

DYNAMICS FOR A RAYLEIGH-LIÉNARD-DUFFING OSCILLATOR IN WING ROCK

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ABSTRACT. In this paper we study the dynamics for a classical type of Rayleigh-Liénard-Duffing oscillator, that is utilized to simulate the wing rock motion with a single degree of freedom. This hybrid oscillator can exhibit rich bifurcations phenomena, including pitchfork bifurcation, Hopf bifurcation and heteroclinic bifurcation. By applying the qualitative theory of differential equations and the averaging theory, we prove that a lower bound for the maximum number of small-amplitude limit cycles (isolate periodic oscillations) for the corresponding piecewise differential system is 2. Furthermore, we classify for this oscillator its phase portraits in the Poincaré disc.

During the past few decades there has been significant scholarly interest in the investigation of piecewise differential systems, primarily because these systems facilitate the modeling of various complex phenomena. One nonlinear aerodynamical model causing wing rock can be simplified to a Rayleigh-Liénard-Duffing oscillator, more precisely to a family cubic piecewise differential system depending on three parameters. To effectively characterize the dynamics of this differential system, it is essential to analyze its local and global dynamical behaviors, with particular emphasis on the limit cycles. Here we obtain that the cubic piecewise differential system possesses abundant nonlinear dynamics. We provide the lower bound of the number of limit cycles for this piecewise cubic system using the Poincaré-Lyapunov method and averaging theory of discontinuous systems. Moreover we characterize the global dynamics of this piecewise system, taking into account their behavior near the infinity using the Poincaré compactification, and present the corresponding global phase portraits in the Poincaré disc.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In recent decades several authors have studied the dynamics of the second-order oscillator

$$(1) \quad \ddot{\phi} + f(\dot{\phi}, \phi)\dot{\phi} + g(\phi) = 0,$$

where the dot denotes the derivative with respect to the time t . This oscillator frequently appears in astronautics, mechanical engineering and biology, see [4, 18,

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20, 21, 22, 23, 28] and the references therein. The well known Rayleigh oscillator can be written as

$$(2) \quad \ddot{\phi} + \mu(a_3 + a_4\dot{\phi}^2)\dot{\phi} + a_1\phi = 0,$$

which was proposed in the theory of sound by Rayleigh [25]. Chen and Zou [10] discussed firstly the global dynamics for a class of Rayleigh oscillators with a Duffing term ϕ^3 . In the recent work [9] the authors studied the center conditions at a singular point and the dynamics near infinity for a generalized polynomial Rayleigh-Duffing oscillator. By replacing the term $a_4\dot{\phi}^2$ in (2) by $a_5\phi^2$ we have the famous Liénard-Duffing oscillator. As it is well known many mathematical models can be transformed into Liénard type systems for analyzing their dynamical behavior, see for instance [17, 27].

The more important problem for Rayleigh and Liénard oscillators is to know if these hybrid oscillators can have isolated periodic oscillations, i.e. if the corresponding plane differential systems can exhibit limit cycles. By combining Rayleigh and Liénard oscillators in [3] the authors obtained the Rayleigh-Liénard-Duffing oscillator

$$(3) \quad \ddot{\phi} + \mu(a_3 + a_4\dot{\phi}^2 + a_5\phi^2)\dot{\phi} + a_1\phi + a_2\phi^3 = 0.$$

Recently Cândido et al. [7] studied the non-existence, existence and uniqueness of limit cycles for the oscillator (3) with $a_1 > 0$ and $a_2 > 0$. Chen et al. [8] found that the Rayleigh-Liénard-Duffing oscillator (3) exhibits rich bifurcation phenomena and has two limit cycles when $a_1 = 1$ and $a_2 = 0$. By using the first order Melnikov functions the authors of [29] proved that the maximal number of limit cycles for the oscillator (3) is 2.

Wing rock refers to an oscillatory rolling motion of an aircraft, which occurs at high angles of attack. This oscillatory phenomenon can be observed in high-speed civil transport and military aircrafts. To characterize mathematically this dynamics Hsu et al. [14], Elzebda et al. [13], and Rong et al. [26] created suitable nonlinear aerodynamic models to analyze the primary aerodynamic nonlinearities responsible for wing rock through the following equation

$$(4) \quad \ddot{\phi} + D\dot{\phi} + \frac{\rho U_\infty^2 S b}{2I_{xx}} C_l = 0,$$

where ϕ is the roll angle, ρ is the density of air, U_∞ is the speed of the freestream, S is the wing surface, b is the chord, I_{xx} is the mass moment of inertia of the wing around the midspan axis, C_l is the roll-moment coefficient, and D is the damping coefficient. Recently, the authors of [6] consider the state space model of the wing rock system (4) with the following roll-moment coefficient

$$(5) \quad C_l = a_1\phi + a_2\phi^3 + a_3\dot{\phi} + a_4|\dot{\phi}|\dot{\phi} + a_5\phi^2\dot{\phi},$$

where coefficients $a_1, a_2, \dots, a_5$ are parameters related to the aircrafts operating conditions. If $(x, y) = (\phi, \dot{\phi})$ then system (4) with (5) can be written as the following planar first-order differential system

$$(6) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -b_1x - b_2x^3 - (b_3 + b_4|y| + b_5x^2)y, \end{aligned}$$

where $b_1 = Ca_1$, $b_2 = Ca_2$, $b_3 = D + Ca_3$, $b_4 = Ca_4$, $b_5 = Ca_5$ and $C = \rho U_\infty^2 Sb / (2I_{xx})$ are all constants.

In this paper we study the dynamics of the differential system (6) in the case $b_1 > 0$ and $b_2 < 0$ using the qualitative theory of differential equations. Applying the change of coordinates

$$\bar{x} = -\frac{\sqrt{b_1}}{\sqrt{-b_2}}x, \quad \bar{y} = \frac{b_1}{\sqrt{-b_2}}y, \quad \bar{t} = -\frac{1}{\sqrt{b_1}}t,$$

we obtain the continuous piecewise differential system

$$(7) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y, \\ -x + x^3 - (a_{01} + a_{02}y + a_{21}x^2)y \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} y, \\ -x + x^3 - (a_{01} - a_{02}y + a_{21}x^2)y \end{pmatrix} & \text{if } y \leq 0, \end{cases}$$

where $a_{01} = -b_3/\sqrt{b_1}$, $a_{02} = -b_4\sqrt{b_1}/\sqrt{-b_2}$ and $a_{21} = b_5\sqrt{b_1}/b_2$, and we rename the new variables $(\bar{x}, \bar{y}, \bar{t})$ as (x, y, t) .

In what follows the differential system in $y \geq 0$ will be the upper system, and the differential system in $y \leq 0$ will be the lower system.

The main results of this paper are the following three theorems.

Theorem 1. *From the singular point localized at the origin of coordinates of the piecewise differential system (7) can bifurcate two small limit cycles.*

Now we shall study the limit cycles that can bifurcate from the Hamiltonian system $\dot{x} = y$, $\dot{y} = -x + x^3$ with Hamiltonian

$$(8) \quad H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}x^4,$$

when this differential system is perturbed inside the class of piecewise differential systems (7) as follows

$$(9) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y, \\ -x + x^3 - \varepsilon(a_{01} + a_{02}y + a_{21}x^2)y \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} y, \\ -x + x^3 - \varepsilon(a_{01} - a_{02}y + a_{21}x^2)y \end{pmatrix} & \text{if } y \leq 0. \end{cases}$$

Theorem 2. *For $\varepsilon > 0$ sufficiently small using the averaging theory of first order the maximum number of limit cycles of the piecewise differential system (9) bifurcating from the set of periodic orbits of the center of the Hamiltonian system with the Hamiltonian $H(x, y)$, given in (8), is 2. Furthermore, this number can be reached.*

We recall that roughly speaking the Poincaré disc $\mathbb{D}$ is the closed unit disc centered at the origin of coordinates whose interior has been identified with the whole plane $\mathbb{R}^2$ and its boundary, the circle of radius one $\mathbb{S}^1$, is identified with the infinity of $\mathbb{R}^2$. Of course, in the plane $\mathbb{R}^2$ we can go to infinity in as many directions as points has the circle $\mathbb{S}^1$. For more details on the Poincaré disc see the subsection 2.3.

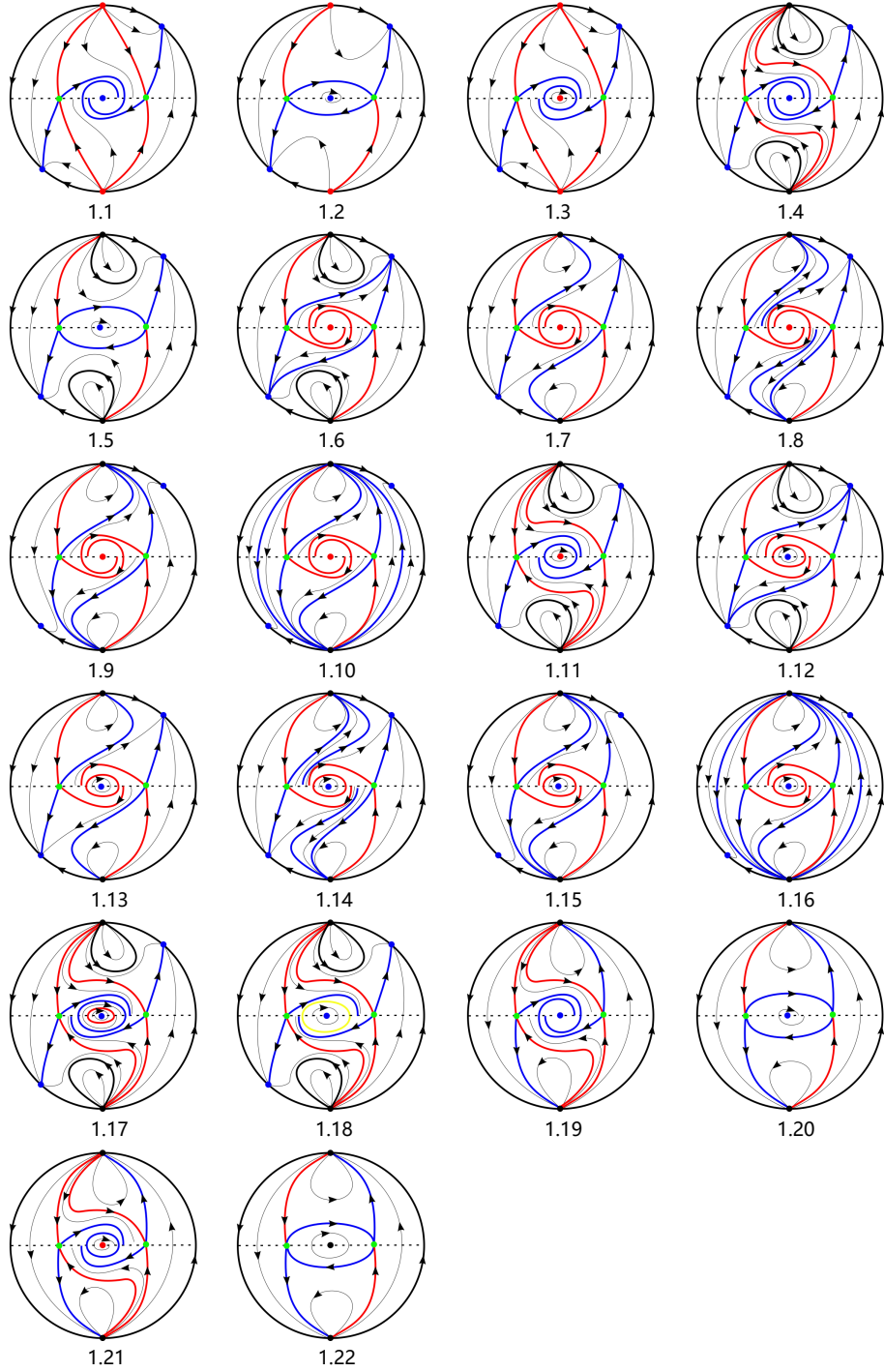


FIGURE 1. The distinct topological phase portraits of the differential system (7) in the Poincaré disc.

Since the continuous piecewise differential systems (7) are invariant under the symmetry $(x, y) \mapsto (-x, -y)$, their phase portraits are symmmatric with respect to the origin of coordinates.

We say that two phase portraits in the Poincaré disc $\mathbb{D}$ are topologically equivalent if there is a homeomorphism $h : \mathbb{D} \rightarrow \mathbb{D}$ that sends $\mathbb{S}^1$ into $\mathbb{S}^1$ and orbits of one phase portrait onto orbit of the other phase portrait, preserving or reveresing the sense of all the orbits.

Theorem 3. *The distinct topological phase portraits of the differential system (7) are shown in Figure 1.*

Theorem 4. *The bifurcation diagram of the differential system (7) consists of the following bifurcation surfaces and curves:*

- (i) *The pitchfork bifurcation surface for singular point at infinity is*

$$\mathcal{P} := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{21} = 0\}.$$

- (ii) *The generalized Hopf bifurcation surfaces and curves are*

$$\mathcal{H}_1^+ := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{01} = 0, a_{02} > 0\},$$

$$\mathcal{H}_1^- := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{01} = 0, a_{02} < 0\},$$

$$\mathcal{H}_2^+ := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{01} = a_{02} = 0, a_{21} > 0\},$$

$$\mathcal{H}_2^- := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{01} = a_{02} = 0, a_{21} < 0\}.$$

- (iii) *The heteroclinic bifurcation surface is*

$$\mathcal{HC} := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{02} = \varphi_1(a_{01}, a_{21})\}.$$

- (iv) *The double limit cycle bifurcation surface is*

$$\mathcal{DC} := \{(a_{01}, a_{02}, a_{21}) \in \mathbb{R}^3 : a_{02} = \varphi_2(a_{01}, a_{21})\}.$$

We recall that a heteroclinic orbit of a differential system is an orbit whose α -limit is a singular point and its ω -limit is a distinct singular point. Here a double limit cycle is a semi-stable limit cycle that attracts the orbits from one side and repels the orbits in the other side, i.e. it can appear from the coalescence of a stable limit cycle with an unstable one.

The proofs of Theorems 1, 2, 3 and 4 are given in sections 3, 4 and 5.

In section 2 we describe some basic results that we shall need for proving the theorems.

2. PRELIMINARY RESULTS

2.1. Singular points. A point (x_0, y_0) is a singular of the differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ if $P(x_0, y_0) = Q(x_0, y_0) = 0$. A singular (x_0, y_0) is *hyperbolic* when the eigenvalues of the Jacobian matrix of the differential system evaluated at this singular point have their real parts non-zero, and it is *semi-hyperbolic* if the Jacobian matrix has only one eigenvalue equal to zero.

The local phase portraits of the hyperbolic and semi-hyperbolic have been classified, see for instance Theorems 2.15 and 2.19 of [12], respectively. In this paper

the results on the local phase portraits of hyperbolic or semi-hyperbolic singular points are obtained using these two theorems. If the Jacobian matrix evaluated at a singular point is identically zero, we must analyze the local phase portrait at this singular point using special changes of variables called blow up's, see for instance [1]), such singular points are called *linearly zero*.

2.2. Lyapunov constants for discontinuous systems. As described in [11, 19], we consider the following piecewise differential system

$$(10) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \sum_{i=2}^n P_i^+(x, y) \\ -x + \delta y + \sum_{i=2}^n Q_i^+(x, y) \end{pmatrix}, & y \geq 0, \\ \begin{pmatrix} y + \sum_{i=2}^n P_i^-(x, y) \\ -x + \delta y + \sum_{i=2}^n Q_i^-(x, y) \end{pmatrix}, & y \leq 0, \end{cases}$$

where $P_i^\pm(x, y)$ and $Q_i^\pm(x, y)$ are homogeneous polynomials in x and y of degree i . The origin is a common singular point in both upper and lower systems of (10).

Under the transformation $x = r \cos \theta$ and $y = r \sin \theta$, system (10) can be written in the polar coordinates as

$$(11) \quad \frac{dr}{d\theta} = \begin{cases} \frac{G^+(r, \theta)}{-1 + H^+(r, \theta)}, & \theta \in [0, \pi], \\ \frac{G^-(r, \theta)}{-1 + H^-(r, \theta)}, & \theta \in [\pi, 2\pi], \end{cases}$$

where $G^\pm(r, \theta)$ and $H^\pm(r, \theta)$ are polynomials in r of degrees n and $n-1$, respectively. Let

$$r^+(\rho, \theta) = \sum_{i=1}^{+\infty} v_i^+(\theta) \rho^i \quad (\text{resp. } r^-(\rho, \theta) = \sum_{i=1}^{+\infty} v_i^-(\theta) \rho^i),$$

be the solution of (11) satisfying the initial condition $r^+(\rho, 0) = \rho$ (resp. $r^-(\rho, \pi) = \rho$) for $\theta \in (0, \pi)$ (resp. $\theta \in (\pi, 2\pi)$), where v_i^+ and v_i^- are the Taylor's coefficients. Then we denote the half-return maps

$$\mathcal{P}^+(\rho) := r^+(\rho, \pi) = \sum_{i=1}^{+\infty} v_i^+(\pi) \rho^i \quad \text{and} \quad \mathcal{P}^-(\rho) := r^-(\rho, 2\pi) = \sum_{i=1}^{+\infty} v_i^-(2\pi) \rho^i,$$

for the upper and lower systems of (10), respectively. Further, we have the displacement function

$$(12) \quad D(\rho) := \mathcal{P}^-(\mathcal{P}^+(\rho)) - \rho = \sum_{i=1}^{+\infty} V_i \rho^i,$$

in which V_k is called the k th Lyapunov constant of system (10).

Definition 5. The origin of the piecewise differential system (10) is called a *weak focus of order k* if $V_1 = \dots = V_k = 0$, $V_{k+1} \neq 0$.

We get that the weak focus is unstable, stable if the first non-zero Lyapunov constant of system (10) is negative, positive, respectively. Actually, from the displacement function (12), the origin of system (10) is a center if and only if $D(\rho) \equiv 0$, i.e. $V_i = 0$ for all i . The limit cycles that bifurcate from a weak focus are called *small amplitude limit cycles*. As mentioned in [5], if system (10) has a weak focus at the origin of order k , then at most k small amplitude limit cycles can bifurcate from the origin when the system is perturbed.

2.3. Vertical blow up. Consider the following polynomial differential system

$$(13) \quad \dot{x} = P(x, y) = P_i(x, y) + \dots, \quad \dot{y} = Q(x, y) = Q_i(x, y) + \dots,$$

in $\mathbb{R}^2$ where P and Q are coprime polynomials, P_i and Q_i are homogeneous polynomials of degree i and as usual the dots denote higher order terms in the variables x and y . The origin is a singular point of system (13) because we assume that n is a positive integer. The real linear factors of the homogeneous polynomial $P_n(x, y)y - Q_n(x, y)x$ define the straight lines through the origin called the *characteristic directions* at the singular point localized at the origin of coordinates. We know that the orbits that start or end at the singular point localized at origin of coordinates start or end tangent to the straight lines of the characteristic directions. For more information on the characteristic directions see for instance [2].

If the singular point at the origin of coordinates of the differential system (13) is linearly zero, in this paper we shall study its local phase portrait doing vertical blow up's.

The vertical blow up in the y -direction is done with the change of variables $(u, v) = (x, y/x)$. This change sends the origin of coordinates of the differential system (13) in the whole straight line $u = 0$. So studying the dynamics of the new differential system in the variables u and v in a neighbourhood of the straight line $u = 0$ we are studying the local phase portrait at the singular point localized at the origin of coordinates of the differential system (13).

Before doing a vertical blow up we must verify that the direction $x = 0$ is not a characteristic direction of the singular point at the origin of coordinates, otherwise we can lost information of the possible orbits of the differential system (13) tangent to the straight line $x = 0$. Consequently, if $x = 0$ is a characteristic direction we must do a convenient twist $(x, y) = (u + \alpha v, v)$ with $\alpha \neq 0$ before doing a vertical blow taking care that the new straight line $u = 0$ not be again a characteristic direction of the differential system in the variables u and v .

2.4. The Poincaré compactification. Let $X = (P^\pm, Q^\pm)$ be the polynomial vector field of degree n in the plane $\mathbb{R}^2$. The Poincaré compactified vector field $p(X)$ of the vector field X will be an analytic vector field on the 2-dimensional sphere $\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3; y_1^2 + y_2^2 + y_3^2 = 1\}$.

We identify the plane $\mathbb{R}^2$ with the tangent plane $y_3 = 1$ at the point $(0, 0, 1)$ of the sphere $\mathbb{S}^2$. By definition the northern hemisphere of the sphere $\mathbb{S}^2$ is $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$, the southern hemisphere is $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$, and the equator is $\mathbb{S}^1 = \{y \in \mathbb{S}^2; y_3 = 0\}$.

We project the plane $y_3 = 1$, and consequently the dynamics on this plane defined by the vector field X , onto the northern and southern hemispheres of the sphere $\mathbb{S}^2$ using the central projections $f_{\pm} : \mathbb{R}^2 \rightarrow H_{\pm}$, $f_{\pm}(y_1, y_2, 1) = (y_1^2 + y_2^2 + 1)^{-1/2}(y_1, y_2, 1)$. In this way we get two copies of the vector field X , one X_+ in H_+ , and another X_- in H_- . Therefore the vector field $X' = X_+ \cup X_-$ is defined in $\mathbb{S}^2 \setminus \mathbb{S}^1$. Note that the infinity of the plane $y_2 = 1$, i.e. of the plane $\mathbb{R}^2$, can be identified with the equator $\mathbb{S}^1$ of the sphere $\mathbb{S}^2$.

The analytic extension of X' from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$ doing $p(X) = y_3^{n-1}X'$ is called the *Poincaré compactified vector field* $p(X)$ of the polynomial vector field X .

The projection of the closed northern hemisphere into the plane $y_3 = 0$ under the projection $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc* $\mathbb{D}$. The dynamics exhibited by the vector field $p(X)$ in a neighborhood of $\mathbb{S}^1$ is the dynamics in a neighborhood of the infinity of the vector field X .

In order to study the dynamics of the compactified vector field $p(X)$ near the infinity of the Poincaré disc, or of the sphere $\mathbb{S}^2$, we shall work with the local charts $U_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i > 0\}$ and $V_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i < 0\}$ and the diffeomorphisms $\phi_i : U_i \rightarrow \mathbb{R}^2$ and $\psi_i : V_i \rightarrow \mathbb{R}^2$ for $i = 1, 2$, defined by $\phi_i(y_1, y_2, y_3) = \psi_i(y_1, y_2, y_3) = (y_{\ell}/y_i, y_k/y_i) = (u, v)$, with $\ell \neq i$, $k \neq i$ and $\ell < k$.

In the local chart U_1 the vector field $p(X)$ writes

$$\dot{u} = v^n \left(Q^{\pm} \left(\frac{1}{v}, \frac{u}{v} \right) - u P^{\pm} \left(\frac{1}{v}, \frac{u}{v} \right) \right), \quad \dot{v} = -v^{n+1} P^{\pm} \left(\frac{1}{v}, \frac{u}{v} \right),$$

with $u \geq 0$ and $u < 0$, respectively. In the chart V_1 the expressions of the compactified vector field $p(X)$ is the same than in the chart U_1 but multiplied by $(-1)^{n-1}$. The expressions in the local charts U_2 and V_2 are

$$\dot{u} = v^n \left(P^+ \left(\frac{u}{v}, \frac{1}{v} \right) - u Q^+ \left(\frac{u}{v}, \frac{1}{v} \right) \right), \quad \dot{v} = -v^{n+1} Q^+ \left(\frac{u}{v}, \frac{1}{v} \right),$$

and

$$\dot{u} = (-1)^{n-1} v^n \left(P^- \left(\frac{u}{v}, \frac{1}{v} \right) - u Q^- \left(\frac{u}{v}, \frac{1}{v} \right) \right), \quad \dot{v} = (-1)^n v^{n+1} Q^- \left(\frac{u}{v}, \frac{1}{v} \right),$$

respectively.

The points of the infinity in the charts U_i and V_i for $i = 1, 2$ are of the form $(u, 0)$. We note that the infinity $\mathbb{S}^1$ is invariant under the flow of $p(X)$, i.e. the orbit of the vector field $p(X)$ starting at a point of the infinity remains at infinity. For more information on the Poincaré compactification see [12, Chapter 5].

We remark as it is usual that the *finite* singular points of X or of $p(X)$ are the singular points of the vector field X , and that the *infinite* singular points of X or of $p(X)$ are the singular points of the vector field $p(X)$ in $\mathbb{S}^1$.

2.5. Chebyshev systems. The set of functions $(g_0, \dots, g_n)$ defined on an interval I define an *Extended Chebyshev system* on I if and only if any linear combination of these functions with finitely many zeros has at most n zeros counting their multiplicities and such number is realizable.

The set of functions $(g_0, \dots, g_n)$ define an *Extendend Complete Chebyshev system* on I if and only if for any $k \in \{0, 1, \dots, n\}$, the set of functions $(g_0, \dots, g_k)$ form an Extendend Chebyshev system.

Theorem 6. *Let $g_0, \dots, g_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then the set of functions $(f_0, \dots, f_n)$ is an Extendend Complete Chebyshev system on I if and only if for each $k \in \{0, 1, \dots, n\}$ and all $y \in I$ the Wronskian*

$$W(g_0, \dots, g_k)(y) = \begin{vmatrix} g_0(y) & g_1(y) & \cdots & g_k(y) \\ g'_0(y) & g'_1(y) & \cdots & g'_k(y) \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{(k)}(y) & g_1^{(k)}(y) & \cdots & g_k^{(k)}(y) \end{vmatrix}$$

is different from zero.

For more details on the Chebyshev systems and proofs see [16].

2.6. Averaging theory for discontinuous systems. In this subsection we recall the averaging theory of first order for computing the periodic orbits of the discontinuous piecewise differential systems separated by one straight line that we need for proving some of the results of this paper. This improvement of the classical averaging theory for computing periodic orbits of discontinuous piecewise differential systems in the plane separated by one straight line was developed in [15].

We consider the following discontinuous differential systems

$$(14) \quad \frac{dr(\theta)}{d\theta} = \begin{cases} F^+(\theta, r, \varepsilon) & \text{if } 0 \leq \theta \leq \pi, \\ F^-(\theta, r, \varepsilon) & \text{if } -\pi \leq \theta \leq 0, \end{cases}$$

where

$$(15) \quad F^\pm(\theta, r, \varepsilon) = \varepsilon F_1^\pm(\theta, r) + O(\varepsilon^2),$$

with $\theta \in \mathbb{S}^1$ and $r \in D$, being ε a sufficiently small parameter, and D an open interval $(0, \rho)$ with $\rho > 0$ of $\mathbb{R}$.

From [15] we have the functions

$$f_1^+(r) = \int_0^\pi F_1^+(\theta, r) d\theta, \quad f_1^-(r) = \int_0^{-\pi} F_1^-(\theta, r) d\theta.$$

The function $f_1(r) = f_1^+(r) - f_1^-(r)$ is called the averaged function of first order of the piecewise differential system (14). Then each simple positive real root of the function $f_1(r)$ provides a limit cycle of the piecewise differential system (14), for more details see [15].

3. PROOFS OF THEOREMS 1 AND 2, AND STATEMENTS (ii) AND (iv) OF THEOREM 4

Now we consider qualitative properties of the finite singular points of system (7) and have the following result.

Lemma 7. *The continuous piecewise differential system (7) has three singular points $O = (0, 0)$, $E_l = (-1, 0)$ and $E_r = (1, 0)$, where E_l and E_r are two saddles.*

- (i) If $|a_{01}| \geq 2$ the singular point O is a hyperbolic node, unstable if $a_{01} \leq -2$, and stable if $a_{01} \geq 2$.
- (ii) If $|a_{01}| < 2$ and $a_{01} \neq 0$ the singular point O is a hyperbolic focus, unstable if $-2 < a_{01} < 0$, and stable if $0 < a_{01} < 2$.
- (iii) If $a_{01} = 0$ and $a_{02} \neq 0$ the singular point O is a weak focus of order one, unstable if $a_{02} < 0$, and stable if $a_{02} > 0$.
- (iv) If $a_{01} = a_{02} = 0$ and $a_{21} \neq 0$ the singular point O is a weak focus of order two, unstable if $a_{21} < 0$, and stable if $a_{21} > 0$.
- (v) If $a_{01} = a_{02} = a_{21} = 0$ the singular point O is a center.

Proof. The Jacobian matrix at the singular point (x, y) of system (7) is

$$(16) \quad J = \begin{bmatrix} 0 & 1 \\ -1 + 3x^2 - 2a_{21}xy & -a_{01} - a_{21}x^2 - 2a_{02}y \end{bmatrix}.$$

The eigenvalues of J at $E_{l,r}$ are a positive real root and a negative real root, so $E_{l,r}$ are two saddles. When $a_{01} \neq 0$, the eigenvalues of J at O are two roots with nonzero real parts. This implies the statements (i) and (ii) based on the signs of the real parts of the two roots. For more details on the hyperbolic singular points see for instance Theorem 2.15 of [12].

When $a_{01} = 0$ the Jacobian matrix (16) at O has the eigenvalues $\pm i$, so the singular point O is either a weak focus or a center. In order to distinguish between a weak focus and a center we apply the Bautin's algorithm described in section 2.2.

We write the upper system of (7) in the polar coordinates (r, θ) through the change $x = r \cos \theta$ and $y = r \sin \theta$, and we obtain

$$(17) \quad \begin{aligned} \frac{dr}{d\theta} &= \frac{r^2 \sin \theta (r \cos^3 \theta - a_{21} r \cos^2 \theta \sin \theta - a_{02} \sin^2 \theta)}{-1 + r^2 \cos^4 \theta - a_{21} r^2 \cos^3 \theta \sin \theta - a_{02} r \cos \theta \sin^2 \theta} \\ &= M_2(\theta)r^2 + M_3(\theta)r^3 + M_4(\theta)r^4 + \dots, \end{aligned}$$

where

$$\begin{aligned} M_2(\theta) &= a_{02} \sin^3 \theta, \\ M_3(\theta) &= -\cos \theta \sin \theta (\cos^2 \theta - a_{21} \cos \theta \sin \theta - a_{02}^2 \sin^4 \theta), \\ M_4(\theta) &= a_{02} \sin^3 \theta \cos^2 \theta (a_{02}^2 \sin^4 \theta - 2a_{21} \sin \theta \cos \theta + 2 \cos^2 \theta). \end{aligned}$$

By transformation $(x, y, t) \rightarrow (x, -y, -t)$, the lower system of (7) becomes

$$(18) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + x^3 - (-a_{02}y - a_{21}x^2)y. \end{aligned}$$

Similarly using the polar coordinates transformation, system (18) can be written as

$$(19) \quad \begin{aligned} \frac{dr}{d\theta} &= \frac{r^2 \sin \theta (r \cos^3 \theta + a_{21} r \cos^2 \theta \sin \theta + a_{02} \sin^2 \theta)}{-1 + r^2 \cos^4 \theta + a_{21} r^2 \cos^3 \theta \sin \theta + a_{02} r \cos \theta \sin^2 \theta} \\ &= N_2(\theta)r^2 + N_3(\theta)r^3 + N_4(\theta)r^4 + \dots, \end{aligned}$$

where

$$\begin{aligned} N_2(\theta) &= -a_{02} \sin^3 \theta, \\ N_3(\theta) &= -\cos \theta \sin \theta (a_{02} \cos^2 \theta + a_{21} \cos \theta \sin \theta + a_{02}^2 \sin^4 \theta), \\ N_4(\theta) &= a_{02} \sin^3 \theta \cos^2 \theta (a_{02}^2 \sin^4 \theta + 2a_{21} \sin \theta \cos \theta + 2 \cos^2 \theta). \end{aligned}$$

By the formulas of the Lyapunov constants in [11, 19], we can obtain the first Lyapunov constant $V_1 = a_{01}\pi = 0$. Then the second Lyapunov constant has the form

$$(20) \quad V_2 = \int_0^\pi M_2(\theta) - N_2(\theta) d\theta = \frac{8}{3} a_{02}.$$

Therefore the singular point $O(0, 0)$ is a stable weak focus of order one if and only if $a_{01} = 0$ and $a_{02} > 0$. It is an unstable weak focus of order one if and only if $a_{01} = 0$ and $a_{02} < 0$. Statement (iii) is proved.

Furthermore, $V_2 = 0$ if and only if $a_{02} = 0$. To determine the local phase portrait at the origin O of system (7) with $a_{02} = 0$, we need to calculate the third Lyapunov constant

$$(21) \quad V_3 = \int_0^\pi \left((M_3(\theta) + 2r_{31}(\theta)M_2(\theta)) - (N_3(\theta) + 2r_{32}(\theta)N_2(\theta)) \right) d\theta = \frac{1}{4} a_{21}\pi,$$

where

$$r_{31}(\theta) = \int_0^\theta M_3(\theta) d\theta, \quad r_{32}(\theta) = \int_0^\theta N_3(\theta) d\theta.$$

Consequently, the singular point O is a stable weak focus of order two if and only if $a_{01} = a_{02} = 0$, $a_{21} > 0$, and it is an unstable weak focus of order two if and only if $a_{01} = a_{02} = 0$, $a_{21} < 0$. So statement (iv) is proved.

When $a_{01} = a_{02} = a_{21} = 0$ the continuous piecewise differential system (7) becomes the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = -x + x^3,$$

with Hamiltonian $H = (x^2 + y^2)/2 - x^4/4$. So the singular point O is a center. Hence statement (v) is proved. $\square$

Proof of Theorem 1 and the statements (ii) and (iv) of Theorem 4. Now we shall study the Hopf bifurcation and the double limit cycle bifurcation that take place in a small neighborhood U of the point $P = (0, 0, a_{21})$ in the parameter space (a_{01}, a_{02}, a_{21}) when $-2 < a_{01} < 2$.

In a similar way to the proof of Lemma 7 we can obtain the displacement function $d(\theta, \xi)$ for the initial conditions $r^+(0, \xi) = r^-(\pi, \xi) = \xi$ of systems (17) and (19), expressed in convergent power series of the variable ξ

$$(22) \quad d(\theta, \xi) = r_1(\theta)\xi + r_2(\theta)\xi^2 + r_3(\theta)\xi^3 + O(\xi^4),$$

for $\xi > 0$ is sufficiently small. By considering the truncated 3-jet of equation (22) and computing the displacement function at $\theta = 2\pi$, we have

$$(23) \quad d(2\pi, \xi) = V_1(a_{01}, a_{02}, a_{21})\xi + V_2(a_{01}, a_{02}, a_{21})\xi^2 + V_3(a_{01}, a_{02}, a_{21})\xi^3,$$

where $V_i(a_{01}, a_{02}, a_{21}) = r_i(2\pi)$ for $i = 1, 2, 3$. From the proof of Lemma 7 we obtain $V_1(a_{01}, a_{02}, a_{21}) = a_{01}\pi$, $V_2(0, a_{02}, a_{21}) = 8a_{02}/3$ when $V_1(a_{01}, a_{02}, a_{21}) = 0$,

and $V_3(0, 0, a_{21}) = \pi a_{21}/4$ when $V_1(a_{01}, a_{02}, a_{21}) = V_2(a_{01}, a_{02}, a_{21}) = 0$. Hence, for ξ sufficiently small the positive roots of the equation

$$(24) \quad V_1(a_{01}, a_{02}, a_{21}) + V_2(a_{01}, a_{02}, a_{21})\xi + V_3(a_{01}, a_{02}, a_{21})\xi^2 = 0,$$

correspond to limit cycles of the piecewise differential system (7). Notice that if equation (24) has two distinct positive roots (resp. one positive root, no positive roots), system (7) has two limit cycles (resp. one limit cycle, no limit cycles).

We shall show that (24) has at most two distinct positive roots. If $V_3 = 0$ then equation (24) has no positive roots or a positive root when $V_1V_2 \geq 0$ or $V_1V_2 < 0$, respectively. If $V_3 \neq 0$, using the Vieta's theorem, we assume that equation (24) has two roots ξ_1 and ξ_2 such that

$$\xi_1 + \xi_2 = -\frac{V_2}{V_3}, \quad \xi_1\xi_2 = \frac{V_1}{V_3},$$

and its discriminant is

$$(25) \quad \Delta_0 = V_2^2 - 4V_1V_3.$$

If $\Delta_0 < 0$, i.e. $64a_{02}^2 - 9a_{01}a_{21}\pi^2 < 0$, equation (24) has no real roots in the parameter region.

If $\Delta_0 = 0$ then $64a_{02}^2 - 9a_{01}a_{21}\pi^2 = 0$. If $\xi_1 + \xi_2 \leq 0$, i.e. $a_{02}a_{21} \geq 0$, equation (24) has a double negative root. If $\xi_1 + \xi_2 > 0$, i.e. $a_{02}a_{21} < 0$, equation (24) has a double positive root, that corresponds to the existence of the double limit cycle bifurcation curve $\mathcal{DC}$.

If $\Delta_0 > 0$ then $64a_{02}^2 - 9a_{01}a_{21}\pi^2 > 0$. If $\xi_1 + \xi_2 < 0$ and $\xi_1\xi_2 > 0$, i.e. $a_{02}a_{21} > 0$ and $a_{01}a_{21} > 0$, equation (24) has two non-positive roots. If $\xi_1\xi_2 \leq 0$, i.e. $a_{01}/a_{21} \leq 0$, equation (24) has at most one positive root. If $\xi_1 + \xi_2 > 0$ and $\xi_1\xi_2 > 0$, i.e. $a_{02}a_{21} < 0$ and $a_{01}a_{21} > 0$, equation (24) has two positive roots.

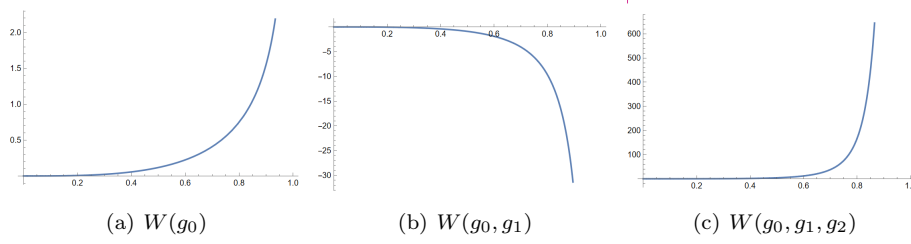
In summary, we have shown that the piecewise differential system (7) has at most two limit cycles in a small neighbourhood of the origin. These two limit cycles can collide into a semistable limit cycle. And this semistable limit cycle can disappears when parameters move from the one side of the double limit cycles bifurcation surface $\mathcal{DC}$. Therefore, we have proved Theorem 1, and the statements (ii) and (iv) of Theorem 4. $\square$

Proof of Theorem 2. We write the upper system of (9) in polar coordinates (r, θ) where $x = r \cos \theta$ and $y = r \sin \theta$, and taking as the new time the variable θ this upper system becomes the differential equation

$$(26) \quad \begin{aligned} \frac{dr}{d\theta} &= \varepsilon \frac{r \sin^2 \theta (2a_{01} + 2a_{02}r \sin \theta + a_{21}r^2 \cos(2\theta) + a_{21}r^2)}{2(1 - r^2 \cos^4 \theta)^2} + O(\varepsilon^2) \\ &= \varepsilon F_1^+(\theta, r) + O(\varepsilon^2). \end{aligned}$$

Doing the same the lower system becomes the differential equation

$$(27) \quad \begin{aligned} \frac{dr}{d\theta} &= \varepsilon \frac{r \sin^2 \theta (2a_{01} - 2a_{02}r \sin \theta + a_{21}r^2 \cos(2\theta) + a_{21}r^2)}{2(1 - r^2 \cos^4 \theta)^2} + O(\varepsilon^2) \\ &= \varepsilon F_1^-(\theta, r) + O(\varepsilon^2). \end{aligned}$$

FIGURE 2. The graphics of the three Wronskians in the interval $(0, 1)$.

Now computing the averaged function of first order $f_1(r) = f_1^+(r) - f_1^-(r)$ defined in subsection 2.6 we obtain

$$f_1(r) = a_{21}g_0(r) + a_{01}g_1(r) + a_{02}g_2(r),$$

where

$$\begin{aligned} g_0(r) &= -\frac{\pi r^2 (\sqrt{1-r^2} + r - 1)}{4(r-1)\sqrt{r+1}}, \\ g_1(r) &= -\frac{\pi (-3r^2 + 3\sqrt{1-r^2}r - 2\sqrt{1-r^2} + r + 2)}{4(r-1)\sqrt{r+1}}, \\ g_2(r) &= \frac{1}{2}\sqrt{r} \left((3r+1) \tan^{-1}(\sqrt{r}) + (3r-1) \tanh^{-1}(\sqrt{r}) \right). \end{aligned}$$

We apply the Chebyshev theory described in subsection 2.5 for studying the zeros of the function $f_1(r)$. We consider the Wronskians

$$\begin{aligned} W(g_0) &= -\frac{\pi r^2 (\sqrt{1-r^2} + r - 1)}{4(r-1)\sqrt{r+1}}, \\ W(g_0, g_1) &= -\frac{\pi^2 r ((3\sqrt{1-r^2} - 1)r^2 - 4\sqrt{1-r^2} + 4)}{8(1-r^2)^{3/2}}, \\ W(g_0, g_1, g_2) &= -\frac{\pi^2}{64\sqrt{r}(1-r^2)^{7/2}} \left((12(\sqrt{1-r^2} - 1) + r(36(\sqrt{1-r^2} - 1) \right. \\ &\quad + r(r(9(\sqrt{1-r^2} - 3) + r(r(-33\sqrt{1-r^2} + 4r + 63) \\ &\quad - 3\sqrt{1-r^2} - 23)) - 5\sqrt{1-r^2} + 31))) \tan^{-1}(\sqrt{r}) \\ &\quad + (r(36(\sqrt{1-r^2} - 1) + r(r(9(\sqrt{1-r^2} - 3) + r(r(-33\sqrt{1-r^2} \\ &\quad - 4r + 63) + 3\sqrt{1-r^2} + 23)) + 5\sqrt{1-r^2} - 31)) \\ &\quad - 12(\sqrt{1-r^2} - 1)) \tanh^{-1}(\sqrt{r}) + 8((r^2 + 8\sqrt{1-r^2} - 12)r^2 \\ &\quad \left. - 8\sqrt{1-r^2} + 8)r^{3/2} \right). \end{aligned}$$

In the interval $(0, 1)$ for the variable r we have that $W(g_0) > 0$, $W(g_0, g_1) < 0$ and $W(g_0, g_1, g_2) > 0$, see Figure 2. Therefore from subsection 2.5 it follows that the function $f_1(r)$ has two zeros in the interval $(0, 1)$ for convenient values of a_{21} , a_{01} and a_{02} . This completes the proof of the theorem. $\square$

4. PROOF OF THEOREM 3

First we analyse the infinite singular points of the piecewise differential system (7), and after we provide their phase portraits in the Poincaré disc.

For the definition of a rotated vector field see for instance Chapter 7 of [12] and [24].

Lemma 8. *The vector field of the piecewise differential system (7) is rotated with respect to the parameters a_{01} , a_{02} or a_{21} . Consequently, when the parameter a_{01} , a_{02} or a_{21} increases (resp. decreases) a stable limit cycles tends to contract (resp. expand) monotonically, while an unstable limit cycle tends to expand (resp. contract) monotonically.*

Proof. Let $(P(x, y, \alpha), Q(x, y, \alpha))$ be a vector field in $\mathbb{R}^2$ depending on a parameter α . If the singular points of this vector field does not depend on α , and the determinant

$$\begin{vmatrix} P & Q \\ \frac{\partial P}{\partial \alpha} & \frac{\partial Q}{\partial \alpha} \end{vmatrix}$$

does not vanish or vanish in a set of zero Lebesgue measure, then the vector field $(P(x, y, \alpha), Q(x, y, \alpha))$ is a *rotated vector field*.

From Lemma 7 it follows that the three singular points of the continuous piecewise differential system (7) do not depend of the parameters a_{01} , a_{02} and a_{21} . Moreover, by direct computations we have

$$\begin{vmatrix} y & -x + x^3 - (a_{01} + a_{02}|y| + a_{21}x^2)y \\ 0 & -y \end{vmatrix} = -y^2 \leq 0,$$

$$\begin{vmatrix} y & -x + x^3 - (a_{01} + a_{02}|y| + a_{21}x^2)y \\ 0 & -y|y| \end{vmatrix} = -y^2|y| \leq 0,$$

and

$$\begin{vmatrix} y & -x + x^3 - (a_{01} + a_{02}|y| + a_{21}x^2)y \\ 0 & -x^2y \end{vmatrix} = -y^2x^2 \leq 0.$$

Fixing two parameters in the piecewise differential system (7) this system is rotated with respect the remaining parameter. Furthermore, from the properties of the rotated vector fields when either a_{01} , or a_{02} , or a_{21} increases (resp. decreases), a stable limit cycles contracts (resp. expands) monotonically while an unstable limit cycles expands (resp. contracts) monotonically. $\square$

For studying the dynamics at the circle of the infinity, we use the Poincaré compactification in the Poincaré disc, see for more details, formulas and notation Chapter 5 of [12].

4.1. The local charts U_1 and U_2 . The piecewise differential system (7) in the local chart U_1 and U_2 are given by

$$(28) \quad \begin{aligned} \dot{u} &= 1 - a_{21}u - v^2 \mp a_{02}u^2v - a_{01}uv^2 - u^2v^2, \\ \dot{v} &= -uv^3, \end{aligned}$$

(the minus in the monomial $a_{02}u^2$ correspond to $y \geq 0$ and the plus to $y \leq 0$), and

$$(29) \quad \begin{aligned} \dot{u} &= a_{02}uv + v^2 + a_{01}uv^2 + a_{21}u^3 - u^4 + u^2v^2, \\ \dot{v} &= v(a_{02}v + a_{01}v^2 + a_{21}u^2 - u^3 + uv^2), \end{aligned}$$

respectively.

Although the infinite singular points of the vector field (28) in U_1 with $u < 0$ are virtual points, due to the symmetry of system (7) there are the corresponding infinite singular points in V_1 with $u > 0$. Hence we need to study all the infinite singular points of system (28), and the origin of system (29) if it is a singular point.

When $a_{21} = 0$ there are no infinite singular points on U_1 , because $\dot{u}|_{v=0} = 1 > 0$. When $a_{21} \neq 0$ there is the singular point $A_1 = (1/a_{21}, 0)$ in U_1 . Since the linear part of system (28) at A_1 is

$$\begin{pmatrix} -a_{21} & \mp a_{02}/a_{21}^2 \\ 0 & 0 \end{pmatrix},$$

the singular point A_1 is a semi-hyperbolic singular point of system (28). From Theorem 2.19 of [12] we obtain that this infinite singular point is a stable node or an unstable node when $a_{21} > 0$ or $a_{21} < 0$, respectively. So we have proved the next proposition.

Proposition 9. *If $a_{21} = 0$ there are no infinite singular points in the local chart U_1 . If $a_{21} \neq 0$ there is a unique node, unstable if $a_{21} < 0$ and stable if $a_{21} > 0$.*

In order to complete the study of the local phase portraits of the infinite singular points we only need to determine the local phase portrait at the infinite singular point localized at the origin of the local chart U_2 because the phase portrait of system (7) is symmetry with respect to the origin.

Since system (7) is invariant under the change $y \rightarrow -y$, $t \rightarrow -t$, $a_{21} \rightarrow -a_{21}$, $a_{02} \rightarrow -a_{02}$. Thus, we only need to analyze the case $a_{21} \geq 0$. We define the following sets in the parameter plane (a_{21}, a_{02}) :

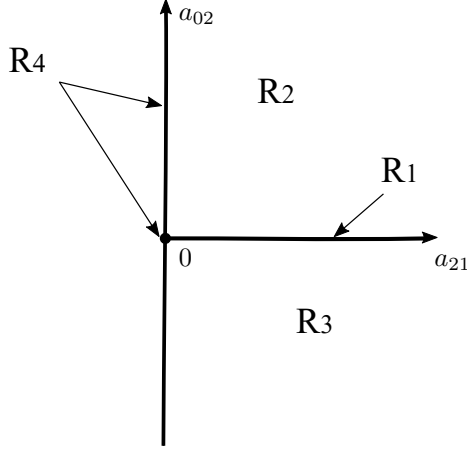
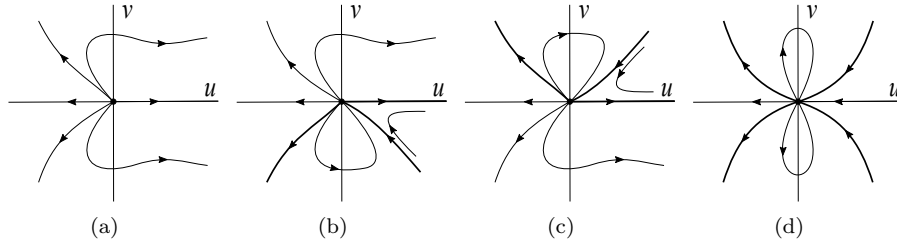
$$\begin{aligned} R_1 &= \{(a_{21}, a_{02}) : a_{21} > 0, a_{02} = 0\}, \\ R_2 &= \{(a_{21}, a_{02}) : a_{21} > 0, a_{02} > 0\}, \\ R_3 &= \{(a_{21}, a_{02}) : a_{21} > 0, a_{02} < 0\}, \\ R_4 &= \{(a_{21}, a_{02}) : a_{21} = 0\}. \end{aligned}$$

See the regions R_k for $k = 1, 2, 3, 4$ in Figure 3.

Proposition 10. *The topological local phase portrait at the singular point localized at the origin O_1 of the local chart U_2 with $a_{21} \geq 0$ is given in*

- Figure 4(a) if $(a_{21}, a_{02}) \in R_1$,
- Figure 4(b) if $(a_{21}, a_{02}) \in R_2$,
- Figure 4(c) if $(a_{21}, a_{02}) \in R_3$,
- Figure 4(d) if $(a_{21}, a_{02}) \in R_4$.

Proof. The singular point at the origin O_1 of the local chart U_2 is linearly zero, so we shall study its local phase portrait using vertical blow ups, see subsection 2.3.

FIGURE 3. The regions R_i for $k = 1, 2, 3, 4$.FIGURE 4. The topological local phase portraits at the origin of the local chart U_2 with $a_{21} \geq 0$.

Since the unique characteristic direction at the origin of U_2 is $v = 0$, we can do the vertical blow up $(u, v) = (u_1, u_1 v_1)$ without loss information, and system (29) becomes

$$(30) \quad \begin{aligned} \dot{u}_1 &= u_1^2(a_{21}u_1 - u_1^2 + a_{02}v_1 + v_1^2 + a_{01}u_1v_1^2 + u_1^2v_1^2), \\ \dot{v}_1 &= -u_1v_1^3. \end{aligned}$$

Doing a rescaling of the time we eliminate the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ and we obtain the system

$$(31) \quad \begin{aligned} \dot{u}_1 &= u_1(a_{21}u_1 - u_1^2 + a_{02}v_1 + v_1^2 + a_{01}u_1v_1^2 + u_1^2v_1^2), \\ \dot{v}_1 &= -v_1^3. \end{aligned}$$

The unique singular point on the straight line $u_1 = 0$ is the origin, that again is linearly zero. Since the characteristic directions at the origin of system (31) are given by the linear factors of $u_1 v_1(a_{21}u_1 + a_{02}v_1)$, we have that the straight line $u_1 = 0$ is a characteristic direction. So before doing another vertical blow up we must do a twist. So we do the change of variables $(u_1, v_1) = (u_2 - v_2, v_2)$. In the

new variables (u_2, v_2) the system (31) becomes

$$(32) \quad \begin{aligned} \dot{u}_2 &= a_{21}u_2^2 + (a_{02} - 2a_{21})u_2v_2 + (a_{21} - a_{02})v_2^2 - u_2^3 + 3u_2^2v_2 - 2u_2v_2^2 \\ &\quad - v_2^3 + a_{01}u_2^2v_2^2 - 2a_{01}u_2v_2^3 + a_{01}v_2^4 + u_2^3v_2^2 - 3u_2^2v_2^3 + 3u_2v_2^4 - v_2^5, \\ \dot{v}_2 &= -v_2^3. \end{aligned}$$

Now we do the vertical blow up $(u_2, v_2) = (u_3, u_3v_3)$ and system (32) goes over the system

$$(33) \quad \begin{aligned} \dot{u}_3 &= u_3^2(a_{21} - u_3 + (a_{02} - 2a_{21})v_3 + 3u_3v_3 + (a_{21} - a_{02})v_3^2 - 2u_3v_3^2 \\ &\quad + a_{01}u_3^2v_3^2 - u_3v_3^3 + u_3^3v_3^2 - 2a_{01}u_3^2v_3^3 - 3u_3^3v_3^3 + a_{01}u_3^2v_3^4 + 3u_3^3v_3^4 \\ &\quad - u_3^3v_3^5), \\ \dot{v}_3 &= u_3v_3(v_3 - 1)(a_{21} - u_3 + (a_{02} - a_{21})v_3 + 2u_3v_3 + u_3v_3^2 + a_{01}u_3^2v_3^2 \\ &\quad + u_3^3v_3^2 - a_{01}u_3^2v_3^3 - 2u_3^3v_3^3 + u_3^3v_3^4). \end{aligned}$$

Eliminating the common factor u_3 between $\dot{u}_3$ and $\dot{v}_3$ doing a new rescaling of the time we obtain the system

$$(34) \quad \begin{aligned} \dot{u}_3 &= u_3(a_{21} - u_3 + (a_{02} - 2a_{21})v_3 + 3u_3v_3 + (a_{21} - a_{02})v_3^2 - 2u_3v_3^2 \\ &\quad + a_{01}u_3^2v_3^2 - u_3v_3^3 + u_3^3v_3^2 - 2a_{01}u_3^2v_3^3 - 3u_3^3v_3^3 + a_{01}u_3^2v_3^4 + 3u_3^3v_3^4 \\ &\quad - u_3^3v_3^5), \\ \dot{v}_3 &= v_3(v_3 - 1)(a_{21} - u_3 + (a_{02} - a_{21})v_3 + 2u_3v_3 + u_3v_3^2 + a_{01}u_3^2v_3^2 \\ &\quad + u_3^3v_3^2 - a_{01}u_3^2v_3^3 - 2u_3^3v_3^3 + u_3^3v_3^4). \end{aligned}$$

On the straight line $u_3 = 0$ system (34) has three possible singular points $(0, 0)$, $(0, 1)$ and $(0, a_{21}/(a_{21} - a_{02}))$ if $a_{02}a_{21}(a_{21} - a_{02}) \neq 0$. The eigenvalues of the Jacobian matrix of system (34) at $(0, 0)$ are $\pm a_{21}$, at $(0, 1)$ are a_{02} and 0, at $(0, a_{21}/(a_{21} - a_{02}))$ are $a_{02}a_{21}/(a_{02} - a_{21})$ and 0.

Now we need to distinguish seven cases.

Case 1: $a_{21} > 0$ and $a_{02} = 0$. In this case system (34) has the two singular points $(0, 0)$ and $(0, 1)$, where $(0, 0)$ is a hyperbolic saddle. But the singular point $(0, 1)$ is linearly zero, so must study its local phase portraits using blow up's.

First we translate the singular point $(0, 1)$ at the origin of coordinates doing the translation $(u_3, v_3) = (u_4, 1 + v_4)$ and system (34) becomes

$$(35) \quad \begin{aligned} \dot{u}_4 &= -u_4(u_4 + 4u_4v_4 - a_{21}v_4^2 + 5u_4v_4^2 - a_{01}u_4^2v_4^2 + u_4v_4^3 - 2a_{01}u_4^2v_4^3 \\ &\quad + u_4^3v_4^3 - a_{01}u_4^2v_4^4 + 2u_4^3v_4^4 + u_4^3v_4^5), \\ \dot{v}_4 &= v_4(1 + v_4)(2u_4 - a_{21}v_4 + 4u_4v_4 - a_{01}u_4^2v_4 + u_4v_4^2 - 2a_{01}u_4^2v_4^2 \\ &\quad + u_4^3v_4^2 - a_{01}u_4^2v_4^3 + 2u_4^3v_4^3 + u_4^3v_4^4). \end{aligned}$$

Since the characteristic directions are given by the linear factors of the homogeneous polynomial $u_4v_4(3u_4 - a_{21}v_4)$, the straight line $u_4 = 0$ is a characteristic direction and we do the twist $(u_4, v_4) = (u_5 - v_5, v_5)$, then system (35) in the new variables

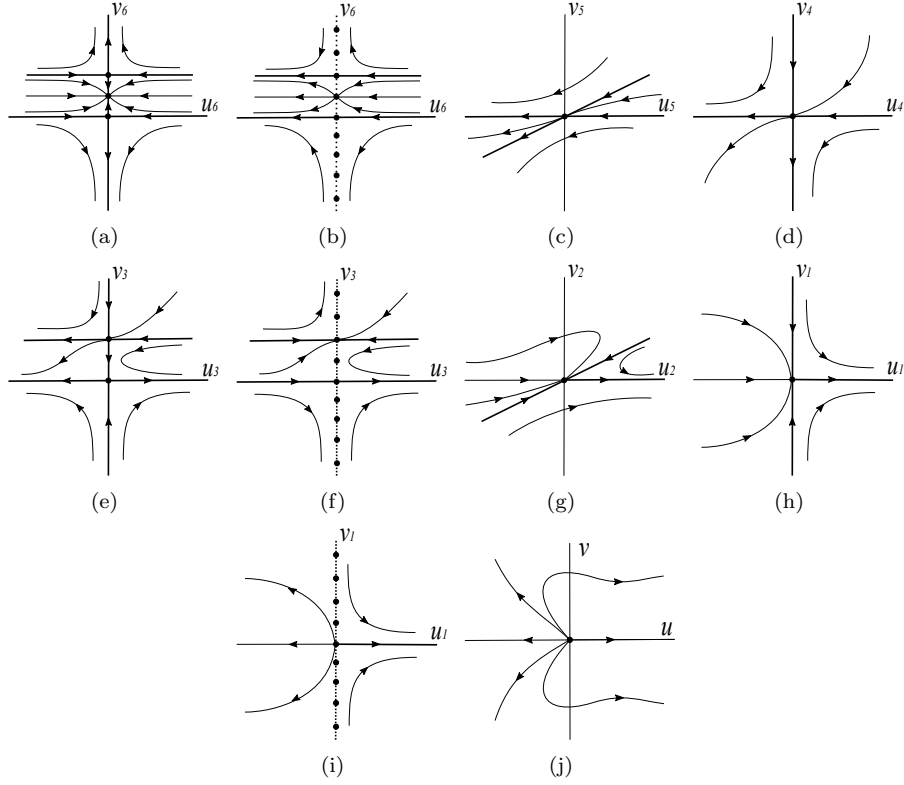


FIGURE 5. The blow ups at the origin of the local chart U_2 when $a_{21} > 0$ and $a_{02} = 0$.

(u_5, v_5) writes

$$\begin{aligned}
 \dot{u}_5 = & -u_5^2 + 4u_5v_5 - (3 + a_{21})v_5^2 - 4u_5^2v_5 + (14 + a_{21})u_5v_5^2 - 2(5 + a_{21})v_5^3 \\
 & - (5 + a_{01})u_5^2v_5^2 + (15 + 2a_{01})u_5v_5^3 - (10 + a_{01})v_5^4 + a_{01}u_5^3v_5^2 \\
 & - (1 + 6a_{01})u_5^2v_5^3 + 3(1 + 3a_{01})u_5v_5^4 - 2(1 + 2a_{01})v_5^5 + (1 + 2a_{01})u_5^3v_5^3 \\
 (36) \quad & - 3(1 + 3a_{01})u_5^2v_5^4 + 3(1 + 4a_{01})u_5v_5^5 - (1 + 5a_{01})v_5^6 - u_5^4v_5^3 \\
 & + (7 + a_{01})u_5^3v_5^4 - (15 + 4a_{01})u_5^2v_5^5 + (13 + 5a_{01})u_5v_5^6 - 2(2 + a_{01})v_5^7 \\
 & - 2u_5^4v_5^4 + 11u_5^3v_5^5 - 21u_5^2v_5^6 + 17u_5v_5^7 - 5v_5^8 - u_5^4v_5^5 + 5u_5^3v_5^6 - 9u_5^2v_5^7 \\
 & + 7u_5v_5^8 - 2v_5^9,
 \end{aligned}$$

$$\begin{aligned}
 \dot{v}_5 = & -v_5(1 + v_5)(-2u_5 + (2 + a_{21})v_5 - 4u_5v_5 + 4v_5^2 + a_{01}u_5^2v_5 \\
 & - (1 + 2a_{01})u_5v_5^2 + (1 + a_{01})v_5^3 + 2a_{01}u_5^2v_5^2 - 4a_{01}u_5v_5^3 + 2a_{01}v_5^4 \\
 & - u_5^3v_5^2 + (3 + a_{01})u_5^2v_5^3 - (3 + 2a_{01})u_5v_5^4 + (1 + a_{01})v_5^5 - 2u_5^3v_5^3 \\
 & + 6u_5^2v_5^4 - 6u_5v_5^5 + 2v_5^6 - u_5^3v_5^4 + 3u_5^2v_5^5 - 3u_5v_5^6 + v_5^7).
 \end{aligned}$$

Now we do the blow up $(u_5, v_5) = (u_6, u_6 v_6)$ and system (36) becomes

$$\begin{aligned}
 (37) \quad \dot{u}_6 = & -u_6^2(1 - 4v_6 + 4u_6v_6 + (3 + a_{21})v_6^2 - (14 + a_{21})u_6v_6^2 + (5 + a_{01})u_6^2v_6^2 \\
 & + 2(5 + a_{21})u_6v_6^3 - a_{01}u_6^3v_6^2 - (15 + 2a_{01})u_6^2v_6^3 + (1 + 6a_{01})u_6^3v_6^3 \\
 & - (1 + 2a_{01})u_6^4v_6^3 - 3(1 + 3a_{01})u_6^3v_6^4 + u_6^5v_6^3 + 3(1 + 3a_{01})u_6^4v_6^4 \\
 & + 2(1 + 2a_{01})u_6^3v_6^5 + (10 + a_{01})u_6^2v_6^4 - (7 + a_{01})u_6^5v_6^4 - 3(1 + 4a_{01})u_6^4v_6^5 \\
 & + 2u_6^6v_6^4 + (15 + 4a_{01})u_6^5v_6^5 + (1 + 5a_{01})u_6^4v_6^6 - 11u_6^6v_6^5 - (13 + 5a_{01})u_6^5v_6^6 \\
 & + u_6^7v_6^5 + 21u_6^6v_6^6 + 2(2 + a_{01})u_6^5v_6^7 - 5u_6^7v_6^6 - 17u_6^6v_6^7 + 9u_6^7v_6^7 + 5u_6^6v_6^8 \\
 & - 7u_6^7v_6^8 + 2u_6^7v_6^9), \\
 \dot{v}_6 = & u_6v_6(v_6 - 1)(-3 + (3 + a_{21})v_6 - 10u_6v_6 + a_{01}u_6^2v_6 + 2(5 + a_{21})u_6v_6^2 \\
 & - 2(5 + a_{01})u_6^2v_6^2 + 4a_{01}u_6^3v_6^2 + (10 + a_{01})u_6^2v_6^3 - u_6^4v_6^2 - 2(1 + 4a_{01})u_6^3v_6^3 \\
 & + (3 + 5a_{01})u_6^4v_6^3 + 2(1 + 2a_{01})u_6^3v_6^4 - 4u_6^5v_6^3 - (3 + 10a_{01})u_6^4v_6^4 \\
 & + 2(6 + a_{01})u_6^5v_6^4 + (1 + 5a_{01})u_6^4v_6^5 - 5u_6^6v_6^4 - 4(3 + a_{01})u_6^5v_6^5 + 15u_6^6v_6^5 \\
 & + 2(2 + a_{01})u_6^5v_6^6 - 2u_6^7v_6^5 - 15u_6^6v_6^6 + 6u_6^7v_6^6 + 5u_6^6v_6^7 - 6u_6^7v_6^7 + 2u_6^7v_6^8).
 \end{aligned}$$

We eliminate the common factor u_6 between $\dot{u}_6$ and $\dot{v}_6$ rescaling again the time and we have the system

$$\begin{aligned}
 (38) \quad \dot{u}_6 = & -u_6(1 - 4v_6 + 4u_6v_6 + (3 + a_{21})v_6^2 - (14 + a_{21})u_6v_6^2 + (5 + a_{01})u_6^2v_6^2 \\
 & + 2(5 + a_{21})u_6v_6^3 - a_{01}u_6^3v_6^2 - (15 + 2a_{01})u_6^2v_6^3 + (1 + 6a_{01})u_6^3v_6^3 \\
 & - (1 + 2a_{01})u_6^4v_6^3 - 3(1 + 3a_{01})u_6^3v_6^4 + u_6^5v_6^3 + 3(1 + 3a_{01})u_6^4v_6^4 \\
 & + 2(1 + 2a_{01})u_6^3v_6^5 + (10 + a_{01})u_6^2v_6^4 - (7 + a_{01})u_6^5v_6^4 - 3(1 + 4a_{01})u_6^4v_6^5 \\
 & + 2u_6^6v_6^4 + (15 + 4a_{01})u_6^5v_6^5 + (1 + 5a_{01})u_6^4v_6^6 - 11u_6^6v_6^5 - (13 + 5a_{01})u_6^5v_6^6 \\
 & + u_6^7v_6^5 + 21u_6^6v_6^6 + 2(2 + a_{01})u_6^5v_6^7 - 5u_6^7v_6^6 - 17u_6^6v_6^7 + 9u_6^7v_6^7 + 5u_6^6v_6^8 \\
 & - 7u_6^7v_6^8 + 2u_6^7v_6^9), \\
 \dot{v}_6 = & v_6(v_6 - 1)(-3 + (3 + a_{21})v_6 - 10u_6v_6 + a_{01}u_6^2v_6 + 2(5 + a_{21})u_6v_6^2 \\
 & - 2(5 + a_{01})u_6^2v_6^2 + 4a_{01}u_6^3v_6^2 + (10 + a_{01})u_6^2v_6^3 - u_6^4v_6^2 - 2(1 + 4a_{01})u_6^3v_6^3 \\
 & + (3 + 5a_{01})u_6^4v_6^3 + 2(1 + 2a_{01})u_6^3v_6^4 - 4u_6^5v_6^3 - (3 + 10a_{01})u_6^4v_6^4 \\
 & + 2(6 + a_{01})u_6^5v_6^4 + (1 + 5a_{01})u_6^4v_6^5 - 5u_6^6v_6^4 - 4(3 + a_{01})u_6^5v_6^5 + 15u_6^6v_6^5 \\
 & + 2(2 + a_{01})u_6^5v_6^6 - 2u_6^7v_6^5 - 15u_6^6v_6^6 + 6u_6^7v_6^6 + 5u_6^6v_6^7 - 6u_6^7v_6^7 + 2u_6^7v_6^8).
 \end{aligned}$$

System (38) on the straight line $u_6 = 0$ has the three singular points $(0, 0)$, $(0, 1)$ and $(0, 3/(3 + a_{21}))$. The corresponding eigenvalues at $(0, 0)$ are 1 and -3 , so the singular point $(0, 0)$ is a hyperbolic saddle. The eigenvalues at $(0, 1)$ are $\pm a_{21}$, so the singular point $(0, 1)$ is also a hyperbolic saddle. For the singular point $(0, 3/(3 + a_{21}))$ the eigenvalues are $-a_{21}/(3 + a_{21})$ and $-3a_{21}/(3 + a_{21})$, so this singular point is a hyperbolic stable node. With this information we have that the local phase portrait in a neighborhood of the straight line $u_6 = 0$ for the system (38) is the one described in Figure 5(a), consequently for the system (37) this local phase portrait is shown in Figure 5(b). Undoing the blow up $(u_5, v_5) = (u_6, u_6 v_6)$ and

taking into account that $\dot{u}_5|_{u_5=0} = -(3 + a_{21})v_5^2$ we obtain the local phase portrait at the origin of system (36) in the Figure 5(c). Going back through the twist $(u_4, v_4) = (u_5 - v_5, v_5)$ we get the local phase portrait at the origin of the system (35), see Figure 5(d). Translating the origin of the system (35) at the singular point $(0, 1)$ of the system (34) and taking into account that at the origin of system (34) we have a saddle we obtain the local phase portrait in a neighborhood of the straight line $u_3 = 0$ of the system (34), see Figure 5(e). Therefore the local phase portrait near $u_3 = 0$ of system (33) is the one of Figure 5(f). Undoing the blow up $(u_2, v_2) = (u_3, u_3 v_3)$ and taking into account that $\dot{u}_2|_{u_2=0} = a_{21}v_2^2 + O(v_2^3)$, we obtain the local phase portrait at the origin of system (32) in the Figure 5(g). Going back through the twist $(u_1, v_1) = (u_2 - v_2, v_2)$ we have the local phase portrait at the origin of system (31) in the Figure 5(h), consequently we get the local phase portrait at the origin of system (30) in the Figure 5(i). Finally undoing the blow up $(u, v) = (u_1, u_1 v_1)$ and taking into account that $\dot{u}|_{u=0} = v^2 + O(v^3)$, we obtain the local phase portrait at the origin of the local chart U_2 in the Figure 5(j) or in the Figure 4(a), that is an unstable degenerate node.

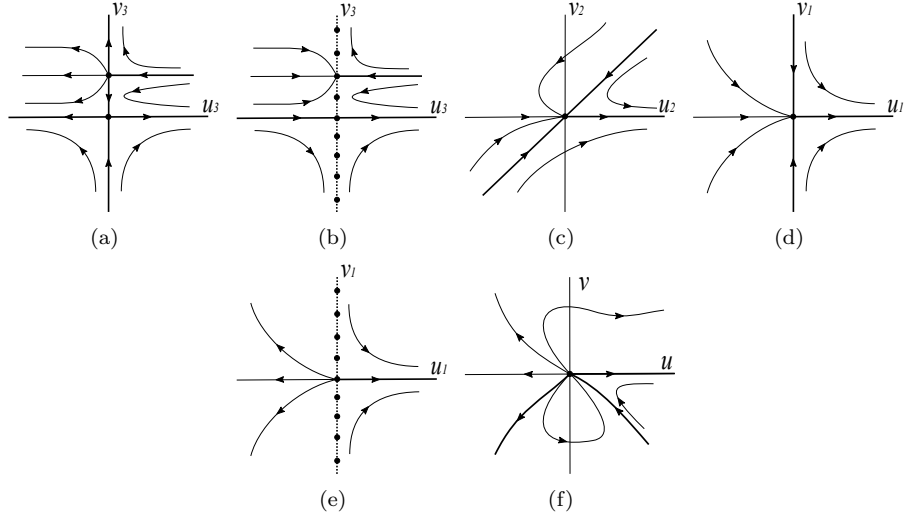


FIGURE 6. The blow ups at the origin of the local chart U_2 when $a_{21} > 0$ and $a_{02} = a_{21}$.

Case 2: $a_{21} > 0$ and $a_{21} = a_{02}$. Then system (34) has only the two singular points $(0, 0)$ and $(0, 1)$ on the straight line $u_3 = 0$. We remark that the position of the saddle-node $(0, 1)$ respect to the axes of coordinates is obtained from Theorem 2.19 of [12]. The local phase portrait near this straight line is shown in Figure 6(a), and consequently the local phase portrait near $u_3 = 0$ of system (33) is the one of Figure 6(b). Undoing the blow up $(u_2, v_2) = (u_3, u_3 v_3)$ and taking into account that $\dot{u}_2|_{u_2=0} = -v_2^3$, we obtain the local phase portrait at the origin of system (32) in the Figure 6(c). Going back through the twist $(u_1, v_1) = (u_2 - v_2, v_2)$ we have the local phase portrait at the origin of system (31) in the Figure 6(d), consequently we get the local phase portrait at the origin of system (30) in the Figure 6(e). Finally undoing the blow up $(u, v) = (u_1, u_1 v_1)$ and taking into account that $\dot{u}|_{u=0} = v^2$,

we obtain the local phase portrait at the origin of the local chart U_2 in the Figure 6(f) or in the Figure 4(b).

Case 3: $a_{21} > 0$ and $a_{21} \neq a_{02} \neq 0$. Now we have the three singular points $(0, 0)$, $(0, 1)$ and $(0, a_{21}/(a_{21} - a_{02}))$ in the straight line $u_3 = 0$. Again working in a similar way to the Case 2 we should obtain the local phase portrait at the origin of the local chart U_2 in the Figures 4(b) and 4(c) when $a_{02} > 0$ and $a_{02} < 0$, respectively.

Case 4: $a_{21} = 0$ and $a_{02} < -1$. Now system (34) has only the two singular points $(0, 0)$ and $(0, 1)$. While the singular point $(0, 1)$ continues being a semihyperbolic saddle-node, the singular point $(0, 0)$ is linearly zero and we shall study its local phase portraits doing vertical blow up's.

Since the characteristic directions at the origin of system (34) are given by the linear factors of $u_3 v_3 (a_{02} v_3 - u_3)$, the straight line $u_3 = 0$ is a characteristic direction, consequently we do the twist $(u_3, v_3) = (u_4 - v_4, v_4)$ and system (34) in the coordinates (u_4, v_4) writes

$$\begin{aligned}
 \dot{u}_4 = & -u_4^2 + (3 + a_{02})u_4 v_4 - 2(1 + a_{02})v_4^2 + 3u_4^2 v_4 - (9 - a_{02})u_4 v_4^2 \\
 & + 2(3 + a_{02})v_4^3 - 2u_4^2 v_4^2 + 5u_4 v_4^3 - 3v_4^4 + a_{01}u_4^3 v_4^2 - (1 + 4a_{01})u_4^2 v_4^3 \\
 & + (3 + 5a_{01})u_4 v_4^4 - 2(1 + a_{01})v_4^5 + u_4^4 v_4^2 - (5 + 2a_{01})u_4^3 v_4^3 \\
 & + (9 + 8a_{01})u_4^2 v_4^4 - (7 + 10a_{01})u_4 v_4^5 + 2(1 + 2a_{01})v_4^6 - 3u_4^4 v_4^3 \\
 & + (15 + a_{01})u_4^3 v_4^4 - (27 + 4a_{01})u_4^2 v_4^5 + (21 + 5a_{01})u_4 v_4^6 \\
 & - 2(3v_4^7 + a_{01})v_4^7 + 3u_4^4 v_4^4 - 15u_4^3 v_4^5 + 27u_4^2 v_4^6 - 21u_4 v_4^7 + 6v_4^8 \\
 & - u_4^4 v_4^5 + 5u_4^3 v_4^6 - 9u_4^2 v_4^7 + 7u_4 v_4^8 - 2v_4^9, \\
 \dot{v}_4 = & v_4(1 - v_4)(u_4 - (1 + a_{02})v_4 - 2u_4 v_4 + 2v_4^2 - u_4 v_4^2 + v_4^3 - a_{01}u_4^2 v_4^2 \\
 & + 2a_{01}u_4 v_4^3 - a_{01}v_4^4 - u_4^3 v_4^2 + (3 + a_{01})u_4^2 v_4^3 - (3 + 2a_{01})u_4 v_4^4 \\
 & + (1 + a_{01})v_4^5 + 2u_4^3 v_4^3 - 6u_4^2 v_4^4 + 6u_4 v_4^5 - 2v_4^6 - u_4^3 v_4^4 + 3u_4^2 v_4^5 \\
 & - 3u_4 v_4^6 + v_4^7).
 \end{aligned} \tag{39}$$

Now we do the blow up $(u_4, v_4) = (u_5, u_5 v_5)$ and system (39) becomes

$$\begin{aligned}
 \dot{u}_5 = & -u_5^2(1 - (3 + a_{02})v_5 - 3u_5 v_5 + 2(1 + a_{02})v_5^2 + (9 + a_{02})u_5 v_5^2 + 2u_5^2 v_5^2 \\
 & - 2(3 + a_{02})u_5 v_5^3 - a_{01}u_5^3 v_5^2 - 5u_5^2 v_5^3 - u_5^4 v_5^2 + (1 + 4a_{01})u_5^3 v_5^3 + 3u_5^2 v_5^4 \\
 & + (5 + 2a_{01})u_5^4 v_5^3 - (3 + 5a_{01})u_5^3 v_5^4 + 3u_5^5 v_5^3 - (9 + 8a_{01})u_5^4 v_5^4 + 7u_5^4 v_5^5 \\
 & + 2(1 + a_{01})u_5^3 v_5^5 - (15 + a_{01})u_5^5 v_5^4 + 10a_{01}u_5^4 v_5^5 - 3u_5^6 v_5^4 + (27 + 4a_{01})u_5^5 v_5^5 \\
 & - 2(1 + 2a_{01})u_5^4 v_5^6 + 15u_5^6 v_5^5 - (21 + 5a_{01})u_5^5 v_5^6 + u_5^7 v_5^5 - 27u_5^6 v_5^6 \\
 & + 2(3 + a_{01})u_5^5 v_5^7 - 5u_5^7 v_5^6 + 21u_5^6 v_5^7 + 9u_5^7 v_5^7 - 6u_5^6 v_5^8 - 7u_5^7 v_5^8 + 2u_5^7 v_5^9), \\
 \dot{v}_5 = & u_5 v_5 (v_5 - 1)(-2 + 2(1 + a_{02})v_5 + 6u_5 v_5 - 2(3 + a_{02})u_5 v_5^2 - 3u_5^2 v_5^2 \\
 & + 2a_{01}u_5^3 v_5^2 + 3u_5^2 v_5^3 + 2u_5^4 v_5^2 - 2(1 + 2a_{01})u_5^3 v_5^3 - 2(3 + 2a_{01})u_5^4 v_5^3 \\
 & + 2(1 + a_{01})u_5^3 v_5^4 - 6u_5^5 v_5^3 + 2(3 + 4a_{01})u_5^4 v_5^4 + 2(9 + a_{01})u_5^5 v_5^4 \\
 & - 2(1 + 2a_{01})u_5^4 v_5^5 + 6u_5^6 v_5^4 - 2(9 + 2a_{01})u_5^5 v_5^5 - 18u_5^6 v_5^5 + 2(2 + a_{01})u_5^5 v_5^6 \\
 & - 2u_5^7 v_5^5 + 18u_5^6 v_5^6 + 6u_5^7 v_5^6 - 6u_5^6 v_5^7 - 6u_5^7 v_5^7 + 2u_5^7 v_5^8).
 \end{aligned} \tag{40}$$

Doing a rescaling we eliminate the common factor between $\dot{u}_5$ and $\dot{v}_5$ and we get the system

$$\begin{aligned}
 (41) \quad \dot{u}_5 = & -u_5(1 - (3 + a_{02})v_5 - 3u_5v_5 + 2(1 + a_{02})v_5^2 + (9 + a_{02})u_5v_5^2 + 2u_5^2v_5^2 \\
 & - 2(3 + a_{02})u_5v_5^3 - a_{01}u_5^3v_5^2 - 5u_5^2v_5^3 - u_5^4v_5^2 + (1 + 4a_{01})u_5^3v_5^3 + 3u_5^2v_5^4 \\
 & + (5 + 2a_{01})u_5^4v_5^3 - (3 + 5a_{01})u_5^3v_5^4 + 3u_5^5v_5^3 - (9 + 8a_{01})u_5^4v_5^4 + 7u_5^4v_5^5 \\
 & + 2(1 + a_{01})u_5^3v_5^5 - (15 + a_{01})u_5^5v_5^4 + 10a_{01}u_5^4v_5^5 - 3u_5^6v_5^4 + (27 + 4a_{01})u_5^5v_5^5 \\
 & - 2(1 + 2a_{01})u_5^4v_5^6 + 15u_5^6v_5^5 - (21 + 5a_{01})u_5^5v_5^6 + u_5^7v_5^5 - 27u_5^6v_5^6 \\
 & + 2(3 + a_{01})u_5^5v_5^7 - 5u_5^7v_5^6 + 21u_5^6v_5^7 + 9u_5^7v_5^7 - 6u_5^6v_5^8 - 7u_5^7v_5^8 + 2u_5^7v_5^9), \\
 \\
 \dot{v}_5 = & v_5(v_5 - 1)(-2 + 2(1 + a_{02})v_5 + 6u_5v_5 - 2(3 + a_{02})u_5v_5^2 - 3u_5^2v_5^2 \\
 & + 2a_{01}u_5^3v_5^2 + 3u_5^2v_5^3 + 2u_5^4v_5^2 - 2(1 + 2a_{01})u_5^3v_5^3 - 2(3 + 2a_{01})u_5^4v_5^3 \\
 & + 2(1 + a_{01})u_5^3v_5^4 - 6u_5^5v_5^3 + 2(3 + 4a_{01})u_5^4v_5^4 + 2(9 + a_{01})u_5^5v_5^4 \\
 & - 2(1 + 2a_{01})u_5^4v_5^5 + 6u_5^6v_5^4 - 2(9 + 2a_{01})u_5^5v_5^5 - 18u_5^6v_5^5 + 2(2 + a_{01})u_5^5v_5^6 \\
 & - 2u_5^7v_5^5 + 18u_5^6v_5^6 + 6u_5^7v_5^6 - 6u_5^6v_5^7 - 6u_5^7v_5^7 + 2u_5^7v_5^8).
 \end{aligned}$$

System (41) on the straight line $u_5 = 0$ has the three singular points $(0, 0)$, $(0, 1)$ and $(0, 1/(1 + a_{02}))$. Since the eigenvalues of the Jacobian matrix of system (41) at $(0, 0)$ are -1 and 2 this singular point is a hyperbolic saddle. The corresponding eigenvalues of $(0, 1)$ are $-a_{02}$ and $2a_{02}$ so the singular point $(0, 1)$ also it is a hyperbolic saddle. But the eigenvalues of $(0, 1/(1 + a_{02}))$ are $-2a_{02}/(1 + a_{02})$ and 0 , by Theorem 2.19 of [12], this singular point is a semihyperbolic stable node.

Taking into account the local phase portraits of the three singular points on the straight line $u_5 = 0$ we obtain the local phase portrait in a neighborhood of $u_3 = 0$ for the system (41), see Figure 7(a), consequently we also have the local phase portrait in a neighborhood of $u_3 = 0$ for the system (40) in the Figure 7(b). Undoing the blow up $(u_4, v_4) = (u_5, u_5v_5)$ and taking into account that $\dot{u}_4|_{u_4=0} = -2(1 + a_{02})v_4^2$ we get the local phase portrait at the origin of system (39), Figure 7(c). Going back to system (34) undoing the twist $(u_3, v_3) = (u_4 - v_4, v_4)$ and taking into account the two singular points $(0, 0)$ and $(0, 1)$ of that system on the straight line $u_3 = 0$ we obtain the local phase portrait in a neighborhood of $u_3 = 0$, see Figure 7(d), and consequently we have the local phase portrait in a neighborhood of $u_3 = 0$ of system (33) in the Figure 7(e). Going back through the vertical blow up $(u_2, v_2) = (u_3, u_3v_3)$ and taking into account that $\dot{u}_2|_{u_2=0} = -v_2^3$, we obtain the local phase portrait at the origin of system (32) in the Figure 7(f). Going back through the twist $(u_1, v_1) = (u_2 - v_2, v_2)$ we have the local phase portrait at the origin of system (31) in the Figure 7(g), consequently we get the local phase portrait at the origin of system (30) in the Figure 7(h). Finally undoing the blow up $(u, v) = (u_1, u_1v_1)$ and taking into account that $\dot{u}|_{u=0} = v^2$, we obtain the local phase portrait at the origin of the local chart U_2 in the Figure 7(i) or in the Figure 4(d).

Case 5: $a_{21} = 0$ and $a_{02} = -1$. Working as in the Case 4 also in this case we obtain that the local phase portrait at the origin of the local chart U_2 is the one of Figure 4(d).

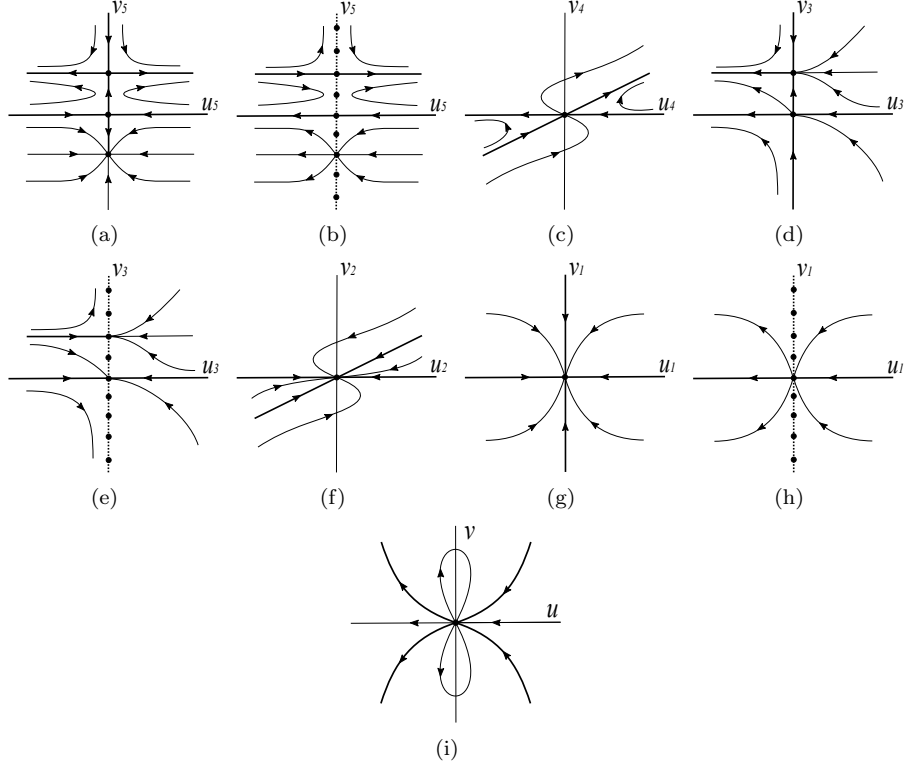


FIGURE 7. The blow ups of the origin of the local chart U_2 when $a_{21} = 0$ and $a_{02} < -1$.

Case 6: $a_{21} = 0$, $a_{02} > -1$ and $a_{02} \neq 0$. Again working as in the Case 4 in this case we obtain that the local phase portrait at the origin of the local chart U_2 is the one of Figure 4(d).

Case 7: $a_{02} = a_{21} = 0$. Under this assumptions we can remove another additional common factor u_3 from $\dot{u}_3$ and $\dot{v}_3$ of the system (34) obtaining the system

$$\begin{aligned}
 \dot{u}_3 &= u_3(-1 + 3v_3 - 2v_3^2 + a_{01}u_3v_3^2 - v_3^3 + u_3^2v_3^2 - 2a_{01}u_3v_3^3 - 3u_3^2v_3^3 \\
 &\quad + a_{01}u_3v_3^4 + 3u_3^2v_3^4 - u_3^2v_3^5), \\
 \dot{v}_3 &= v_3(v_3 - 1)(-1 + 2v_3 + v_3^2 + a_{01}u_3v_3^2 + u_3^2v_3^2 - a_{01}u_3v_3^3 - 2u_3^2v_3^3 \\
 &\quad + u_3^2v_3^4).
 \end{aligned}
 \tag{42}$$

This system has the following four singular points on $u_3 = 0$: $(0, -1 - \sqrt{2})$, $(0, 0)$, $(0, -1 + \sqrt{2})$ and $(0, 1)$. These singular points are all hyperbolic, and are stable node, saddle, stable node and saddle, respectively. Then the local phase portrait in a neighborhood of the straight line $u_3 = 0$ for the system (42) is the one of Figure 8(a), and consequently in a neighborhood of the straight line $u_3 = 0$ for the system (33) is the one of Figure 8(b). Undoing the blow up $(u_2, v_2) = (u_3, u_3v_3)$ and taking into account that $\dot{u}_2|_{u_2=0} = -v_2^3$, we obtain the local phase portrait at the origin of system (32) in the Figure 8(c). Going back through the twist

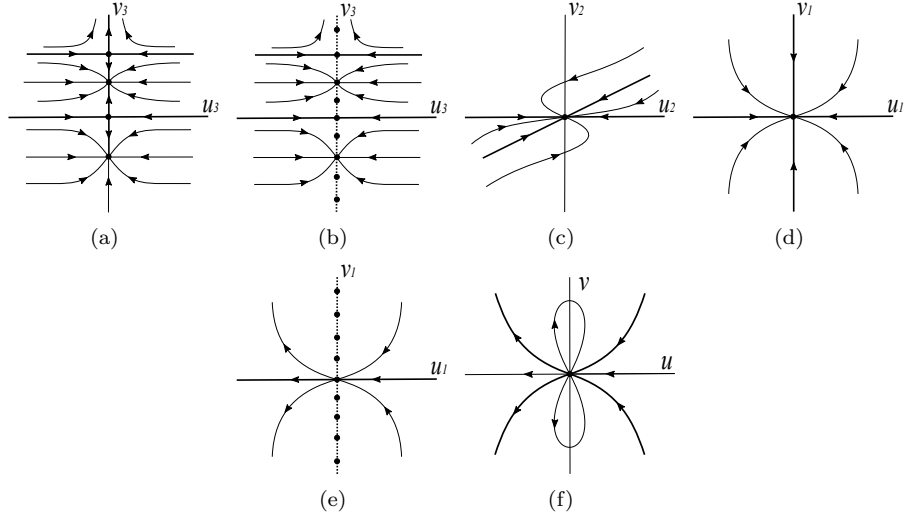


FIGURE 8. The blow ups of the origin of the local chart U_2 when $a_{02} = a_{21} = 0$.

$(u_1, v_1) = (u_2 - v_2, v_2)$ we have the local phase portrait at the origin of system (31) in the Figure 8(d), consequently we get the local phase portrait at the origin of system (30) in the Figure 8(e). Finally undoing the blow up $(u, v) = (u_1, u_1 v_1)$ and taking into account that $\dot{u}|_{u=0} = v^2$, we obtain the local phase portrait at the origin of the local chart U_2 in the Figure 8(f) or in the Figure 4(d). $\square$

Now we shall study the phase portraits of the piecewise differential system (7) in the Poincaré disc.

Lemma 11. *If either $a_{01} \geq 0$, $a_{02} \geq 0$, $a_{21} \geq 0$, or $a_{01} \leq 0$, $a_{02} \leq 0$, $a_{21} \leq 0$, and $a_{01}^2 + a_{02}^2 + a_{21}^2 > 0$, then the piecewise differential system (7) has no periodic orbits, and consequently no limit cycles.*

Proof. The divergence of the piecewise differential system (7) is

$$\text{div}(y, -x + x^3 - a_{01}y - a_{02}|y|y - a_{21}x^2y) = -a_{01} - 2a_{02}|y| - a_{21}x^2.$$

The Bendixson criterium states: Assume that the divergence of a planar differential system has constant sign in a simply connected region R , and that it is not identically zero on any subregion of R . Then the differential system does not have a periodic orbit which lies entirely in R . For more details on this criterium see Theorem 7.10 of [12]. Note that in [12] the Bendixson criterium is proved for C^1 differential systems, while we shall apply it to the piecewise differential system (7) that it is C^1 in the whole plane except in the straight line $y = 0$ where it is only continuous. But looking at the proof of Theorem 7.10 of [12], such a proof can be extended to the piecewise differential system (7).

Applying the Bendixson criterium to the piecewise differential system (7) the lemma follows. $\square$

From Lemma 7 the piecewise differential system (7) has three finite singular points O and $E_{l,r}$, where $E_{l,r}$ are two saddles with separatrices S_i ($i = 1, 2, 3, 4$) and S_j ($j = 5, 6, 7, 8$), respectively. The local phase portraits of the singular point O are described in Lemma 7. From Propositions 9 and 10 we obtain the local phase portraits at the two possible infinite singular points $A_{1,2}$ of the local charts U_1 and V_1 , and of the origins $O_{1,2}$ of the local charts U_2 and V_2 , respectively. Taking into account the four distinct possible local phase portraits at the infinite singular points described in Proposition 10 we divide the study of the phase portraits of the piecewise differential system (7) into four cases.

Case 1: The parameters $(a_{21}, a_{02}) \in R_1$. Since $a_{02} = 0$ the parameter space (a_{01}, a_{21}, a_{02}) is reduced to the parameter half plane (a_{01}, a_{21}) with $a_{21} > 0$. From Lemma 7 in the quadrant $a_{01} < 0, a_{21} > 0$ the origin O is unstable, while in the quadrant $a_{01} > 0, a_{21} > 0$ the origin O is stable. Moreover on the straight line $a_{01} = 0, a_{21} > 0$ the origin is a stable weak focus. Consequently when the parameter a_{01} crosses decreasing the half-straight line $a_{01} = 0$ a stable small limit cycle bifurcates from the origin O .

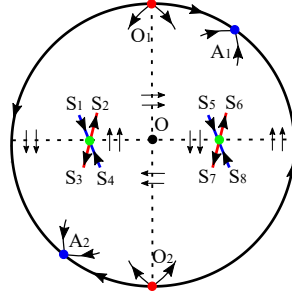


FIGURE 9. The local phase portraits at the finite and infinite singular points in the parameter straight line R_1 .

In Figure 9 we have described the local phase portraits at the finite and infinite singular points of the piecewise differential system (7) when the parameters are in R_1 . From Figure 9 it is clear that the separatrix S_1 must come from the infinite singular point O_1 , the separatrix S_3 must go to the infinite singular point A_2 . By the symmetry the separatrix S_8 must come from the infinite singular point O_2 , the separatrix S_6 must go to the infinite singular point A_1 .

From Figure 9 the remaining separatrices S_2, S_4, S_5 and S_7 only can have one of the three behaviors described in Figure 10. Thus the phase portraits of (a), (b) and (c) of Figure 10 are realizable for the values $(1, 1)$, $(a_*, 1)$ ($a_* \in (0.2, 0.21)$) and $(-1, 1)$ of the parameters (a_{01}, a_{21}) , respectively. By the continuity of the phase portraits with respect to the parameters, we give the convenient function $\varphi_1(a_{01}, a_{21}) = 0$ to describe the phase portrait in Figure 10(b), whose existence is given by continuity going from the phase portrait in Figure 10(a) to the phase portrait in Figure 10(c). In this case system (7) has two heteroclinic orbits.

Taking into account all the previous results in the parameter region R_1 , system (7) has at most one small-amplitude limit cycle, which appears when the stability

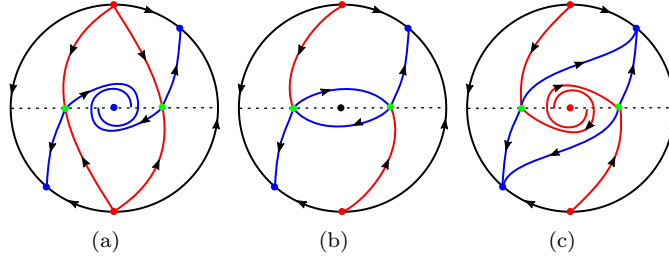


FIGURE 10. The three possible phase portraits for the separatrices S_2, S_4, S_5, S_7 of Figure 9.

of the focus reverses. Then when the parameters are in R_1 there are the three distinct topological phase portraits 1.1-1.3 of Figure 1.

Case 2: The parameters $(a_{21}, a_{02}) \in R_2$. The parabolic sector of the infinite singular point O_1 in U_2 is inside the Poincaré disc and its hyperbolic sector and elliptic sector are outside the Poincaré disc. By the symmetry of the piecewise differential system (7) the parabolic sector of O_2 in V_2 is inside the Poincaré disc and its hyperbolic sector and elliptic sector are outside the Poincaré disc. Then we obtain the local phase portraits of all singular points that are also described in Figure 9. Working as in Case 1 also for these values of the parameters we obtain the three global phase portraits 1.1-1.3 of Figure 1.

Case 3: The parameters $(a_{21}, a_{02}) \in R_3$. From Lemma 7 in the half-space $a_{01} < 0$ of the parameter space (a_{01}, a_{21}, a_{02}) , the origin O is unstable, while in the half-space $a_{01} > 0$ the origin O is stable. From Figure 4(c) the hyperbolic sector and elliptic sector of the infinite singular points $O_{1,2}$ are inside the Poincaré disc and their parabolic sectors are outside the Poincaré disc. By the symmetry we have described the local phase portraits at the finite and infinite singular points of the piecewise differential system (7) in Figure 12 for the parameter vector R_3 .

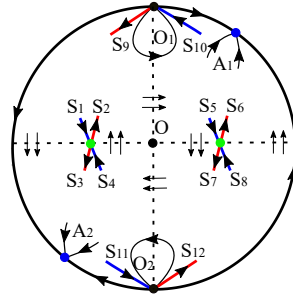


FIGURE 11. The local phase portraits at the finite and infinite singular points in the parameter space R_3 .

It is clear that the separatrix S_1 must come from the separatrix S_9 of the equilibrium O_1 , then the separatrix S_8 must come from the separatrix S_{12} of the equilibrium point O_2 . When the separatrix S_3 goes to the infinite singular point

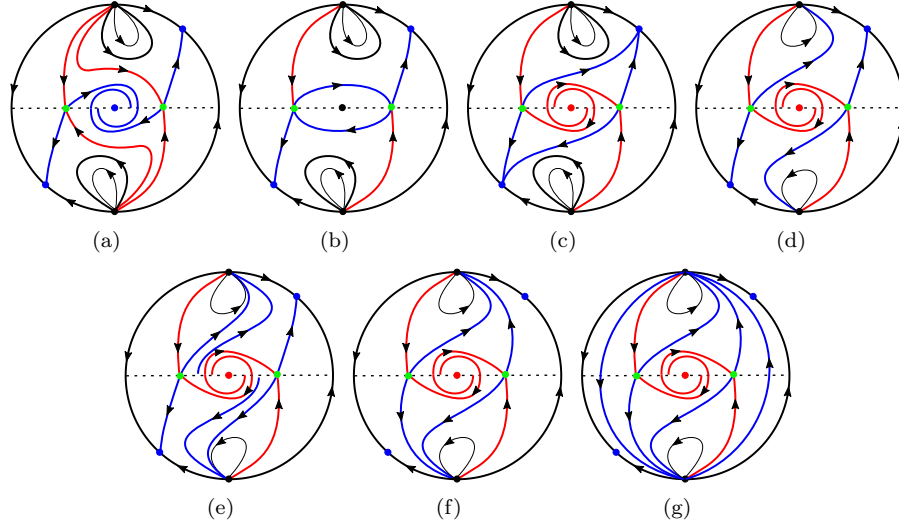


FIGURE 12. The five possible behavior phase portraits for the separatrices S_4 , S_6 , S_7 , S_9 of the Figure 11.

A_2 , the separatrix S_6 must go to the infinite singular point A_1 . From the Figure 11 the remaining separatrices S_4 , S_6 , S_7 and S_9 only can have one of the phase portraits of (a)-(e) of Figure 12, which are realizable for the values $(1, -1, 1)$, $(a^*, -1, 1)$ ($a^* \in (0.363, 0.364)$), $(-0.2, -1, 1)$, $(a^{**}, -1, 1)$ ($a^{**} \in (-0.382, -0.381)$) and $(-1, -1, 1)$ of the parameters (a_{01}, a_{02}, a_{21}) , respectively.

Actually, if the separatrix S_4 connects the infinite singular point O_2 , the separatrix S_2 connects the origin O . By the symmetry we obtain the phase portrait in the Figure 12(a). If the separatrix S_2 connects the infinite singular point A_1 , the separatrix S_4 connects the origin O . By the symmetry we obtain the phase portrait of Figure 12(c). From the phase portraits of Figures 12(a) and 12(c) it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait of Figure 12(b), that is the separatrices $S_{2,4}$ connect with the separatrices $S_{5,7}$. Then there exists a convenient function $a_{02} = \varphi_1(a_{01}, a_{21})$ such that system (7) has a heteroclinic loop. If the separatrix S_2 connects the separatrix S_{10} of the origin O , by the symmetry, the phase portrait in this case turn out to the Figure 12(d). If the separatrices S_2 and S_{12} connect the origin O , we have a phase portrait topologically equivalent to the one of Figure 12(d). If the separatrices S_2 and S_{12} connect the origin O , we have a phase portrait topologically equivalent to the one of Figure 12(e).

When the separatrix S_3 goes to the separatrix S_{11} , the separatrix S_6 must go to the separatrix S_{10} . Then the remaining separatrices S_4 , S_6 , S_7 and S_9 only can have the phase portraits of Figures 12(f) and 12(g), which are realizable for the values $(a^{***}, -1, 1)$ ($a^{***} \in (-3.51, -3.52)$) or $(-4, -1, 1)$ of the parameters (a_{01}, a_{02}, a_{21}) .

Taking into account all the previous results in the parameter region R_3 system (7) has at most one and two small limit cycles in the neighbourhood of the origin

when $a_{01} < 0$ and $a_{01} > 0$, respectively. Actually we have the global phase portrait 1.4 of Figure 1 with a stable singular point at origin, which comes from the phase portrait without limit cycles in the Figure 12(a). When the stability of the origin reverses, and a Hopf bifurcation occurs, and a stable limit cycle bifurcates from the origin, whose phase portrait is topologically equivalent to the phase portrait 1.11 of Figure 1. Then the stability of the origin reverses again, a Hopf bifurcation occurs, and an unstable limit cycle bifurcates from the origin. Hence there exist exactly two limit cycles in the phase portrait 1.17 of Figure 1, one is unstable and one is stable. When the parameters are in the bifurcation surface $\mathcal{DC}$, the two limit cycles of the phase portrait 1.17 collide producing a semi-stable double limit cycle in the phase portrait 1.18 of Figure 1. Similarly we have the phase portraits 1.5-1.10 of Figure 1, which have no limit cycles, and the phase portraits 1.12-1.16 of Figure 1, which have one limit cycle.

Case 4: The parameters $(a_{21}, a_{02}) \in R_4$. Since $a_{21} = 0$ the parameter space is reduced to the parameter plane (a_{01}, a_{02}) . From Lemma 7 in the half parameter plane $a_{01} < 0$ the origin O is unstable, while in the half parameter plane $a_{01} > 0$ the origin O is stable. Moreover, on the straight line $a_{01} = 0$ except at the point $(a_{21}, a_{02}) = (0, 0)$ the origin is a weak focus, stable if $a_{02} > 0$ and unstable if $a_{02} < 0$. Consequently when the parameter a_{01} crosses $a_{01} = 0$ with $a_{02} > 0$ decreasing a stable small limit cycle bifurcates from the origin O .

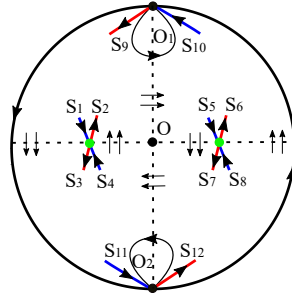


FIGURE 13. The local phase portraits at all finite and infinite singular points in parameter line of R_4 .

In Figure 13 we have described the local phase portraits at the finite and infinite singular points of the piecewise differential system (7) in the parameter plane (a_{01}, a_{02}) with $a_{21} = 0$. From Figure 13 it is clear that the separatrix S_1 must come from an orbit S_9 , the separatrix S_3 must go to an orbit S_{11} , the separatrix S_6 must go to an orbit S_{10} , and the separatrix S_8 must come from an orbit S_{12} .

From Figure 13 the remaining separatrices S_2 , S_4 , S_5 and S_7 only can have one of the three behaviors described in the Figure 14. These three behaviors occur. Thus, the phase portraits of (a), (b) and (c) of Figure 14 are realizable for the values $(1, 1)$, $(0, 0)$ and $(-1, 1)$ of the parameters (a_{01}, a_{02}) with $a_{21} = 0$, respectively.

Taking into account all the previous results in the parameter line R_4 there are the four distinct topological phase portraits 1.19-1.22 of Figure 1.

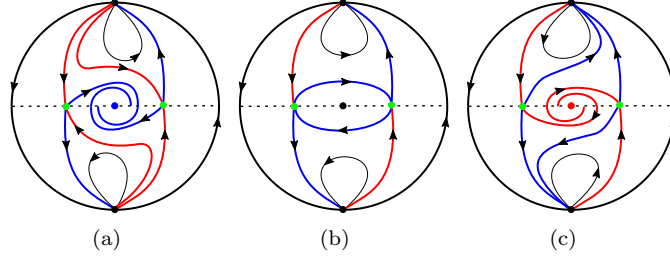


FIGURE 14. The three possible phase portraits for the separatrices S_2 , S_4 , S_5 and S_7 of the Figure 13.

5. PROOF OF THEOREM 4

The pitchfork bifurcation surface $\mathcal{P}$ for the infinite singular point is obtained via Proposition 9. Then we use Lemma 7 to obtain the generalized Hopf bifurcation surfaces and curves $\mathcal{H}_{1,2}^\pm$. Further, by the properties of the rotated vector field with respect to the parameters a_{01} , a_{02} and a_{21} , we can obtain the heteroclinic bifurcation surface $\mathcal{HC}$ and the double limit cycles bifurcation surface $\mathcal{DC}$, as they are described in Sections 3 and 4.

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