeomorphism and $\mathcal{X}(x, y) = (x, y)$, then the vector field $JF \cdot \mathcal{Y}_{\mathcal{X}}$ has the following explicit expression

$$JF \cdot \mathcal{Y}_{\mathcal{X}} = (fg_y - gf_y, -fg_x + gf_x). \quad (4)$$

A vector field G in the plane is said to be globally asymptotically stable if it has an equilibrium point p and every other orbit of the vector field converges to p in forward time.

The problem of global asymptotic stability is closely related to the Markus-Yamabe Conjecture (Markus and Yamabe 1960).

Conjecture 5 (Markus-Yamabe) *Let $Y : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 vector field having the origin as an equilibrium point. If the eigenvalues of the Jacobian matrix of Y at every point $(x_1, \dots, x_n) \in \mathbb{R}^n$ have negative real parts, then Y is globally asymptotically stable.*

The Markus-Yamabe Conjecture was proved to be true in dimension two independently by Feßler (1995), Glutsyuk (1995) and Gutiérrez (1995). However, if n is greater than or equal to three, the conjecture was proved to be false. See the articles Bernat and Llibre (1996), and Cima et al. (1997). For an analysis of the behavior of vector fields defined in plane strips, see the article (Gutiérrez 1992). On the other hand, for an analysis of the behavior of planar vector fields with a unique equilibrium point of hyperbolic saddle type, see the article (Gutiérrez et al. 2012).

In the planar case, the condition on the eigenvalues of the Jacobian matrices of the vector field Y is equivalent to

$$\operatorname{tr} DY(x, y) < 0, \quad \det DY(x, y) > 0, \quad \forall (x, y) \in \mathbb{R}^2, \quad (5)$$

where tr denotes the trace or equivalently the divergence of the vector field.

In the proofs of Theorems 1 and 2, we shall apply Theorem 4 with the radial vector field $\mathcal{X}$. More precisely, we shall show that, under the assumptions of these theorems, the vector field (4), which coincides with the vector field (2), is globally asymptotically stable. It is important to emphasize that, in general, proving that a planar vector field is globally asymptotically stable is one of the major problems in the qualitative theory of differential equations.

3 Proofs of Theorems 1 and 2

We begin this section by studying some general properties of the vector field (2). The following propositions were established for the polynomial case in Llibre and Mello (2025) and are included here for the sake of completeness.

Proposition 6 *Let $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^2 map such that $JF < 0$. Then the equilibrium points of the vector field G in (2) are stable nodes.*

Proof Since the vector field G can be written as

$$G = \begin{pmatrix} g_y & -f_y \\ -g_x & f_x \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

and the above matrix is nonsingular, the equilibrium points of the vector field G are precisely the intersection points of the curves $f^{-1}(\{0\})$ and $g^{-1}(\{0\})$. Denote by (x_0, y_0) an equilibrium point of the vector field G and by $JF(x_0, y_0)$ the value of the determinant of the Jacobian matrix of the map F evaluated at (x_0, y_0) . Then the Jacobian matrix of the vector field G evaluated at (x_0, y_0) is

$$DG(x_0, y_0) = \begin{pmatrix} JF(x_0, y_0) & 0 \\ 0 & JF(x_0, y_0) \end{pmatrix}.$$

So this matrix has the eigenvalue $JF(x_0, y_0) < 0$ with multiplicity 2, and consequently the equilibrium point (x_0, y_0) is a stable node. $\square$

Proposition 7 *Let $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^2 map such that $JF < 0$. Then the vector field G in (2) has no periodic orbits.*

Proof Under the hypotheses of the proposition, it follows that

$$\begin{aligned} \operatorname{div}(G) &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial(fg_y - gf_y)}{\partial x} + \frac{\partial(-fg_x + gf_x)}{\partial y} \\ &= f_x g_y + f g_{y_x} - g_x f_y - g f_{y_x} - f_y g_x - f g_{x_y} + g_y f_x + g f_{x_y} \\ &= 2(f_x g_y - g_x f_y) = 2 JF < 0. \end{aligned}$$

By Bendixson's Theorem (see, for instance, Theorem 7.10 in Dumortier et al. (2006)), the vector field G has no periodic orbits. $\square$

The proof of Theorem 1 is presented below.

Proof of Theorem 1 Assume the hypotheses of Theorem 1. We first prove that if the map F is injective, then the vector field G is bounded. By Theorem 4, the vector field G is topologically equivalent to the radial vector field $\mathcal{X}(x, y) = (x, y)$. This implies

that G is globally asymptotically stable and, consequently, that the vector field G is bounded.

We now prove the converse. Assume that the vector field G is bounded. Considering the radial vector field $\mathcal{X}(x, y) = (x, y)$, we prove that $\mathcal{X}$ and G given in (2) are topologically equivalent. Hence, by Theorem 4, the map F is injective.

We claim that $(0, 0)$ is the unique equilibrium point of the vector field G . Indeed, assume that G has an equilibrium point different from $(0, 0)$. By Proposition 6, every equilibrium point of G is a stable node. Hence, the basin of attraction of $(0, 0)$ is an open set with a nonempty boundary. Let p be a point on this boundary. Since the basin of attraction of $(0, 0)$ is open, by the continuous dependence of a solution of an ordinary differential equation and by the Poincaré-Bendixson Theorem (see, for instance, Corollary 1.30 in Dumortier et al. (2006)), the ω -limit set of the orbit of p cannot be a periodic orbit (because the vector field G has no periodic orbits, see Proposition 7), cannot be a graphic (because the vector field G has no graphics, for the definition of a graphic, see Dumortier et al. 2006), and therefore can only be an equilibrium point or the empty set (corresponding to a point at the infinity). However, this is impossible because the point p lies on the boundary of a basin of attraction. Hence, its ω -limit set cannot be an equilibrium point. On the other hand, it cannot be the empty set either, since the vector field G is bounded and therefore the ω -limit set of every point is nonempty and compact. Consequently, the basin of attraction of the origin has no boundary in the plane, and the origin is globally asymptotically stable. By Theorem 4, the map F is injective.

In short, the proof of Theorem 1 is complete. $\square$

We now prove Theorem 2.

Proof of Theorem 2 Assume the hypotheses of Theorem 2. We consider the radial vector field $\mathcal{X}(x, y) = (x, y)$ and show that the corresponding vector field G is globally asymptotically stable.

By hypothesis, $F(0, 0) = (0, 0)$. By Proposition 6, the origin $(0, 0)$ is an equilibrium point of G and is a stable node. By Proposition 7, the divergence of the vector field G is given by

$$\operatorname{div}(G)(x, y) = 2 JF(x, y) < 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

So the trace of the Jacobian matrix of the vector field G is negative at every point in the plane. By (3), $JG(x, y) = \det DG(x, y) > 0$, for all $(x, y) \in \mathbb{R}^2$. It follows from (5) that the vector field G is a Markus-Yamabe vector field and hence is globally asymptotically stable. Therefore, by Theorem 4, the proof of Theorem 2 is complete. $\square$

Below, we present four examples for which global injectivity follows from Theorem 2.

Example 8 Consider the surjective rational map

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x, y) = \left(\frac{y}{1 + 3x^2}, x + x^3 \right).$$

It follows that $F(0, 0) = (0, 0)$ and $JF(x, y) = -1$, for all $(x, y) \in \mathbb{R}^2$. The vector field G defined in (2) has the form

$$G(x, y) = \left(-\frac{x + x^3}{1 + 3x^2}, -\left(1 + \frac{6x(x^3 + x)}{(1 + 3x^2)^2}\right)y \right).$$

By a direct calculation, we obtain $\text{tr } DG(x, y) = -2 < 0$ and

$$JG(x, y) = \frac{(3x^4 + 1)(15x^4 + 12x^2 + 1)}{(3x^2 + 1)^4} > 0,$$

for all $(x, y) \in \mathbb{R}^2$. It follows from Theorem 2 that F is a global diffeomorphism.

The following example was studied in Campbell (2000), where a generalization of this example can also be found.

Example 9 Define the analytic surjective map

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x, y) = (y - 3 \cos(3x + 5y) + 3, x + 5 \cos(3x + 5y) - 5).$$

It follows that $F(0, 0) = (0, 0)$ and $JF(x, y) = -1$, for all $(x, y) \in \mathbb{R}^2$. The vector field G defined in (2) has the form $G(x, y) = (P(x, y), Q(x, y))$, where

$$\begin{aligned} P(x, y) &= -5(3x + 5y) \sin(3x + 5y) - 5 \cos(3x + 5y) - x + 5, \\ Q(x, y) &= 3(3x + 5y) \sin(3x + 5y) + 3 \cos(3x + 5y) - y - 3. \end{aligned}$$

By a direct calculation, $\text{tr } DG(x, y) = -2 < 0$ and $JG(x, y) = 1 > 0$, for all $(x, y) \in \mathbb{R}^2$. Therefore, by Theorem 2, F is a global diffeomorphism.

The following example was studied in Coomes (1989), where F was proved to be a global diffeomorphism by showing that the Jacobian matrix of F is strictly row diagonally dominant.

Example 10 Define the analytic surjective map

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x, y) = (2x + 5y + x^3, 3x - 16y + \sin(x)).$$

It follows that $F(0, 0) = (0, 0)$ and

$$JF(x, y) = -47 - 48x^2 - 5 \cos(x) < 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

The vector field G defined in (2) has the form $G(x, y) = (P(x, y), Q(x, y))$, where

$$\begin{aligned} P(x, y) &= -47x - 16x^3 - 5 \sin(x), \\ Q(x, y) &= (2 + 3x^2)(3x - 16y + \sin(x)) - (2x + 5y + x^3)(3 + \cos(x)). \end{aligned}$$

By a direct calculation, we obtain

$$\begin{aligned}\operatorname{tr} DG(x, y) &= -2(47 + 48x^2 + 5\cos(x)) < 0, \\ JG(x, y) &= (47 + 48x^2 + 5\cos(x))^2 > 0,\end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$. It follows from Theorem 2 that F is a global diffeomorphism.

The following example was studied in Chamberland and Meisters (1998), where F was proved to be a global diffeomorphism by applying Hadamard's theorem for proper maps.

Example 11 Define the analytic surjective map

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x, y) = (2y + e^{\cos^2(y)} - e, x - e^{\cos^2(y)} + e).$$

It follows that $F(0, 0) = (0, 0)$ and $JF(x, y) = -2 + e^{\cos^2(y)} \sin(2y) < 0$, for all $(x, y) \in \mathbb{R}^2$. The vector field G defined in (2) has the form $G(x, y) = (P(x, y), Q(x, y))$, where

$$\begin{aligned}P(x, y) &= -2(e + x) + e^{\cos^2(y)}(2 + (x + 2y) \sin(2y)), \\ Q(x, y) &= e - 2y - e^{\cos^2(y)}.\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{tr} DG(x, y) &= -4 + 2e^{\cos^2(y)} \sin(2y) < 0, \\ JG(x, y) &= (-2 + e^{\cos^2(y)} \sin(2y))^2 > 0,\end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$. Therefore, by Theorem 2, F is a global diffeomorphism.

Below, we present an example in which global injectivity is established by means of Theorem 1, whereas Theorem 2 does not apply.

Example 12 Consider the polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$f(x, y) = -x + y - x^3 + y^3, \quad g(x, y) = x + y + x^3 + y^3 + y^5.$$

It follows that $F(0, 0) = (0, 0)$. The map F is a local orientation reversing diffeomorphism because

$$JF(x, y) = -((1 + 3x^2)(2 + 6y^2 + 5y^4)) < 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

The vector field $G = (P, Q)$ given in (2) can be written as

$$\begin{aligned}P(x, y) &= -2x - 2x^3 - 6xy^2 - 6x^3y^2 - 5xy^4 - 5x^3y^4 + 4y^5 + 2y^7, \\ Q(x, y) &= -2y - 6x^2y - 2y^3 - 6x^2y^3 - y^5 - 3x^2y^5.\end{aligned} \quad (6)$$

The map F does not satisfy the hypotheses of Theorem 2, since the Jacobian of G changes sign: $JG(-2, 0) = 676$ and $JG(-2, 2) = -8060$. We shall prove that the vector field G is bounded and, consequently, by Theorem 1, the map F is injective.

Consider the Poincaré compactification of the vector field G , see Chapter 5 of Dumortier et al. (2006). As the polynomial vector field G has degree 7, the stability of the infinite equilibrium points on the local charts V_1 and V_2 follows directly from the study on the local charts U_1 and U_2 . The origin is not an infinite equilibrium point of the Poincaré compactification of G in the local chart U_2 . On the local chart U_1 , the Poincaré compactification of G defines the system

$$\begin{aligned}\dot{u} &= -2(-1 + u)u(u^4 + u^5 + u^6 + 2u^4v^2 - 2v^4 - 2uv^4), \\ \dot{v} &= -v(-5u^4 + 2u^7 - 6u^2v^2 - 5u^4v^2 + 4u^5v^2 - 2v^4 - 6u^2v^4 - 2v^6).\end{aligned}\quad (7)$$

There are two infinite equilibrium points covered by this chart: $E_1 = (0, 0)$ and $E_2 = (1, 0)$. The Jacobian matrix of the vector field that defines (7) evaluated at E_2 has the form

$$\begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix}.$$

So the infinite equilibrium point E_2 is a hyperbolic saddle. On the other hand, the Jacobian matrix of the vector field that defines (7) evaluated at E_1 is the zero matrix. Thus, the infinite equilibrium point E_1 is degenerate. Consider the polar blow-up $(u, v) = (r \cos(\theta), r \sin(\theta))$. After dividing by the common factor r^4 , system (7) can be written as

$$\begin{aligned}\dot{r} &= \frac{1}{128} r \mathcal{R}(r, \theta), \\ \dot{\theta} &= -\frac{1}{64} (21 \sin(2\theta) - 4 \sin(4\theta) + \sin(6\theta)) (r^2 \cos(2\theta) - r^2 - 6),\end{aligned}\quad (8)$$

where

$$\begin{aligned}\mathcal{R}(r, \theta) &= 12 \cos(2\theta) + 40 \cos(4\theta) + 3 (-4 \cos(6\theta) + 49r^2 + 72) \\ &\quad + r^2 (\cos(8\theta) - 20(6 \cos(2\theta) + \cos(4\theta)) - 8 \cos(6\theta) \\ &\quad + 128r(\cos(2\theta) - 3) \cos^5(\theta)).\end{aligned}$$

The infinite equilibrium point E_1 is blown up into the circle $r = 0$. The equilibrium points belonging to this circle are

$$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{2}, \quad \theta_3 = \pi, \quad \theta_4 = \frac{3\pi}{2}.$$

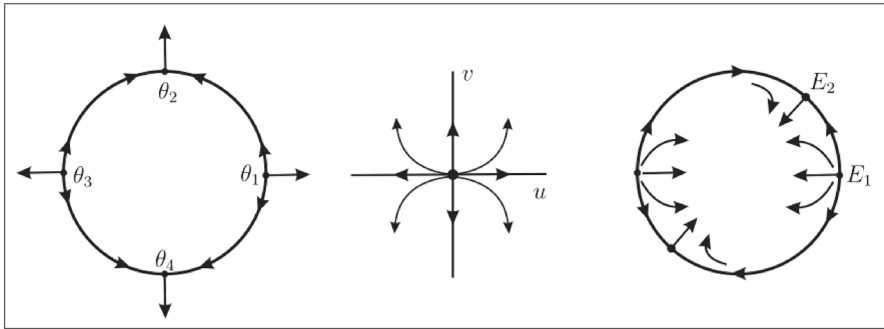
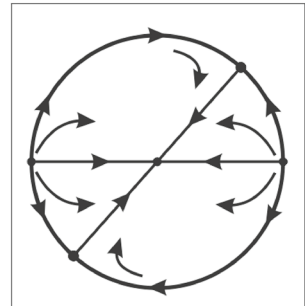


Fig. 1 **Left:** Desingularization of system (7) using polar blow-up. **Center:** Topological local phase portrait at the origin of system (7). **Right:** Dynamics close to the infinity of the Poincaré compactification of the vector field G (6) in the Poincaré disk

Fig. 2 Global phase portrait of the vector field G (6) in the Poincaré disk



The Jacobian matrices of the vector field that defines system (8) at these equilibrium points are, respectively,

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}.$$

So the equilibrium points $(0, \theta_1)$ and $(0, \theta_3)$ are hyperbolic stable nodes, while the equilibrium points $(0, \theta_2)$ and $(0, \theta_4)$ are hyperbolic saddles repelling in the radial direction and attracting in the polar direction. See Fig. 1.

In summary the vector field G is bounded. It is easy to show that the straight lines

$$\{(x, y) \in \mathbb{R}^2 : y = 0, x \in \mathbb{R}\}, \quad \{(x, y) \in \mathbb{R}^2 : y = x, x \in \mathbb{R}\},$$

are invariant by the flow of G . The global phase portrait of G is illustrated in Fig. 2. In summary, by Theorem 1, the map F is injective.

In the following example, we consider a polynomial local diffeomorphism that is not a global diffeomorphism.

Example 13 Consider the polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where

$$\begin{aligned} f(x, y) &= (1 - x + x^2y)(-x + y - 2xy + 3x^2y + 2x^2y^2 - 3x^3y^2 + x^4y^3), \\ g(x, y) &= 5/2 - 8x + (15x^2)/2 - 3x^3 + 5y - 21xy + 44x^2y - 42x^3y + 18x^4y \\ &\quad - 5xy^2 + (69x^2y^2)/2 - 84x^3y^2 + 93x^4y^2 - 45x^5y^2 - 16x^3y^3 + 68x^4y^3 \\ &\quad - 102x^5y^3 + 60x^6y^3 - 20x^5y^4 + (111x^6y^4)/2 - 45x^7y^4 - 12x^7y^5 \\ &\quad + 18x^8y^5 - 3x^9y^6. \end{aligned}$$

From Fernandes (2022), $JF(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$. So F is a local diffeomorphism. We analyze this example in the context of the results established in this article. Firstly, consider $\tilde{F} = (\tilde{f}, \tilde{g}) = (g - 5/2, f)$. Thus $\tilde{F}(0, 0) = (0, 0)$ and $J\tilde{F}(x, y) < 0$, for all $(x, y) \in \mathbb{R}^2$. It follows that $\tilde{F}$ is injective if and only if F is injective. Let G be the pullback of the radial vector field by $\tilde{F}$. Note that $G(0, 0) = (0, 0)$.

By a direct calculation, the vector field G in (2) can be written as

$$\left(\left(\frac{\tilde{g}}{\tilde{f}} \right)_y \tilde{f}^2, - \left(\frac{\tilde{g}}{\tilde{f}} \right)_x \tilde{f}^2 \right)$$

which is topologically equivalent to the vector field

$$\left(\left(\frac{\tilde{g}}{\tilde{f}} \right)_y, - \left(\frac{\tilde{g}}{\tilde{f}} \right)_x \right), \quad (9)$$

outside the curve $\tilde{f}^{-1}(\{0\})$. The vector field (9) is a Hamiltonian vector field defined in $\mathbb{R}^2 \setminus \tilde{f}^{-1}(\{0\})$ with Hamiltonian $\tilde{g}/\tilde{f}$. Similarly, it follows that the vector field G in (2) is topologically equivalent to the vector field

$$\left(- \left(\frac{\tilde{f}}{\tilde{g}} \right)_y, \left(\frac{\tilde{f}}{\tilde{g}} \right)_x \right), \quad (10)$$

outside the curve $\tilde{g}^{-1}(\{0\})$. The vector field (10) is a Hamiltonian vector field defined in $\mathbb{R}^2 \setminus \tilde{g}^{-1}(\{0\})$ with Hamiltonian $\tilde{f}/\tilde{g}$.

From the above analysis the set

$$\left(\frac{\tilde{g}}{\tilde{f}} \right)^{-1}(\{0\}) = f^{-1}(\{0\})$$

is formed by orbits of G . The set $f^{-1}(\{0\})$ contains the curve defined by the graph of

$$h(x) = \frac{x-1}{x^2}, \quad x > 0.$$

By a straightforward calculation, we obtain

$$\lim_{x \rightarrow 0^+} h(x) = -\infty, \quad \lim_{x \rightarrow \infty} h(x) = 0.$$

Thus, the ends of the graph of h are the origin of the chart V_2 and the origin of the chart U_1 . It follows that the ω -limit set of this orbit is not $\{(0, 0)\}$. We conclude that G is not bounded and, by Theorem 1, $\tilde{F}$ is not injective. Therefore, F is not a global diffeomorphism.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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