



Invited

Inverse problems for the Stäckel mechanical systems

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ARTICLE INFO

Communicated by Dr A Celletti

2020 MSC:

Primary 70H06

Keywords:

Stäckel theorem

Nambu bracket

Inverse problem

Integrability by separation of variables

Korteweg de Vries equation

Stark problem

Pseudo Riemann space of constant curvature

Kepler problem

Homogeneous symmetric polynomial

Levi-Civita theorem

ABSTRACT

The study of the integrability by separation of variables of the Hamilton-Jacobi equations is a classical problem in Mechanics, dating back to the foundational works of Liouville, Jacobi, Stäckel, Levi-Civita and others. Recently this kind integrability received a big attention due to its applications to the theory of integrable partial differential equations of Korteweg de Vries type and to the theory of quantum integrable systems.

The most relevant development to the solution of the separation of variables due to Stäckel was to give the general solution in orthogonal coordinates.

In the present paper by applying the Nambu bracket and its new properties we developed the Stäckel ideas and state and solve two inverse problems in ordinary differential equations. The first one is related with the problem of the construction of the Stäckel potential from a given special family of curves, and the second one is related with the problem of the construction of the Stäckel matrix from a given Riemann-Stäckel metric and consequently we construct the Stäckel mechanical system.

The obtained results are illustrated solving some relevant applications. In particular we solve the second inverse Stäckel problem for the Kepler-Coulomb system, the Stark system in arbitrary dimension and for the first time we introduce and study the Stark-Coloumb system. For these Hamiltonian systems we prove that they are integrable by separation of variables. Moreover we solve the second Stäckel problem in the pseudo-Riemann space of constant curvature and prove that the Stäckel potential in this case is a complete homogeneous symmetric polynomial.

1. Introduction

The study of the integrability by separation of variables of the Hamilton-Jacobi equations is a classical problem in Mechanics, dating back to the foundational works of Liouville, Jacobi, Stäckel, Levi-Civita and others. This question was studied intensively in the second half of the last century (see for instance [1–9]). For an outline of the theory of separation of variables we refer to the book of Kalnins [10]. The integrability by separation of variables of the Hamilton-Jacobi equations has recently received a big attention due to its applications to the theory of integrable partial differential equations of Korteweg de Vries type and to the theory of quantum integrable systems (see for instance [8,11–13]). In the review *Separation of Variables. New Trends* see [12] Sklyanin argued that separation of variables could be the most universal tool to solve integrable models of the classical and quantum mechanics.

If a solution of the non-linear partial differential equation (Hamilton-Jacobi equation)

$$H\left(z_1, \dots, z_M, \frac{\partial S}{\partial z_1}, \dots, \frac{\partial S}{\partial z_M}\right) = h = \alpha_M. \quad (1)$$

can be written as follows $S = \alpha_M t + W$, where

$$W = W_1(z_1, \alpha_1, \dots, \alpha_M) + \dots + W_M(z_M, \alpha_1, \dots, \alpha_M),$$

$$\det \left(\frac{\partial^2 W}{\partial z_j \partial \alpha_k} \right) \neq 0, \quad (2)$$

then we say that the Hamiltonian system is *additive separable* or *integrable by separation of variables* or *integrable in the Jacobi sense*.

The definition of the integrable in the Liouville sense is the following. Consider a Hamiltonian mechanical system defined by the Hamiltonian $H = H_M$ with M degrees of freedom. If there exists a family of $M - 1$ first integrals $H_1, H_2, \dots, H_{M-1}$ such that

(a) they are in involution with respect to the Poisson bracket, that is

$$\{H_j, H_k\}^* = \sum_{j=1}^M \left(\frac{\partial H_j}{\partial P_j} \frac{\partial H_k}{\partial z_j} - \frac{\partial H_j}{\partial z_j} \frac{\partial H_k}{\partial P_j} \right) = 0,$$

for $j, k = 1, \dots, M$,

(b) they are independent, i.e. the rank of the matrix formed by the gradients of $H_1, \dots, H_M = H$ is M except (perhaps) in a set of zero Lebesgue measure.

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One of the first works dedicated to the theory of separation of variables is the Liouville's work [14] in which he studied a dynamical system with two degree of freedom (z_1, z_2) and Hamiltonian function

$$H = \frac{P_1^2 + P_2^2 + 2(\mu_1(z_1) + \mu_2(z_2))}{2(\kappa_1(z_1) + \kappa_2(z_2))}, \text{ called of Liouville type.}$$

The most relevant development for the solution of the additive separation of variables problem due to Stäckel [15,16] was to give the general solution in orthogonal coordinates. He proved the following two theorems.

Theorem 1. [Stäckel Theorem I] *The necessary and sufficient conditions in order that the Hamilton-Jacobi equation $\sum_{j=1}^M \frac{1}{A_j} \left(\frac{\partial S}{\partial z_j} \right)^2 = U(z_1, \dots, z_M) + h$, with h a constant be integrable by separation of variables (or equivalently in the Jacobi sense), i.e. there exists a function $S = \sum_{j=1}^M S_j(z_j, \alpha_1, \dots, \alpha_M)$ with α_j constant for $j = 1, \dots, M$ such that $\det \left(\frac{\partial^2 S}{\partial z_j \partial \alpha_k} \right) \neq 0$ and*

- (i) *there exists a Stäckel matrix $\Delta = (\Delta_{ij})$ such that $A_j = \frac{\Delta^{jM}}{\det(\Delta)}$ for $j = 1, \dots, M$ where Δ^{jM} is the (j, M) component of the inverse matrix Δ^{-1} ;*
- (ii) *there are functions $\Psi_j(z_j)$ for $j = 1, \dots, M$ such that*

$$U(z_1, \dots, z_M) = \sum_{j=1}^M \frac{\Delta^{jM}}{|\Delta|} \Psi_j(z_j). \quad (3)$$

The function (3) is called the Stäckel potential.

Theorem 2. [Stäckel Theorem II] *The Hamilton-Jacobi equations with Hamiltonian function $H = F_M = \sum_{j=1}^M \frac{\Delta^{jM}}{|\Delta|} (P_j^2/2 + \Psi_j(z_j))$, has $M - 1$ independent first integrals in involution*

$$F_j = \sum_{k=1}^M \frac{\Delta^{jk}}{|\Delta|} (P_k^2/2 + \Psi_k(z_k)), \quad (4)$$

for $j = 1, \dots, M - 1$.

After the publication of the Stäckel's results the following question arise: *what is the necessary and sufficient conditions under which the Hamilton-Jacobi equation is integrable in the Jacobi sense in the given set of coordinates, not necessarily orthogonal?* Levi-Civita provided the answer to this question and proved the following result:

Theorem 3 (Levi-Civita Theorem). *The Hamilton-Jacobi equations are integrable in the Jacobi sense, if and only if the Hamiltonian H satisfy the $M(M - 1)/2$ second-order partial differential equations*

$$L_{jk}(H, z, P) := \frac{\partial H}{\partial P_j} \frac{\partial H}{\partial P_k} \frac{\partial^2 H}{\partial z_j \partial z_k} + \frac{\partial H}{\partial z_j} \frac{\partial H}{\partial z_k} \frac{\partial^2 H}{\partial P_j \partial P_k} - \frac{\partial H}{\partial P_j} \frac{\partial H}{\partial z_k} \frac{\partial^2 H}{\partial z_j \partial P_k} - \frac{\partial H}{\partial P_k} \frac{\partial H}{\partial z_j} \frac{\partial^2 H}{\partial P_j \partial z_k} = 0,$$

for $j, k = 1, \dots, M$ with $j \neq k$.

In this paper we develop the Stäckel ideas and state an solve two inverse problems. The first one is related with the problem of the construction of the Stäckel potential function from a given family of curves given by the intersection of $M - 1$ hypersurfaces in $\mathbb{R}^M$, and the second one is related with the problem of the construction from a given Riemann-Stäckel metric of the Stäckel matrix that is a nondegenerate matrix whose j columns depend only on the coordinate z_j and the Hamiltonian-Stäckel function.

As we observe one of the most relevant results in the theory of integrability in the Jacobi sense is due to Stäckel, [16] who gives the general solution of the additive separation of variables problem in orthogonal

coordinates (see Stäckel Theorems I and II). The aim of this paper is to state and solve the inverse problem in dynamics which we call the Stäckel inverse problem I, which is a particular case of the Suslov problem [17,18] and the Stäckel inverse problem II, which is a new inverse problem in dynamics.

First we introduce the following notations and definitions.

For the independent functions H_j for $j = 1, \dots, N$ the determinant of the Jacobian matrix of these functions, i.e.

$$\left\{ H_1, H_2, \dots, H_{M-1}, H_M \right\} := \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \dots & \frac{\partial H_1}{\partial x_M} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \dots & \frac{\partial H_2}{\partial x_M} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial H_{M-1}}{\partial x_1} & \frac{\partial H_{M-1}}{\partial x_2} & \dots & \frac{\partial H_{M-1}}{\partial x_M} \\ \frac{\partial H_M}{\partial x_1} & \frac{\partial H_M}{\partial x_2} & \dots & \frac{\partial H_M}{\partial x_M} \end{vmatrix}.$$

is called the *Nambu bracket* (see for instance [17,19–21]).

We shall use the following four identities proved, in particular, in [17,19]

- (i) $\sum_{j=1}^M \frac{\partial f}{\partial x_j} \{g_1, \dots, g_{n-1}, x_j, g_{n+1}, \dots, g_N\} = \{g_1, \dots, g_{n-1}, f, g_{n+1}, \dots, g_M\}.$
- (ii) $\sum_{j=1}^M \frac{\partial}{\partial x_j} \{g_1, \dots, g_{n-1}, x_j, g_{n+1}, \dots, g_M\} = 0$, for $n = 1, 2, \dots, M.$
- (iii) $-\{f_1, \dots, f_{M-1}, G\} \{g_1, \dots, g_M\} + \sum_{n=1}^M \{f_1, \dots, f_{M-1}, g_n\} \{g_1, \dots, g_{n-1}, G, g_{n+1}, \dots, g_M\} \equiv 0.$
- (iv) $-\{f_1, \dots, f_{M-1}, \lambda \{g_1, \dots, g_M\}\} + \sum_{n=1}^M \{g_1, \dots, g_{n-1}, \lambda \{f_1, \dots, f_{M-1}, g_n\}, g_{n+1}, \dots, g_M\} \equiv 0.$

Corollary 1. *The identities (iii) and (iv) for $M = 3$ becomes*

$$\begin{aligned} &\{f_1, f_2, G\} \{g_1, g_2, g_3\} = \\ &\{f_1, f_2, g_1\} \{G, g_2, g_3\} + \{f_1, f_2, g_2\} \{g_1, G, g_3\} + \{f_1, f_2, g_3\} \{g_1, g_2, G\}, \\ &\{f_1, f_2, \lambda \{g_1, g_2, g_3\}\} = \\ &\{\lambda \{f_1, f_2, g_1\}, g_2, g_3\} + \{g_1, \lambda \{f_1, f_2, g_2\}, g_3\} + \{g_1, g_2, \lambda \{f_1, f_2, g_3\}\} \end{aligned} \quad (5)$$

The matrix

$$\Delta = \begin{pmatrix} \varphi_{11}(z_1) & \varphi_{12}(z_1) & \dots & \varphi_{1M}(z_1) \\ \varphi_{21}(z_2) & \varphi_{22}(z_2) & \dots & \varphi_{2M}(z_2) \\ \vdots & \vdots & \dots & \vdots \\ \varphi_{M1}(z_M) & \varphi_{M2}(z_M) & \dots & \varphi_{MM}(z_M) \end{pmatrix},$$

is called the *Stäckel matrix* where φ_{ij} are functions for $i, j = 1, \dots, M$ (for

more details see [22]). By introducing the 1-forms $d\varphi_\alpha = \sum_{k=1}^M \varphi_{k\alpha}(z_k) dz_k$,

for $\alpha = 1, \dots, M$, then we get that the matrix Δ can be written as

$$\Delta = \begin{pmatrix} \frac{\partial \varphi_1}{\partial z_1} & \frac{\partial \varphi_2}{\partial z_1} & \dots & \frac{\partial \varphi_M}{\partial z_1} \\ \frac{\partial \varphi_1}{\partial z_2} & \frac{\partial \varphi_2}{\partial z_2} & \dots & \frac{\partial \varphi_M}{\partial z_2} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial \varphi_1}{\partial z_M} & \frac{\partial \varphi_2}{\partial z_M} & \dots & \frac{\partial \varphi_M}{\partial z_M} \end{pmatrix}, \quad (6)$$

and consequently the inverse matrix $\Delta^{-1} = \left(\frac{\Delta^{jk}}{|\Delta|} \right)$ can be written as

$$\begin{aligned} & \begin{pmatrix} \frac{\{z_1, \varphi_2, \dots, \varphi_M\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} & \dots & \frac{\{z_M, \varphi_2, \dots, \varphi_M\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \\ \frac{\{\varphi_1, z_1, \dots, \varphi_M\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} & \dots & \frac{\{\varphi_1, z_M, \dots, \varphi_M\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \\ \vdots & \dots & \vdots \\ \frac{\{\varphi_1, \dots, \varphi_{M-1}, z_1\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} & \dots & \frac{\{\varphi_1, \dots, \varphi_{M-1}, z_M\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \end{pmatrix} \\ & := \begin{pmatrix} \lambda a_{11}(z_2, \dots, z_M) & \dots & \lambda a_{1M}(z_1, \dots, z_{M-1}) \\ \lambda a_{21}(z_2, \dots, z_M) & \dots & \lambda a_{2M}(z_1, \dots, z_{M-1}) \\ \vdots & \dots & \vdots \\ \lambda a_{M1}(z_2, \dots, z_M) & \dots & \lambda a_{MM}(z_1, \dots, z_{M-1}) \end{pmatrix}, \end{aligned} \quad (7)$$

where $\frac{1}{\lambda} = \{\varphi_1, \varphi_2, \dots, \varphi_M\} = |\Delta| = \det(\Delta) \neq 0$.

We introduce the following definition. The diagonal Riemann metric G such that

$$G^{jj} = \frac{\Delta^{jM}}{|\Delta|} = \frac{\{\varphi_1, \dots, \varphi_{M-1}, z_j\}}{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}}, \quad (8)$$

we call it the *Riemann-Stäckel metric* G , where $G^{-1} = \text{diag}(G^{11}, \dots, G^{NN})$ is the inverse matrix to the matrix $G = \text{diag}(G_{11} = 1/G^{11}, \dots, G_{NN} = 1/G^{NN})$.

We call the Hamiltonian system with Hamiltonian

$$\begin{aligned} H &= \sum_{k=1}^M \frac{p_k^2}{2} \frac{\{\varphi_1, \varphi_2, \dots, z_j\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} + \frac{\{\varphi_1, \varphi_2, \dots, \Psi\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}}, \\ &= \sum_{k=1}^M \frac{\{\varphi_1, \varphi_2, \dots, z_j\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \left(\frac{p_k^2}{2} + \Psi_j(z_j) \right) \end{aligned}$$

the *Hamiltonian-Stäckel system*, where $\Psi = \sum_{j=1}^M \int_{z_{j0}}^{z_j} \Psi_j(z_j) dz_j$.

In [17,19,23] we study and solve the following inverse problem in ordinary differential equations.

Inverse Stäckel problem I. Assume that we have a mechanical system with kinetic energy $T = \frac{1}{2} \sum_{j=1}^M \frac{\dot{z}_j^2}{A_j} := \frac{1}{2} \sum_{j=1}^M G_{jj} \dot{z}_j^2$, where $G^{-1} = (G^{jj})$

is the *Riemann-Stäckel metric* with $G^{jj} = A_j = \frac{\{\varphi_1, \dots, \varphi_{M-1}, z_j\}}{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}}$, $j = 1, \dots, M$. We require to determine the Stäckel potential U in such a way that the $M-1$ parametric family of curves defined by the intersection of the hypersurfaces $f_j = f_{j1}(z_1, \alpha_1, \dots, \alpha_M) + \dots + f_{jM}(z_M, \alpha_1, \dots, \alpha_M) = c_j$ for $j = 1, \dots, M-1$, are described by this mechanical system.

The second inverse problem is the following.

Inverse Stäckel problem II. Assume that the Riemann-Stäckel metric is given. We require to determine the Stäckel matrix Δ . Moreover we construct the Hamiltonian-Stäckel function.

The solution of this problem is our main objective and it is given in the next section.

The main result is the statement and solution of the inverse Stäckel problem II and the applications of this result to solve the non-trivial problems, such as the integrability by variable separation, such as the inverse problem of two fixed centers, such as the extended Kepler-Coulomb system, and such as the Stark problem in arbitrary dimension. All these results are completely new.

2. Solution of the Stäckel inverse problem II and new proof of Theorem Stäckel II

The solution of the Stäckel inverse problem II requires to determine the functions $\varphi_1, \varphi_2, \dots, \varphi_M$ and Ψ such that the following relations hold,

$$G^{jj} = \frac{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, z_j\}}{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \varphi_M\}}, \quad U = \frac{\{\varphi_1, \dots, \varphi_{M-1}, \Psi\}}{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}}, \quad (9)$$

for $j = 1, \dots, M$. Thus in view of property (i) of the Nambu bracket we get that

$$\begin{aligned} \frac{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_k\}}{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}} &= \sum_{j=1}^M G^{jj} \frac{\partial \varphi_k}{\partial z_j} = 0, \quad k = 1, \dots, M-1, \\ \frac{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}}{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}} &= \sum_{j=1}^M G^{jj} \frac{\partial \varphi_M}{\partial z_j} = 1, \\ \frac{\{\varphi_1, \dots, \varphi_{M-1}, \Psi\}}{\{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}} &= \sum_{j=1}^M G^{jj} \frac{\partial \Psi}{\partial z_j} = U. \end{aligned} \quad (10)$$

From (9) it follows that $G^{jj} \{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \varphi_M\} = \{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, z_j\}$. Thus from property (ii) we have $\sum_{j=1}^M \frac{\partial}{\partial z_j} (\{\varphi_1, \dots, \varphi_{M-1}, z_j\}) = 0$, and we get that

$$\sum_{j=1}^M \frac{\partial}{\partial z_j} (G^{jj} \{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\}) = 0. \quad (11)$$

By solving these partial differential equations we should obtain the Stäckel matrix Δ and the Stäckel potential. Consequently we should have the Hamilton Stäckel function. We shall do these computations in several applications.

On the other hand, by considering that the partial differential system (10) is linear, then if the function $\varphi := (\varphi_1, \varphi_2, \dots, \varphi_M)^T$ is a solution of (10), then the functions $\tilde{\varphi} := (\tilde{\varphi}_1, \tilde{\varphi}_2, \dots, \tilde{\varphi}_M)^T$ such that $\tilde{\varphi} = B \varphi$, where B is an arbitrary constant matrix. It is also a solution of (10).

The function

$$S = \sum_{k=1}^M \int_{z_{k0}}^{z_k} \sqrt{-2\Psi_k(z_k) + 2 \sum_{j=1}^M \alpha_j \varphi_{kj}(z_k)} dz_k, \quad (12)$$

is called the *Jacobi-Stäckel function* where $\Psi_k(z_k)$, $\varphi_{kj}(z_k)$ are arbitrary function on the variable z_k for $j, k = 1, \dots, M$, and $\alpha_1, \alpha_2, \dots, \alpha_M = h$ are arbitrary constants see [22]. We define the functions

$$P_k = \frac{\partial S}{\partial z_k}, \quad k = 1, \dots, M. \quad (13)$$

Now we provide a new simple proof and shorter of the following theorem. We observe that in the proof of this theorem plays an important role the first integrals (12). Note that the proof of the involution of the given first integrals will be obtained by applying the new identities of the Nambu bracket. Since these identities are original our proof is different of the Stäckel one. The proof of Theorem 5 will be also applied in the study of the integrability of the given examples.

Theorem 4. The Stäckel-Hamiltonian system is integrable in the Liouville sense.

Proof. First we prove that the Hamiltonian-Stäckel system admits M independent first integrals. Indeed, from (12) and (13) it follows that

$$P_k^2 = \left(\frac{\partial S}{\partial z_k} \right)^2 = -2\Psi_k(z_k) + 2 \sum_{j=1}^M \alpha_j \Delta_{kj},$$

hence $\frac{1}{2} P_k^2 + \Psi_k(z_k) = \sum_{j=1}^M \Delta_{kj} \alpha_j$. Therefore denoting by

$$\mathbf{a} := (\alpha_1, \dots, \alpha_M)^T, \quad \mathbf{R} := (P_1^2/2 + \Psi_1(z_1), \dots, P_M^2/2 + \Psi_M(z_M))^T,$$

we get from (7) that $\mathbf{a} = \Delta^{-1} \mathbf{R}$. Consequently from (4) we get the functions $F_1, \dots, F_M = H$ where

$$\begin{aligned} F_1 &= \sum_{j=1}^M \frac{P_j^2}{2} \frac{\{\varphi_1, \varphi_2, \dots, \varphi_N\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} + \frac{\{\psi, \varphi_2, \dots, \varphi_N\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \\ &= \sum_{j=1}^M \frac{\{\varphi_1, \varphi_2, \dots, \varphi_N\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \left(\frac{P_j^2}{2} + \Psi_j(z_j) \right), \\ F_2 &= \sum_{j=1}^M \frac{P_j^2}{2} \frac{\{\varphi_1, z_j, \dots, \varphi_M\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} + \frac{\{\varphi_1, \psi, \varphi_3, \dots, \varphi_N\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \\ &= \sum_{j=1}^M \frac{\{\varphi_1, z_j, \dots, \varphi_N\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \left(\frac{P_j^2}{2} + \Psi_j(z_j) \right), \\ &\vdots \\ F_M &= H = \sum_{j=1}^M \frac{P_j^2}{2} \frac{\{\varphi_1, \varphi_2, \dots, z_j\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} + \frac{\{\varphi_1, \varphi_2, \dots, \psi\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \\ &= \sum_{j=1}^M \frac{\{\varphi_1, \varphi_2, \dots, z_j\}}{\{\varphi_1, \varphi_2, \dots, \varphi_M\}} \left(\frac{P_j^2}{2} + \Psi_j(z_j) \right), \end{aligned} \quad (14)$$

are first integrals of the Hamiltonian-Stäckel system.

The independence of these first integrals follows from the fact that the matrix Δ is non degenerated with $|\Delta| = \{\varphi_1, \dots, \varphi_{M-1}, \varphi_M\} \neq 0$.

Finally in order to prove that these first integrals are in involution, i.e. $\{F_j, F_k\}^* = 0$, for $j, k = 1, \dots, M$, where $\{F_j, F_k\}^*$ denotes the Poisson bracket of the functions F_j and F_k . We shall study the involution for the case $M = 3$, and when $M > 3$ the proof is analogous. We observe that this proof of the involution of the given first integrals is obtained by applying the new identities of the Nambu bracket.

Now we give a proof of the main identities that we apply in this paper. These proofs are given in particular in our monograph J. Llibre and R. Ramírez, Inverse problems in ordinary differential equations and applications, Progress in Math. 313, Birkhäuser, 2016.

The next identities are well known

$$\begin{aligned} (I) \sum_{j=1}^N \frac{\partial f}{\partial x_j} \{g_1, \dots, g_{n-1}, x_j, g_{n+1}, \dots, g_N\} &= \{g_1, \dots, g_{n-1}, f, g_{n+1}, \dots, g_N\}, \\ (II) \sum_{j=1}^N \frac{\partial}{\partial x_j} \{g_1, \dots, g_{n-1}, x_j, g_{n+1}, \dots, g_N\} &= 0, \text{ for } n = 1, 2, \dots, N, \end{aligned}$$

and Filippov Identity

$$\begin{aligned} (III) \quad \Gamma(f_1, \dots, f_{N-1}, g_1, \dots, g_N) : \\ = \sum_{n=1}^N \{g_1, \dots, g_{n-1}, \{f_1, \dots, f_{N-1}, g_n\}, g_{n+1}, \dots, g_N\} \\ - \{f_1, \dots, f_{N-1}, \{g_1, \dots, g_N\}\} \equiv 0 \end{aligned}$$

The following identity is new

$$\begin{aligned} (IV) \quad \Pi_G(f_1, \dots, f_{N-1}, g_1, \dots, g_N) := -\{f_1, \dots, f_{N-1}, G\} \{g_1, \dots, g_N\} \\ + \sum_{n=1}^N \{f_1, \dots, f_{N-1}, g_n\} \{g_1, \dots, g_{n-1}, G, g_{n+1}, \dots, g_N\} \equiv 0, \end{aligned}$$

The proof of this identity is the following. First (V) is equivalently to

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_N} & \{f_1, \dots, f_{N-1}, g_1\} \\ \vdots & \dots & \vdots & \vdots \\ \frac{\partial g_N}{\partial x_1} & \dots & \frac{\partial g_N}{\partial x_N} & \{f_1, \dots, f_{N-1}, g_N\} \\ \frac{\partial G}{\partial x_1} & \dots & \frac{\partial G}{\partial x_N} & \{f_1, \dots, f_{N-1}, G\} \end{vmatrix}.$$

Using identity (I) the Nambu bracket we obtain that

$$\begin{aligned} &\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_N} & \sum_{j=1}^N \{f_1, \dots, f_{N-1}, x_j\} \frac{\partial g_1}{\partial x_j} \\ \vdots & \dots & \vdots & \vdots \\ \frac{\partial g_N}{\partial x_1} & \dots & \frac{\partial g_N}{\partial x_N} & \sum_{j=1}^N \{f_1, \dots, f_{N-1}, x_j\} \frac{\partial g_N}{\partial x_j} \\ \frac{\partial G}{\partial x_1} & \dots & \frac{\partial G}{\partial x_N} & \sum_{j=1}^N \{f_1, \dots, f_{N-1}, x_j\} \frac{\partial G}{\partial x_j} \end{vmatrix} \\ &= - \sum_{j=1}^N \{f_1, \dots, f_{N-1}, x_j\} \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_N} & \frac{\partial g_1}{\partial x_j} \\ \vdots & \dots & \vdots & \vdots \\ \frac{\partial g_N}{\partial x_1} & \dots & \frac{\partial g_N}{\partial x_N} & \frac{\partial g_N}{\partial x_j} \\ \frac{\partial G}{\partial x_1} & \dots & \frac{\partial G}{\partial x_N} & \frac{\partial G}{\partial x_j} \end{vmatrix} \equiv 0. \end{aligned}$$

we denote by

$$\begin{aligned} \Gamma_\lambda(f_1, \dots, f_{N-1}, g_1, \dots, g_N) &:= -\{f_1, \dots, f_{N-1}, \lambda\{g_1, \dots, g_N\}\} \\ &+ \sum_{n=1}^N \{g_1, \dots, g_{n-1}, \lambda\{f_1, \dots, f_{N-1}, g_n\}, g_{n+1}, \dots, g_N\}, \end{aligned}$$

and we get the following identity (generalization of the Filippov identity)

$$(V) \quad \Gamma_\lambda(f_1, \dots, f_{N-1}, g_1, \dots, g_N) \equiv 0$$

Indeed, by applying the property of the determinants (Leibniz rule) we get that

$$\begin{aligned} &-\{f_1, \dots, f_{N-1}, \lambda\{g_1, \dots, g_N\}\} \\ &+ \left(\sum_{n=1}^N \{g_1, \dots, g_{n-1}, \lambda\{f_1, \dots, f_{N-1}, g_n\}, g_{n+1}, \dots, g_N\} \right) \\ &= \lambda \left(-\{f_1, \dots, f_{N-1}, \{g_1, \dots, g_N\}\} \right. \\ &+ \sum_{n=1}^N \{g_1, \dots, g_{n-1}, \{f_1, \dots, f_{N-1}, g_n\}, g_{n+1}, \dots, g_N\} \Big) \\ &+ \left(-\{f_1, \dots, f_{N-1}, \lambda\}\{g_1, \dots, g_N\} \right. \\ &+ \sum_{n=1}^N \{g_1, \dots, g_{n-1}, \lambda, g_{n+1}, \dots, g_N\} \{f_1, \dots, f_{N-1}, g_n\} \Big) \\ &= \lambda \left(\Gamma_\lambda(f_1, \dots, f_{N-1}, g_1, \dots, g_N) \right) \Big|_{\lambda=1} \\ &+ \left(\Pi_G(f_1, \dots, f_{N-1}, g_1, \dots, g_N) \right) \Big|_{G=\lambda} \equiv 0. \end{aligned}$$

In short the new identities are proved.

First we analyze the existence of the quadratic first integrals in the momenta for the Hamiltonian system with three degrees of freedom

$$H = \frac{1}{2} \sum_{j=1}^3 B_j(z_1, z_2, z_3) (p_j^2 + 2\Psi_j(z_1, z_2, z_3)) := \frac{1}{2} \sum_{j=1}^3 B_j p_j^2 + U_1. \quad (15)$$

We prove that the Hamiltonian system (15) admits the quadratic first

$$\text{integral } F = \frac{1}{2} \sum_{j=1}^3 C_j(z_1, z_2, z_3) (p_j^2 + 2\nu_j(z_1, z_2, z_3)) := \frac{1}{2} \sum_{j=1}^3 C_j p_j^2 + U_2.$$

$$\text{From the Poisson bracket } \{H, F\}^* := \sum_{j=1}^3 \left(\frac{\partial H}{\partial p_j} \frac{\partial F}{\partial z_j} - \frac{\partial H}{\partial z_j} \frac{\partial F}{\partial p_j} \right) = 0$$

we get that F is a first integral if and only if the following partial

differential system holds

$$\begin{aligned} B_1 \frac{\partial C_1}{\partial z_1} - C_1 \frac{\partial B_1}{\partial z_1} &= 0, & B_2 \frac{\partial C_1}{\partial z_2} - C_2 \frac{\partial B_1}{\partial z_2} &= 0, & B_3 \frac{\partial C_1}{\partial z_3} - C_3 \frac{\partial B_1}{\partial z_3} &= 0, \\ B_2 \frac{\partial C_2}{\partial z_2} - C_2 \frac{\partial B_2}{\partial z_2} &= 0, & B_1 \frac{\partial C_2}{\partial z_1} - C_1 \frac{\partial B_2}{\partial z_1} &= 0, & B_3 \frac{\partial C_2}{\partial z_3} - C_3 \frac{\partial B_2}{\partial z_3} &= 0, \\ B_3 \frac{\partial C_3}{\partial z_3} - C_3 \frac{\partial B_3}{\partial z_3} &= 0, & B_2 \frac{\partial C_3}{\partial z_2} - C_2 \frac{\partial B_3}{\partial z_2} &= 0, & B_1 \frac{\partial C_3}{\partial z_1} - C_1 \frac{\partial B_3}{\partial z_1} &= 0, \end{aligned} \quad (16)$$

and

$$B_1 \frac{\partial U_1}{\partial z_1} - C_1 \frac{\partial U_2}{\partial z_1} = 0, \quad B_2 \frac{\partial U_1}{\partial z_2} - C_2 \frac{\partial U_2}{\partial z_2} = 0, \quad B_3 \frac{\partial U_1}{\partial z_3} - C_3 \frac{\partial U_2}{\partial z_3} = 0.$$

Simple computations prove that the partial differential system (16) is equivalent to the system

$$\begin{aligned} C_1 &= \Lambda_1(z_2, z_3)B_1, & C_2 &= \Lambda_2(z_1, z_3)B_2, & C_3 &= \Lambda_3(z_1, z_2)B_3, \\ \frac{\partial}{\partial z_2}(C_1 - \Lambda_1(z_1, z_3)B_1) &= 0, & \frac{\partial}{\partial z_3}(C_1 - \Lambda_3(z_1, z_2)B_1) &= 0, \\ \frac{\partial}{\partial z_1}(C_2 - \Lambda_1(z_2, z_3)B_2) &= 0, & \frac{\partial}{\partial z_3}(C_2 - \Lambda_3(z_1, z_2)B_2) &= 0, \\ \frac{\partial}{\partial z_2}(C_3 - \Lambda_2(z_1, z_3)B_3) &= 0, & \frac{\partial}{\partial z_1}(C_3 - \Lambda_1(z_2, z_3)B_3) &= 0, \end{aligned} \quad (17)$$

We apply this result when the first integrals are

$$\begin{aligned} F_1 &= H = \sum_{j=1}^3 \frac{P_j^2}{2} \frac{\{z_j, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} + \frac{\{\psi, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, \\ F_2 &= F = \sum_{j=1}^3 \frac{P_j^2}{2} \frac{\{\varphi_1, z_j, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} + \frac{\{\varphi_1, \psi, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}. \end{aligned}$$

Hence we get that

$$\begin{aligned} C_1 &= \frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, & C_2 &= \frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, & C_3 &= \frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, \\ B_1 &= \frac{\{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, & B_2 &= \frac{\{\varphi_1, z_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, & B_3 &= \frac{\{\varphi_1, z_3, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}, \end{aligned}$$

and consequently the relations (17) become

$$\begin{aligned} \frac{\partial}{\partial z_3} \left(\frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} - \frac{\frac{\{z_3, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \frac{\{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}}{\frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}} \right) &= 0 \\ \frac{\partial}{\partial z_2} \left(\frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} - \frac{\frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \frac{\{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}}{\frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}} \right) &= 0 \\ \frac{\partial}{\partial z_1} \left(\frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} - \frac{\frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \frac{\{\varphi_1, z_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}}{\frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}} \right) &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial z_3} \left(\frac{\frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} - \frac{\frac{\{z_3, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \frac{\{\varphi_1, z_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}}{\frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}} \right) &= 0 \\ \frac{\partial}{\partial z_2} \left(\frac{\frac{\{z_3, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} - \frac{\frac{\{z_2, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \frac{\{\varphi_1, z_3, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}}{\frac{\{z_3, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}} \right) &= 0, \\ \frac{\partial}{\partial z_1} \left(\frac{\frac{\{z_3, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} - \frac{\frac{\{z_1, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \frac{\{\varphi_1, z_3, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}}{\frac{\{z_3, \varphi_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}}} \right) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial z_3} \left(\frac{\{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\} - \{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \right) &= 0, \\ \frac{\partial}{\partial z_2} \left(\frac{\{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\} - \{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \right) &= 0, \\ \frac{\partial}{\partial z_1} \left(\frac{\{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\} - \{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\}}{\{\varphi_3, \varphi_2, \varphi_3\}} \right) &= 0, \\ \frac{\partial}{\partial z_3} \left(\frac{\{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\} - \{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \right) &= 0, \\ \frac{\partial}{\partial z_2} \left(\frac{\{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\} - \{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \right) &= 0, \\ \frac{\partial}{\partial z_1} \left(\frac{\{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\} - \{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\}}{\{\varphi_3, \varphi_2, \varphi_3\}} \right) &= 0, \end{aligned} \quad (18)$$

Finally in view of the first identity (5) we get that

$$\begin{aligned} &\frac{\{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\} - \{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \\ &= \{z_1, \varphi_3, z_3\} = \frac{\partial \varphi_3}{\partial z_2}, \\ &\frac{\{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\} - \{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \\ &= \{z_1, \varphi_3, z_2\} = -\frac{\partial \varphi_3}{\partial z_3}, \\ &\frac{\{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\} - \{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\}}{\{\varphi_3, \varphi_2, \varphi_3\}} \\ &= \{z_2, \varphi_3, z_1\} = \frac{\partial \varphi_3}{\partial z_3}, \\ &\frac{\{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\} - \{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \\ &= \{z_2, \varphi_3, z_3\} = \frac{\partial \varphi_3}{\partial z_1}, \\ &\frac{\{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_2, \varphi_3\} - \{z_2, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\}}{\{\varphi_1, \varphi_2, \varphi_3\}} \\ &= \{z_3, \varphi_3, z_2\} = -\frac{\partial \varphi_3}{\partial z_1}, \\ &\frac{\{z_3, \varphi_2, \varphi_3\} \{\varphi_1, z_1, \varphi_3\} - \{z_1, \varphi_2, \varphi_3\} \{\varphi_1, z_3, \varphi_3\}}{\{\varphi_3, \varphi_2, \varphi_3\}} \\ &= \{z_3, \varphi_3, z_1\} = -\frac{\partial \varphi_3}{\partial z_2} \end{aligned}$$

Thus in view of (18) we get that

$$\frac{\partial^2 \varphi_3}{\partial z_3 \partial z_2} = \frac{\partial^2 \varphi_3}{\partial z_3 \partial z_1} = \frac{\partial^2 \varphi_3}{\partial z_1 \partial z_2} = 0,$$

then $\varphi_3 = \varphi_3(z_3)$. Consequently $\{F_1, F_2\}^* = 0$.

If there exists the first integral

$$F_3 = \sum_{j=1}^3 \frac{P_j^2}{2} \frac{\{\varphi_1, \varphi_2, z_j\}}{\{\varphi_1, \varphi_2, \varphi_3\}} + \frac{\{\varphi_1, \varphi_3, \psi\}}{\{\varphi_1, \varphi_2, \varphi_3\}},$$

then in an analogously way we can prove that $\{F_1, F_3\}^* = 0$ and $\{F_3, F_2\}^* = 0$. if and only if $\frac{\partial^2 \varphi_j}{\partial z_3 \partial z_2} = \frac{\partial^2 \varphi_j}{\partial z_3 \partial z_1} = \frac{\partial^2 \varphi_j}{\partial z_1 \partial z_2} = 0$, for $j = 1, 2$. Consequently $\varphi_1 = \varphi_1(z_1)$ and $\varphi_2 = \varphi_2(z_2)$. Thus the involution of the given first integrals is proved and consequently the theorem follows. $\square$

3. Applications of the solution of the Stäckel inverse problem II.

We shall illustrated the solution of the Inverse Stäckel problem II in the next examples.

Example 1. The following problem is the *inverse problem of two fixed centers* (for more details see [17,19]).

Let P be a particle of infinitesimal mass which is attracted by two fixed centers C_0 and C_1 with masses m_1 and m_2 , respectively. Selecting coordinates so that the origin coincides with the center of mass and the abscissa passing through the points C_0 and C_1 and designating by r_0, r_1 and $2c$ the distances between $C_0(x_0, 0, 0) - P(x, y, z)$, $C_1(x_1, 0, 0) - P(x, y, z)$ and $C_0(x_0, 0, 0) - C_1(x_1, 0, 0)$ respectively, we obtain that

$$r_0 = \sqrt{(x - x_0)^2 + y^2 + z^2}, \quad r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}, \quad 2c = |x_1 - x_0|.$$

With these notations the particle moving in the configuration space $\mathbb{R}^3$ with the Lagrangian function $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \left(\frac{m_0}{r_0} + \frac{m_1}{r_1}\right)f$, where f is the attraction constant (see [11]).

After the change of coordinates

$$x = \frac{m_0 - m_1}{m_1 + m_0}c + c\lambda\mu, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}\cos w,$$

$$z = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}\sin w,$$

we obtain that the Lagrangian function L writes

$$L = c^2 \left(\frac{(\lambda^2 - \mu^2)\lambda^2}{4(\lambda^2 - 1)} - \frac{(\lambda^2 - \mu^2)\dot{\mu}^2}{4(1 - \mu^2)} + \frac{(\lambda^2 - 1)(1 - \mu^2)\dot{w}^2}{2} \right) - \frac{f((m_0 + m_1)\lambda + (m_1 - m_0)\mu)}{c(\lambda^2 - \mu^2)},$$

and $r_0 = c(\lambda + \mu)$, $r_1 = c(\lambda - \mu)$, where $1 \leq \lambda < +\infty$, $-1 \leq \mu \leq 1$ and $0 \leq w \leq 2\pi$. Hence we get that the generalized momenta are

$$p_\lambda = c^2 \frac{(\lambda^2 - \mu^2)\dot{\lambda}}{2(\lambda^2 - 1)}, \quad p_\mu = c^2 \frac{(\lambda^2 - \mu^2)\dot{\mu}}{2(1 - \mu^2)}, \quad p_w = c^2 \frac{(\lambda^2 - 1)(1 - \mu^2)\dot{w}}{2}$$

Clearly that the matrix G^{-1} in this case is

$$\begin{pmatrix} \frac{2(\lambda^2 - 1)}{c^2(\lambda^2 - \mu^2)} & 0 & 0 \\ 0 & \frac{2(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} & 0 \\ 0 & 0 & \frac{1}{c^2(\lambda^2 - 1)(1 - \mu^2)} \end{pmatrix}.$$

Now we solve the inverse Stäckel problem II for the problem is the of two fixed centers.

Hence the Eq. (10) now write

$$\begin{aligned} 0 &= \frac{2(\lambda^2 - 1)}{c^2(\lambda^2 - \mu^2)} \frac{\partial \varphi_j}{\partial \lambda} + \frac{2(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} \frac{\partial \varphi_j}{\partial \mu} + \frac{1}{c^2(\lambda^2 - 1)(1 - \mu^2)} \frac{\partial \varphi_j}{\partial w}, \\ j &= 1, 2, \\ 1 &= \frac{2(\lambda^2 - 1)}{c^2(\lambda^2 - \mu^2)} \frac{\partial \varphi_3}{\partial \lambda} + \frac{2(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} \frac{\partial \varphi_3}{\partial \mu} + \frac{1}{c^2(\lambda^2 - 1)(1 - \mu^2)} \frac{\partial \varphi_3}{\partial w}, \\ 0 &= \frac{\partial}{\partial \lambda} \left(\frac{2(\lambda^2 - 1)}{c^2(\lambda^2 - \mu^2)} \{\varphi_1, \varphi_2, \varphi_3\} \right) + \frac{\partial}{\partial \mu} \left(\frac{2(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} \{\varphi_1, \varphi_2, \varphi_3\} \right) \\ &\quad + \frac{\partial}{\partial w} \left(\frac{2(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} \{\varphi_1, \varphi_2, \varphi_3\} \right), \end{aligned} \quad (19)$$

and

$$\frac{2(\lambda^2 - 1)}{c^2(\lambda^2 - \mu^2)} \frac{\partial \Psi}{\partial \lambda} + \frac{2(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} \frac{\partial \Psi}{\partial \mu} + \frac{1}{c^2(\lambda^2 - 1)(1 - \mu^2)} \frac{\partial \Psi}{\partial w} = U.$$

The functions

$$\begin{aligned} \varphi_1 &= \log \left| \frac{\lambda - 1}{\lambda + 1} \right| + \log \left| \frac{\mu - 1}{\mu + 1} \right|, \quad \varphi_2 = \frac{1}{\lambda + 1} + \frac{1}{\lambda - 1} + \frac{1}{\mu - 1} + \frac{1}{\mu + 1}, \\ \varphi_3 &= c^2(\lambda + \mu), \quad \Psi = fc \left(\frac{m_1 + m_0}{4} \ln |\lambda^2 - 1| + \frac{-m_1 + m_0}{4} \ln |1 - \mu^2| \right) \end{aligned}$$

satisfy the Eq. (19). Thus $|\Delta| = \{\varphi_1, \varphi_2, \varphi_3\} = \frac{4c^2(\lambda^2 - \mu^2)}{(\mu^2 - 1)(\lambda^2 - 1)}$,

Thus the Stäckel matrix becomes

$$\Delta = \begin{pmatrix} \frac{1}{(\lambda^2 - 1)^2} & -\frac{1 + \lambda^2}{(\lambda^2 - 1)^2} & c^2 \\ \frac{1}{(\mu^2 - 1)^2} & -\frac{1 + \mu^2}{(\mu^2 - 1)^2} & c^2 \\ 0 & 2 & 0 \end{pmatrix}.$$

After some computations we obtain that the function (12) in this case is

$$\begin{aligned} S &= \int_{\lambda_0}^{\lambda} \sqrt{\frac{1}{(\lambda^2 - 1)^2} \alpha_1 - \frac{1 + \lambda^2}{(\lambda^2 - 1)^2} \alpha_2 + c^2 \alpha_3 + \frac{(m_1 + m_0)}{2} \frac{\lambda}{\lambda^2 - 1} \alpha_1} d\lambda \\ &\quad \int_{\mu_0}^{\mu} \sqrt{\frac{1}{(\mu^2 - 1)^2} \alpha_1 - \frac{1 + \mu^2}{(\mu^2 - 1)^2} \alpha_2 + c^2 \alpha_3 - \frac{(m_1 - m_0)}{2} \frac{\mu}{\mu^2 - 1} \alpha_1} d\mu \\ &\quad + \sqrt{2} w, \end{aligned}$$

α_1, α_2 , and α_3 are constants such that $F_j = \alpha_j$ for $j = 1, 2, 3$. This function is the classical Jacobi function for the problem of the two fixed centers.

By considering that the inverse matrix Δ^{-1} is

$$\begin{pmatrix} \frac{(1 - \mu^2)(\lambda^2 - 1)}{2(\lambda^2 - \mu^2)} & -\frac{(1 - \mu^2)(\lambda^2 - 1)}{2(\lambda^2 - \mu^2)} & -\frac{\lambda^2 \mu^2 + \lambda^2 + \mu^2 - 3}{4(1 - \mu^2)(\lambda^2 - 1)} \\ 0 & 0 & \frac{1}{2} \\ \frac{(\lambda^2 - 1)}{c^2(\lambda^2 - \mu^2)} & \frac{(1 - \mu^2)}{c^2(\lambda^2 - \mu^2)} & \frac{1}{c^2(\lambda^2 - 1)(1 - \mu^2)} \end{pmatrix}.$$

Consequently

$$\begin{aligned} \Delta^{-1} &= \begin{pmatrix} \frac{(m_1 + m_0)}{2} \frac{\lambda}{\lambda^2 - 1} \\ -\frac{(m_1 - m_0)}{2} \frac{\mu}{\mu^2 - 1} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{fc}{\lambda^2 - \mu^2} ((m_1 + m_0)(1 - \mu^2)\lambda + (-m_1 + m_0)\mu(\lambda^2 - 1)) \\ 0 \\ \frac{fc}{\lambda^2 - \mu^2} ((m_1 + m_0)\lambda + (m_1 - m_0)\mu) \end{pmatrix}. \end{aligned}$$

Therefore the first integrals (14) are

$$\begin{aligned}
F_1 &= \frac{(1-\mu^2)(\lambda^2-1)}{2(\lambda^2-\mu^2)} \left(p_\lambda^2 - p_\mu^2 \right) - \frac{\lambda^2\mu^2 + \lambda^2 + \mu^2 - 3}{4(1-\mu^2)(\lambda^2-1)} p_w^2 \\
&\quad + \frac{fc}{\lambda^2 - \mu^2} \left((m_1 + m_0)(1-\mu^2)\lambda + (-m_1 + m_0)\mu(\lambda^2-1) \right) \\
F_2 &= \frac{p_w^2}{2}, \\
F_3 &= \frac{(\lambda^2-1)}{2(\lambda^2-\mu^2)} p_\lambda^2 + \frac{(1-\mu^2)}{2(\lambda^2-\mu^2)} p_\mu^2 + \frac{1}{c^2(\lambda^2-1)(1-\mu^2)} p_w^2 \\
&\quad + \frac{fc}{\lambda^2 - \mu^2} \left((m_1 + m_0)\lambda + (m_1 - m_0)\mu \right),
\end{aligned}$$

3.1. Solution of the inverse Stäckel problem II for the extended kepler-Coulomb system

We shall study the Hamiltonian system with Hamiltonian

$$H = \sum_{j=1}^M y_j^2/2 + V_1(\sqrt{x_1^2 + x_2^2 + \dots + x_M^2}) - \frac{1}{2} \sum_{j=1}^M \frac{\beta_j}{x_j^2},$$

where $\beta_1, \beta_2, \dots, \beta_M$ are constants.

We recall that the Hamiltonian system with M degrees of freedom and having $S > M$ independent first integrals is called superintegrable.

Proposition 1. *The extended Kepler-Coulomb system is superintegrable for $M > 2$.*

Proof. Indeed, after some computations we can prove that this system have $(M - 1)M/2$ first integrals

$$F_{jk} = (x_j y_k - x_k y_j)^2 + (x_j^2 + x_k^2) \left(\frac{\beta_j}{x_j^2} + \frac{\beta_k}{x_k^2} \right), \quad j, k = 1, \dots, M, \quad j < k. \quad (20)$$

By considering that the matrix (F_{jk}) has the rank equal to $2M - 3$, see [24]. Therefore the system has $2M - 2$ independent first integrals taking into account the Hamiltonian. Since for $M > 2$ we have that $2M - 2 > M$. $\square$

Now we solve the inverse Stäckel problem *II* for the extended Kepler-Coulomb system and consequently we prove that this system when $\beta_1 = 0$ is integrable by separation of variables.

Proposition 2. *The extended Kepler-Coulomb system with $\beta_1 = 0$, i.e. the Hamiltonian system with Hamiltonian*

$$H = \sum_{j=1}^M y_j^2/2 + V_1(\sqrt{x_1^2 + x_2^2 + \dots + x_M^2}) - \frac{1}{2} \sum_{j=2}^M \frac{\beta_j}{x_j^2}, \quad (21)$$

is integrable by separation of variables.

Proof. Thus after the change of coordinates

$$x_1 = r \sin \vartheta_{M-1} \prod_{j=1}^{M-2} \sin \vartheta_j \neq 0,$$

$$x_2 = r \cos \vartheta_{M-1} \prod_{i=1}^{M-2} \sin \vartheta_j,$$

$$x_3 = r \cos \vartheta_{M-2} \prod_{j=3}^{M-3} \sin \vartheta_j,$$

$$x_4 = r \cos \vartheta_{M-3} \prod_{j=1}^{M-4} \sin \vartheta_j,$$

$$\begin{array}{ccccccc} & & & & & & \\ \vdots & & \vdots & & \vdots & & \vdots \\ & & & & & & \end{array}$$

$$x_{M-2} = r \cos \vartheta_3 \sin \vartheta_1 \sin \vartheta_2,$$

$$x_{M-1} = r \cos \vartheta_2 \sin \vartheta_1,$$

$$x_M = r \cos \vartheta_1,$$

we get that $\sum_{j=1}^M x_j^2 = r^2$ and the Hamiltonian function (21) becomes

$$\sum_{j=1}^M G^{jj} P_j^2 / 2 + V_1(r) + \frac{\beta_M}{r^2 \cos^2 \vartheta_1^2} + \frac{\beta_{M-1}}{r^2 \cos^2 \vartheta_2 \sin^2 \vartheta_1} + \dots$$

$$+ \frac{\beta_2}{r^2 \cos^2 \vartheta_{M-1}^2 \prod_{j=1}^{M-2} \sin^2 \vartheta_j},$$

where $G^{-1} = (G^{jj})$ is the inverse Riemann metric given by

$$G^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{r^2} & 0 & \dots & \vdots \\ 0 & 0 & \frac{1}{r^2 \sin^2(\vartheta_1)} & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{r^2 \prod_{k=1}^{M-2} \sin^2(\vartheta_j)} \end{pmatrix}$$

Now we solve the inverse Stäckel problem II. Therefore the differential system (10) in this case can be written as

$$\frac{\partial \varphi_k}{\partial r} + \frac{1}{r^2} \frac{\partial \varphi_k}{\partial \theta_1} + \frac{1}{r^2 \sin^2(\theta_1)} \frac{\partial \varphi_k}{\partial \theta_2} + \dots$$

$$+ \frac{1}{r^2 \prod_{j=1}^{M-2} \sin^2(\theta_j)} \frac{\partial \varphi_k}{\partial \theta_{M-1}} = 0, \quad k = 2, \dots, M,$$

$$\frac{\partial \varphi_1}{\partial r} + \frac{1}{r^2} \frac{\partial \varphi_1}{\partial \vartheta_1} + \frac{1}{r^2 \sin^2(\vartheta_1)} \frac{\partial \varphi_1}{\partial \vartheta_2} + \dots + \frac{1}{r^2 \prod_{k=1}^{M-2} \sin^2(\vartheta_j)} \frac{\partial \varphi_1}{\partial \vartheta_M} = 1,$$

$$\frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial \Psi}{\partial \vartheta_1} + \frac{1}{r^2 \sin^2(\vartheta_1)} \frac{\partial \Psi}{\partial \vartheta_2} + \dots + \frac{1}{r^2 \prod_{k=1}^{M-2} \sin^2(\vartheta_j)} \frac{\partial \Psi}{\partial \vartheta_{M-1}} = U,$$

$$\begin{aligned} \frac{\partial}{\partial r}(\{\varphi_1, \dots, \varphi_M\}) + \frac{\partial}{\partial \vartheta_1} \left(\frac{1}{r^2} \{\varphi_1, \dots, \varphi_M\} \right) + \dots \\ + \frac{\partial}{\partial \vartheta_{M-1}} \left(\frac{\{\varphi_1, \dots, \varphi_M\}}{r^2} \prod_{k=1}^{M-2} \frac{1}{\sin^2(\vartheta_i)} \right) = 0. \end{aligned}$$

The particular solutions of this system are

$$\varphi_1 = r, \varphi_2 = \frac{1}{r} + \vartheta_1, \varphi_3 = \cot(\vartheta_1) + \vartheta_2, \dots, \varphi_M = \cot(\vartheta_{M-1}) + \vartheta_{M-1},$$

$$\Psi = \int V(r)dr + \sum_{j=2}^{M-1} \beta_{M+1-j} \tan \vartheta_j,$$

then the Stäckel matrix becomes

$$\Delta = \begin{pmatrix} 1 & -\frac{1}{r^2} & 0 & \dots & 0 & 0 \\ 0 & 1 & -\frac{1}{\sin^2(\theta_1)} & \dots & 0 & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{1}{\sin^2(\theta_{M-1})} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$|\Delta| = (\varphi_1, \dots, \varphi_{M-1}) = 1.$$

Therefore the inverse matrix is

$$\Delta^{-1} = \begin{pmatrix} 1 & \frac{1}{r^2 \sin^2(\theta_1)} & \frac{1}{r^2} \frac{1}{\sin^2(\theta_1) \sin^2(\theta_2)} & \cdots & \frac{1}{r^2} \prod_{j=1}^{M-1} \frac{1}{\sin^2(\theta_j)} \\ 0 & 1 & \frac{1}{\sin^2(\theta_1)} & \cdots & \prod_{j=2}^{M-1} \frac{1}{\sin^2(\theta_j)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{\sin^2(\theta_{M-1})} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Consequently the first integrals (14) becomes

$$\begin{aligned} F_1 = H = P_r^2 + \frac{P_{\theta_1}^2}{r^2 \sin^2(\theta_1)} + \cdots + \frac{P_{\theta_{M-1}}^2}{r^2} \prod_{j=1}^{M-2} \frac{1}{\sin^2(\theta_j)} + U, \\ F_2 = P_{\theta_1}^2 + \frac{\beta_3}{\cos^2(\theta_1)} + \frac{F_3}{\sin^2(\theta_1)}, \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots, \\ F_{M-1} = P_{\theta_{M-2}}^2 + \frac{\beta_3}{\cos^2(\theta_{M-2})} + \frac{F_M}{\sin^2(\theta_{M-2})}, \\ F_M = P_{\theta_{M-1}}^2 + \frac{\beta_2}{\cos^2(\theta_{M-1})}. \end{aligned}$$

Finally we observe that if $V_1(r) = \frac{1}{r}$ then the studied Hamiltonian system is the extended Kepler-Coulomb system, and if $V_1(r) = r^2$ then we deduce the extended harmonic oscillator (see for instance [25,26]). $\square$

3.2. The Stark problem in arbitrary dimension

In this subsection we illustrate the solution of the inverse Stäckel problem II in the study of the Stark problem for arbitrary dimension (see [27]).

The main integrable problems of Celestial Mechanics is the Kepler problem and the Stark problem. This small number of integrable problems explains why the Stark problem, corresponding to the motion in a Newtonian gravitational field subjected to an additional uniform force of constant magnitude and direction, has received special attention over almost two and one-half centuries (see for instance [28–31]). The integrability in the Jacobi sense of the two and three dimensional Stark problems was first established in [30,32]. In [27] we extend this problem to arbitrary dimension and show that it continues being integrable in the Jacobi sense. In this section we solve the Stäckel problem II for this mechanical system.

Two dimensional Stark problem. In the (x, y) plane we consider the motion of an arbitrary point P in the gravitational field induced by a body at the origin, and subjected to an additional constant inertial force. Without loss of generality we can assume that this force is in the y -direction since the arbitrary direction can be arrived by means of a trivial coordinate rotation. The corresponding planar equations of motion are

$$\ddot{x} = -\frac{\mu x}{r^3}, \quad \ddot{y} = -\frac{\mu y}{r^3} + \varepsilon,$$

where $r = \sqrt{x^2 + y^2}$ and ε is a parameter fixing the value of the constant force.

The Lagrangian function is $\tilde{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{r} - \varepsilon y$. Under the change of variables

$$\sigma : \quad x = \xi \eta, \quad y = \frac{\xi^2 - \eta^2}{2},$$

we get that the function $\tilde{L}$ becomes

$$L = \tilde{L}|_{\sigma} = \frac{1}{2}(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) - \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2}(\xi^2 - \eta^2).$$

Hence the Hamiltonian function $\tilde{H}$ in coordinates (ξ, η) can be written as follows

$$H = \tilde{H}|_{\sigma} = \frac{p_{\xi}^2 + p_{\eta}^2}{2(\xi^2 + \eta^2)} - \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2}(\xi^2 - \eta^2).$$

Consequently, from (9) we obtain that $G^{11} = \frac{1}{\xi^2 + \eta^2}$ and $G^{22} = \frac{1}{\xi^2 + \eta^2}$.

Now we solve the invers Stäckel problem II for the two dimensional Stark problem. Thus we have

$$\frac{\{\varphi_1, \xi\}}{\{\varphi_1, \varphi_2\}} = \frac{1}{\xi^2 + \eta^2}, \quad \frac{\{\varphi_1, \eta\}}{\{\varphi_1, \varphi_2\}} = \frac{1}{\xi^2 + \eta^2}.$$

Therefore the partial differential Eqs. (10) and (11) in this case are

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \xi} + \frac{\partial \varphi_1}{\partial \eta} &= 0, \\ \frac{\partial \varphi_2}{\partial \xi} + \frac{\partial \varphi_2}{\partial \eta} &= \xi^2 + \eta^2, \\ \frac{\partial}{\partial \xi} \left(\frac{\{\varphi_1, \varphi_2\}}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \eta} \left(\frac{\{\varphi_1, \varphi_2\}}{\xi^2 + \eta^2} \right) &= 0. \end{aligned}$$

The solutions of these equations are $\varphi_1 = \xi - \eta$ and $\varphi_2 = \frac{\xi^3 + \eta^3}{3}$. Therefore $\{\varphi_1, \varphi_2\} = \xi^2 + \eta^2$.

On the other hand, by solving the equation

$$\{\varphi_1, \Psi\} = -2\mu - \frac{\varepsilon}{2}(\xi^4 - \eta^4),$$

we get that $\Psi = -\mu(\xi + \eta) - \frac{\varepsilon}{10}(\xi^5 - \eta^5)$. Therefore

$$\Psi_{\xi} = -\mu - \frac{\varepsilon}{2}\xi^4, \quad \Psi_{\eta} = -\mu + \frac{\varepsilon}{2}\eta^4.$$

The Stäckel matrix Δ and Δ^{-1} are

$$\Delta = \begin{pmatrix} 1 & \xi^2 \\ -1 & \eta^2 \end{pmatrix}, \quad \Delta^{-1} = \begin{pmatrix} \frac{\eta^2}{\xi^2 + \eta^2} & -\frac{\xi^2}{\xi^2 + \eta^2} \\ \frac{1}{\xi^2 + \eta^2} & \frac{1}{\xi^2 + \eta^2} \end{pmatrix}.$$

Therefore, the first integrals are

$$F_1 = H_1|_{\sigma} = \frac{\eta^2 p_{\xi}^2}{2(\xi^2 + \eta^2)} - \frac{\xi^2 p_{\eta}^2}{2(\xi^2 + \eta^2)} - \frac{\xi^2 - \eta^2}{\xi^2 + \eta^2} \mu - \frac{\varepsilon}{2} \eta^2 \xi^2,$$

$$F_2 = H = \frac{p_{\xi}^2 + p_{\eta}^2}{2(\xi^2 + \eta^2)} + \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2}(\xi^2 - \eta^2).$$

Clearly that Jacobi-Stäckel function is

$$S = -\alpha t + \int_{\xi_0}^{\xi} \sqrt{\alpha_1 + \alpha_2 \xi^2} d\xi + \int_{\eta_0}^{\eta} \sqrt{-\alpha_1 + \alpha_2 \eta^2} d\eta.$$

Returning back to Cartesian coordinates the first integral H_1 can be written as

$$H_1 = \dot{x}(y\dot{x} - x\dot{y}) - \frac{2\mu y}{r} + \frac{\varepsilon x^2}{2} = \frac{\partial F}{\partial y} + \frac{\varepsilon x^2}{2},$$

where $F = (y\dot{x} - x\dot{y})^2/2 + 2\mu\sqrt{x^2 + y^2}$. In [30] the integrals of motion are determined, and the resulting quadratures are analytically given. A complete list of exact closed-form solutions is deduced in terms of elliptic functions.

Three dimensional Stark problem. As in the 2D case we can assume without loss of generality that this force is in the z -direction. The corresponding equations of motion are

$$\ddot{x} = -\frac{\mu x}{r^3}, \quad \ddot{y} = -\frac{\mu y}{r^3}, \quad \ddot{z} = -\frac{\mu z}{r^3} + \varepsilon. \quad (22)$$

Clearly that the functions

$$H_1 = x\dot{y} - y\dot{x},$$

$$H_2 = \dot{x}(z\dot{x} - x\dot{z}) - \dot{y}(z\dot{y} - y\dot{z}) - \frac{\mu z}{\sqrt{x^2 + y^2 + z^2}} + \frac{\varepsilon}{2}(x^2 + y^2)$$

$$= \frac{\partial F}{\partial z} + \varepsilon \frac{x^2 + y^2}{2},$$

where $2F = (x^2 + y^2 + z^2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (x\dot{x} + y\dot{y} + z\dot{z})^2$, are first integrals of the differential system (22).

After the change of variables through the following relations:

$$\sigma : \quad x = \xi\eta \cos \Phi, \quad y = \xi\eta \sin \Phi, \quad z = \frac{\xi^2 - \eta^2}{2},$$

and doing some computations we get that the Hamiltonian function H becomes

$$H|_{\sigma} = \frac{p_{\xi}^2 + p_{\eta}^2}{2(\xi^2 + \eta^2)} + \frac{p_{\Phi}^2}{2\xi^2\eta^2} - \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2}(\xi^2 - \eta^2).$$

where $p_{\xi} = (\xi^2 + \eta^2) \frac{d\xi}{dt}$, and $p_{\eta} = (\xi^2 + \eta^2) \frac{d\eta}{dt}$. Thus, from (8) we obtain that

$$G^{11} = G^{22} = \frac{1}{\xi^2 + \eta^2}, \quad G^{33} = \frac{1}{\xi^2\eta^2}.$$

Now we solve the inverse Stäckel problem II for the three dimensional Stark problem. Since

$$\frac{\{\varphi_1, \xi\}}{\{\varphi_1, \varphi_2\}} = \frac{1}{\xi^2 + \eta^2}, \quad \frac{\{\varphi_1, \eta\}}{\{\varphi_1, \varphi_2\}} = \frac{1}{\xi^2 + \eta^2}, \quad \frac{\{\varphi_1, \Phi\}}{\{\varphi_1, \varphi_2\}} = \frac{1}{\xi^2\eta^2},$$

for determining the Stäckel matrix we need to solve the partial differential equations from (10):

$$\frac{\partial \varphi_j}{\partial \xi} + \frac{\partial \varphi_j}{\partial \eta} + \frac{\partial \varphi_j}{\partial \Phi} = 0, \quad j = 1, 2,$$

$$\frac{\partial \varphi_3}{\partial \xi} + \frac{\partial \varphi_3}{\partial \eta} + \frac{\partial \varphi_3}{\partial \Phi} = 1,$$

$$\frac{\partial}{\partial \xi} \left(\frac{\{\varphi_1, \varphi_2, \varphi_3\}}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \eta} \left(\frac{\{\varphi_1, \varphi_2, \varphi_3\}}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \Phi} \left(\frac{\{\varphi_1, \varphi_2, \varphi_3\}}{\xi^2\eta^2} \right) = 0,$$

Clearly that this system can be rewritten as follows

$$\frac{\partial \varphi_j}{\partial \xi} + \frac{\partial \varphi_j}{\partial \eta} + \left(\frac{1}{\xi^2} + \frac{1}{\eta^2} \right) \frac{\partial \varphi_j}{\partial \Phi} = 0, \quad j = 1, 2,$$

$$\frac{\partial \varphi_3}{\partial \xi} + \frac{\partial \varphi_3}{\partial \eta} + \left(\frac{1}{\xi^2} + \frac{1}{\eta^2} \right) \frac{\partial \varphi_3}{\partial \Phi} = \xi^2 + \eta^2,$$

$$\frac{\partial}{\partial \xi} \left(\frac{\{\varphi_1, \varphi_2, \varphi_3\}}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \eta} \left(\frac{\{\varphi_1, \varphi_2, \varphi_3\}}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \Phi} \left(\frac{\{\varphi_1, \varphi_2, \varphi_3\}}{\xi^2\eta^2} \right) = 0,$$

Particular solutions of this system are

$$\varphi_1 = \frac{1}{\xi} + \frac{1}{\eta} + \Phi, \quad \varphi_2 = \xi - \eta, \quad \varphi_3 = \frac{\xi^3 + \eta^3}{3}, \quad \{\varphi_1, \varphi_2, \varphi_3\} = \xi^2 + \eta^2.$$

Therefore the Stäckel matrix is

$$\Delta = \begin{pmatrix} -\frac{1}{\xi^2} & 1 & \xi^2 \\ -\frac{1}{\eta^2} & -1 & \eta^2 \\ 1 & 0 & 0 \end{pmatrix},$$

with $|\Delta| = \xi^2 + \eta^2$. Consequently the inverse matrix is

$$\Delta^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{\eta^2}{\eta^2 + \xi^2} & -\frac{\xi^2}{\eta^2 + \xi^2} & \frac{\eta^2 - \xi^2}{\eta^2\xi^2} \\ \frac{1}{\eta^2 + \xi^2} & \frac{1}{\eta^2 + \xi^2} & \frac{1}{\eta^2\xi^2} \end{pmatrix}.$$

The first integrals (14) in this case become

$$F_1 = H|_{\sigma} = p_{\Phi}^2,$$

$$F_2 = H_2|_{\sigma} = \frac{\eta^2}{\xi^2 + \eta^2} p_{\xi}^2 - \frac{\xi^2}{\xi^2 + \eta^2} p_{\eta}^2 + \frac{\eta^2 - \xi^2}{\xi^2\eta^2} p_{\Phi}^2 + \mu \frac{(\eta^2 - \xi^2)}{\eta^2 + \xi^2} - \frac{\varepsilon}{2} \xi^2\eta^2,$$

$$F_3 = H|_{\sigma} = \frac{1}{\xi^2 + \eta^2} p_{\xi}^2 + \frac{1}{\xi^2 + \eta^2} p_{\eta}^2 + \frac{1}{\xi^2\eta^2} p_{\Phi}^2 + \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2).$$

The Jacobi-Stäckel function is

$$S = -\alpha t + \Phi + \int_{\xi_0}^{\xi} \sqrt{-\frac{\alpha_1}{\xi^2} + \alpha_2 + \xi^2 \alpha_3} d\xi + \int_{\eta_0}^{\eta} \sqrt{-\frac{\alpha_1}{\eta^2} - \alpha_2 + \eta^2 \alpha_3} d\eta$$

M-dimensional Stark problem. Now we shall illustrate the integrability in the Jacobi sense for the Stark problem in dimension $N > 3$ whose

kinetic energy for unit of mass is $T = \sum_{j=1}^M \dot{x}_j^2$ and its potential function

is $U = -\frac{\mu}{r} - \varepsilon x_N$, where $r = \sqrt{\sum_{j=1}^N x_j^2}$. Hence the Lagrangian function is $\tilde{L} = T - U$.

The equations of motion are

$$\ddot{x}_j = -\frac{\mu x_j}{r^3}, \quad \ddot{x}_N = -\frac{\mu x_N}{r^3} + \varepsilon,$$

for $j = 1, \dots, N-1$. The Lagrangian function $\tilde{L}$ is

$$\tilde{L} = \sum_{j=1}^M \frac{\dot{x}_j^2}{2} - \frac{\mu}{r} + \varepsilon x_M$$

The Hamiltonian equations are

$$\dot{x}_j = p_j, \quad j = 1, \dots, M,$$

$$\dot{p}_k = -\frac{\mu x_k}{r^3}, \quad k = 1, \dots, M-1,$$

$$\dot{p}_M = -\frac{\mu x_M}{r^3} + \varepsilon, \quad (23)$$

whose Hamiltonian is

$$\tilde{H} = \sum_{j=1}^M \frac{p_j^2}{2} + \frac{\mu}{r} - \varepsilon x_M. \quad (24)$$

This Hamiltonian system is not integrable in Jacobi sense in cartesian coordinates. However, as we see in the previous results, the authors of [28,31] and others have found that it becomes separable in parabolic coordinates when $M = 2$ and $M = 3$. We develop these results and prove that (23) is integrable in the Jacobi sense for all M .

The Hamiltonian system with Hamiltonian (24) is called the *Stark system*.

Proposition 3. *The Stark system for $M > 3$ has $(M-1)(M-2)/2$ first integrals $F_{jk} = x_j y_k - x_k y_j$, $j, k = 1, \dots, M-1$, the Hamiltonian H and*

$$H_1 = \frac{\partial G}{\partial x_M} - \frac{\varepsilon}{2} \sum_{j=1}^{M-1} x_j^2,$$

where $G = ||x||^2 ||y||^2 - \langle x, y \rangle^2 + \mu ||x||$. Consequently for $M > 3$ the Stark system is superintegrable.

Proof. Trivially it follows after some computations and by considering that the matrix (F_{jk}) for $j, k = 1, \dots, M-1$ has the rank equal to $2M-5$, see [24]. Therefore the system has $2M-3$ independent first integrals. Since for $M > 3$ we have that $2M-3 > M$. $\square$

Proposition 4. The Stark system in the coordinates $\sigma := (\vartheta_1, \vartheta_2, \dots, \vartheta_{M-2}, \xi, \eta)$ where

$$\begin{aligned} x_1 &= \xi \eta \sin \vartheta_{M-2} \prod_{j=1}^{M-3} \sin \vartheta_j \neq 0, \\ x_2 &= \xi \eta \cos \vartheta_{M-2} \prod_{j=1}^{M-3} \sin \vartheta_j, \\ x_3 &= \xi \eta \cos \vartheta_{M-3} \prod_{j=1}^{M-4} \sin \vartheta_j, \\ x_4 &= \xi \eta \cos \vartheta_{M-4} \prod_{j=1}^{M-5} \sin \vartheta_j, \\ &\vdots \\ x_{M-2} &= \xi \eta \cos \vartheta_2 \sin \vartheta_1, \\ x_{M-1} &= \xi \eta \cos \vartheta_1, \\ x_M &= \frac{\xi^2 - \eta^2}{2}, \end{aligned} \quad (25)$$

is integrable in the Jacobi sense.

Proof. First we prove that Hamiltonian function (24) in the new coordinates becomes

$$H = \tilde{H}|_\sigma = \sum_{j=1}^M G^{jj} p_j^2 + \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2), \quad (26)$$

where $G^{-1} = (G^{jj})$ is the matrix

$$G^{-1} = \begin{pmatrix} \frac{1}{\xi^2 + \eta^2} & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\xi^2 + \eta^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{\xi^2 \eta^2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \frac{1}{\xi^2 \eta^2 \sin^2 \vartheta_1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{\xi^2 \eta^2 \prod_{j=1}^{M-3} \sin^2 \vartheta_j} \end{pmatrix}. \quad (27)$$

Indeed by considering that

$$\sum_{j=1}^{M-1} x_j^2 + x_M^2 = \xi^2 \eta^2 + \left(\frac{\xi^2 - \eta^2}{2} \right)^2 = \left(\frac{\xi^2 + \eta^2}{2} \right)^2,$$

after some computations we get that the Lagrangian function becomes

$$\begin{aligned} L = \tilde{L}|_\sigma &= -\frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2) + (\xi^2 + \eta^2) (\dot{\xi}^2 + \dot{\eta}^2) + \\ &\xi^2 \eta^2 \left(\dot{\vartheta}_1^2 + (\sin^2 \vartheta_1) \dot{\vartheta}_2^2 + (\sin^2 \vartheta_1 \sin^2 \vartheta_2) \dot{\vartheta}_3^2 \right. \\ &\left. + \dots + \left(\prod_{j=1}^{M-3} \sin^2 \vartheta_j \right) \dot{\vartheta}_{M-2}^2 \right). \end{aligned} \quad (28)$$

Since the Riemann metric G is

$$G = \begin{pmatrix} \xi^2 + \eta^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & \xi^2 + \eta^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \xi^2 \eta^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \xi^2 \eta^2 \sin^2 \vartheta_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \xi^2 \eta^2 \prod_{j=1}^{M-3} \sin^2 \vartheta_j \end{pmatrix},$$

and its inverse G^{-1} is given in (27). Hence after Legendre transformation we obtain the Hamiltonian (26).

Now we solve the inverse Stäckel problem II. From the relations

$$\begin{aligned} \frac{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \xi\}}{\{\varphi_1, \varphi_2, \varphi_3, \varphi_M\}} &= \frac{1}{\xi^2 + \eta^2}, & \frac{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \eta\}}{\{\varphi_1, \varphi_2, \varphi_3, \varphi_M\}} &= \frac{1}{\xi^2 + \eta^2}, \\ \frac{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \vartheta_1\}}{\{\varphi_1, \varphi_2, \varphi_3, \varphi_M\}} &= \frac{1}{\xi^2 \eta^2}, & \frac{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \vartheta_2\}}{\{\varphi_1, \varphi_2, \varphi_3, \varphi_M\}} &= \frac{1}{\xi^2 \eta^2 \sin^2 \vartheta_1}, \\ & & & \vdots \\ \frac{\{\varphi_1, \dots, \varphi_{M-1}, \vartheta_{M-3}\}}{\{\varphi_1, \varphi_2, \varphi_3, \varphi_M\}} &= \frac{1}{\xi^2 \eta^2 \prod_{j=1}^{M-4} \sin^2 \vartheta_j}, \\ \frac{\{\varphi_1, \dots, \varphi_{M-1}, \vartheta_{M-2}\}}{\{\varphi_1, \varphi_2, \varphi_3, \varphi_M\}} &= \frac{1}{\xi^2 \eta^2 \prod_{j=1}^{M-3} \sin^2 \vartheta_j}. \end{aligned}$$

we get that the partial differential systems (10) and (11)

$$\begin{aligned} 0 &= \frac{\frac{\partial \varphi_j}{\partial \xi}}{\xi^2 + \eta^2} + \frac{\frac{\partial \varphi_j}{\partial \eta}}{\xi^2 + \eta^2} + \\ &\frac{1}{\xi^2 \eta^2} \left(\frac{\frac{\partial \varphi_j}{\partial \vartheta_1}}{\sin^2 \vartheta_1} + \frac{\frac{\partial \varphi_j}{\partial \vartheta_2}}{\sin^2 \vartheta_1 \sin^2 \vartheta_2} + \dots + \frac{\frac{\partial \varphi_j}{\partial \vartheta_{M-2}}}{\prod_{j=1}^{M-3} \sin^2 \vartheta_j} \right), \\ 1 &= \frac{\frac{\partial \varphi_M}{\partial \xi}}{\xi^2 + \eta^2} + \frac{\frac{\partial \varphi_M}{\partial \eta}}{\xi^2 + \eta^2} + \\ &\frac{1}{\xi^2 \eta^2} \left(\frac{\frac{\partial \varphi_M}{\partial \vartheta_1}}{\sin^2 \vartheta_1} + \frac{\frac{\partial \varphi_M}{\partial \vartheta_2}}{\sin^2 \vartheta_1 \sin^2 \vartheta_2} + \dots + \frac{\frac{\partial \varphi_M}{\partial \vartheta_{M-2}}}{\prod_{j=1}^{M-3} \sin^2 \vartheta_j} \right), \\ &\frac{\frac{\partial \Psi}{\partial \xi}}{\xi^2 + \eta^2} + \frac{\frac{\partial \Psi}{\partial \eta}}{\xi^2 + \eta^2} + \\ &\frac{1}{\xi^2 \eta^2} \left(\frac{\frac{\partial \Psi}{\partial \vartheta_1}}{\sin^2 \vartheta_1} + \frac{\frac{\partial \Psi}{\partial \vartheta_2}}{\sin^2 \vartheta_1 \sin^2 \vartheta_2} + \dots + \frac{\frac{\partial \Psi}{\partial \vartheta_{M-2}}}{\prod_{j=1}^{M-3} \sin^2 \vartheta_j} \right) \end{aligned}$$

$$\begin{aligned} &= -\frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2), \\ 0 &= \frac{\partial}{\partial \xi} \left(\frac{|\Delta|}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \eta} \left(\frac{|\Delta|}{\xi^2 + \eta^2} \right) + \frac{\partial}{\partial \vartheta_1} \left(\frac{|\Delta|}{\xi^2 \eta^2} \right) \\ &+ \frac{\partial}{\partial \vartheta_2} \left(\frac{|\Delta|}{\xi^2 \eta^2 \sin^2 \vartheta_1} \right) + \end{aligned}$$

$$\frac{\partial}{\partial \vartheta_3} \left(\frac{|\Delta|}{\xi^2 \eta^2 \sin^2 \vartheta_1 \sin^2 \vartheta_2} \right) + \frac{\partial}{\partial \vartheta_{M-2}} \left(\frac{|\Delta|}{\xi^2 \eta^2 \prod_{j=1}^{M-3} \sin^2 \vartheta_j} \right).$$

After tedious computations we get that the following functions are solutions of the previous system of linear first order partial differential

equations

$$\begin{aligned}\varphi_j &= \vartheta_{j+1} + \cot \vartheta_j, \quad j = 1, \dots, M-3 \\ \varphi_{M-2} &= \frac{1}{\xi} + \frac{1}{\eta} + \vartheta_1, \quad \varphi_{M-1} = \xi - \eta, \quad \varphi_M = \frac{\xi^3 + \eta^3}{3}, \\ \Psi &= \mu(\xi + \eta) - \frac{\varepsilon}{10}(\xi^5 - \eta^5).\end{aligned}$$

Hence we deduce the Stäckel matrix.

By considering that $F_j = \sum_{k=1}^M \Delta^{jk} \Lambda_k / 2$, where $\Lambda = (p_{\vartheta_1}^2, p_{\vartheta_2}^2, \dots, p_{\vartheta_{M-2}}^2, p_{\xi}^2 + \mu - \xi^4/2, p_{\eta}^2 + \mu + \eta^4/2)^T$ then the first integrals (14) becomes

$$\begin{aligned}F_1 &= p_{\vartheta_{M-2}}^2, \\ F_2 &= p_{\vartheta_{M-3}}^2 + \frac{p_{\vartheta_{M-2}}^2}{\sin^2 \vartheta_{M-3}} = p_{\vartheta_{M-3}}^2 + \frac{F_1}{\sin^2 \vartheta_{M-3}}, \\ F_3 &= p_{\vartheta_{M-4}}^2 + \frac{1}{\sin^2 \vartheta_{M-4}} \left(p_{\vartheta_{M-3}}^2 + \frac{p_{\vartheta_{M-2}}^2}{\sin^2 \vartheta_{M-3}} \right) = p_{\vartheta_{M-4}}^2 + \frac{F_2}{\sin^2 \vartheta_{M-4}}, \\ &\vdots \\ F_{M-3} &= p_{\vartheta_2}^2 + \frac{F_{M-4}}{\sin^2 \vartheta_2}, \\ F_{M-2} &= p_{\vartheta_1}^2 + \frac{F_{M-3}}{\sin^2 \vartheta_1} = F_{M-2}(p_{\vartheta_1}, \vartheta_1, F_{M-3}(F_{M-4}(p_{\vartheta_2}, \vartheta_2, F_{M-5} \dots))), \\ F_{M-1} &= \frac{\eta^2 p_{\xi}^2 - \xi^2 p_{\eta}^2}{\xi^2 + \eta^2} - \frac{\xi^2 - \eta^2}{\xi^2 \eta^2} F_{M-2} + \mu \frac{(\eta^2 - \xi^2)}{\eta^2 + \xi^2} - \frac{\varepsilon}{2} \xi^2 \eta^2, \\ F_M &= H = \frac{p_{\xi}^2 + p_{\eta}^2}{\xi^2 + \eta^2} + \frac{F_{M-2}}{\xi^2 \eta^2} + \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2).\end{aligned}$$

Since the Stark system is integrable in Jacobi sense for arbitrary dimension. Thus the proposition is proved. $\square$

By determining the inverse transformation (σ^{-1}) of (25) and going back to cartesian coordinates $(x_1, x_2, \dots, x_M)$. The first integral $\tilde{F}_j = F_j|_{\sigma^{-1}}$ for $j = 1, \dots, M > 4$ can be written as

$$\begin{aligned}\tilde{F}_1 &= F_{12}^2, \\ \tilde{F}_2 &= F_{12}^2 + F_{23}^2 + F_{31}^2, \\ \tilde{F}_3 &= F_{12}^2 + F_{23}^2 + F_{31}^2 + F_{42}^2 + F_{51}^2 + F_{34}^2, \\ &\vdots \\ \tilde{F}_{M-1} &= \frac{\partial G}{\partial x_M} - \frac{\varepsilon}{2} \sum_{j=1}^{M-1} x_j^2, \\ \tilde{F}_M &= \sum_{j=1}^M \dot{x}_j^2 / 2 + \frac{2\mu}{\sqrt{\sum_{j=1}^M x_j^2}} + \varepsilon x_M,\end{aligned}$$

where $G = ||\mathbf{x}||^2 ||\mathbf{y}||^2 - \langle \mathbf{x}, \mathbf{y} \rangle^2 + \mu ||\mathbf{x}||$.

Corollary 2. The parametric expressions of the solution $\xi = \xi(t)$ and $\eta = \eta(t)$ are

$$\xi^2 = \mathfrak{P}(\tau - \tau_0) - \frac{4h}{3\varepsilon}, \quad \eta^2 = \mathfrak{P}(\tau - \tau_1) + \frac{4h}{3\varepsilon}, \quad (29)$$

where $\mathfrak{P}$ is the Weierstrass function, τ_0, τ_1 are constants, t and τ are

$$t = \int_{\tau_0}^{\tau} (\mathfrak{P}(\tau - \tau_0) + \mathfrak{P}(\tau - \tau_1)) d\tau.$$

Proof. By solving the equations

$$\begin{aligned}F_{M-2} &= \alpha_{M-2}, \quad F_{M-1} = \alpha_{M-1}, \quad F_M = h, \\ \frac{\eta^2 p_{\xi}^2 - \xi^2 p_{\eta}^2}{\xi^2 + \eta^2} + \frac{\xi^2 - \eta^2}{\xi^2 \eta^2} \alpha_{M-2} + \mu \frac{(\eta^2 - \xi^2)}{\eta^2 + \xi^2} - \frac{\varepsilon}{2} \xi^2 \eta^2 - \alpha_{M-1} &= 0, \\ \frac{p_{\xi}^2 + p_{\eta}^2}{\xi^2 + \eta^2} + \frac{\alpha_{M-2}}{\xi^2 \eta^2} + \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2) - h &= 0.\end{aligned}$$

with respect to p_{η}^2 and p_{ξ}^2 we get

$$\begin{aligned}p_{\xi}^2 &= \frac{\varepsilon \xi^6 + 4h\xi^4 + 2(-\mu + 2\alpha_{M-1})\xi^2 - 2\alpha_{M-2}}{2\xi^2}, \\ p_{\eta}^2 &= \frac{-\varepsilon \eta^6 + 4h\eta^4 - 2(\mu + \alpha_{M-1})\eta^2 - 2\alpha_{M-2}}{2\eta^2},\end{aligned}$$

Since

$$\begin{aligned}4\xi^2 p_{\xi}^2 &= (2\xi(\xi + \eta^2)\xi)^2 = 2(\varepsilon \xi^6 + 4h\xi^4 + 2(-\mu + \alpha_{M-1})\xi^2 - 2\alpha_{M-2}) : \\ &= S(\xi^2), \\ 4\eta^2 p_{\eta}^2 &= (2\eta(\eta^2 + \eta^2)\eta)^2 = 2(-\varepsilon \eta^6 + 4h\eta^4 + 2(\mu + \alpha_{M-1})\eta^2 - 2\alpha_{M-2}) : \\ &= R(\eta^2),\end{aligned}$$

By introducing the new time τ through $dt = (\xi^2 + \eta^2)d\tau$ we get that

$$\begin{aligned}\frac{d\xi_1}{d\tau} &= \sqrt{S(\xi_1)}, \quad \xi_1 = \xi^2, \\ \frac{d\eta_1}{d\tau} &= \sqrt{R(\eta_1)}, \quad \eta_1 = \eta^2.\end{aligned}$$

Thus after the integration the following relations hold

$$\int \frac{d\eta_1}{\sqrt{R(\eta_1)}} = \tau - \tau_0, \quad \int \frac{d\xi_1}{\sqrt{S(\xi_1)}} = \tau - \tau_1, \quad (30)$$

where τ_0 and τ_1 are arbitrary constants.

After the change

$$\pi : \quad \xi_1 = X - \frac{4h}{3\varepsilon}, \quad \eta_1 = Y + \frac{4h}{3\varepsilon}$$

we get that the polynomials $R(\eta_1)$ and $S(\xi_1)$ become

$$P_3(X) = R(\eta_1)|_{\pi}, \quad Q_3(Y) = S(\xi_1)|_{\pi}.$$

Consequently (30) can be rewritten as

$$\int \frac{dX}{\sqrt{P_3(X)}} = \tau - \tau_0, \quad \int \frac{dY}{\sqrt{Q_3(Y)}} = \tau - \tau_1. \quad (31)$$

Thus the problem is reduced to quadratures, more specifically to elliptic integrals. The key point now is to invert these integrals to find parametric expressions in the variables X and Y of the functions and with the new time τ . By inverting the integrals (31) we get that

$$X = \mathfrak{P}(\tau - \tau_0), \quad Y = \mathfrak{P}(\tau - \tau_1),$$

where $\mathfrak{P}$ is the Weierstrass function. Therefore

$$\xi_1 = \xi^2 = \mathfrak{P}(\tau - \tau_0) - \frac{4h}{3\varepsilon}, \quad \eta_1 = \eta^2 = \mathfrak{P}(\tau - \tau_1) + \frac{4h}{3\varepsilon}.$$

On the other hand, from the relation $dt = (\xi^2 + \eta^2)d\tau$ we get that

$$dt = (\mathfrak{P}(\tau - \tau_0) + \mathfrak{P}(\tau - \tau_1))d\tau.$$

Hence after integration we get (29). In short the corollary is proved. $\square$

From (30) it follows that the trajectories of the Stark system lies on the family of curves of the plane (ξ_1, η_1)

$$\int \frac{d\eta_1}{\sqrt{R(\eta_1)}} - \int \frac{d\xi_1}{\sqrt{S(\xi_1)}} = C = \tau_1 - \tau_0.$$

4. The Stark-Coulomb problem

In this section for the first time we introduce and study the Stark-Coulomb Hamiltonian

$$H = \sum_{j=1}^M y_j^2 + \frac{2\mu}{\sqrt{x_1^2 + x_2^2 + \dots + x_M^2}} - \frac{1}{2} \sum_{j=1}^{M-1} \frac{\beta_j}{x_j^2} + \varepsilon x_M,$$

where $\beta_1, \beta_2, \dots, \beta_{M-1}$ are constants.

Proposition 5. *The Stark-Coulomb Hamiltonian is superintegrable when $M > 3$.*

Proof. After some computations we can prove that this Hamiltonian system admits $(M-1)(M-2)/2$ first integrals

$$F_{jk} = (x_j y_k - x_k y_j)^2 + (x_j^2 + x_k^2) \left(\frac{\beta_j}{x_j^2} + \frac{\beta_k}{x_k^2} \right), \quad j, k = 1, \dots, M-1, \quad j < k. \quad (32)$$

It is possible to prove that the rang of the matrix $(F_{jk})|_{j,k=1,\dots,M-1}$ is $2M-5$, see [24]. Therefore the maximum number of independent functions of the set (32) is $2M-5$. On the other hand, it is easy to show the existence of the first integral

$$H_1 = \frac{\partial F}{\partial x_M} - \frac{\varepsilon}{2} \sum_{j=1}^{M-1} x_j^2 = 2(x_M ||y||^2 - y_M < x, y >) + \frac{2\mu}{\sqrt{x_1^2 + x_2^2 + \dots + x_M^2}} - \frac{\varepsilon}{2} \sum_{j=1}^{M-1} x_j^2 + 2x_M \sum_{j=1}^{M-1} \frac{\beta_j}{x_j^2}, \quad (33)$$

where

$$F := \sum_{j,k=1, j>k}^M \left((x_j y_k - x_k y_j)^2 + (x_j^2 + x_k^2) \left(\frac{\beta_j}{x_j^2} + \frac{\beta_k}{x_k^2} \right) \right) \Big|_{\beta_M=0} + 2\mu \sqrt{\sum_{j=1}^M x_j^2} \\ = ||x||^2 ||y||^2 - < x, y >^2 + \sum_{j=1}^{M-1} \left(\frac{\beta_j}{x_j^2} ||x||^2 + (M-2)\beta_j \right) + 2\mu ||x||.$$

By adding the Hamiltonian we get that the system has $2M-3$ independent first integrals. Thus $2M-3 > M$ if and only if $M > 3$. In short the proposition is proved. $\square$

Proposition 6. *The Stark-Coulomb Hamiltonian is integrable by separation of variables.*

Proof. First we prove the proposition for $M = 2$. In this case the Hamiltonian H and the function H_1 given in (33) become

$$\tilde{H} = y_1^2 + y_2^2 + \frac{2\mu}{\sqrt{x_1^2 + x_2^2}} - \frac{\beta_1}{2x_1^2} + \varepsilon x_2, \quad H_1 = \frac{\partial F}{\partial x_2} - \frac{\varepsilon}{2} x_1^2,$$

where $F = (x_1 y_2 - x_2 y_1)^2 + (x_1^2 + x_2^2) \frac{\beta_1}{x_1^2} + 2\mu \sqrt{x_1^2 + x_2^2}$. Under the change of variables

$$\sigma : \quad x_1 = \xi \eta, \quad x_2 = \frac{\xi^2 - \eta^2}{2},$$

we get that the functions F_1 and F_2 become

$$F_1 := H_1|_{\sigma} = \frac{\eta^2 p_{\xi}^2}{\xi^2 + \eta^2} - \frac{\xi^2 p_{\eta}^2}{\xi^2 + \eta^2} - \frac{\xi^2 - \eta^2}{\xi^2 + \eta^2} \mu - \frac{\varepsilon}{2} \eta^2 \xi^2 - \frac{\xi^2 - \eta^2}{\xi^2 \eta^2} \beta_1,$$

$$F_2 := H = \tilde{H}|_{\sigma} = \frac{p_{\xi}^2 + p_{\eta}^2}{2(\xi^2 + \eta^2)} + \frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2} (\xi^2 - \eta^2) + \frac{\beta_1}{\xi^2 \eta^2},$$

where $p_{\xi} = (\xi^2 + \eta^2) \frac{d\xi}{dt}$ and $p_{\eta} = (\xi^2 + \eta^2) \frac{d\eta}{dt}$. By solving the relations $F_j = \alpha_j$ for $j = 1, 2$ with respect to p_{ξ} and p_{η} we get

$$p_{\xi}^2 = \varepsilon \xi^4 + 4\alpha_2 \xi^2 - 2(\alpha_1 + \mu) - \frac{\beta_1}{\xi^2},$$

$$p_{\eta}^2 = -\varepsilon \eta^4 + 4\alpha_2 \eta^2 - 2(\alpha_1 + \mu) - \frac{\beta_1}{\eta^2}.$$

By introducing the change $X = \xi^2$, $Y = \eta^2$ we get the following differential system

$$\left(\frac{dX}{d\tau} \right)^2 = -\frac{\varepsilon X^3}{2} - \alpha_2 X^2 - (\alpha_1 + 2\mu)X - \beta_1,$$

$$\left(\frac{dY}{d\tau} \right)^2 = \frac{\varepsilon Y^3}{2} + \alpha_2 Y^2 - (-\alpha_1 + 2\mu)Y - \beta_1.$$

Clearly the solutions of this system can be expressed through elliptic functions.

For $M = 3$ we get that the first integrals are

$$H_1 = F_{12} = (x_1 y_2 - x_2 y_1)^2 + (x_1^2 + x_2^2) \left(\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} \right),$$

$$H_2 = \frac{\partial F}{\partial x_4} - \frac{\varepsilon}{2} (x_1^2 + x_2^2),$$

$$\tilde{H} = y_1^2 + y_2^2 + y_3^2 + \frac{2\mu}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \frac{1}{2} \left(\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} \right) + \varepsilon x_3,$$

where $F = F_{12} + F_{23} + F_{31}|_{\beta_3=0} + 2\mu \sqrt{x_1^2 + x_2^2 + x_3^2}$, the functions F_{12} , F_{23} and F_{31} are given in (32).

After the change of coordinates

$$\sigma : \quad x_1 = \xi \eta \cos \vartheta_1, \quad x_2 = \xi \eta \sin \vartheta_1, \quad x_3 = \frac{\xi^2 - \eta^2}{2},$$

we get that the first integrals H_1 , H_2 and H_3 become

$$F_1 = H_1|_{\sigma} = p_{\vartheta_1}^2 + \frac{\beta_1}{\sin^2 \vartheta_1} + \frac{\beta_2}{\cos^2 \vartheta_1},$$

$$F_2 = H_2|_{\sigma} = \frac{\xi^2 p_{\eta}^2 - \eta^2 p_{\xi}^2}{\eta^2 + \xi^2} + \frac{2\mu(\xi^2 - \eta^2)}{\eta^2 + \xi^2} - \frac{\varepsilon \xi^2 \eta^2}{2} + \frac{(\xi^2 - \eta^2)}{\eta^2 \xi^2} F_1,$$

$$F_3 = H = \tilde{H}|_{\sigma} = \frac{p_{\eta}^2 + p_{\xi}^2}{\eta^2 + \xi^2} + \frac{2\mu}{\eta^2 + \xi^2} + \frac{\varepsilon}{2} (\xi^2 - \eta^2) + \frac{1}{\eta^2 \xi^2} F_1,$$

where $p_{\xi} = (\xi^2 + \eta^2) \frac{d\xi}{dt}$, $p_{\eta} = (\xi^2 + \eta^2) \frac{d\eta}{dt}$ and $p_{\vartheta} = \xi^2 \eta^2 \frac{d\vartheta_1}{dt}$.

By solving the equations $F_2 = \alpha_2$ and $F_3 = \alpha_3$ with respect to p_{ξ}^2 and p_{η}^2 , and taking $F_1 = \alpha_1$ we get

$$\xi^2 p_{\xi}^2 = \varepsilon \xi^6 + 4\alpha_3 \xi^2 - 2(\alpha_1 + 2\mu)\xi^2 - 2\alpha_1,$$

$$\eta^2 p_{\eta}^2 = -\varepsilon \eta^6 + 4\alpha_3 \eta^4 - 2(\alpha_2 + \mu)\eta^2 - 2\alpha_1.$$

By introducing the change $(\xi^2 + \eta^2) \frac{d}{dt} = \frac{d}{d\tau}$ and $X = \xi^2$, $Y = \eta^2$ we obtain the following ordinary differential system

$$\left(\frac{dX}{d\tau} \right)^2 = -\frac{\varepsilon X^3}{2} + \alpha_2 X^2 - (\alpha_1 + 2\mu)X + \alpha_3,$$

$$\left(\frac{dY}{d\tau} \right)^2 = \frac{\varepsilon Y^3}{2} + \alpha_2 Y^2 + (\alpha_1 - 2\mu)Y + \alpha_3.$$

Denoting by $X(\tau)$ and $Y(\tau)$ the solutions of these systems and inserting into the equation $p_{\vartheta}^2 + \frac{\beta_1}{\sin^2 \vartheta} + \frac{\beta_2}{\cos^2 \vartheta} = \alpha_1$ we deduce that

$$\left(\frac{d\vartheta}{d\tau} \right)^2 = \frac{(X(\tau) + Y(\tau)) \left(\alpha_1 - \frac{\beta_1}{\sin^2 \vartheta} - \frac{\beta_2}{\cos^2 \vartheta} \right)}{(X(\tau)Y(\tau))}.$$

Hence the solutions of this system can be expressed through elliptic functions.

For $M > 3$ the problem on the separation of variables for the Stark-Coulomb Hamiltonian system can be obtained in an analogous form than in the Stark Hamiltonian system by applying the transformation (25).

Indeed, the Stäckel first integrals become

$$\begin{aligned}
 F_1 &= F_{12}|_\sigma = p_{\theta_{M-2}}^2 + \frac{\beta_{M-4}}{\sin^2 \theta_{M-2}} + \frac{\beta_{M-3}}{\cos^2 \theta_{M-2}}, \\
 F_2 &= (F_{12} + F_{23} + F_{31})|_\sigma = p_{\theta_{M-3}}^2 + \frac{\beta_{M-2}}{\cos^2 \theta_{M-3}} + \frac{F_1}{\sin^2 \theta_{M-3}}, \\
 F_3 &= (F_{12} + F_{23} + F_{31} + F_{42} + F_{51} + F_{34})|_\sigma = p_{\theta_{M-4}}^2 + \frac{\beta_{M-1}}{\cos^2 \theta_{M-4}} \\
 &\quad + \frac{F_3}{\sin^2 \theta_{M-4}}, \\
 F_j &= \dots = p_{\theta_{M-j-1}}^2 + \frac{\beta_{M-j}}{\cos^2 \theta_{M-1-j}} + \frac{F_{j-1}}{\sin^2 \theta_{M-1-j}}, \quad j = 1, \dots, M-2, \\
 F_{M-1} &= \frac{\partial F}{\partial x_M} - \frac{\epsilon}{2} \sum_{j=1}^{M-1} x_j^2 \Big|_\sigma = \frac{\eta^2 p_\xi^2 - \xi^2 p_\eta^2}{\xi^2 + \eta^2} - \frac{\mu(\xi^2 - \eta^2)}{\xi^2 + \eta^2} - \frac{\epsilon}{2} \xi^2 \eta^2 \\
 &\quad - \frac{\xi^2 - \eta^2}{\xi^2 \eta^2} F_{M-2}, \\
 F_M &= H = \tilde{H}|_\sigma = \frac{p_\xi^2 + p_\eta^2}{\xi^2 + \eta^2} - \frac{\epsilon}{2}(\xi^2 - \eta^2) + \frac{2\mu}{\xi^2 + \eta^2} + \frac{F_{M-2}}{\xi^2 \eta^2},
 \end{aligned}$$

where $F = \sum_{j,k=1, j>k}^M F_{jk} \Big|_{\beta_{M-0}} + \mu \sqrt{\sum_{j=1}^M x_j^2}$. As in the previous results by solving the equations $F_{M-1} = \alpha_{M-1}$ and $F_M = \alpha_M$ and taking $F_{M-2} = \alpha_{M-2}$, we deduce that the trajectories on the plane (ξ, η) can be expressed through elliptic functions. Therefore the proposition is proved. $\square$

5. Inverse Stäckel problem II in the Riemann spaces of constant curvature

In this section we study the inverse Stäckel problem II in the Riemann spaces of constant curvature. First we introduce the following well known concepts and notations.

Let (M, G) be a Riemannian (pseudo Riemann) manifold, and $\mathfrak{X}(M)$ be the space of all vector fields on M . We define the Riemann curvature tensor as the map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),$$

that send the vector fields (X, Y, Z) to the vector field

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z,$$

where ∇ is the Levi-Civita connection (see for instance [5,33]).

A Riemannian manifold is a space of constant curvature if the Riemann tensor is of the form

$$R_{abcd} = K (G_{ac} G_{db} - G_{ad} G_{cb}),$$

where K is a constant and G_{ij} are the components of the Riemann metric. The three most fundamental Riemannian manifolds with constant curvature are the Euclidean n -dimensional space E^n , the n -dimensional sphere S^n , and the hyperbolic space H^n .

For S^n the curvature is $K = 1$, for E^n the curvature is $K = 0$, and for H^n the $K = -1$.

Now we give the Eisenhart results related with the Riemann spaces of constant curvature. Eisenhart see [5,25] proved that the necessary and sufficient conditions in order that the $H = \sum_{j=1}^n G^{jj} p_j^2$ is in the Stäckel form are

$$\frac{\partial^2 \tau_i}{\partial z_j \partial z_k} = \frac{\partial \tau_i}{\partial z_j} \frac{\partial \tau_i}{\partial z_k} - \frac{\partial \tau_j}{\partial z_k} \frac{\partial \tau_i}{\partial z_j} - \frac{\partial \tau_i}{\partial z_k} \frac{\partial \tau_k}{\partial z_j}, \quad (34)$$

where $i, j, k = 1, \dots, n$ with i, j and k distinct, $\tau_j = \frac{\ln G^{jj}}{\sigma}$ being $\sigma \neq 0$ a constant. An orthogonal coordinate system on a Riemann space of constant curvature satisfies these conditions. In particular this coordinate

system in the Riemann spaces E^n , S^n and H^n always exist. In [25] the author gives a classification of separable coordinate systems on these spaces.

Thus we get that the variable separation in

(I) E^n the Riemann metric is:

$$\begin{aligned}
 &\text{In elliptic coordinates } G_{jj} = \frac{\prod_{k \neq j} (z_j - z_k)}{\prod_{m=1}^n (z_j - e_m)}, \text{ where } e_1 < z_1 < e_2 < \dots < e_n < z_n. \\
 &\text{In parabolic coordinates } G_{jj} = \frac{\prod_{k \neq j} (z_j - z_k)}{\prod_{m=1}^{n-1} (z_j - e_m)}, \text{ where } z_1 < e_1 < z_2 < e_2 < \dots < z_{n-1} < e_{n-1} < z_n.
 \end{aligned}$$

(II) S^n the Riemann metric is $G_{jj} = \frac{\prod_{k \neq j} (z_j - z_k)}{-4 \prod_{m=1}^{n+1} (z_j - e_m)}$, where $e_1 < z_1 < e_2 < \dots < e_n < z_n < e_{n+1}$.

(III) H^n the Riemann metric is $G_{jj} = \frac{\prod_{k \neq j} (z_j - z_k)}{4 \prod_{m=1}^{n+1} (z_j - e_m)}$.

In all these previous expressions the e_j 's are constants.

Now we solve the inverse Stäckel problem II in the Riemann (pseudo-Riemann) spaces of constant curvature.

We determine the Stäckel matrix when $G^{jj} = \frac{f(z_j)}{\prod_{k \neq j}^M (z_j - z_k)}$ for $j =$

$1, \dots, M$ where $f(z_j) = a \prod_{m=1}^l (z_j - e_m)$, $l \geq 1$ is an integer, $a \neq 0$ and $e_1, \dots, e_l$ are convenient constants. Clearly that the metric of the given space of constant curvature is a particular case of this metric.

Consequently

$$G^{jj} = \frac{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, z_j\}}{\{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \varphi_M\}} \quad (35)$$

Therefore to solve the inverse Stäckel problem II in this case we need to solve the first order partial differential equations from (10)

$$\begin{aligned}
 \sum_{j=1}^M \frac{f(z_j)}{\prod_{k \neq j}^M (z_j - z_k)} \frac{\partial \varphi_k}{\partial z_j} &= 0, \quad j = 1, \dots, M-1, \\
 \sum_{j=1}^M \frac{f(z_j)}{\prod_{k \neq j}^M (z_j - z_k)} \frac{\partial \varphi_M}{\partial z_j} &= 1, \\
 \sum_{j=1}^M \frac{\partial}{\partial z_j} \left(\frac{f(z_j)}{\prod_{k \neq j}^M (z_j - z_k)} \{\varphi_1, \varphi_2, \dots, \varphi_{M-1}, \varphi_M\} \right) &= 0.
 \end{aligned}$$

The solutions of these equations are

$$\begin{aligned}\varphi_j &= \int \sum_{k=1}^M \frac{\alpha_{M-1}^j z_k^{M-1} + \dots + \alpha_1^j z_k + \alpha_0^j}{f(z_k)} dz_k, \quad j = 1, \dots, M-1, \\ \varphi_M &= \int \sum_{k=1}^M \frac{\alpha_{M-1}^M z_k^M + \dots + \alpha_1^M z_k + \alpha_0^M}{f(z_k)} dz_k \\ \{\varphi_1, \varphi_2, \dots, \varphi_M\} &= |\Delta| = \det \Omega \prod_{k=1}^M f(z_k),\end{aligned}$$

where $\alpha_{M-1}^j, \dots, \alpha_0^j$ for $j = 1, \dots, M$ are constants.

In particular we have the following solutions

$$\begin{aligned}\varphi_1 &= \int \sum_{k=1}^M \frac{1}{f(z_k)} dz_k, \quad \varphi_2 = \int \sum_{k=1}^M \frac{z_k}{f(z_k)} dz_k, \dots, \\ \varphi_M &= \int \sum_{k=1}^M \frac{z_k^{M-1}}{f(z_k)} dz_k,\end{aligned}$$

then the Stäckel matrix in this case becomes

$$\Delta = \begin{pmatrix} \frac{1}{f(z_1)} & \frac{z_1}{f(z_1)} & \dots & \dots & \frac{z_1^{M-1}}{f(z_1)} \\ \frac{1}{f(z_2)} & \frac{z_2}{f(z_2)} & \dots & \dots & \frac{z_2^{M-1}}{f(z_2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{f(z_{M-1})} & \frac{z_{M-1}}{f(z_{M-1})} & \dots & \dots & \frac{z_{M-1}^{M-1}}{f(z_{M-1})} \\ \frac{1}{f(z_M)} & \frac{z_M}{f(z_M)} & \dots & \dots & \frac{z_M^{M-1}}{f(z_M)} \end{pmatrix}.$$

The inverse Stäckel matrix is

$$\Delta^{-1} = \begin{pmatrix} \frac{f(z_1) \prod_{k \neq 1}^M z_k}{\prod_{k \neq 1}^M (z_1 - z_k)} & \frac{f(z_2) \prod_{k \neq 2}^M z_k}{\prod_{k \neq 2}^M (z_2 - z_k)} & \dots & \dots & \frac{f(z_M) \prod_{k \neq M}^M z_k}{\prod_{k \neq M}^M (z_M - z_k)} \\ \vdots & \vdots & \dots & \dots & \vdots \\ \frac{f(z_1) \sum_{k \neq 1}^M z_k}{\prod_{k \neq 1}^M (z_1 - z_k)} & \frac{f(z_2) \sum_{k \neq 2}^M z_k}{\prod_{k \neq 2}^M (z_2 - z_k)} & \dots & \dots & \frac{f(z_M) \sum_{k \neq M}^M z_k}{\prod_{k \neq M}^M (z_M - z_k)} \\ \frac{f(z_1)}{\prod_{k \neq 1}^M (z_1 - z_k)} & \frac{f(z_2)}{\prod_{k \neq 2}^M (z_2 - z_k)} & \dots & \dots & \frac{f(z_M)}{\prod_{k \neq M}^M (z_M - z_k)} \end{pmatrix}.$$

The first integrals can be obtained from formula (14). Consequently

$$\begin{aligned}F_1 &= p_1^2/2 \frac{f(z_1) \prod_{k \neq 1}^M z_k}{\prod_{k \neq 1}^M (z_1 - z_k)} + p_2^2/2 \frac{f(z_2) \prod_{k \neq 2}^M z_k}{\prod_{k \neq 2}^M (z_2 - z_k)} \\ &\quad + \dots + p_M^2/2 \frac{f(z_M) \prod_{k \neq M}^M z_k}{\prod_{k \neq M}^M (z_M - z_k)}, \\ &\quad \vdots \\ F_{M-1} &= p_1^2/2 \frac{f(z_2) \sum_{k \neq 2}^M z_k}{\prod_{k \neq 1}^M (z_1 - z_k) \prod_{k \neq 2}^M (z_2 - z_k)} \\ &\quad + \dots + p_M^2/2 \frac{f(z_M) \sum_{k \neq M}^M z_k}{\prod_{k \neq M}^M (z_M - z_k)}, \\ F_M &= H = p_1^2/2 \frac{f(z_1)}{\prod_{k \neq 1}^M (z_1 - z_k)} + p_2^2/2 \frac{f(z_2)}{\prod_{k \neq 2}^M (z_2 - z_k)} \\ &\quad + \dots + p_M^2/2 \frac{f(z_M)}{\prod_{k \neq M}^M (z_M - z_k)},\end{aligned}$$

5.1. The Hamiltonian Stäckel function in the spaces of constant curvature

From the previous results we get that the geodesic Hamiltonian in the Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, G)$ is $F_M = \sum_{j=1}^M \frac{p_j^2}{2} \frac{f(z_j)}{\prod_{k \neq j}^M (z_j - z_k)}$.

The complete homogeneous symmetric polynomial of degree k in M variables $(z_1, \dots, z_M)$ is the sum of all monomials of total degree k in these variables, i.e.

$$P_k(z_1, \dots, z_M) = \sum_{n_1 + \dots + n_M = k} z_1^{n_1} z_2^{n_2} \dots z_M^{n_M}.$$

Now we determine the potential function U such that $H = F_M + U$ is the Hamiltonian-Stäckel function.

From (10) and (35) we get that the potential U is

$$\sum_{j=1}^M \frac{f(z_j)}{\prod_{k \neq j}^M (z_j - z_k)} \frac{\partial \Psi}{\partial z_j} = U(z_1, \dots, z_M), \quad (36)$$

$$\text{where } \Psi = \sum_{j=1}^M \int_{z_{j0}}^{z_j} \Psi_j(z_j) dz_j.$$

Proposition 7. The Stäckel potential function U that is a solution of the partial differential Eq. (36) is the complete homogeneous symmetric polynomial of degree k in the variables $(z_1, \dots, z_M)$. Moreover the functions $\Psi_j(z_j)$ are

$$\Psi_j(z_j) := \Psi_j^{(k)}(z_j) = \nu \frac{z_j^{k+M-1}}{f(z_j)}, \quad k \geq 1,$$

where ν is a convenient non-zero constant.

Proof. Indeed, after tedious computations from (36) we get that

$$\begin{aligned} & \nu \sum_{j=1}^M \frac{f(z_j)}{\prod_{k \neq j} (z_j - z_k)} \frac{\partial}{\partial z_j} \left(\int_{z_0}^{z_j} \frac{z_j^{k+M-1}}{f(z_j)} dz_j \right) \\ &= \nu \sum_{n_1+\dots+n_M=k} z_1^{n_1} \dots, z_M^{n_M} = U_k(z_1, \dots, z_M), \end{aligned}$$

for $k \geq 1$. Therefore we get

$$\sum_{j=1}^M \frac{z_j^{k+M-1}}{\prod_{k \neq j} (z_j - z_k)} = \sum_{n_1+\dots+n_M=k} z_1^{n_1} \dots, z_M^{n_M}, \quad (37)$$

for $k \geq 1$. We observe that this equality was proved in [34]. $\square$

Therefore we deduce that the require Hamiltonian Stäckel function is

$$\begin{aligned} H &= \sum_{j=1}^M \frac{f(z_j)}{\prod_{k \neq j} (z_j - z_k)} \left(p_j^2/2 + \nu \frac{z_j^{k+M-1}}{f(z_j)} \right) \\ &= \sum_{j=1}^M p_j^2/2 \frac{f(z_j)}{\prod_{k \neq j} (z_j - z_k)} + \nu \sum_{n_1+\dots+n_M=k} z_1^{n_1} \dots, z_M^{n_M}, \end{aligned}$$

for $k \geq 1$.

Now we shall study the case when $f(z_j) = z_j$. Thus we determine the Stäckel matrix when $G^{ij} = \frac{z_j}{\prod_{k \neq j} (z_j - z_k)}$ for $j = 1, \dots, M$. Thereby

we have the following solutions

$$\varphi_1 = \log \left| \prod_{j=1}^M z_j \right|, \quad \varphi_2 = \sum_{k=1}^M z_k, \quad \dots, \quad \varphi_M = \sum_{k=1}^M \frac{z_k^{M-1}}{M-1}.$$

Then the Stäckel matrix in this case becomes

$$\Delta = \begin{pmatrix} \frac{1}{z_1} & 1 & \dots & \dots & z_1^{M-2} \\ \frac{1}{z_2} & 1 & \dots & \dots & z_2^{M-2} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{1}{z_M} & 1 & \dots & \dots & z_M^{M-2} \end{pmatrix}.$$

The inverse Stäckel matrix is

$$\Delta^{-1} = \begin{pmatrix} \frac{\prod_{k=1}^M z_k}{\prod_{k \neq 1} (z_1 - z_k)} & \frac{\prod_{k=1}^M z_k}{\prod_{k \neq 2} (z_2 - z_k)} & \dots & \dots & \frac{\prod_{k=1}^M z_k}{\prod_{k \neq M} (z_M - z_k)} \\ \frac{z_1 \sum_{k \neq 1}^M z_k}{\prod_{k \neq 1} (z_1 - z_k)} & \frac{z_2 \sum_{k \neq 2}^M z_k}{\prod_{k \neq 2} (z_2 - z_k)} & \dots & \dots & \frac{z_M \sum_{k \neq M}^M z_k}{\prod_{k \neq M} (z_M - z_k)} \\ \frac{z_1}{\prod_{k \neq 1} (z_1 - z_k)} & \frac{z_2}{\prod_{k \neq 2} (z_2 - z_k)} & \dots & \dots & \frac{z_M}{\prod_{k \neq M} (z_M - z_k)} \end{pmatrix}.$$

We take in this case $\Psi_j(z_j) := \Psi_j^{(M)}(z_j) = \nu z_j^{2M-1}$. Therefore the first integrals (14) become

$$\begin{aligned} F_1 &= \sum_{k \neq 1}^M \frac{\prod_{k=1}^M z_k}{\prod_{k \neq j} (z_j - z_k)} p_j^2/2 + \nu \tilde{U}_{2M}, \\ &\vdots \\ F_{M-1} &= \frac{\sum_{j=1}^M p_j^2/2 \sum_{k \neq j}^M z_k}{\prod_{k \neq j} (z_j - z_k)} + \nu \tilde{U}_{M+2}, \\ F_M &= H = \frac{\sum_{j=1}^M p_j^2/2 \sum_{k \neq j}^M z_k}{\prod_{k \neq j} (z_j - z_k)} + \nu U_{M+1}, \end{aligned}$$

where $\tilde{U}_j$ for $j = 2M, \dots, M+2$ are symmetric polynomials of degree j and U_{M+1} is a complete symmetric polynomial.

The Jacobi-Stäckel function can be written as

$$W = \sum_{k=1}^M \int_{z_0^k}^{z_k} \sqrt{\nu z^{2M-1} + \sum_{n=0}^M z_k^{n-1} \alpha_n} dz_k.$$

In particular for $M = 3$ we have that the previous first integrals are

$$\begin{aligned} F_1 &= \frac{z_1 z_2 z_3}{(z_1 - z_2)(z_1 - z_3)} p_1^2/2 + \frac{z_1 z_2 z_3}{(z_2 - z_1)(z_2 - z_3)} p_2^2/2 \\ &\quad + \frac{z_1 z_2 z_3}{(z_3 - z_1)(z_3 - z_2)} p_3^2/2 - \nu z_1 z_2 z_3 U_3, \\ F_2 &= \frac{z_1(z_2 + z_3)p_1^2/2}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2(z_1 + z_3)p_2^2/2}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3(z_2 + z_1)p_3^2/2}{(z_3 - z_1)(z_3 - z_2)} \\ &\quad + \nu (z_1 + z_2)(z_1 + z_3)(z_3 + z_4)(z_1^2 + z_2^2 + z_3^2), \\ F_3 &= H = \frac{z_1 p_1^2/2}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2 p_2^2/2}{(z_2 - z_1)(z_2 - z_3)} \\ &\quad + \frac{z_3 p_3^2/2}{(z_3 - z_1)(z_3 - z_2)} + \nu U_4, \end{aligned}$$

where U_3 and U_4 are complete symmetric polynomials of degree 3 and 4, respectively.

In [35] the authors present two quasi-biHamiltonian systems with three and four degrees of freedom, they proved that they belong to a family of integrable flows obtained in [36] as stationary flows of the Korteweg-de Vries hierarchy [37]. This family contains the classical Hénon-Heiles system as its second member, so the higher members can be considered as multi-dimensional extensions of Hénon-Heiles. The third member of this family, which is a stationary reduction of the seventh order KdV flow, is defined in a six dimensional phase space with coordinates (q_1, q_2, q_3) and (μ_1, μ_2, μ_3) by the Hamiltonian

$$\tilde{H}_3 = (\mu_3^2 + 2\mu_2\mu_1)/2 - 5q_1^4/8 + 5q_1^2q_2/2 + q_1q_3^2/2 - q_2^2/2.$$

After the change of coordinates $\sigma_3(q_1, q_2, q_3, \mu_1, \mu_2, \mu_3) = (z_1, z_2, z_3, p_1, p_2, p_3)$ given by

$$\begin{aligned} q_1 &= -\frac{1}{2}(z_1 + z_2 + z_3), \\ q_2 &= -\frac{1}{8}((z_1 + z_2 + z_3)^2 - 4(z_1 z_2 + z_1 z_3 + z_2 z_3)), \\ q_3 &= \sqrt{z_1 z_2 z_3}, \\ \mu_1 &= -2\left(\frac{z_1(-z_1 + z_2 + z_3)p_1}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2(z_1 - z_2 + z_3)p_2}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3(z_1 + z_2 - z_3)p_3}{(z_3 - z_2)(z_3 - z_1)}\right), \\ \mu_2 &= -2\left(\frac{z_1 p_1}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2 p_2}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3 p_3}{(z_3 - z_2)(z_3 - z_1)}\right), \\ \mu_3 &= 2\sqrt{z_1 z_2 z_3}\left(\frac{p_1}{(z_1 - z_2)(z_1 - z_3)} + \frac{p_2}{(z_2 - z_1)(z_2 - z_3)} + \frac{p_3}{(z_3 - z_2)(z_3 - z_1)}\right), \end{aligned}$$

and by applying the identity (37) we get the Hamiltonian

$$F_3 = \frac{\sum_{j=1}^3 p_j^2 / 2 z_j}{\prod_{k \neq j} (z_j - z_k)} + \nu U_4 = \tilde{H}_3|_{\sigma_3}.$$

The second example is a Hénon-Heiles system with four degrees of freedom. It can be constructed as a stationary reduction of the ninth order KdV flow. Its phase space is eight dimensional, and the Hamiltonian is

$$\begin{aligned} \tilde{H}_4 &= \left(\mu_4^2 + 2\mu_1\mu_3 + \mu_2^2\right)/2 + 3q_1^5/4 - 5q_1^3 q_2/2 + 2q_1 q_2^2 \\ &\quad + 5q_1^2 q_3/2 + q_1 q_4^2/2 - q_2 q_3. \end{aligned}$$

After a convenient change $\sigma_4(q_1, q_2, q_3, q_4, \mu_1, \mu_2, \mu_3, \mu_4) = (z_1, z_2, z_3, z_4, p_1, p_2, p_3, p_4)$, see [35] and by applying the identity (37) we get that

$$F_4 = \frac{\sum_{j=1}^4 p_j^2 / 2 z_j}{\prod_{k \neq j} (z_j - z_k)} + \nu U_5 = \tilde{H}_4|_{\sigma_4}.$$

The following question arise: Is it possible to construct the Hénon-Heiles system with M degree of freedom related with the Hamiltonian F_M as a stationary reduction of the $2M + 1$ order KdV flow, i.e.

$$F_M = \frac{\sum_{j=1}^M p_j^2 / 2 z_j}{\prod_{k \neq j} (z_j - z_k)} + \nu U_{M+1} = \tilde{H}_M|_{\sigma_M}?$$

CRedit authorship contribution statement

Jaume Llibre: Writing – original draft, Investigation, Formal analysis, Conceptualization; **Rafael Ramírez:** Writing – original draft, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Jaume Llibre reports administrative support was provided by Royal Academy of Science and Arts of Barcelona. Jaume Llibre reports financial support was provided by Agencia Estatal de Investigación of Spain.

If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The first author is partially supported by the **Agencia Estatal de Investigación of Spain** grant PID2022-136613NB-I00, and by the **Reial Acadèmia de Ciències i Arts de Barcelona**.

Data availability

No data was used for the research described in the article.

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