



Hopf Bifurcation for the Prey-Predator Rosenzweig-MacArthur Model

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Abstract

This paper studies the bifurcation of limit cycles for a specific predator-prey model called Rosenzweig-MacArthur, that presents persistent oscillations in the amount of individuals of the population from predator group and from the prey group. This model uses a Holling Type II function response. The averaging theory of third order allows to investigate the Hopf bifurcation that exhibits this model, providing an analytical approximation of the bifurcated periodic orbit and its kind of stability.

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1 Introduction and statement of the main results

In the last decades the mathematical treatment for modeling and understanding biological phenomena gained importance as a fundamental tool, specially in prey- predator models. In [1] many prey-predator models are explained in details, included the model treated here. This mathematical treatment make possible the understanding of the interactions between different populations species in the natural environment. The mathematical models are constructed through experiments and observations with the aim of make predictions. The dynamics of biological phenomena are expressed through differential systems, that play a fundamental role in mathematics and have wide applications in biology, physics, engineering and others. In general, for these systems we do not have flow explicit solutions and we have to apply the qualitative theory of dynamical systems and/or numerical methods to obtain informations about them.

There exists in nature many kind of interactions between different species and one of such interactions is related with predator and prey in a determined ecosystem. A predator is an animal that hunts and kills other animals with the propose of get food and a prey is an animal that is hunted, killed and eaten by the predators. The relationship between predator and prey is based on interactions of two species and the effects produced by these interactions on each other. Moreover, the predator prey populations have the chance to form a cycle along time. Initially, the predators reduce the prey population and then, due to the lack of food resources, the predator population decreases. As a consequence

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of this, the reduced number of predators enables the recovery of prey populations. The mathematical models arise in this situation to understand ecosystem principles and the realistic behaviors of natural ecosystems. Many mathematical models are created along the time and in this paper, particularly, we study the Rosenzweig-MacArthur model. Before we define this model, we need to introduce some concepts for a better understanding.

1.1 Structure of the article

This article starts in this section with some definitions necessary to the understanding of the main results, related with Hopf bifurcation and prey-predator models. Right away, the main results (the Proposition 1 and the Theorem 2) are enunciated. After this, the prove of the Proposition and of the Theorem is developed in the section 2. Then a conclusion about the work is made and, finally, an appendix 4 is put as a final section, explaining the averaging theory of third order, necessary to the understanding of the proofs.

1.2 Definitions

A Hopf Bifurcation takes place in an equilibrium point that changes its stability, becoming possible the emerging of periodic solutions. More precisely, it is a local bifurcation in which an equilibrium point loses stability through a pair of complex conjugate eigenvalues that cross the complex plane in the imaginary axis. The arising and the disappearance of a periodic orbit by a local change in the stability properties of an equilibrium point is known as Hopf bifurcation and it occurs in differential systems with two or more dimensions.

The relation of a predator rate, the number of prey eaten per predator per unit time and the prey density. It varies depending on the predator prey model considered. The total number of prey eaten in a determined period considered (one year or one prey generation, for example) is obtained by the product of the functional response by the predator population density and a time factor. It is possible to predict the future of prey-predator populations densities through additional information, such as an energy conversion factor and the current predator density. The general type of prey-predator interaction model is showed below.

$$\begin{aligned}\dot{u} &= F(u, v), \\ \dot{v} &= G(u, v),\end{aligned}$$

where the dot in the variables u and v represents time derivation, $F(u, v)$ and $G(u, v)$ depend on the species interactions and is given by

$$F(u, v) = \Phi(u) - f(u, v)v, \quad G(u, v) = \gamma f(u, v)v - \Psi(v).$$

The functions $\Phi(u)$ and $\Psi(v)$ represent the growth and decay of the prey and predator populations respectively and the functional response is given by $f(u, v)$, representing the interactions between prey and predator. One of the most usual functional responses used to describe the averaging feeding rate of a predator is the Holling type functional response. In [2] Holling did a categorization of the functional responses in three different types, called as type I, II and III and later the reference [3] introduced another type denominated Holling type IV. These models depend on u and the functions $f(u, v) = f(u)$ are given below

$$\begin{aligned}\text{Holling type I} &: f_1(u, v) = \delta u, \\ \text{Holling type II} &: f_2(u, v) = \frac{\delta u}{(\beta + \alpha u)}, \\ \text{Holling type III} &: f_3(u, v) = \frac{\delta u^2}{(\beta + \alpha u^2)}, \\ \text{Holling type IV} &: f_4(u, v) = \frac{\delta u}{(\beta + u)(1 + \alpha u)},\end{aligned}$$

where u is the density of the prey population, δ and α, β are positive constants that represent the effects of capture rate and handling time respectively. It is not hard to see that the four functional responses are prey-dependent and generally are not affected by predator absence in a determined ecosystem. Now we begin the study of the particular case treated here.

In this paper we study the bifurcation of periodic orbits from the two dimensional differential system called as the Rosenzweig-MacArthur model (see [4])

$$\begin{aligned}\dot{x} &= rx\left(1 - \frac{x}{k}\right) - \frac{cxy}{a+x}, \\ \dot{y} &= \frac{bcxy}{a+x} - dy,\end{aligned}\tag{1}$$

where x and y represent dynamical variables, the dot denotes derivative related to the time variable t and the parameters r, k, a, b, c and d are positive constants. The importance of system (1) emerges because it is about a structurally stable model that can exhibit persistent oscillations. The biologist Holling [2] suggested the introduction of the term $\frac{cx}{a+x}$, which represents a monotonically increasing bounded function representing a functional response of Holling Type II for invertebrate predators. Here c denotes the maximum rate of predation and a is associated in a proportional way with the necessary time for the predator search and find a prey. System (1) and some extensions of it has been studied by several authors, some of the most recent ones are [5–10].

System (1) has the equilibrium point

$$P = \left(-\frac{ad}{-bc+d}, -\frac{ab(ad-bck+dk)r}{(-bc+d)^2k}\right),$$

and we assume that P is located in the first quadrant. Moreover, the point $(0,0)$ also is an equilibrium point, independent of the parameters values. We study here the Hopf bifurcation that may occurs at the equilibrium point P and the following proposition supplies condition under which P is a Hopf equilibrium point.

1.3 Results obtained

In this paper we got two results, one enunciated as a Proposition and the main result enunciated as a Theorem.

Proposition 1. *If $k = \frac{a(bc+d)}{bc-d}$, then the differential system (1) presents a Hopf equilibrium at the point $P = \left(-\frac{ad}{-bc+d}, -\frac{ab(ad-bck+dk)r}{(-bc+d)^2k}\right)$, i.e., the eigenvalues of the Jacobian matrix of system (1) at P are purely imaginary. In this case, the equilibrium point P becomes $P_0 = \left(-\frac{ad}{-bc+d}, \frac{ab^2cr}{b^2c^2-d^2}\right)$.*

Proposition 1 is proved in section 2.

Under the assumptions of proposition 1, we study the conditions for the existence of a Hopf bifurcation at the equilibrium point P_0 for system (1) using of the averaging theory of third order. In order to apply the averaging theory of third order, we take the parameter k equal to $k(\varepsilon) = \frac{a(bc+d)}{bc-d} + \varepsilon k_1 + \varepsilon^2 k_2$, with $k_1, k_2 \geq 0$ and $\varepsilon > 0$ sufficiently small. Under these assumptions, we study the Hopf bifurcation of system (1). So system (1) becomes

$$\begin{aligned}\dot{x} &= rx\left(1 - \frac{x}{k(\varepsilon)}\right) - \frac{cxy}{a+x}, \\ \dot{y} &= \frac{bcxy}{a+x} - dy.\end{aligned}\tag{2}$$

The main result of this paper is the following.

Theorem 2. Consider the differential system (2) satisfying the conditions $bc - d$ and $bc + d$ positive, $k_1 = 0$ and $k_2 > 0$. Then system (2) has a Hopf bifurcation at the equilibrium point P_0 , i.e., under these conditions for $\varepsilon > 0$ sufficiently small, system (2) presents a periodic orbit $(x(t, \varepsilon), y(t, \varepsilon))$ such that $(x(t, \varepsilon), y(t, \varepsilon)) \rightarrow P_0$ as $\varepsilon \rightarrow 0$. Furthermore, this periodic orbit is stable when $-bc + d < 0$. If $-bc + d > 0$ the equilibrium point P_0 is deslocated from the first quadrant.

Theorem 2 is proved in section 2.

In fact in the proof of Theorem 2 we show that an analytic approximation of the periodic orbit $(x(t, \varepsilon), y(t, \varepsilon))$ of Theorem 2 is

$$\begin{aligned} x(t, \varepsilon) &= -\frac{ad}{d-bc} + \varepsilon \sqrt{\frac{2adk_2}{bc-d}} \cos\left(\frac{ab^2cdr}{\sqrt{d(b^2c^2-d^2)}r}t\right) + O(\varepsilon^2), \\ y(t, \varepsilon) &= \frac{ab^2cr}{b^2c^2-d^2} + \varepsilon br \sqrt{\frac{2ak_2}{r(bc+d)}} \sin\left(abcr \sqrt{\frac{d}{r(b^2c^2-d^2)}}t\right) + O(\varepsilon^2). \end{aligned}$$

In the paper [11] is studied a Hopf bifurcation of a modified Rosenzweig-MacArthur model where the prey population was divided into two sub-populations, and consequently the model is three-dimensional instead of the original two-dimensional model.

2 Proof of proposition 1 and theorem 2

Proof of Proposition 1. This proposition follows from the fact that the characteristic polynomial of linear part of the system (1) at P presents the form

$$p(\lambda) = \lambda^2 + \frac{ad(a(bc+d) + (-bc+d)k)r}{(-bc+d)^2k} \lambda - \frac{a^2bcd(ad + (-bc+d)k)r}{(-bc+d)^2k}.$$

The equilibrium point P has eigenvalues of the form $\pm i\omega$ if and only if $k = \frac{a(bc+d)}{bc-d}$. ■

Proof of Theorem 2. For becoming the application of Theorem 3 viable, we must transform the differential system (2) into the normal form of the appendix. In other words, we must transform system (2) in a non-autonomous time periodic system, with a small parameter ε . For achieving this we begin translating the equilibrium point P_0 to the origin of coordinates, and after writing the linear part of system (2) with $\varepsilon = 0$ at the origin of coordinates in its real Jordan normal form. The next step will be to apply a polar change of coordinates $(\varepsilon r, \theta)$ to system. After this we adopt the new time variable θ and the system transforms in a differential equation of the form $\frac{dr}{d\theta} = \varepsilon f(\theta, r, \varepsilon)$, that will be in the normal form (A1), ready for applying the averaging theory.

Initially we multiply components of system (2) by $a+x$ and then the equilibrium point P_0 writes as

$$\left(\frac{ad}{bc-d}, \frac{ab(bck-ad-dk)r}{(-bc+d)^2k}\right).$$

We assume that P_0 is located at the positive quadrant of the plane. So we calculated the Jacobian matrix of the system (2) (multiplied by $a+x$ in the two components) and its characteristic polynomial P in the variable λ , forcing P to be equal to $\lambda^2 + \omega^2$, which is the characteristic polynomial for having a Hopf bifurcation, we obtain the two solutions.

$$S_{1,2} = (k, \omega) = \left(\frac{a(bc+d)}{bc-d}, \pm \frac{abc\sqrt{dr}}{\sqrt{(bc-d)(bc+d)}}\right).$$

Now we perturb the parameter $k(\varepsilon) = \frac{a(bc+d)}{bc-d} + \varepsilon k_1 + \varepsilon^2 k_2$ in order to study the possible limit cycles bifurcating from the Hopf equilibrium point P_0 located at the origin of coordinates. Now we translate the equilibrium point P_0 for $\varepsilon = 0$ to the origin of coordinates doing the change of coordinates.

$$x = \frac{ad}{bc-d} + x_1 \quad \text{and} \quad y = \frac{ab(bck(\varepsilon) - ad - dk(\varepsilon))r}{(-bc + d^2)k(\varepsilon)} + y_1.$$

We write the linear part of the new differential system in its real Jordan normal form doing the change of coordinates

$$x_1 = x_2 \quad \text{and} \quad y_1 = \frac{\sqrt{b^2(bc-d)d(bc+d)ry_2}}{d(bc+d)},$$

the the system takes the following form, where $W = \frac{\sqrt{b^2(bc-d)d(bc+d)r}}{bc+d}$, $W_1 = (a + \varepsilon(k_1 + k_2\varepsilon))$ and $W_2 = (a - \varepsilon(k_1 + k_2\varepsilon))$,

$$\begin{aligned} \dot{x}_2 &= -\frac{(ad + bcx_2 - dx_2)(bc(rx_2(x_2 - \varepsilon(k_1 + k_2\varepsilon)) + \frac{cW_{y_2}W_1}{d}) + d(\frac{cW_{y_2}W_2}{d} + rx_2(-x_2 + \varepsilon(k_1 + k_2\varepsilon))))}{(bc-d)(a(bc+d) + (bc-d)\varepsilon(k_1 + k_2\varepsilon))} \\ \dot{y}_2 &= \frac{d(bc+d)x_2(abr - Wy_2 + \frac{bcW_{y_2}}{d} - \frac{a^2bdr}{a(b+c) + (bc-d)\varepsilon(k_1 + k_2\varepsilon)})}{\sqrt{b^2(bc-d)d(bc+d)r}}. \end{aligned}$$

Now we have to put this differential system into the normal form (A1) doing the polar change of coordinates $x_2 = \varepsilon R \cos \theta$, $y_2 = \varepsilon R \sin \theta$, and so the differential system becomes

$$\begin{aligned} \dot{R} &= \frac{\varepsilon^2 R \cos \theta (-dr \sqrt{b^2 d(b^2 c^2 - d^2)} r R \cos^2 \theta + r \cos \theta (dk_1 \sqrt{b^2 d(b^2 c^2 - d^2)} r}{(bc+d) \sqrt{b^2 d(b^2 c^2 - d^2)} r} \\ &\quad + b^2 c (-b^2 c^2 + d^2) R \sin \theta) + (bc-d) \sin \theta (bd^2 k_1 r + (bc+d) \\ &\quad (bc+d) \sqrt{b^2 d(b^2 c^2 - d^2)} r \\ &\quad \sqrt{b^2 d(b^2 c^2 - d^2)} r + b^2 c (-b^2 c^2 + d^2) R \sin \theta)) \\ &\quad (bc+d) \sqrt{b^2 d(b^2 c^2 - d^2)} r \\ &\quad + \varepsilon^3 R \cos \theta (r \cos \theta (d(-bc+d)k_1^2 + ad(bc+d)k_2 - (bc-d)R \cos \theta (-((bc+2d)k_1) \\ &\quad + (bc+d)R \cos \theta)) + \frac{d(bc(-k_1^2 + ak_2) + d(k_1^2 + ak_2)) \sqrt{b^2 d(b^2 c^2 - d^2)} r \sin \theta}{b(bc+d)}), \\ \dot{\theta} &= \frac{ab^2 cdr}{\sqrt{b^2 d(b^2 c^2 - d^2)} r} + \frac{(\varepsilon \cos \theta (-r \sin \theta (dk_1 \sqrt{b^2 (bc-d)d(bc+d)r + b^2 c}{(bc+d) \sqrt{b^2 (bc-d)d(bc+d)r}} \\ &\quad (-b^2 c^2 + d^2) R \sin \theta) + \cos \theta (b(bc-d)d^2 k_1 r + \sqrt{b^2 (bc-d)d(bc+d)r} \\ &\quad (bc+d) \sqrt{b^2 (bc-d)d(bc+d)r} \\ &\quad (b^2 c^2 + d(-d+r)R \sin \theta)))}{(bc+d) \sqrt{b^2 (bc-d)d(bc+d)r}} + \varepsilon^2 \frac{r \cos \theta}{a(bc+d)^2} (-\frac{b(bc-d)d^2}{\sqrt{b^2 (bc-d)d(bc+d)r}} \\ &\quad \frac{((bc-d)k_1^2 - a(bc+d)k_2) \cos \theta}{\sqrt{b^2 (bc-d)d(bc+d)r}} - (d(-bc+d)k_1^2 + ad(bc+d)k_2 \\ &\quad - (bc-d)R \cos \theta (-((bc+2d)k_1) + (bc+d)R \cos \theta)) \sin \theta) + \varepsilon^3 \frac{r \cos \theta}{2a^2(bc+d)^3} \\ &\quad (-\frac{2bd^2(-bc+d)^2 k_1 ((-bc+d)k_1^2 + 2a(bc+d)k_2) \cos \theta}{\sqrt{b^2 (bc-d)d(bc+d)r}} \end{aligned}$$

$$+ (bc - d)(2(bc + 2d)((bc - d)k_1^2 - a(bc + d)k_2)R \cos \theta + k_1(-2bcd(k_1^2 - 2ak_2) - b^2c^2R^2 + d^2(2k_1^2 + 4ak_2 + R^2) + (-b^2c^2 + d^2)R^2 \cos 2\theta)) \sin \theta).$$

Now we take the variable θ as the new time variable and we obtain the differential equation

$$\begin{aligned} R' = & \varepsilon^2 \frac{1}{ab^2cd(bc+d)r} R \cos \theta (-dr \sqrt{b^2(bc-d)d(bc+d)r} R \cos^2 \theta \\ & + r \cos \theta (dk_1 \sqrt{b^2(bc-d)d(bc+d)r} + b^2c(-b^2c^2 + d^2)R \sin \theta) \\ & + (bc-d) \sin \theta (bd^2k_1r + (bc+d) \sqrt{b^2(bc-d)d(bc+d)r} R \sin \theta)) \\ & + \varepsilon^3 \frac{1}{a^2b^4c^2d^2(bc+d)^2r^2} R \varepsilon^3 \cos \theta (b^2cdr^2 (\sqrt{b^2(bc-d)d(bc+d)r} \cos \theta (d(-bc+d)k_1^2 \\ & + ad(bc+d)k_2 + (bc-d)R \cos \theta ((bc+2d)k_1 - (bc+d)R \cos \theta)) \\ & - b(bc-d)d^2((bc-d)k_1^2 - a(bc+d)k_2) \sin \theta) \\ & - \cos \theta (-dr \sqrt{b^2(bc-d)d(bc+d)r} R \cos^2 \theta + r \cos \theta (dk_1 \sqrt{b^2(bc-d)d(bc+d)r} \\ & + b^2c(-b^2c^2 + d^2)R \sin \theta) + (bc-d) \sin \theta (bd^2k_1r + (bc+d) \\ & \sqrt{b^2(bc-d)d(bc+d)r} R \sin \theta)) + (-r \sin \theta (dk_1 \sqrt{b^2(bc-d)d(bc+d)r} \\ & + b^2c(-b^2c^2 + d^2)R \sin \theta) + \cos \theta (b(bc-d)d^2k_1r \\ & + \sqrt{b^2(bc-d)d(bc+d)r} (b^2c^2 + d(-d+r))R \sin \theta))), \end{aligned}$$

where the prime represents the derivative with respect to the new time θ , which is written in the normal form for applying averaging theory of third order.

$$F(\theta, R) = \varepsilon^2 F_2(\theta, R) + \varepsilon^3 F_3(\theta, R) + O(\varepsilon^4).$$

First we calculate the average function of first order

$$f_1(R) = \int_0^{2\pi} F_1(\theta, R) d\theta = 0, \quad (3)$$

because $F_1(\theta, R) = 0$. Moreover, we have that $y_1(\theta, R) = \int_0^\theta F_1(s, R) ds = 0$ and we are able to apply the averaging theory of second order. Computing $f_2(r)$ given by

$$f_2(R) = \int_0^T [F_2(\theta, R) + \frac{\partial F_1}{\partial r}(\theta, R) y_1(\theta, y)] d\theta,$$

and we obtain

$$f_2(R) = \int_0^{2\pi} F_2(\theta, R) dt = \frac{(bc-d)dk_1\pi r R}{ac\sqrt{b^2(bc-d)d(bc+d)r}}. \quad (4)$$

$R = 0$ is a solution of (5), but this solution does not provide a periodic solution, only an equilibrium point. So we take $k_1 = 0$ to get $f_2(r) \equiv 0$ allowing the application of the averaging theory of third order.

$$f_3(R) = \int_0^{2\pi} \left(\frac{1}{2} \frac{\partial^2 F_1}{\partial \theta^2}(\theta, R) (y_1(\theta, R))^2 + \frac{1}{2} \frac{\partial F_1}{\partial \theta}(\theta, R) y_2(\theta, R) + \frac{\partial F_2}{\partial \theta}(\theta, R) y_1(\theta, R) + F_3(\theta, R) \right) d\theta.$$

As long as we have $F_1(\theta, R) = 0$ and $y_1(\theta, R) = 0$, $f_3(R)$ becomes

$$f_3(\theta, R) = \int_0^{2\pi} F_3(\theta, R) d\theta = \frac{(bc-d)\pi r R (2adk_2 + (-bc+d)R^2)}{2a^2c\sqrt{b^2(bc-d)d(bc+d)r}}.$$

The unique positive solution for $f_3(\theta, R) = 0$ found is $R = \frac{\sqrt{2adk_2}}{\sqrt{bc-d}}$, ensuring that $k_2 > 0$. We define $R^* = \frac{\sqrt{2adk_2}}{\sqrt{bc-d}}$. By the averaging theory of third order, we obtain for $k_2 > 0$ and for the differential equation R' that a limit cycle $R(\theta, r)$ bifurcates from the zero Hopf equilibrium point and from Theorem 3 we have that $R(\theta, r) = \frac{\sqrt{2adk_2}}{\sqrt{bc-d}} + O(\varepsilon)$.

In the differential system $(\dot{R}, \dot{\theta})$, the limit cycle becomes

$$(R(t, \varepsilon), \theta(t, \varepsilon)) = \left(\frac{\sqrt{2adk_2}}{\sqrt{bc-d}}, \frac{ab^2cdr}{\sqrt{b^2d(b^2c^2-d^2)}r}t \right) + O(\varepsilon).$$

In the differential system $(\dot{x}_2, \dot{y}_2)$, the limit cycle becomes

$$(x_2(t, \varepsilon), y_2(t, \varepsilon)) = \left(\varepsilon \frac{\sqrt{2adk_2}}{\sqrt{bc-d}} \cos \frac{ab^2cdr}{\sqrt{b^2d(b^2c^2-d^2)}r}t, \varepsilon \frac{\sqrt{2adk_2}}{\sqrt{bc-d}} \sin \frac{ab^2cdr}{b^2d\sqrt{bd(b^2c^2-d^2)}r}t \right) + O(\varepsilon^2).$$

Through the change of variables made $x_1 = x_2$ and $y_1 = \frac{\sqrt{b^2(bc-d)d(bc+d)}r}{d(bc+d)}y_2$, in the differential system $(\dot{x}_1, \dot{y}_1)$, the limit cycle becomes

$$(x_1(t, \varepsilon), y_1(t, \varepsilon)) = \left(\varepsilon \frac{\sqrt{2adk_2}}{\sqrt{bc-d}} \cos \frac{ab^2cdr}{\sqrt{d(b^2c^2-d^2)}r}t, \varepsilon \frac{b\sqrt{2adk_2d(bc+d)}r}{d(bc+d)} \sin \frac{ab^2cdr}{\sqrt{d(b^2c^2-d^2)}r}t \right) + O(\varepsilon^2).$$

Moreover, the derivative of the averaging function $f_3(r)$ on the variable R at the zero $R^* = \frac{\sqrt{2adk_2}}{\sqrt{bc-d}}$ is

$$\frac{\partial f_3}{\partial R}(R^*) = \frac{2d(-bc+d)k_2\pi r}{ac\sqrt{b^2(bc-d)d(bc+d)}r}, \quad (5)$$

which is different of zero by the assumptions of the theorem. Therefore, under the hypotheses of Theorem 2, the averaging function of third order supplies a positive simple zero satisfying the assumptions of Theorem (3), and then we have a limit cycle bifurcating from the equilibrium point P_0 of the differential system (1). From (5) when $-bc+d < 0$, the periodic orbit obtained is stable. If $-bc+d > 0$, the equilibria P and P_0 are deslocated from the first quadrant. ■

3 An application of theorem 2

We shall apply Theorem 2 to the next prey-predator Rosenzweig-MacArthur model

$$\begin{aligned} \dot{x} &= x \left(1 - \frac{3000000x^2}{3560003} \right) - \frac{82xy}{5(x+1)}, \\ \dot{y} &= \frac{164(75x-7)y}{75(x+1)}, \end{aligned}$$

for computing the limit cycle that bifurcates from its Hopf equilibrium. The analytic approximation of this limit cycle given in (3) for $e = 1/1000$ is

$$(x(t, \varepsilon), y(t, \varepsilon)) = \left(\frac{7}{75} + \frac{1}{2500} \sqrt{\frac{7}{6}} \cos \left(\frac{26896}{1125} \sqrt{\frac{287}{89}}t \right), \frac{82}{1335} + \frac{1}{250} \sqrt{\frac{41}{534}} \sin \left(\frac{164}{75} \sqrt{\frac{287}{89}}t \right) \right) + O(\varepsilon^2).$$

Taking the value

$$(x(0, \varepsilon), y(0, \varepsilon)) = \left(\frac{7}{75} + \frac{1}{2500} \sqrt{\frac{7}{6}}, \frac{82}{1335} \right) + O(\varepsilon^2),$$

obtained analytically from (7) as the initial condition for computing numerically the limit cycle, we obtain the bifurcated limit cycle of the differential system (6) in Fig.1.

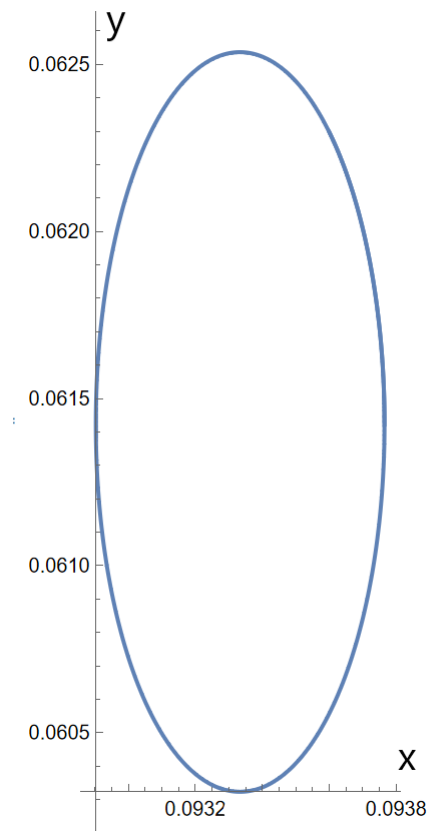


Fig. 1 The bifurcated limit cycle of the Hopf equilibrium of the prey-predator Rosenzweig-MacArthur model (6).

4 Conclusion

In this work we show analytically that the Rosenzweig-MacArthur model exhibits a Hopf bifurcation at the equilibrium point P_0 . This type of bifurcation exists when the conditions under the parameter $bc - d$ and $bc + d$ are positive, $k_1 = 0$ and $k_2 > 0$. Moreover, when this bifurcation exists, the bifurcated limit cycle is stable. Moreover, we provide an analytic approximation of this limit cycle.

From the biological point of view this shows that the populations of predators and preys can enter in a stable periodic behaviour determined by the bifurcated limit cycle from the Hopf equilibrium.

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Appendix

We present in this section the basic results from the averaging theory used in the proof of Theorem 2. For obtain periodic orbits bifurcating from a Hopf equilibrium point of a differential system, the third order averaging theory provides a method when the first and the second averaged functions $f_1(y)$ and $f_2(y)$ are identically zero. This is known in the literature like the third order averaging theory and it is a particular case of the k -th order averaging theory, which is provided in [12].

Theorem 3. *We consider the differential system*

$$x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 R(t, x, \varepsilon), \quad (\text{A1})$$

and suppose that $F_i(t, \cdot) \in C^{3-i}$, for all $t \in \mathbb{R}$, $i = 1, 2, 3$, $\partial^{k-i} F_i$ and R are locally Lipschitz in the variable x for $i = 1, 2, 3$, and that F_i and R are T -periodic in the variable t . Define

$$f_1(y) = \int_0^T F_1(t, y) dt \equiv 0, \quad f_2(y) = \int_0^T (F_2(t, y) + \partial F_1(t, y) y_1(t, y)) dt \equiv 0,$$

with $y_1(t, y) = \int_0^t F_1(s, y) ds$,

$$f_3(y) = \int_0^T (F_3(t, y) + \partial F_2(t, y) y_1(t, y) + \frac{1}{2} \partial^2 F_1(t, y) y_1(t, y)^2 + \frac{1}{2} \partial F_1(t, y) y_2(t, y)) dt \neq 0,$$

where $y_2(t, y) = \int_0^t (2F_2(s, y) + 2\partial F_1(s, y) y_1(s, y)) ds$. Assume that there exists $p \in D$ with $f_3(p) = 0$ and there exists a neighborhood $V \subset D$ such that $f_3(y) \neq 0$ for all $y \in \bar{V} \setminus \{p\}$, and that the Brouwer degree $d_B(f_3(y), V, p) \neq 0$. Then, for ε small enough, there exists a T -periodic solution $x(t, \varepsilon)$ of system (A1), such that $x(t, \varepsilon) \rightarrow p$ when $\varepsilon \rightarrow 0^+$.

Remark 1. Let $f : D \rightarrow \mathbb{R}^n$ be a C^1 function with $f(p) = 0$, where D is an open subset of $\mathbb{R}^n$ and $p \in D$. If p is a simple zero of f , i.e., the Jacobian $J_f(p)$ of f at p is non zero, then there exists a neighborhood V of p such that $f(z) \neq 0$ for all $z \in \bar{V} \setminus \{p\}$ and $d_B(h, V, p) \in \{-1, 1\}$. For details see reference [13].

By using the averaging theory, we get information about the location of the periodic orbits in the sense that when p is a zero of the averaged function f_3 with $d_B(f_3, V, p) \neq 0$, then there exists a limit cycle $x(t, \varepsilon)$ of system (A1) satisfying the condition $x(0, \varepsilon) \rightarrow p$ when $\varepsilon \rightarrow 0$. Therefore, near p there is the initial condition for the periodic orbit provided by Theorem 3.