

THE DYNAMICS OF THE EHRHARD-MÜLLER SYSTEM WITH INVARIANT ALGEBRAIC SURFACES

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ABSTRACT. In this paper we study the global dynamics of the Ehrhard-Müller differential system

$$\dot{x} = s(y - x), \quad \dot{y} = rx - xz - y + c, \quad \dot{z} = xy - z,$$

where s , r and c are real parameters, and x , y , and z are real variables. We classify the invariant algebraic surfaces of degree 2 of this differential system. After we describe the phase portraits in the Poincaré ball of this differential system having one of this invariant algebraic surfaces.

The Poincaré ball is the closed unit ball in $\mathbb{R}^3$ whose interior has been identified with $\mathbb{R}^3$, and his boundary, the 2-dimensional sphere $\mathbb{S}^2$, has been identified with the infinity of $\mathbb{R}^3$. Note that in the space $\mathbb{R}^3$ we can go to infinity in as many as directions as points has the sphere $\mathbb{S}^2$. A polynomial differential system as the Ehrhard-Müller system can be extended analytically to the Poincaré ball, in this way we can study its dynamics in a neighborhood of infinity. Providing these phase portraits in the Poincaré ball we are describing the dynamics of all orbits of the Ehrhard-Müller system having an invariant algebraic surface of degree 2.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Consider the polynomial differential system

$$(1) \quad \dot{x} = s(y - x), \quad \dot{y} = rx - xz - y + c, \quad \dot{z} = xy - z,$$

where r, s, c are real parameters, $(x, y, z) \in \mathbb{R}^3$ and the dot $\cdot$ denotes the derivative of the functions $x(t)$, $y(t)$ and $z(t)$ with respect to the real variable t , usually called the time. Furthermore, we will denote by

$$(2) \quad X = s(y - x) \frac{\partial}{\partial x} + (rx - xz - y + c) \frac{\partial}{\partial y} + (xy - z) \frac{\partial}{\partial z}$$

the vector field associated to system (1). This differential system if $c = 0$ becomes the classical Lorenz system with $b = 1$. In what follows we will focus our attention when c is non-zero.

The nonlinear dynamic properties of system (1) have been extensively analyzed in various articles; see, for instance, [7, 8, 9, 12, 13, 14]. Specifically, [7] established the stability and bifurcation behaviors of the Ehrhard-Müller system, consistent with both symmetric and non-symmetric heating experiments. Meanwhile, [12] investigated the periodicity of the system, nonlinear dynamics, and chaos, deriving a periodicity diagram. Additional studies [8, 9, 13, 14] explored the chaotic properties of system (1) through numerical and experimental approaches. However, the full range of dynamical behaviors exhibited by the Ehrhard-Müller system has not yet been fully characterized. This gap motivates our present investigation.

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Comparing the results on the Lorenz system with the ones of the Ehrhard-Müller system some natural questions arise: what are the invariant algebraic surfaces of degree 2 of the Ehrhard-Müller system? What is the global dynamics in the Poincaré ball of the Ehrhard-Müller system having some of these invariant surfaces? The objective of this paper is to answer these two questions.

Let $f = f(x, y, z)$ be a real polynomial. We recall that $f(x, y, z) = 0$ is an *invariant algebraic surface* of the differential system (1) or of its vector field X if

$$(3) \quad Xf = kf,$$

where $k = k(x, y, z)$ is a real polynomial of degree at most 1, called the *cofactor* of f . This means that if an orbit of the differential system (1) passes through a point on this surface, then the whole orbit is contained in the surface.

Theorem 1. *All the invariant algebraic surfaces $f(x, y, z) = 0$ of degree 2 of the Ehrhard-Müller system (1) are given in the Table 1.*

Case	s, r, c	$f(x, y, z)$	k
(a)	$s = 1/2$	$x^2 - z$	-1
(b)	$s = 2, r = 0$	$y^2 + z^2 - cx$	-2
(c)	$r = 0, c = 0$	$y^2 + z^2$	-2
(d)	$s = 1, c = 0$	$y^2 + z^2 - rx^2$	-2

TABLE 1. The invariant algebraic surfaces $f(x, y, z) = 0$ of degree 2 of system (1).

A function of the form $I = I(x, y, z)e^{st}$ is a *Darboux invariant* for the differential system (1) if $dI/dt = 0$ on the solutions of this differential system. In other words, a Darboux invariant is a kind of first integral depending on the time.

Proposition 2. *The Ehrhard-Müller system (1) has the Darboux invariant*

- (a) $(x^2 - z)e^t$ if $s = 1/2$;
- (b) $(y^2 + z^2 - cx)e^{2t}$ if $s = 2$ and $r = 0$;
- (c) $(y^2 + z^2)e^{2t}$ if $r = c = 0$;
- (d) $(y^2 + z^2 - rx^2)e^{2t}$ if $s = 1$ and $c = 0$.

For a definition of the Poincaré ball see Subsubsection 2.1.2.

Proposition 3. *For all values of the parameters $r, s, c \in \mathbb{R}$ the phase portrait of the Ehrhard-Müller system (1) on the sphere $\mathbb{S}^2$ of the infinity is given in Figure 1. On this figure there are two centers at the endpoints of the x -axis and a circle of equilibria at the endpoints of the plane $x = 0$.*

In the next theorem we describe the phase portrait in the Poincaré ball of the Ehrhard-Müller system (1) having the invariant algebraic surface $x^2 - z = 0$. For the usual definitions ω -limit set and α -limit set of an orbit of the differential system (1) see Subsection 2.4.

Theorem 4. *The following statements hold for the Ehrhard-Müller system (1) having the invariant algebraic surface $x^2 - z = 0$.*

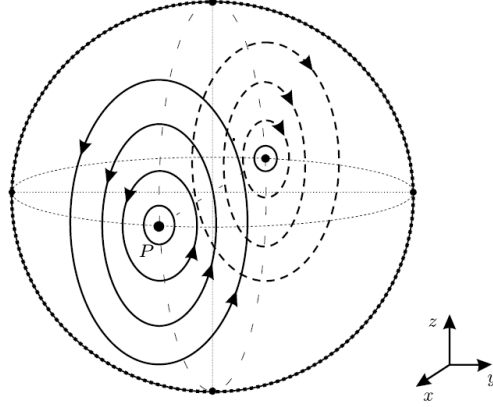
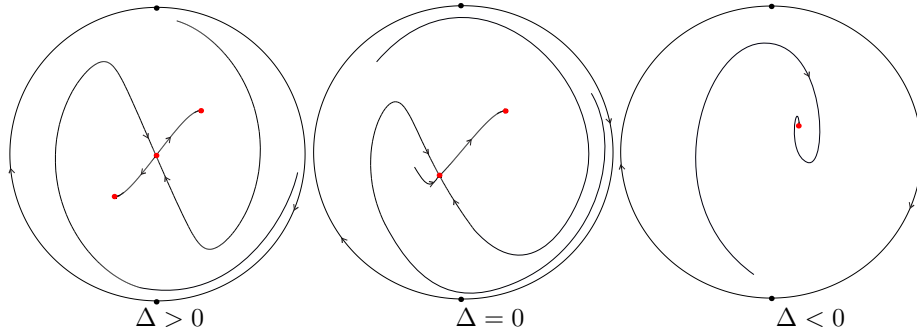


FIGURE 1. Phase portrait of the Ehrhard-Müller system (1) at the infinity sphere.

- (i) The phase portraits on the invariant algebraic surface $x^2 - z = 0$ are topological equivalent to one of the phase portraits of Figure 2.
- (ii) The α -limit of an orbit contained in the interior of the Poincaré ball and in the region $x^2 - z < 0$ is the infinite equilibrium point at the origin of the local chart U_3 , while the α -limit of an orbit contained in the interior of the Poincaré ball and in the region $x^2 - z > 0$ is the infinite equilibrium point at the origin of the local chart V_3 .
- (iii) The ω -limit of an orbit contained in the interior of the Poincaré ball and outside of the invariant surface $x^2 - z = 0$ is either one of the stable finite equilibrium points contained in the surface $x^2 - z = 0$, or one of the two infinite equilibrium points at the endpoints of the y -axis contained in the surface $x^2 - z = 0$.

FIGURE 2. The topological phase portraits on the invariant algebraic surface $x^2 - z = 0$.

In the following theorem we describe the phase portrait in the Poincaré ball of the Ehrhard-Müller system (1) having the invariant algebraic surface $y^2 + z^2 - cx = 0$.

Theorem 5. *The following statements hold for the Ehrhard-Müller system (1) having the invariant algebraic surface $y^2 + z^2 - cx = 0$.*

- (i) The phase portraits on the invariant algebraic surface $y^2 + z^2 - cx = 0$ are topologically equivalent to the phase portrait in Figure 3. So all the orbits start at infinity and end in the stable focus Q .

- (ii) The α -limit of an orbit contained in the interior of the Poincaré ball and outside of $y^2 + z^2 - cx = 0$ is the infinity equilibrium point P at the origin of the chart U_1 , and it is located at the endpoint of the positive x -axis.
- (iii) The ω -limit of an orbit contained in the interior of the Poincaré ball and outside of the invariant surface $y^2 + z^2 - cx = 0$ is the stable focus Q .

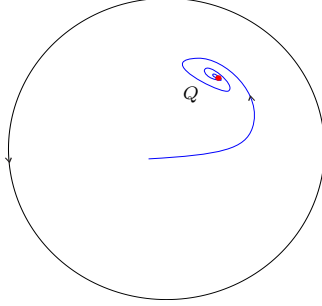


FIGURE 3. The topological phase portraits on the invariant algebraic surface $y^2 + z^2 - cx = 0$. The Q point is the stable focus of system (1) on the algebraic surface $y^2 + z^2 - cx = 0$.

We note that the Ehrhard-Müller system (1) having the invariant straight line $y^2 + z^2 = 0$, i.e. the x -axis, coincides with the Lorenz system $\dot{x} = s(y - x)$, $\dot{y} = -xz - y$, $\dot{z} = xy - z$, while the Ehrhard-Müller system (1) having the invariant surface $y^2 + z^2 - rx^2 = 0$, coincides with the Lorenz system $\dot{x} = y - x$, $\dot{y} = rx - xz - y$, $\dot{z} = xy - z$. Hence the phase portrait in the Poincaré ball of the Ehrhard-Müller system (1) having these two invariant algebraic surfaces have been studied in the papers [3, 10].

The paper is organized as follows. In Subsection 2.1 we introduce the Poincaré compactification in $\mathbb{R}^2$ and $\mathbb{R}^3$. Subsection 2.2 is dedicated to the vertical blow-ups that are essential for analyzing the local dynamics of the planar polynomial differential systems near equilibrium points having its linear part identically zero. In Subsection 2.3 we examine the Bendixson and the Bendixson-Dulac criteria, that aids in analyzing the non-existence of periodic orbits in the planar differential systems. Subsection 2.4 is dedicated to introduce the Darboux invariants. The proofs of Theorem 1 and Propositions 2 and 3 are given in Section 3. In Sections 4 and 5 we prove Theorems 4 and 5, respectively.

2. PRELIMINARIES

This section is devoted to establish some basic results that will be used throughout the paper. We divide it in four subsections.

2.1. Poincaré Compactification. For determining the global dynamics of a polynomial differential system in $\mathbb{R}^3$ or in an invariant algebraic surface we need to know the local phase portraits of its finite and infinite equilibrium points in the Poincaré ball and in the Poincaré disc, respectively.

2.1.1. Poincaré Compactification in $\mathbb{R}^2$. Let $\mathbb{S}^2 = \{\mathbf{z} \in \mathbb{R}^3 : \|\mathbf{y}\| = 1\}$ be the unit sphere of $\mathbb{R}^3$ centered at the origin of coordinates. From [6], we know that a polynomial vector field X in $\mathbb{R}^2$ induces two copies of X in the sphere $\mathbb{S}^2$, one in the open northern hemisphere and the other in the open southern hemisphere of the sphere $\mathbb{S}^2$. This vector field on $\mathbb{S}^2 \setminus \mathbb{S}^1$ can be extended

analytically to a vector field $p(X)$ on the whole sphere $\mathbb{S}^2$. Here $\mathbb{S}^1 = \{\mathbf{y} \in \mathbb{S}^2 : y_3 = 0\}$ denotes the equator of $\mathbb{S}^2$, that can be identified with the infinity of $\mathbb{R}^2$. The vector field $p(X)$ allows to study the dynamics of the vector field X in the neighbourhood of infinity, i.e. in the neighbourhood of the equator $\mathbb{S}^1$.

To get the analytical expression for $p(X)$ we should consider the sphere $\mathbb{S}^2$ as a smooth manifold. In this context, it is enough to choose the six local charts given by $U_i = \{\mathbf{y} \in \mathbb{S}^2 : y_i > 0\}$ and $V_i = \{\mathbf{y} \in \mathbb{S}^2 : y_i < 0\}$, for $i = 1, 2, 3$, with the corresponding coordinate maps $\varphi_i : U_i \rightarrow \mathbb{R}^2$ and $\psi_i : V_i \rightarrow \mathbb{R}^2$, defined by $\varphi_k(\mathbf{y}) = \psi_k(\mathbf{y}) = (y_m/y_k, y_n/y_k)$ for $m < n$ and $m, n \neq k$. Denote by (u, v) the local coordinates on U_i and V_i for $i = 1, 2, 3$. If $X = (P, Q)$ from [6] the vector field $p(X)$ in these local charts is

$$(4) \quad \begin{aligned} (\dot{u}, \dot{v}) &= \left(v^n \left(-uP \left(\frac{1}{v}, \frac{u}{v} \right) + Q \left(\frac{1}{v}, \frac{u}{v} \right) \right), -v^{n+1}P \left(\frac{1}{v}, \frac{u}{v} \right) \right) \quad \text{in } U_1; \\ (\dot{u}, \dot{v}) &= \left(v^n \left(-uQ \left(\frac{u}{v}, \frac{1}{v} \right) + P \left(\frac{u}{v}, \frac{1}{v} \right) \right), -v^{n+1}Q \left(\frac{u}{v}, \frac{1}{v} \right) \right) \quad \text{in } U_2; \\ (\dot{u}, \dot{v}) &= (P(u, v), Q(u, v)) \quad \text{in } U_3, \end{aligned}$$

where n is the degree of the polynomial vector field X . We emphasize that the expressions of the vector field $p(X)$ in the local chart (V_i, ψ_i) is the same that as in the local card (U_i, φ_i) multiplied by $(-1)^{n-1}$ for $i = 1, 2, 3$.

The points of the infinity in all local charts are of the form $(u, 0)$. The infinity $\mathbb{S}^1$ is invariant under the flow of $p(X)$.

2.1.2. Poincaré Compactification in $\mathbb{R}^3$. Let $\mathbb{S}^3 = \{\mathbf{y} \in \mathbb{R}^4 : \|\mathbf{y}\| = 1\}$ be the unit sphere in $\mathbb{R}^4$ centered at the origin of coordinates. From [4], we know that a polynomial vector field X of degree n in $\mathbb{R}^3$ induces two copies on the sphere $\mathbb{S}^3$, one in the open northern hemisphere and the other in the southern hemisphere of the sphere $\mathbb{S}^3$. This induced vector field on $\mathbb{S}^3 \setminus \mathbb{S}^2$ can be extended analytically to a vector field $p(X)$ on the whole sphere $\mathbb{S}^3$. Here $\mathbb{S}^2 = \{\mathbf{y} \in \mathbb{S}^3 : y_4 = 0\}$, the equator of $\mathbb{S}^3$, is identified with the infinity of $\mathbb{R}^3$. The vector field $p(X)$ allows to study the dynamics of the vector field X in the neighborhood of infinity, i.e., in the neighborhood of the equator $\mathbb{S}^2$.

To get the analytical expression for $p(X)$ we shall consider the sphere $\mathbb{S}^3$ as a smooth manifold. It is enough to choose the 8 local charts $U_i = \{\mathbf{y} \in \mathbb{S}^3 : y_i > 0\}$ and $V_i = \{\mathbf{y} \in \mathbb{S}^3 : y_i < 0\}$, for $i = 1, 2, 3, 4$, with the corresponding coordinate maps $\varphi_i : U_i \rightarrow \mathbb{R}^3$ and $\psi_i : V_i \rightarrow \mathbb{R}^3$, defined by $\varphi_k(\mathbf{y}) = \psi_k(\mathbf{y}) = (y_\ell/y_k, y_m/y_k, y_n/y_k)$ for $\ell < m < n$ and $\ell, m, n \neq k$. Denote by (z_1, z_2, z_3) the local coordinates on U_i and V_i for $i = 1, 2, 3, 4$. If $X = (P_1, P_2, P_3)$ from [5] the vector field $p(X)$ in these local charts is

$$(5) \quad \begin{aligned} z_3^n (-z_1 P_1^1 + P_1^2, -z_2 P_1^1 + P_1^3, -z_3 P_1^1) &\quad \text{in } U_1; \\ z_3^n (-z_1 P_2^2 + P_2^1, -z_2 P_2^2 + P_2^3, -z_3 P_2^2) &\quad \text{in } U_2; \\ z_3^n (-z_1 P_3^3 + P_3^1, -z_2 P_3^3 + P_3^2, -z_3 P_3^3) &\quad \text{in } U_3; \\ (P_4^1, P_4^2, P_4^3) &\quad \text{in } U_4, \end{aligned}$$

where $P_1^i = P^i \left(\frac{1}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3} \right)$, $P_2^i = P^i \left(\frac{z_1}{z_3}, \frac{1}{z_3}, \frac{z_2}{z_3} \right)$, $P_3^i = P^i \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}, \frac{1}{z_3} \right)$, $P_4^i = P^i(z_1, z_2, z_3)$, $\Delta(z) = (1 + \sum_{i=1}^3 z_i^2)^{1/2}$ and n is the degree of the polynomial vector field X .

The expression for $p(X)$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$ for $i = 1, 2, 3, 4$.

The points of the infinity in all local charts are of the form $(z_1, z_2, 0)$. The infinity $\mathbb{S}^2$ is invariant under the flow of $p(X)$.

The proof of the following result can be found in [10, Lemma 2.1].

Lemma 6. *Let $f(x_1, x_2, x_3) = 0$ be an algebraic surface of degree m in $\mathbb{R}^3$. The extension of this surface to the boundary of the Poincaré ball is the curve described by the intersection of the two surfaces*

$$y_4^m f\left(\frac{x_1}{y_4}, \frac{x_2}{y_4}, \frac{x_3}{y_4}\right) = 0, \quad y_4 = 0.$$

2.2. Vertical Blow Up. Consider a polynomial differential system in $\mathbb{R}^2$ described by the equations:

$$(6) \quad \dot{x} = P(x, y) = P_n(x, y) + \dots, \quad \dot{y} = Q(x, y) = Q_n(x, y) + \dots,$$

where P and Q are coprime polynomials, P_n and Q_n are homogeneous polynomials of degree $n \in \mathbb{N}$, and the dots indicate higher-order terms in x and y . Since $n > 0$, the origin is an equilibrium point of the system (6). The *characteristic directions* at the origin correspond to the straight lines through the origin, determined by the real linear factors of the homogeneous polynomial:

$$(7) \quad R_n(x, y) = P_n(x, y)y - Q_n(x, y)x.$$

It is established that orbits starting or ending at the origin do so tangentially along these characteristic directions. For further details on characteristic directions, refer to [1].

Assuming there is an equilibrium point at the origin, as described in system (6), and that this equilibrium is linearly zero, we will analyze its local phase portrait using vertical blow-ups.

We define a vertical blow-up in the y direction through the variable transformation $(u, v) = (x, y/x)$. This transformation maps the origin of system (6) to the straight line $u = x = 0$. By studying the dynamics of the differential system in a neighborhood of this line, we effectively analyze the local phase portrait of the equilibrium point at the origin of system (6). However, prior to performing a vertical blow-up, we must ensure that the direction $x = 0$ is not a characteristic direction of the origin. If $x = 0$ is indeed a characteristic direction, we perform an appropriate twist using the transformation $(x, y) = (v + \alpha u, v)$ with $\alpha \neq 0$.

2.3. Bendixson-Dulac criterion. In what follows we recall a result that allows to establish the non-existence of closed orbits for a C^1 differential systems in the plane $\mathbb{R}^2$ of the form

$$(8) \quad x' = P(x, y), \quad y' = Q(x, y).$$

The criterion is due to Bendixson [2] (see also Theorem 7.10 of [6]) and says: Let D be a simply-connected domain in $\mathbb{R}^2$. Assume that the divergence of the system

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

has constant sign in D , except perhaps in a subset of D of zero Lebesgue measure. Then the differential system (8) has no closed orbits contained in D .

This criterion was generalized by H. Dulac, see Theorem 7.12 of [6] as follows: If D is a simply-connected domain in $\mathbb{R}^2$, if there exists a C^1 function $f(x, y)$ such that

$$\frac{\partial(fP)}{\partial x} + \frac{\partial(fQ)}{\partial y}$$

has constant sign in D , except perhaps in a subset of D of zero Lebesgue measure. Then the differential system (8) has no closed orbits contained in D .

2.4. Darboux invariants. Let $\varphi_p(t)$ be the solution of the differential system (1) passing through the point $p \in \mathbb{R}^3$, defined on its maximal interval $I_p = (\alpha_p, \omega_p)$. If $\omega_p = \infty$ we define the set

$$\omega(p) = \{q \in \Delta : \text{there exist } \{t_n\} \text{ with } t_n \rightarrow \infty \text{ and } \varphi(t_n) \rightarrow q \text{ when } n \rightarrow \infty\}.$$

In the same way, if $\alpha_p = -\infty$ we define the set

$$\alpha(p) = \{q \in \Delta : \text{there exist } \{t_n\} \text{ with } t_n \rightarrow -\infty \text{ and } \varphi(t_n) \rightarrow q \text{ when } n \rightarrow \infty\}.$$

The sets $\omega(p)$ and $\alpha(p)$ are called the ω -limit set (or simply ω -limit) and the α -limit set (or α -limit) of p , respectively.

The existence of a Darboux invariant provides information about the ω - and α -limit sets of all orbits of system (1). More precisely, we have the following result.

Proposition 7. *Let $I(x, y, z, t) = f(x, y, z)e^{st}$ be a Darboux invariant of system (1). Let $p \in \mathbb{R}^3$ and $\varphi_p(t)$ be the solution of system (1) with maximal interval (α_p, ω_p) such that $\varphi_p(0) = p$. Then*

- (i) *If $\omega_p = \infty$ then $\omega(p) \subset \{f(x, y, z) = 0\} \cup \mathbb{S}^2$.*
- (ii) *If $\alpha_p = -\infty$ then $\alpha(p) \subset \{f(x, y, z) = 0\} \cup \mathbb{S}^2$.*

Here $\mathbb{S}^2$ denotes the sphere of the infinity of the Poincaré ball.

For a proof of Proposition 7 see Proposition 5 of [11].

3. PROOF OF THEOREM 1 AND PROPOSITIONS 2 AND 3

In this section, we provide the formal proofs for some results previously stated. First, we demonstrate Theorem 1 that characterizes the invariant algebraic surfaces of degree 2 in the Ehrhard-Müller system.

Proof of Theorem 1. Let X be the vector field defined by the differential system (2). To find all invariant algebraic surfaces $f(x, y, z) = 0$ of degree 2 of system (1), we must find the coefficients a_i , $i = 1, \dots, 9$ and k_j , $j = 1, 2, 3, 4$ that satisfy equation (3) with

$$f(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6xz + a_7y^2 + a_8yz + a_9z^2$$

and

$$k(x, y, z) = k_1x + k_2y + k_3z + k_4.$$

Substituting f and k into equation (3), and equating both polynomials we obtain the following system of equations whose variables are the coefficients a_i , k_j , r , s , and c :

$$\begin{array}{ll}
(i) \ a_2c - a_0k_4 = 0; & (ii) \ a_5c - a_1k_4 - a_0k_1 + a_2r - a_1s = 0; \\
(iii) \ a_2 - 2a_7c + a_2k_4 + a_0k_2 - a_1s = 0; & (iv) \ a_3 - a_8c + a_3k_4 + a_0k_3 = 0; \\
(v) \ a_4k_4 + a_1k_1 - a_5r + 2a_4s = 0; & (vi) \ 2a_7 + a_7k_4 + a_2k_2 - a_5s = 0; \\
(vii) \ 2a_9 + a_9k_4 + a_3k_3 = 0; & (viii) \ 2a_8 + a_8k_4 + a_3k_2 + a_2k_3 - a_6s = 0; \\
(ix) \ a_2 + a_6 + a_6k_4 + a_3k_1 + a_1k_3 - a_8r + a_6s = 0; & (x) \ a_8 - a_7k_1 - a_5k_2 = 0; \\
(xi) \ a_6 - a_5k_1 - a_4k_2 = 0; & (xi) \ a_5 + a_6k_1 + a_4k_3 = 0; \\
(xii) \ 2a_7 - 2a_9 + a_8k_1 + a_6k_2 + a_5k_3 = 0; & (xiv) \ a_8k_2 + a_7k_3 = 0; \\
(xv) \ a_8 + a_9k_1 + a_6k_3 = 0; & (xvi) \ a_9k_2 + a_8k_3 = 0; \\
(xvii) \ a_4k_1 = 0; & (xviii) \ a_7k_2 = 0; \\
(xix) \ a_9k_3 = 0; & (xx) \ a_3 - a_5 - a_5k_4 - a_2k_1 - a_1k_2 + \\
& \quad + 2a_7r + 2a_4s - a_5s = 0.
\end{array}$$

It is straightforward to verify that $k_1 = k_2 = k_3 = 0$. Thus, the equations (xiv), (xvi), (xvii), (xviii) and (xix) simplify to $0 = 0$ and do not provide any additional information. Then using equations (xii), (xi), (xv), (ix) and (xiii), we obtain that $a_5 = a_6 = a_8 = a_2 = 0$ and $a_7 = a_9$, respectively. Since $a_2 = 0$, equation (i) implies that $a_0 = 0$.

Thus the system that we need to analyze reduces to the following relevant equations concerning the coefficients a_i and k_4 :

$$\begin{array}{ll}
(ii) \ -a_1k_4 - a_1s = 0; & (iii) \ -2a_9c - a_1s = 0; \\
(iv) \ a_3 + a_3k_4 = 0; & (v) \ a_4k_4 + 2a_4s = 0; \\
(vi) \ 2a_9 + a_9k_4 = 0; & (vii) \ 2a_9 + a_9k_4 = 0; \\
(xx) \ a_3 + 2a_9r + 2a_4s = 0.
\end{array}$$

From equation (vii) we conclude that $a_9 = 0$ or $k_4 = -2$.

Case 1: $a_9 = 0$. Since the invariant surface that we are looking for is of degree 2, we have that $a_4 \neq 0$. Thus, using equations (iii), (v), (xx) and (iv) we obtain that $a_1 = 0$, $k_4 = -2s$, $a_3 = -2sa_4$ and $s = 1/2$, respectively. Setting $a_4 = 1$, we get that $f(x, y, z) = x^2 - z$ is the invariant surface (a) given in Table 1.

Case 2: $k_4 = -2$. From equation (iv), we obtain that $a_3 = 0$. In addition, equation (v) implies that $a_4 = 0$ or $s = 1$.

Subcase 2.1: $a_4 = 0$. Since $a_9 \neq 0$, equation (xx) and (iii) implies that $r = 0$ and $sa_1 = -2ca_9$, respectively. Furthermore, by equation (ii), we get $a_1 = 0$ or $s = 2$.

- If $s = 2$, then taking $a_9 = 1$, we get that $f(x, y, z) = -cx + y^2 + z^2$ is the invariant surface (b) of Table 1.
- Otherwise, employing equation (iii) we obtain that $a_1 = 0$ implies that $c = 0$. Again, taking $a_9 = 1$, we obtain that $f(x, y, z) = y^2 + z^2$ is the invariant surface (c) in Table 1.

Subcase 2.2: $s = 1$. By equation (ii), we know that $a_1 = 0$. Thus, (iii) implies that $c = 0$. In addition, from equation (xx) we can conclude that $a_4 = -a_9r$. Setting $a_9 = 1$ we get that $f(x, y, z) = -rx^2 + y^2 + z^2$ is the invariant surface (d) of Table 1. This completes the proof of Theorem 1. $\square$

Having proven Theorem 1 that outlines the invariant algebraic surfaces of degree 2 in the Ehrhard-Müller system, we now turn to Proposition 2. This proposition demonstrates the existence of Darboux invariants for the system, offering further characterization of the system's dynamics

Proof of Proposition 2. We shall prove the statement (a) of this proposition, the other statements are proved in a similar way.

Let $I = I(x, y, z, t) = (x^2 - z)e^t$. Then, when $s = 1/2$ we have

$$\frac{dI}{dt} = \frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial z} \frac{dz}{dt} + \frac{\partial I}{\partial t} = xe^t(y - x) - e^t(xy - z) + (x^2 - z)e^t = 0.$$

Hence $(x^2 - z)e^t$ is a Darboux invariant. $\square$

In what follows, we proceed to prove Proposition 3. This one provides an important geometric insight by describing the phase portrait of the system on the infinity sphere, which reveals new structural features of the system.

Proof of Proposition 3. Using the results stated in Subsubsection 2.1.2 we obtain the expressions of the Ehrhard-Müller system (1) in the local charts U_j for $j = 1, 2, 3$. Thus in the local chart U_1 we have the system

$$\begin{aligned} \dot{z}_1 &= -z_2 + rz_3 + (s - 1)z_1z_3 + cz_3^2 - sz_1^2z_3, \\ \dot{z}_2 &= z_1 + (s - 1)z_2z_3 - sz_1z_2z_3, \\ \dot{z}_3 &= s(1 - z_1)z_3^2. \end{aligned} \tag{9}$$

The unique infinite equilibrium point of system (9) is the $(0, 0, 0)$. Restricting system (9) at infinity, i.e. at $z_3 = 0$, we obtain the system $\dot{z}_1 = -z_2$, $\dot{z}_2 = z_1$. Hence the equilibrium $(0, 0, 0)$ restricted to infinity is a linear center. So we have another linear center at the origin of the local chart V_1 .

In the local chart U_2 we have the system

$$\begin{aligned} \dot{z}_1 &= sz_3 + (1 - s)z_1z_3 + z_1^2z_2 - rz_1^2z_3 - cz_1z_3^2, \\ \dot{z}_2 &= z_1 + z_1z_2^2 - rz_1z_2z_3 - cz_2z_3^2, \\ \dot{z}_3 &= z_3(z_3 + z_1z_2 - rz_1z_3 - cz_3^2). \end{aligned} \tag{10}$$

System (10) has the straight line $z_1 = 0$ filled with infinite equilibria.

Finally in the local chart U_3 we have the system

$$\begin{aligned} \dot{z}_1 &= (1 - s)z_1z_3 + sz_2z_3 - z_1^2z_2, \\ \dot{z}_2 &= -z_1 + rz_1z_3 - z_1z_2^2 + cz_3^2, \\ \dot{z}_3 &= z_3(z_3 - z_1z_2). \end{aligned} \tag{11}$$

In this local chart we only need to see if the origin of coordinates is an infinite equilibrium point, because all the other infinite equilibrium points already have been detected in the local charts U_1 and U_2 . The origin $(0, 0, 0)$ of system (11) is an infinite equilibrium point, that together with the equilibria of the local chart U_2 , V_2 and V_3 form a circle filled with equilibria on the infinite sphere $\mathbb{S}^2$. This completes the proof of Proposition 3. $\square$

4. PROOF OF THEOREM 4

Now we shall study the dynamics of the Ehrhard-Müller systems in the Poincaré ball having the invariant algebraic surface $x^2 - z = 0$.

First we analyze the dynamics of the differential system (1) on the invariant surface $x^2 - z = 0$. Substituting $z = x^2$ in system (1) we obtain the differential system

$$(12) \quad \dot{x} = \frac{1}{2}(y - x), \quad \dot{y} = rx - x^3 - y + c.$$

We begin by studying the local phase portraits of the infinite equilibrium points of system (12).

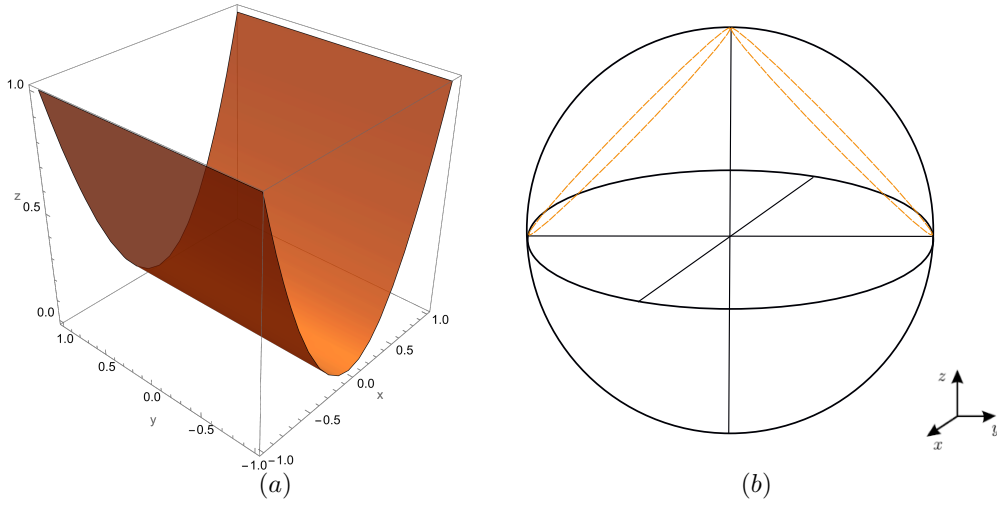


FIGURE 4. (a) The invariant algebraic surface $x^2 - z = 0$. (b) The interception of the invariant algebraic surface $x^2 - z = 0$ with the sphere $\mathbb{S}^2$ of the infinity.

Lemma 8. *The differential system (12) has two diametrically opposed infinite equilibrium points in the Poincaré disc at the origins of the local charts U_2 and V_2 , that are formed by two hyperbolic sectors on the Poincaré sphere. Moreover, the intersection of the invariant surface $x^2 - z = 0$ with the sphere of the infinity $\mathbb{S}^2$ is shown in Figure 4.*

Proof. We study the infinite equilibrium points using the Poincaré compactification. From Subsubsection 2.1.1 system (12) in the local charts U_1 and U_2 becomes

$$(13) \quad \dot{u} = \frac{1}{2}(-2 + 2rv^2 - uv^2 - u^2v^2 + 2cv^3), \quad \dot{v} = \frac{1}{2}(1 - u)v^3,$$

and

$$(14) \quad \dot{u} = \frac{1}{2}(2u^4 + v^2 + uv^2 - 2ru^2v^2 - 2cuv^3), \quad \dot{v} = -v(-u^3 - v^2 + ruv^2 + cv^3),$$

respectively. Since in the local chart U_1 system (12) has no infinite equilibrium points, it is enough to study if the origin of the local chart U_2 is an infinite equilibrium point, and it is a linearly zero equilibrium point. So in order to determine its local phase portrait we must use blow ups.

From (7) we obtain for the differential system (14) that $R_2(u, v) = v^3/2$. So $u = 0$ is not a characteristic direction. Hence we do the vertical blow-up $(u, v) = (u_1, u_1 v_1)$ obtaining the system

$$(15) \quad \dot{u}_1 = -\frac{1}{2}u_1^2(-2u_1^2 - v_1^2 - u_1 v_1^2 + 2ru_1^2 v_1^2 + 2cu_1^2 v_1^3), \quad \dot{v}_1 = \frac{1}{2}u_1 v_1^3(-1 + u_1).$$

Doing a rescaling of the time we eliminate the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ and we obtain the system

$$(16) \quad \dot{u}_1 = -\frac{1}{2}u_1(-2u_1^2 - v_1^2 - u_1 v_1^2 + 2ru_1^2 v_1^2 + 2cu_1^2 v_1^3), \quad \dot{v}_1 = \frac{1}{2}v_1^3(-1 + u_1).$$

Then the unique equilibrium point of system (16) on the straight line $u_1 = 0$ is the $(0, 0)$. We analyze its local phase portrait doing blow-ups. For system (16) $R_2(u_1, v_1) = u_1 v_1(v_1^2/2 + u_1^2)$, so $u_1 = 0$ is a characteristic direction. Consequently before doing a vertical blow-up we translate the direction $u_1 = 0$ to the direction $v_1 - u_1 = 0$ doing the change of variables $(u_1, v_1) = (v_2 - u_2, v_2)$. In the new variables (u_2, v_2) system (16) becomes

$$(17) \quad \begin{aligned} \dot{u}_2 &= \frac{1}{2} \left(2u_2^3 - 6u_2^2 v_2 + 7u_2 v_2^2 - u_2^2 v_2^2 - 2ru_2^3 v_2^2 - 4v_2^3 + u_2 v_2^3 + 6ru_2^2 v_2^3 \right. \\ &\quad \left. - 2cu_2^3 v_2^3 - 6ru_2 v_2^4 + 6cu_2^2 v_2^4 + 2rv_2^5 - 6cu_2 v_2^5 + 2cv_2^6 \right), \\ \dot{v}_2 &= \frac{1}{2}v_2^3(-1 - u_2 + v_2). \end{aligned}$$

Now we do the vertical blow-up $(u_2, v_2) = (u_3, u_3 v_3)$ obtaining the system

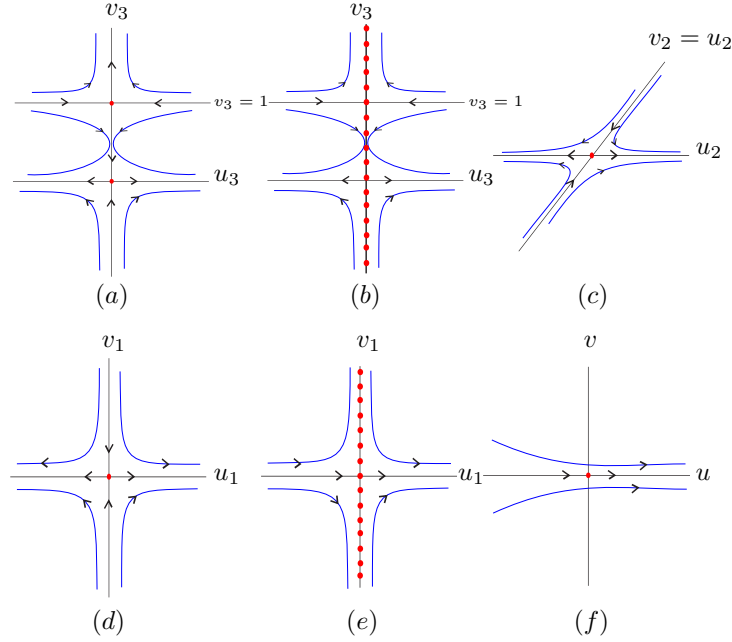
$$(18) \quad \begin{aligned} \dot{u}_3 &= \frac{1}{2}u_3^3 \left(2 - 6v_3 + 7v_3^2 - u_3 v_3^2 - 2ru_3^2 v_3^2 - 4v_3^3 + u_3 v_3^3 + 6ru_3^2 v_3^3 - 2cu_3^3 v_3^3 \right. \\ &\quad \left. - 6ru_3^2 v_3^4 + 6cu_3^3 v_3^4 + 2ru_3^2 v_3^5 - 6cu_3^3 v_3^5 + 2cu_3^3 v_3^6 \right), \\ \dot{v}_3 &= -u_3^2 \left(-1 + v_3 \right) v_3 \left(-1 + 2v_3 - 2v_3^2 + ru_3^2 v_3^2 - 2ru_3^2 v_3^3 + cu_3^3 v_3^3 + ru_3^2 v_3^4 \right. \\ &\quad \left. - 2cu_3^3 v_3^4 + cu_3^3 v_3^5 \right). \end{aligned}$$

Doing a rescaling of the time we eliminate the common factor u_3^2 between $\dot{u}_3$ and $\dot{v}_3$ and we obtain the system

$$(19) \quad \begin{aligned} \dot{u}_3 &= \frac{1}{2}u_3 \left(2 - 6v_3 + 7v_3^2 - u_3 v_3^2 - 2ru_3^2 v_3^2 - 4v_3^3 + u_3 v_3^3 + 6ru_3^2 v_3^3 - 2cu_3^3 v_3^3 \right. \\ &\quad \left. - 6ru_3^2 v_3^4 + 6cu_3^3 v_3^4 + 2ru_3^2 v_3^5 - 6cu_3^3 v_3^5 + 2cu_3^3 v_3^6 \right), \\ \dot{v}_3 &= -(-1 + v_3)v_3 \left(-1 + 2v_3 - 2v_3^2 + ru_3^2 v_3^2 - 2ru_3^2 v_3^3 + cu_3^3 v_3^3 + ru_3^2 v_3^4 \right. \\ &\quad \left. - 2cu_3^3 v_3^4 + cu_3^3 v_3^5 \right). \end{aligned}$$

The two equilibrium points of system (19) are the $(0, 0)$ and the $(0, 1)$, both are hyperbolic saddles because the eigenvalues of the Jacobian matrix of the system at these equilibria are -1 and 1 , and $-1/2$ and 1 , respectively. See Figure 5(a) for the phase portrait in a neighborhood of the straight line $u_3 = 0$. Hence the phase portrait in a neighborhood of the straight line $u_3 = 0$ for the differential system (18) is shown in Figure 5(b).

Going back through the blow up $(u_2, v_2) = (u_3, u_3 v_3)$ and taking into account that $\dot{u}_2|_{u_2=v_2} = -v_2^3/2$, we obtain the local phase portrait at the equilibrium $(0, 0)$ of the differential system (17), see Figure 5(c). Undoing the twist $(u_1, v_1) = (u_2, v_2 - u_2)$ we get the local phase portrait at the equilibrium $(0, 0)$ of the differential system (16), see Figure 5(d). Consequently, the local phase portrait at the equilibrium $(0, 0)$ of the differential system (15), see Figure 5(e). Finally, going back through the blow up $(u, v) = (u_1, u_1 v_1)$ and taking into account that

FIGURE 5. The blow up of the origin of the local chart U_2 .

$\dot{u}|_{u=0} = v^2/2$, the local phase portrait at the origin of the local chart U_2 is shown in Figure 5(f). $\square$

In what follows we will focus on studying the finite equilibrium points of system (12). Recall that these equilibrium points are on the diagonal $D = \{(x, y) : x = y\}$ and satisfy the equation

$$x^3 + (1 - r)x - c = 0.$$

The discriminant of that polynomial is $\Delta(c, r) = -27c^2 + 4(r - 1)^3$, see Figure 6. Using this discriminant we divide the study of the finite equilibria into three lemmas.

Lemma 9. *Consider the region $R^+ = \{(c, r) : \Delta(c, r) > 0\}$. Then system (12) $\big|_{R^+}$ has three hyperbolic equilibrium points on the diagonal $D = \{(x, y) : x = y\}$, one of them is a saddle and the other two are foci or nodes. More precisely,*

- (a) *if $1 + 8r - 24x^2 < 0$, then the equilibrium point (x, x) is an attracting focus;*
- (b) *if $1 + 8r - 24x^2 \in [0, 9)$, then the equilibrium point (x, x) is an attracting node; and*
- (c) *if $1 + 8r - 24x^2 > 9$, then the equilibrium point (x, x) is a saddle.*

See Figure 2 with $\Delta > 0$.

Proof. Note that in the statement of this lemma $1 + 8r - 24x^2$ can not be 9, otherwise $\Delta(c, r) = 0$.

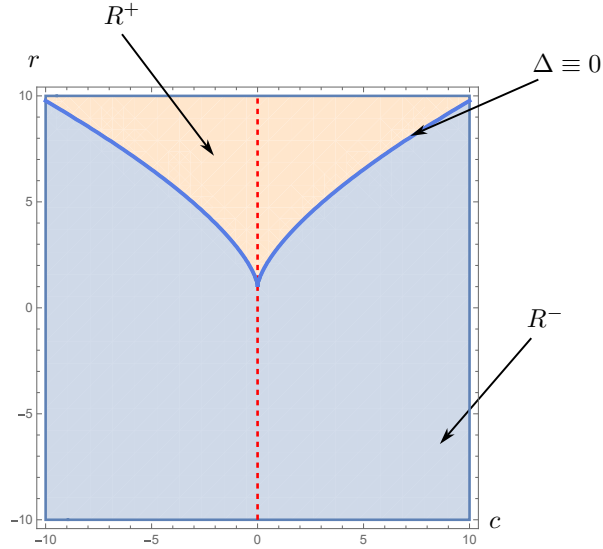


FIGURE 6. Parameter plane (c, r) . The discriminant $\Delta \equiv 0$ is the continuous curve given in the figure, that divides this plane into the regions $R^+ = \{(c, r) : \Delta(c, r) > 0\}$ and $R^- = \{(c, r) : \Delta(c, r) < 0\}$. The dashed line represents the straight line $c = 0$, which was already studied in [3, 10].

Suppose that $p = (x, x)$ is an equilibrium point of $(12)|_{R^+}$. Let F be the vector field associated to system (12). The Jacobian matrix of system (12) at p is

$$JF(p) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ r - 3x^2 & -1 \end{pmatrix}.$$

Moreover, their eigenvalues are given by $\lambda_{\pm} = \frac{1}{4}(-3 \pm \sqrt{1 + 8r - 24x^2})$ and since $\text{Tr}(JF(p)) = -3/2 \neq 0$, then system (12) has no center. So the statements of the lemma follows immediately from the Hartman-Grobman Theorem, see Theorem 2.15 of [6] for the version of this theorem for the planar differential systems. $\square$

Lemma 10. *Consider the curve $\Delta = \{(c, r) : \Delta(c, r) = 0\}$. Then system (12) has one equilibrium point of multiplicity one and a second equilibrium point of multiplicity two on the diagonal $D = \{(x, y) : x = y\}$, the first is a hyperbolic focus or a node, and the second is a semi-hyperbolic saddle-node. More precisely,*

- (a) *if $|c| > 1/(8\sqrt{2})$, then the equilibrium point is an attracting focus; and*
- (b) *if $|c| \leq 1/(8\sqrt{2})$, then the equilibrium point is an attracting node.*

See Figure 2 with $\Delta = 0$.

Proof. Since $\Delta = 0$, then $r = 1 + 3(c/2)^{2/3}$. Thus, the equilibrium points of (12) are given by $p_1 = (-(c/2)^{1/3}, -(c/2)^{1/3})$ and $p_2 = ((4c)^{1/3}, (4c)^{1/3})$. Let F be the vector field associated

to system (12). The Jacobian matrices of system (12) at p_1 and p_2 are

$$JF(p_1) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad JF(p_2) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 - 9\left(\frac{c}{2}\right)^{2/3} & -1 \end{pmatrix},$$

respectively. Moreover, their eigenvalues are $-3/2$ and 0 , and $(3/4)\left(-1 \pm \sqrt{1 - (27c^2)^{1/3}}\right)$, respectively. Since $\text{Tr}(JF(p_2)) = -3/2 \neq 0$, then system (12) has no centers and by the Hartman-Grobman Theorem it follows statements (a) and (b). From [6, Theorem 2.19], we can conclude that p_1 is a semi-hyperbolic saddle. See Figure 2. $\square$

Lemma 11. *Consider the region $R^- = \{(c, r) : \Delta(c, r) < 0\}$. Then system (12) $\big|_{R^-}$ has a unique equilibrium point with multiplicity one on the diagonal $D = \{(x, y) : x = y\}$, which is a focus or node. Moreover,*

- (a) *If $1 + 8r - 24(x^*)^2 < 0$, then the equilibrium point (x^*, x^*) is an attracting focus.*
- (b) *If $1 + 8r - 24(x^*)^2 \in [0, 9)$, then the equilibrium point (x^*, x^*) is an attracting node.*

Here

$$x^* = -\frac{\sqrt[3]{2}(r-1)}{\sqrt[3]{\sqrt{729c^2 + 4(3-3r)^3} - 27c}} - \frac{\sqrt[3]{\sqrt{729c^2 + 4(3-3r)^3} - 27c}}{3\sqrt[3]{2}},$$

with $729c^2 + 4(3-3r)^3 \geq 0$. See Figure 2 with $\Delta < 0$.

Proof. It is easy to see that $p = (x^*, x^*)$ is the unique equilibrium point of system (12) $\big|_{R^-}$. Let F be the vector field associated to system (12). The Jacobian matrix of system (12) at p is

$$JF(p) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ r - 3(x^*)^2 & -1 \end{pmatrix}.$$

Moreover, their eigenvalues are given by $\lambda_{\pm} = \frac{1}{4}(-3 \pm \sqrt{1 + 8r - 24(x^*)^2})$ and since $\text{Tr}(JF(p)) = -3/2 \neq 0$, then system (12) has no center.

Statements (a) and (b) follow immediately from the expression of the eigenvalues $\lambda_{\pm}$ and the Hartman-Grobman Theorem, see Figure 2.

It is easy to show that when $\Delta(c, r) < 0$ and $729c^2 + 4(3-3r)^3 \geq 0$, it is not possible that $1 + 8r - 24(x^*)^2 \geq 9$. $\square$

Lemma 12. *System (12) has no limit cycles.*

Proof. Let F be the vector field associated to system (12). Notice that the divergence of F is $-3/2$, thus from Bendison's criterion (see Section 2.3) the result follows. $\square$

From Lemmas 8, 9, 10, 11 and 12 follows statement (i) of Theorem 4.

Let $\gamma(t) = (x(t), y(t), z(t))$ be an orbit of the differential system (1) in the Poincaré ball outside the invariant surface $x^2 - z = 0$. Then the Darboux invariant $(x^2 - z)e^t$ evaluated on this orbit is a non-zero constant. The interval of definition of an orbit of a differential system

defined in a compact space is $(-\infty, \infty)$, see for instance the first chapter of [6]. So the α -limit and the ω -limit of the orbit $\gamma(t)$ are defined.

For the orbit $\gamma(t)$ and for all $t \in \mathbb{R}$ we have that $(x(t)^2 - z(t))e^t = k$, where k is a non-zero constant. Since $e^t \rightarrow 0$ when $t \rightarrow -\infty$ it follows that $x(t)^2 - z(t) \rightarrow \pm\infty$ when $t \rightarrow -\infty$. Therefore, from Figure 4 we get that the α -limit of the orbit $\gamma(t)$ contained in the interior of the Poincaré ball and in the region $x^2 - z < 0$ is the infinite equilibrium point at the origin of the local chart U_3 , while the α -limit of $\gamma(t)$ contained in the interior of the Poincaré ball and in the region $x^2 - z > 0$ is the infinite equilibrium point at the origin of the local chart V_3 . This completes the proof of statement (ii) of Theorem 4.

As before let $\gamma(t) = (x(t), y(t), z(t))$ be an orbit of the differential system (1) in the Poincaré ball outside the invariant surface $x^2 - z = 0$. For this orbit and for all $t \in \mathbb{R}$ we have that $(x(t)^2 - z(t))e^t = k$, where k is a non-zero constant. Since $e^t \rightarrow \infty$ when $t \rightarrow \infty$ it follows that $x(t)^2 - z(t) \rightarrow 0$ when $t \rightarrow \infty$. Hence, from Figures 2 and 4 we obtain that the ω -limit of the orbit $\gamma(t)$ is either one of the stable finite equilibrium points contained in the surface $x^2 - z = 0$, or one of the two infinite equilibrium points at the endpoints of the y -axis contained in the surface $x^2 - z = 0$. So statement (iii) of Theorem 4 is proved.

5. PROOF OF THEOREM 5

We will analyze the dynamics of the Ehrhard-Müller systems having the invariant algebraic surface $y^2 + z^2 - cx = 0$ in the Poincaré ball.

In this context we study the dynamics of the differential system (1) restricted to the invariant algebraic surface $y^2 + z^2 - cx = 0$, which is given by

$$(20) \quad \dot{y} = -\frac{z}{c}(y^2 + z^2) - y + c, \quad \dot{z} = \frac{y}{c}(y^2 + z^2) - z.$$

where $c \in \mathbb{R} \setminus \{0\}$.

We begin by studying the nature of the points at the infinity of system (20), for that we state the following result.

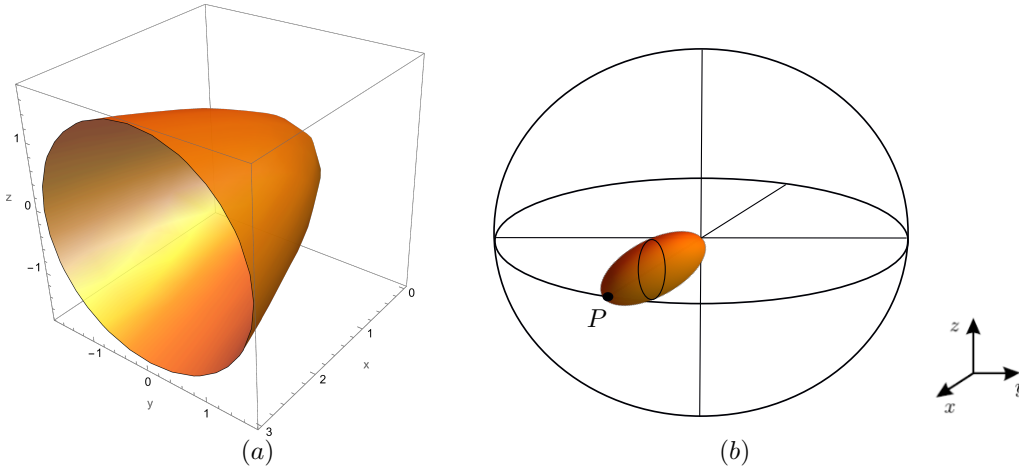


FIGURE 7. (a) The invariant algebraic surface $y^2 + z^2 - cx = 0$. (b) The interception of the invariant algebraic surface $y^2 + z^2 - cx = 0$ with the sphere $\mathbb{S}^2$ of the infinity. The equilibrium point P at infinity is localized at the endpoint of the positive x -axis.

Lemma 13. *The differential system (20) has no infinity equilibria. Furthermore, the intersection of the invariant surface $y^2 + z^2 - cx = 0$ with the sphere of the infinity $\mathbb{S}^2$ is shown in Figure 7.*

Proof. To obtain information about the infinite equilibria we use the Poincaré compactification. From Section 4 system (20) in the local charts U_1 and U_2 becomes

$$\dot{u} = \frac{1}{c}((1+u^2)^2 - c^2uv^3), \quad \dot{v} = -\frac{v}{c}(-u - u^3 - cv^2 + c^2v^3),$$

and

$$\dot{u} = \frac{1}{c}(-(1+u^2)^2 + c^2v^3), \quad \dot{v} = \frac{v}{c}(-u - u^3 + cv^2),$$

respectively. Since the equilibrium points at infinity (if they exist) are on $v = 0$, we obtain that $\dot{u} = \frac{1}{c}(1+u^2)^2 \neq 0$ (resp. $\dot{u} = -\frac{1}{c}(1+u^2)^2 \neq 0$) and $\dot{v} = 0$ in the chart U_1 (resp. U_2). Consequently, in the local charts U_1 and U_2 system (20) has no infinite equilibrium points. $\square$

Now, we analyze the finite equilibrium points of system (20). We shall study the equilibria lying in the set $B = \{(y, z) \in \mathbb{R}^2 \mid y^2 + z^2 - cy = 0, y \neq 0\}$, which describes a circle centered at $(c/2, 0)$ with radius $c/2$ in the yz -plane, excluding the points $(0, 0)$ and $(c, 0)$.

Lemma 14. *System (12) $\Big|_B$ has a unique equilibrium point with multiplicity one, which is a stable focus.*

Proof. Let G be the vector field associated to system (20). The Jacobian matrix of system (20) at p is

$$JG(p) = \begin{pmatrix} -\frac{2yz}{c} - 1 & -\frac{y^2 + 3z^2}{c} \\ \frac{z^2 + 3y^2}{c} & \frac{2yz}{c} - 1 \end{pmatrix}.$$

Moreover, their eigenvalues are given by $\lambda_{\pm} = -1 \pm i\frac{\sqrt{3}}{c}(y^2 + z^2)$ and since $\text{Tr}(JG(p)) = -2 \neq 0$, then system (20) has no center.

The lemma directly follow from the expression of the eigenvalues $\lambda_{\pm}$ and the Hartman-Grobman Theorem. $\square$

Lemma 15. *The system described by equation (20) does not have periodic orbits.*

Proof. Let F denote the vector field corresponding to the system in equation (20). It is important to note that the divergence of F is equal to -2 . Therefore, by applying *Bendixson's criterion* (see Section 2.3), we conclude that the system has no limit cycles. $\square$

From Lemmas 13, 14 and 15 follows statement (i) of Theorem 5.

Let $\gamma(t) = (x(t), y(t), z(t))$ be an orbit of the differential system (1) in the Poincaré ball outside the invariant surface $y^2 + z^2 - cx = 0$. Then, the Darboux invariant $(y^2 + z^2 - cx)e^{2t}$ evaluated on this orbit is a non-zero constant. The interval of definition of an orbit of a differential system defined in a compact space is $(-\infty, \infty)$, see for instance the first chapter of [6]. Therefore, the α -limit and the ω -limit of the orbit $\gamma(t)$ are well-defined.

For the orbit $\gamma(t)$ and for all $t \in \mathbb{R}$, we have that $(y(t)^2 + z(t)^2 - cx(t))e^{2t} = k$, where k is a non-zero constant. Since $e^{2t} \rightarrow 0$ as $t \rightarrow -\infty$, it follows that $y(t)^2 + z(t)^2 - cx(t) \rightarrow \pm\infty$

as $t \rightarrow -\infty$. Hence, from Figure 7, we obtain that the α -limit of the orbit $\gamma(t)$, contained in the interior of the Poincaré ball and in the region $y^2 + z^2 - cx \neq 0$, is the infinity equilibrium point P at the origin of the chart U_1 , which is located at the endpoint of the positive x -axis. This completes the proof of statement (ii) of Theorem 5.

As before, let $\gamma(t) = (x(t), y(t), z(t))$ be an orbit of the differential system (1) in the Poincaré ball outside the invariant surface $y^2 + z^2 - cx = 0$. For this orbit and for all $t \in \mathbb{R}$, we have that $(y(t)^2 + z(t)^2 - cx(t))e^{2t} = k$, where k is a non-zero constant. Since $e^{2t} \rightarrow \infty$ as $t \rightarrow \infty$, it follows that $y(t)^2 + z(t)^2 - cx(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, from Figures 3 and 7, we obtain that the ω -limit of the orbit $\gamma(t)$ is the stable focus Q contained in the surface $y^2 + z^2 - cx = 0$. Thus, statement (iii) of Theorem 5 is proved.

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AUTHOR DECLARATIONS

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Author Contributions:

- Jaume Llibre: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing–original draft, Writing–review & editing.
- Gabriel Rondón: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing–original draft, Writing–review & editing.

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