

An SIS model with importation and removal

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The susceptible-infectious-susceptible (SIS) model is a classical model for describing the spread of diseases without permanent immunity and has been studied by many authors. In this paper, we characterize all topologically distinct phase portraits in the Poincaré disc for a particular SIS epidemic model with a constant importation or removal rate of infected individuals. Thus, for the first time, we provide a complete description of the qualitative dynamics of all orbits of this system, including those tending to or coming from infinity.

Keywords: SIS epidemic model, Poincaré compactification, Phase portrait, Averaging theory, Hopf bifurcation

1. Introduction and Statement of the Main Results

The susceptible-infected-susceptible (SIS) model is one of the classical frameworks for describing the transmission of infectious diseases for which infected individuals do not acquire permanent immunity after recovery [Kermack & McKendrick, 1927; Llibre & Valls, 2008; Nucci & Leach, 2004]. In this model, the population is divided into susceptible and infected compartments, denoted here by x and y , respectively. Many classical SIS models are formulated under the assumption that the population is closed. However, this assumption is not suitable in several epidemiological situations, since infected individuals may enter a community through travel, immigration or temporary mobility, and may also leave it through isolation, treatment, transfer, migration or other removal mechanisms.

The effects of the inflow and removal of infected individuals have been studied in different epidemic models. Brauer and van den Driessche [Brauer & van den Driessche, 2001] investigated disease transmission models with immigration of infectives. Shim [Shim, 2006] considered epidemic models with infective immigrants and vaccination, including SIS and SIR models, and discussed the occurrence of multiple endemic steady states and fold bifurcations. Related models involving immigration, vaccination, delay, nonlinear incidence or population structure have also been studied; see, for instance, [Sigdel & McCluskey, 2014; Henshaw & McCluskey, 2015; Uggenti & McCluskey, 2018; McCluskey, 2021]. These works show that the entrance or removal of infected individuals may substantially affect the qualitative dynamics, especially the existence, uniqueness and stability of equilibria, as well as the bifurcation structure of the model.

More recently, Yaacob, Mohd Siam and Nasir [Yaacob *et al.*, 2025] proposed an SIS model with a constant influx of infective visitors or immigrants and showed that it can fit COVID-19 data. Their model is

$$\begin{aligned}\dot{x} &= a - bxy - \mu x + cy, \\ \dot{y} &= e + bxy - (c + \mu + \mu_1)y,\end{aligned}\tag{1}$$

where a is the influx rate of susceptible individuals, b is the transmission rate, c is the recovery rate, μ is the natural removal rate, μ_1 is the disease-induced death rate, and e is a real constant describing the importation or removal of infected individuals. More precisely, $e > 0$ corresponds to the importation of infected individuals, whereas $e < 0$ corresponds to their removal from the community.

The global phase portraits of system (1) in the special case $\mu_1 = e = 0$ were obtained by Oliveira and Rezende [Oliveira & Rezende, 2013]. Their work shows that Poincaré compactification is a useful tool for studying SIS dynamics from a global point of view. In recent years, compactification and phase portrait methods have also been applied to epidemic models. Ichida [Ichida, 2022] used Poincaré compactification to study the global behavior of an SIR epidemic model, and Llibre and Salhi [Llibre & Salhi, 2024] classified the phase portraits of another SIR epidemic model. These studies illustrate that global methods may reveal dynamical phenomena which are not visible from a purely local analysis near finite equilibria.

The aim of the present paper is to give a complete global qualitative analysis of the quadratic SIS system obtained from (1) after the parameter renaming described below, considering both importation and removal of infected individuals. Introducing the parameter renaming

$$\mu = b, \quad \mu + \mu_1 = d,$$

and keeping the same letters for simplicity of notation, we study the quadratic system

$$\begin{aligned}\dot{x} &= a - bx + cy - bxy := P(x, y), \\ \dot{y} &= e - (c + d)y + bxy := Q(x, y),\end{aligned}\tag{2}$$

where $a, b, c, d > 0$ and $e \neq 0$ is real. The case $e > 0$ corresponds to the constant importation of infected individuals, whereas the case $e < 0$ corresponds to their constant removal from the community. Our purpose is to classify the global phase portraits of (2) in the Poincaré disc and to determine how the parameter e affects the global dynamics.

The Poincaré disc is obtained by compactifying the plane and adding the circle at infinity. In this compactification, finite equilibria remain in the interior of the disc, whereas directions at infinity are represented by points on the boundary circle. Thus finite equilibria, infinite equilibria, separatrices, limit cycles and homoclinic loops can be studied in the same compact phase space. This is particularly useful for distinguishing global phase portraits which may have similar local finite dynamics but different behavior near infinity.

In this paper we first determine all finite and infinite equilibrium points of system (2) and classify their local phase portraits. These local data are then combined with a separatrix analysis to obtain the global phase portraits in the Poincaré disc. In particular, we prove that limit cycles do not occur when $e > 0$, nor when $e < 0$ and $\Delta \leq 0$. In the remaining case $e < 0$ and $\Delta > 0$, the system has at most one limit cycle. Moreover, this limit cycle, when it exists, may bifurcate from a finite equilibrium through a Hopf bifurcation and may disappear when it coalesces with a homoclinic loop.

Compared with previous studies on SIS models, the main contribution of this paper is to provide a complete global classification in the Poincaré disc, including the dynamics at infinity, the degenerate cases where finite equilibria coalesce, and the bifurcation regimes in which a limit cycle or a homoclinic loop appears. In this way, we obtain all topologically distinct global phase portraits of system (2).

For this purpose, set

$$\Delta = 4de + (a - c - d + e)^2.$$

When $e < 0$ and $\Delta = 0$, define

$$a_1 = c + d - e + 2\sqrt{-de}, \quad a_2 = c + d - e - 2\sqrt{-de}.$$

For $e < 0$, the condition $\Delta > 0$ is equivalent to $a > a_1$ or $a < a_2$, whereas $\Delta = 0$ is equivalent to $a = a_1$ or $a = a_2$.

Two phase portraits in the Poincaré disc are said to be topologically equivalent if there exists a homeomorphism of the closed disc onto itself which maps oriented orbits onto oriented orbits. In particular, finite equilibria are mapped to finite equilibria, infinite equilibria to infinite equilibria, and separatrix connections to separatrix connections.

The main result of this paper is the following.

Theorem 1. *System (2) has nineteen topologically distinct phase portraits in the Poincaré disc. More precisely:*

- (1) *if $e > 0$, there is one phase portrait, shown in Fig. 3;*
- (2) *if $e < 0$ and $\Delta < 0$, there is one phase portrait, shown in Fig. 4;*
- (3) *if $e < 0$, $\Delta = 0$ and $a = a_1$, there are five topologically distinct phase portraits, shown in Fig. 5;*
- (4) *if $e < 0$, $\Delta = 0$ and $a = a_2$, there are six topologically distinct phase portraits, shown in Fig. 7;*
- (5) *if $e < 0$ and $\Delta > 0$, there are six topologically distinct phase portraits. They are represented in Fig. 8 for $a > a_1$. The six portraits in Fig. 9 for $a < a_2$ occur in a different parameter region, but they are topologically equivalent to the corresponding portraits in Fig. 8.*

Remark 1.1. The six phase portraits in Fig. 9 are topologically equivalent to the corresponding phase portraits in Fig. 8. More precisely,

$$\text{Fig. 8(a)} \sim \text{Fig. 9(f)}, \quad \text{Fig. 8(b)} \sim \text{Fig. 9(e)}, \quad \text{Fig. 8(c)} \sim \text{Fig. 9(d)},$$

and

$$\text{Fig. 8(d)} \sim \text{Fig. 9(c)}, \quad \text{Fig. 8(e)} \sim \text{Fig. 9(b)}, \quad \text{Fig. 8(f)} \sim \text{Fig. 9(a)}.$$

These pairs are drawn separately because they occur in different parameter regions and because the finite equilibria occupy different positions in the original (x, y) -plane. However, their separatrix configurations in the Poincaré disc are the same. The verification of these equivalences is given at the end of the proof of Theorem 1. In particular, the phase portraits represented by Fig. 8(c) and Fig. 9(d) are topologically equivalent.

Topological classifications of phase portraits in the Poincaré disc have also been carried out for other families of polynomial differential systems; see, for instance, [Llibre & Pantazi, 2023; Artés *et al.*, 2024]. In contrast with those works, the present paper focuses on an SIS epidemic model with constant importation or removal of infected individuals.

The remainder of the paper is organized as follows. Section 2 presents the preliminary material on Poincaré compactification, averaging theory and quadratic polynomial differential systems. Section 3 classifies the finite and infinite equilibrium points of system (2). Section 4 combines the local information, the limit-cycle results and the separatrix analysis to prove Theorem 1. At the end of the paper, we connect the classified phase portraits with their epidemiological implications, distinguishing persistence, extinction and oscillatory regimes.

2. Preliminaries

In this section we collect the preliminary material needed in the proof of the main theorem. We recall the Poincaré compactification in the local charts used to study the behavior at infinity, the local criteria for nilpotent and semi-hyperbolic singularities, and some standard results on averaging theory and quadratic systems. These results will be applied in Sections 3 and 4 to determine the finite and infinite equilibrium points, their local phase portraits, and the global separatrix configurations.

2.1. Poincaré compactification

In this subsection we summarize the equations of the Poincaré compactification that we need in this paper. The Poincaré compactification is a classical tool for studying the behavior of polynomial differential systems near infinity; see Chapter 5 of [Dumortier *et al.*, 2006] for details.

In order to study the polynomial differential system (2) in the Poincaré disc $\mathbb{D}^2$ we need the expressions of this polynomial differential system in the local charts (U_i, ϕ_i) and (V_i, ψ_i) , where $U_i = \{(u, v) \in \mathbb{D}^2 : v > 0\}$, $V_i = \{(u, v) \in \mathbb{D}^2 : v < 0\}$, $\phi_i : U_i \rightarrow \mathbb{R}^2$ and $\psi_i : V_i \rightarrow \mathbb{R}^2$ for $i = 1, 2$, with $\phi_1(u, v) = \psi_1(u, v) = (1/v, u/v)$ and $\phi_2(u, v) = \psi_2(u, v) = (u/v, 1/v)$.

If $\mathcal{X} = (P, Q)$ is the vector field associated to the polynomial differential system (2) defined in $\mathbb{R}^2$ we denote its extension to the Poincaré disc $\mathbb{D}^2$ by $p(\mathcal{X})$. The vector field $p(\mathcal{X})$ is called the *Poincaré compactification* of the vector field $\mathcal{X}$.

The expression of $p(\mathcal{X})$ in the local chart (U_1, ϕ_1) is

$$\dot{u} = v^d \left[-uP \left(\frac{1}{v}, \frac{u}{v} \right) + Q \left(\frac{1}{v}, \frac{u}{v} \right) \right], \quad \dot{v} = -v^{d+1} P \left(\frac{1}{v}, \frac{u}{v} \right), \quad (3)$$

while in the chart (U_2, ϕ_2) is

$$\dot{u} = v^d \left[P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right) \right], \quad \dot{v} = -v^{d+1} Q \left(\frac{u}{v}, \frac{1}{v} \right). \quad (4)$$

In the local chart (V_i, ψ_i) the expressions of $p(\mathcal{X})$ are the same as in the charts (U_i, ϕ_i) multiplied by $(-1)^{n-1}$ for $i = 1, 2$, where n is the degree of the polynomial differential system (2). Note that the circle $\mathbb{S}^1$ constitutes the boundary of the Poincaré disc $\mathbb{D}^2$. In each local chart, the points corresponding to the infinity of $\mathbb{R}^2$ are characterized by $v = 0$, and hence are of the form $(u, 0)$.

From the expressions of the vector field $p(\mathcal{X})$ it follows that the circle $\mathbb{S}^1$ of the infinity is invariant by the flow of $p(\mathcal{X})$. The equilibrium points of $p(\mathcal{X})$ which lie in $\mathbb{S}^1$ are the *infinite equilibrium points* of $\mathcal{X}$.

Note that if $p \in \mathbb{S}^1$ is an infinite equilibrium point, then $-p$ is also an infinite equilibrium point and these two points have the same local phase portrait if the degree of vector field is odd. Such phase portraits change the orientation of their orbits if the degree of the vector field is even.

From the expressions of $p(\mathcal{X})$ in the distinct local charts it follows that for determining the infinite equilibrium points it is sufficient to determine the ones on the local chart U_1 , and to verify if the origin of the local chart U_2 is or not an equilibrium.

2.2. Nilpotent and semi-hyperbolic singularities

In this subsection we recall some basic facts on nilpotent and semi-hyperbolic singularities which will be used in the sequel. We follow the terminology and notation of [Dumortier *et al.*, 2006, Sections 1.5, 2.5 and 5.2].

Theorem 2 [Nilpotent cusp criterion]. *Consider a planar analytic differential system which, after a local analytic change of coordinates, can be written in the form*

$$\dot{X} = Y + P_2(X, Y), \quad \dot{Y} = Q_2(X, Y),$$

where P_2 and Q_2 contain only terms of degree at least two. By the implicit function theorem, there exists a unique analytic function $Y = \varphi(X)$, with $\varphi(0) = 0$, such that

$$\varphi(X) + P_2(X, \varphi(X)) \equiv 0.$$

Define

$$F(X) = Q_2(X, \varphi(X))$$

and

$$G(X) = \left(\frac{\partial P_2}{\partial X} + \frac{\partial Q_2}{\partial Y} \right) (X, \varphi(X)).$$

If

$$F(X) = \alpha X^2 + O(X^3), \quad \alpha \neq 0,$$

and

$$G(X) = \beta X + O(X^2), \quad \beta \neq 0,$$

then the origin is a nilpotent cusp.

This is the case $m = 2$ and $n = 1$ of Theorem 3.5 in [Dumortier *et al.*, 2006], which is the only nilpotent case needed in this paper.

We next recall the corresponding criterion for semi-hyperbolic singularities. Consider a planar analytic differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (5)$$

and assume that the origin is an isolated singular point, that is, $P(0, 0) = Q(0, 0) = 0$. The origin is called a semi-hyperbolic singular point if the Jacobian matrix

$$DX(0, 0) = \begin{pmatrix} P_x(0, 0) & P_y(0, 0) \\ Q_x(0, 0) & Q_y(0, 0) \end{pmatrix}$$

has exactly one zero eigenvalue and one non-zero eigenvalue.

After a linear change of variables, system (5) can be written in the form

$$\dot{x} = F(x, y), \quad \dot{y} = \lambda y + G(x, y), \quad (6)$$

where $\lambda \neq 0$, and $F(x, y)$, $G(x, y)$ contain only terms of degree at least two. By the implicit function theorem, there exists a unique analytic function $y = f(x)$, with $f(0) = 0$, such that

$$\lambda f(x) + G(x, f(x)) \equiv 0.$$

Substituting this function into the first equation of (6), we obtain the one-dimensional reduced differential equation

$$\dot{x} = g(x) := F(x, f(x)).$$

Assume that

$$g(x) = \gamma_m x^m + O(x^{m+1}), \quad \gamma_m \neq 0, \quad m \geq 2. \quad (7)$$

Then the local phase portrait of the semi-hyperbolic singular point is determined by the first non-zero term of this reduced differential equation, together with the sign of the non-zero eigenvalue λ .

In this paper we only need the case $m = 2$. In this case, the reduced differential equation has the form

$$\dot{x} = \gamma_2 x^2 + O(x^3), \quad \gamma_2 \neq 0,$$

and the origin is a semi-hyperbolic saddle-node. The direction corresponding to the zero eigenvalue is the parabolic direction, whereas the eigendirection corresponding to the non-zero eigenvalue is the hyperbolic direction.

More explicitly, suppose that, in suitable local coordinates, the non-zero eigendirection is tangent to the v -axis and the associated reduced differential equation in the direction of the zero eigenvalue is

$$\dot{u} = \eta u^2 + O(u^3), \quad \eta \neq 0. \quad (8)$$

If the non-zero eigenvalue is λ , then the hyperbolic direction separates the two sides of the saddle-node. If

$$\lambda\eta > 0,$$

then the side $u > 0$ is the parabolic, or node-like, side, while the side $u < 0$ is the hyperbolic, or saddle-like, side. If

$$\lambda\eta < 0,$$

then the side $u < 0$ is the parabolic, or node-like, side, while the side $u > 0$ is the hyperbolic, or saddle-like, side.

Equivalently, if a constant change of time $d\tau = \lambda dt$ transforms the system into the form

$$\frac{du}{d\tau} = \tilde{\eta} u^2 + O(u^3), \quad \frac{dv}{d\tau} = v + O(\|(u, v)\|^2),$$

then the parabolic side is $u > 0$ if $\tilde{\eta} > 0$, and it is $u < 0$ if $\tilde{\eta} < 0$. A negative constant change of time reverses only the orientation of the orbits, but it does not change the position of the parabolic and hyperbolic sectors.

For infinite singular points, we shall also use the notation SN_1 and SN_2 introduced in [Dumortier *et al.*, 2006, Section 5.2]. In the local charts of the Poincaré compactification, the equator is represented by $v = 0$. If a semi-hyperbolic saddle-node at infinity has Jacobian matrix of the form

$$\begin{pmatrix} \lambda & * \\ 0 & 0 \end{pmatrix}, \quad \lambda \neq 0,$$

then it is said to be of type SN_1 . If its Jacobian matrix is of the form

$$\begin{pmatrix} 0 & * \\ 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0,$$

then it is said to be of type SN_2 . These two local configurations are illustrated in Fig. 1.

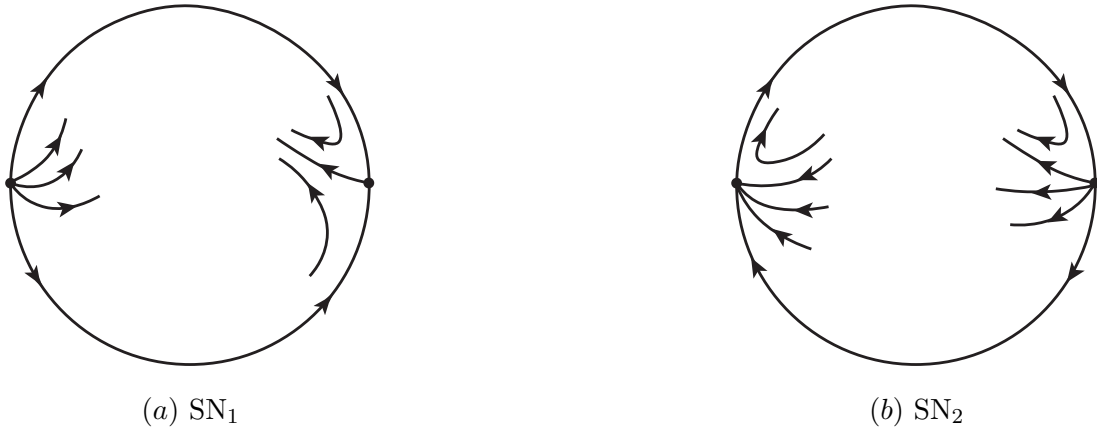


Fig. 1. Schematic local configurations of semi-hyperbolic saddle-nodes of type SN_1 and SN_2 at infinity. The line $v = 0$ represents the equator in a local chart of the Poincaré compactification. In the case SN_1 , the non-zero eigendirection is tangent to the equator, whereas in the case SN_2 , the zero eigendirection is tangent to the equator.

The notation SN_1 or SN_2 records the position of the non-zero eigendirection with respect to the equator. However, it does not by itself determine which side of the equator corresponds to the parabolic, or node-like, side of the saddle-node. This must be determined from the reduced differential equation and from the side of the local chart which represents the Poincaré disc.

For instance, suppose that a saddle-node of type SN_1 is written near the origin in the form

$$\dot{u} = \lambda u + O(\|(u, v)\|^2), \quad \dot{v} = \eta v^2 + O(v^3),$$

where $\lambda\eta \neq 0$, and the equator is $v = 0$. If the Poincaré disc corresponds to the side $v > 0$, then $\lambda\eta > 0$ means that the side $v > 0$ is the parabolic side, while $\lambda\eta < 0$ means that the side $v > 0$ is the hyperbolic side. If the Poincaré disc corresponds to $v < 0$, the conclusion is reversed. The case SN_2 is treated similarly, after interchanging the roles of the two local coordinates.

In the sequel, we shall proceed as follows. First, we compute the Jacobian matrix at the corresponding equilibrium point. If it has exactly one zero eigenvalue and one non-zero eigenvalue, then the point is semi-hyperbolic. Second, by computing the first non-zero term of the reduced differential equation, we decide whether it is a saddle-node. Finally, for infinite saddle-nodes, we use the form of the Jacobian matrix to determine whether the point is of type SN_1 or SN_2 , and use the sign of $\lambda\eta$, together with the side representing the Poincaré disc, to determine which side is node-like and which side is saddle-like.

2.3. Averaging theory of first and second order

Consider the differential system

$$\dot{x}(t) = \varepsilon F(t, x) + \varepsilon^2 G(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad (9)$$

where the functions $F, G : \mathbb{R} \times D \rightarrow \mathbb{R}^n$ and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$ are continuous and T -periodic in t , and D is an open subset of $\mathbb{R}^n$.

Assume that the next two conditions hold.

- (1) For each $t \in \mathbb{R}$ the mappings $F(t, \cdot), G(t, \cdot) \in C^1(D)$. Moreover, the functions $F, G, R, D_x F$ and $D_x G$ are locally Lipschitz with respect to x , and R is differentiable with respect to ε . We define

$$f(z) = \frac{1}{T} \int_0^T F(s, z) ds,$$

$$g(z) = \frac{1}{T} \int_0^T \left[D_z F(s, z) \int_0^s F(t, z) dt + G(s, z) \right] ds.$$

- (2) Let $V \subset D$ be an open and bounded set. For each $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, there exists $p \in V$ such that $f(p) + \varepsilon g(p) = 0$, and

$$\det \left(D_z [f(z) + \varepsilon g(z)] \Big|_{z=p} \right) \neq 0.$$

The *first-order averaging theory* takes place when $f \neq 0$, then for sufficiently small ε the zeros p of $f + \varepsilon g = 0$ are closely related to the zeros of f itself, and for sufficiently small ε there exists a T -periodic solution $\varphi(t, \varepsilon)$ of system (9) such that $\varphi(0, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

The *second-order averaging theory* takes place when $f \equiv 0$ while g is not identically zero, then the zeros p of $f + \varepsilon g$ are the zeros of g , and for sufficiently small ε there exists a T -periodic solution $\varphi(t, \varepsilon)$ of system (9) such that $\varphi(0, \varepsilon) \rightarrow p$ as $\varepsilon \rightarrow 0$.

In the first-order averaging theory we consider the averaged system

$$\dot{y} = f(y),$$

where

$$f(y) = \frac{1}{T} \int_0^T F(t, y) dt.$$

The above theory provides insights into the stability properties of the limit cycle $\varphi(t, \varepsilon)$, that are determined by the stability of the equilibrium point p in the averaged system. The stability type of p corresponds precisely to the stability type of the Poincaré map associated with the limit cycle.

Similarly, in the second-order averaging theory, where $f \equiv 0$ and g is nontrivial, the stability of the periodic solution $\varphi(t, \varepsilon)$ is determined by the stability of the equilibrium point p of the averaged system

$$\dot{y} = g(y),$$

where

$$g(y) = \frac{1}{T} \int_0^T \left[D_y F(s, y) \int_0^s F(t, y) dt + G(s, y) \right] ds,$$

and the stability type of the periodic orbit is again determined by the stability type of the equilibrium point p .

For a comprehensive treatment of the averaging theory and its applications, see [Novaes *et al.*, 2014; Sanders *et al.*, 2007; Verhulst, 1990].

2.4. Some results for quadratic systems

We recall some results on quadratic polynomial differential system that we shall need here. A quadratic polynomial differential system is simply called a quadratic system, written as

$$\dot{x} = p_1(x, y) + p_2(x, y), \quad \dot{y} = q_1(x, y) + q_2(x, y), \quad (10)$$

where

$$\begin{aligned} p_1(x, y) &= a_{10}x + a_{01}y, & p_2(x, y) &= a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ q_1(x, y) &= b_{10}x + b_{01}y, & q_2(x, y) &= b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned}$$

Theorem 3. ([Coppel, 1966]) *An equilibrium point in the interior of a closed path of system (10) must be either a focus or a center.*

Theorem 4. ([Coppel, 1989]) *If the highest degree terms $p_2(x, y)$, $q_2(x, y)$ of system (10) are proportional, then it either has a center or has at most one periodic orbit. Moreover, if there is a unique periodic orbit it is a stable or unstable limit cycle with a non-zero characteristic exponent.*

Theorem 5. ([Zhang, 1992]) *Consider system (10) with $p_1(x, y) = -y$ and $q_1(x, y) = x$ and define Liapunov constants*

$$W_1 = A\alpha - B\beta, \quad W_2 = [\beta(5A - \beta) + \alpha(5B - \alpha)]\gamma, \quad W_3 = (A\beta + B\alpha)\gamma\delta,$$

where

$$A = a_{20} + a_{02}, \quad B = b_{20} + b_{02}, \quad \alpha = a_{11} + 2b_{02}, \quad \beta = b_{11} + 2a_{20},$$

$$\gamma = b_{20}A^3 - (a_{20} - b_{11})A^2B + (b_{02} - a_{11})AB^2 - a_{02}B^3, \quad \delta = a_{02}^2 + b_{20}^2 + a_{02}A + b_{20}B,$$

The origin of (10) is a k th order weak focus ($k = 1, 2, 3$) if and only if the following corresponding k th condition is satisfied

$$(1) \ W_1 \neq 0, \quad (2) \ W_1 = 0, \ W_2 \neq 0, \quad (3) \ W_1 = W_2 = 0, \ W_3 \neq 0.$$

Moreover, the stability of the k th order weak focus is determined by the sign of W_k . The focus is stable if $W_k < 0$, and unstable if $W_k > 0$.

Theorem 6. ([Artés et al., 2018]) *The straight line connection between one finite equilibrium point and a pair of infinite equilibrium points in a quadratic system is either formed by trajectories or it is a line with exactly one contact point. This contact point is the finite equilibrium point. For the latter case the flow goes in the different directions on each half-line.*

We shall also use the Wintner–Perko termination principle: a one-parameter family of limit cycles can be extended to a maximal family, which, if it is not closed, terminates either at a singular point or at a separatrix cycle. See, for instance [Perko, 2001].

3. Equilibrium points of system (2)

We are going to study the finite and infinite equilibrium points of system (2) for the two cases $e > 0$ and $e < 0$.

3.1. Infinite equilibrium points

The following proposition describes the infinite equilibrium points of system (2) and their local phase portraits.

Proposition 1. *For system (2), the infinite equilibrium points on the Poincaré disc are described as follows. In the local chart U_1 , there are two infinite equilibrium points $D_1 = (-1, 0)$ and $D_2 = (0, 0)$, where D_1 is a hyperbolic stable node and D_2 is a semi-hyperbolic saddle-node of type SN_1 . In the local chart U_2 , the origin $D_3 = (0, 0)$ is also a semi-hyperbolic saddle-node of type SN_1 . Moreover, if D'_i denotes the antipodal point of D_i on the circle at infinity, then D'_1 is a hyperbolic unstable node, and D'_2 and D'_3 are semi-hyperbolic saddle-nodes of type SN_1 . In the Poincaré disc, the side of D_2 lying inside the disc corresponds to the parabolic sector, whereas the side of D_3 lying inside the disc corresponds to the hyperbolic sector.*

Proof. We use the Poincaré compactification to study the equilibrium points at infinity. In the local chart U_1 , system (2) becomes

$$\begin{aligned}\frac{du}{dt} &= bu + bu^2 + (b - c - d)uv + ev^2 - cu^2v - auv^2, \\ \frac{dv}{dt} &= (bu + bv - cuv - av^2)v.\end{aligned}\tag{11}$$

The circle at infinity is given by $v = 0$. Since $v = 0$ is invariant, the restriction of (11) to the circle at infinity is

$$\frac{du}{dt} = bu(1 + u).$$

Hence the equilibrium points in the chart U_1 are

$$D_1 = (-1, 0), \quad D_2 = (0, 0).$$

At $D_1 = (-1, 0)$, the Jacobian matrix of (11) is

$$J(D_1) = \begin{pmatrix} -b & d - b \\ 0 & -b \end{pmatrix}.$$

Both eigenvalues are equal to $-b < 0$. Therefore D_1 is a hyperbolic stable-node.

We next study $D_2 = (0, 0)$. The Jacobian matrix at D_2 is

$$J(D_2) = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus D_2 has one non-zero eigenvalue $b > 0$ and one zero eigenvalue. The eigendirection corresponding to the non-zero eigenvalue is tangent to the line $v = 0$, namely to the circle at infinity.

Since the center direction is tangent to the v -axis, we write the center manifold in the form

$$u = h(v), \quad h(0) = h'(0) = 0.$$

The invariance equation gives

$$h(v) = -\frac{e}{b}v^2 + O(v^3).$$

Substituting this expression into the second equation of (11), we obtain the reduced equation on the center manifold:

$$\frac{dv}{dt} = bv^2 + O(v^3).$$

Since the first non-zero term has degree two, D_2 is a semi-hyperbolic saddle-node by the criterion recalled in Subsection 2.2.

Moreover, the Jacobian matrix at D_2 has the form

$$\begin{pmatrix} \lambda & * \\ 0 & 0 \end{pmatrix}, \quad \lambda = b > 0.$$

Therefore D_2 is of type SN_1 . In the chart U_1 , the Poincaré disc corresponds to $v > 0$. For D_2 , the coefficient of the first non-zero term in the reduced equation is $\eta = b > 0$. Hence

$$\lambda\eta = b^2 > 0.$$

Therefore, by the criterion in Subsection 2.2, the side $v > 0$ of D_2 , namely the side lying inside the Poincaré disc, is the parabolic, or node-like, side of the saddle-node.

On the circle at infinity, near D_2 , we have

$$\frac{du}{dt} = bu(1 + u).$$

Thus the arrows on $v = 0$ point away from D_2 on both sides:

$$\leftarrow D_2 \rightarrow.$$

We now consider the local chart U_2 . In this chart, the compactified system is

$$\begin{aligned} \frac{du}{dt} &= -bu - bu^2 + (c + (c + d - b)u)v + (a - eu)v^2, \\ \frac{dv}{dt} &= -buv + (c + d)v^2 - ev^3. \end{aligned} \tag{12}$$

Again, the circle at infinity is given by $v = 0$. On this invariant line,

$$\frac{du}{dt} = -bu(1 + u).$$

Therefore the equilibrium points on $v = 0$ in the chart U_2 are

$$(-1, 0), \quad (0, 0).$$

By the correspondence between the charts U_1 and U_2 on the circle at infinity, the point $(-1, 0)$ represents the antipodal point of D_1 ; we denote it by D'_1 . The origin $(0, 0)$ is denoted by D_3 .

At $D'_1 = (-1, 0)$, the Jacobian matrix of (12) is

$$J(D'_1) = \begin{pmatrix} b & b - d \\ 0 & b \end{pmatrix}.$$

Both eigenvalues are equal to $b > 0$. Hence D'_1 is a hyperbolic unstable node.

It remains to study $D_3 = (0, 0)$. At the origin of (12), the linear part is

$$\begin{pmatrix} -b & c \\ 0 & 0 \end{pmatrix}.$$

To put this linear part into diagonal form, we perform the change of variables

$$u = u_1 + \frac{c}{b}v_1, \quad v = v_1.$$

Then system (12) becomes

$$\begin{aligned} \frac{du_1}{dt} &= -bu_1(1 + u_1 + v_1) + v_1(du_1 + (a - c - eu_1)v_1), \\ \frac{dv_1}{dt} &= -v_1(bu_1 + v_1(-d + ev_1)). \end{aligned} \tag{13}$$

The Jacobian matrix of (13) at the origin is

$$J(D_3) = \begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus D_3 has one non-zero eigenvalue $-b < 0$ and one zero eigenvalue. The eigendirection corresponding to the non-zero eigenvalue is tangent to the line $v_1 = 0$, namely to the circle at infinity.

Since the center direction is tangent to the v_1 -axis, we write the center manifold as

$$u_1 = h_1(v_1), \quad h_1(0) = h'_1(0) = 0.$$

The invariance equation gives

$$h_1(v_1) = \frac{a - c}{b}v_1^2 + O(v_1^3).$$

Substituting this expression into the second equation of (13), we obtain

$$\frac{dv_1}{dt} = dv_1^2 + O(v_1^3).$$

Since the first non-zero term has degree two, D_3 is a semi-hyperbolic saddle-node by the criterion recalled in Subsection 2.2.

Furthermore, the Jacobian matrix at D_3 has the form

$$\begin{pmatrix} \lambda & * \\ 0 & 0 \end{pmatrix}, \quad \lambda = -b < 0.$$

Therefore D_3 is also of type SN_1 . In the chart U_2 , the Poincaré disc corresponds to $v_1 > 0$. For D_3 , the coefficient of the first non-zero term in the reduced equation is $\eta = d > 0$. Hence

$$\lambda\eta = (-b)d < 0.$$

Therefore, by the criterion in Subsection 2.2, the side $v_1 > 0$ of D_3 , namely the side lying inside the Poincaré disc, is the hyperbolic, or saddle-like, side of the saddle-node.

On the circle at infinity, near D_3 , we have

$$\frac{du_1}{dt} = -bu_1(1 + u_1),$$

so the arrows on $v_1 = 0$ point toward D_3 on both sides:

$$\rightarrow \quad D_3 \quad \leftarrow .$$

Finally, the remaining points D'_2 and D'_3 are the antipodal points of D_2 and D_3 , respectively. Since system (2) has degree two, the local phase portrait at each antipodal point is obtained from the corresponding one by reversing the direction of the arrows. The antipodal map preserves the tangency of the non-zero eigendirection to the circle at infinity. Hence D'_2 and D'_3 are also semi-hyperbolic saddle-nodes of type SN_1 . More precisely, the arrows on the circle at infinity near D'_2 point toward D'_2 , whereas the arrows near D'_3 point away from D'_3 .

This proves the proposition. ■

The local phase portrait at infinity described in Proposition 1 is shown schematically in Fig. 2.

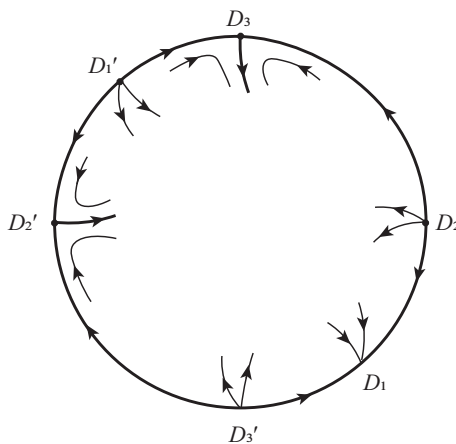


Fig. 2. Local phase portrait at infinity of system (2) on the Poincaré disc. The points D_1 and D'_1 are respectively a hyperbolic stable node and a hyperbolic unstable node. The points D_2, D'_2, D_3 and D'_3 are semi-hyperbolic saddle-nodes of type SN_1 . The side of D_2 lying inside the disc is parabolic, whereas the side of D_3 lying inside the disc is hyperbolic.

3.2. Finite equilibrium points

Solving the equations $P(x, y) = Q(x, y) = 0$, we obtain

$$E_1 = \left(\frac{a + c + d + e + \sqrt{\Delta}}{2b}, \frac{a - c - d + e - \sqrt{\Delta}}{2d} \right),$$

and

$$E_2 = \left(\frac{a + c + d + e - \sqrt{\Delta}}{2b}, \frac{a - c - d + e + \sqrt{\Delta}}{2d} \right),$$

where $\Delta = 4de + (a - c - d + e)^2$.

We first consider the case when $e > 0$. The following proposition gives the properties of the finite equilibrium points for system (2).

Proposition 2. *When $e > 0$ system (2) has exactly two finite equilibrium points E_1 and E_2 , where E_1 is a hyperbolic saddle in the fourth quadrant and E_2 is a hyperbolic stable node or focus in the first quadrant.*

Proof. When $e > 0$ we obtain $\Delta > 0$ and it is easy to find that the point E_1 is in the fourth quadrant. Since

$$(a + c + d + e)^2 - \Delta = 4(ac + ad + ce) > 0,$$

E_2 is in the first quadrant.

The Jacobian matrix at the equilibrium point E_1 is

$$J(E_1) = \begin{pmatrix} -\frac{b}{2d}(a - c + d + e - \sqrt{\Delta}) & -\frac{1}{2}(a - c + d + e + \sqrt{\Delta}) \\ \frac{b}{2d}(a - c - d + e - \sqrt{\Delta}) & \frac{1}{2}(a - c - d + e + \sqrt{\Delta}) \end{pmatrix}.$$

Since $\det(J(E_1)) = -b\sqrt{\Delta} < 0$, E_1 is a hyperbolic saddle, see for instance Theorem 2.15 of [Dumortier *et al.*, 2006].

For the equilibrium point E_2 its Jacobian matrix is

$$J(E_2) = \begin{pmatrix} -\frac{b}{2d}(a - c + d + e + \sqrt{\Delta}) & -\frac{1}{2}(a - c + d + e - \sqrt{\Delta}) \\ \frac{b}{2d}(a - c - d + e + \sqrt{\Delta}) & \frac{1}{2}(a - c - d + e - \sqrt{\Delta}) \end{pmatrix}.$$

Since $\det(J(E_2)) = b\sqrt{\Delta} > 0$, and

$$\text{tr}(J(E_2)) = \frac{1}{2} \left((a - c - d + e - \sqrt{\Delta}) - \frac{b}{d}(a - c - d + e + \sqrt{\Delta}) - 2b \right) < 0,$$

E_2 is a stable node if $\text{tr}(J(E_2))^2 - 4\det(J(E_2)) \geq 0$ and E_2 is a stable focus if $\text{tr}(J(E_2))^2 - 4\det(J(E_2)) < 0$. ■

Note that when $e < 0$ the value of Δ may be negative, zero or positive.

The following proposition gives the properties of finite equilibrium points for system (2) with $e < 0$ when $\Delta \neq 0$.

Proposition 3. *Assume that $e < 0$ and $\Delta \neq 0$. Then the finite equilibrium points of system (2) are as follows:*

- (a) if $\Delta < 0$, then system (2) has no finite equilibrium points;
- (b) if $\Delta > 0$, then system (2) has exactly two finite equilibrium points E_1 and E_2 .

Moreover, E_1 is a hyperbolic saddle. For E_2 , one has

$$\det(J(E_2)) = b\sqrt{\Delta} > 0$$

and

$$\operatorname{tr}(J(E_2)) = \frac{(a - c - d + e)(d - b) - (b + d)\sqrt{\Delta} - 2bd}{2d}.$$

Consequently, if $\operatorname{tr}(J(E_2)) < 0$, then E_2 is a stable node or focus; if $\operatorname{tr}(J(E_2)) > 0$, then E_2 is an unstable node or focus. If $\operatorname{tr}(J(E_2)) = 0$, then E_2 is a first-order weak focus. More precisely, E_2 is stable when $a < c - e$, and unstable when $a > c - e$.

Proof. When $e < 0$, the number of finite equilibrium points of system (2) is determined by the sign of

$$\Delta = 4de + (a - c - d + e)^2.$$

If $\Delta < 0$, then there are no finite equilibrium points. If $\Delta > 0$, then there are two distinct finite equilibrium points E_1 and E_2 .

We first consider E_1 . The Jacobian matrix at E_1 is

$$J(E_1) = \begin{pmatrix} -\frac{b}{2d}(a - c + d + e - \sqrt{\Delta}) & -\frac{1}{2}(a - c + d + e + \sqrt{\Delta}) \\ \frac{b}{2d}(a - c - d + e - \sqrt{\Delta}) & \frac{1}{2}(a - c - d + e + \sqrt{\Delta}) \end{pmatrix}.$$

Thus

$$\det(J(E_1)) = -b\sqrt{\Delta} < 0.$$

Hence E_1 is a hyperbolic saddle.

For the equilibrium point E_2 , its Jacobian matrix is

$$J(E_2) = \begin{pmatrix} -\frac{b}{2d}(a - c + d + e + \sqrt{\Delta}) & -\frac{1}{2}(a - c + d + e - \sqrt{\Delta}) \\ \frac{b}{2d}(a - c - d + e + \sqrt{\Delta}) & \frac{1}{2}(a - c - d + e - \sqrt{\Delta}) \end{pmatrix}.$$

Therefore

$$\det(J(E_2)) = b\sqrt{\Delta} > 0$$

and

$$\begin{aligned} \operatorname{tr}(J(E_2)) &= -\frac{b}{2d}(a - c + d + e + \sqrt{\Delta}) + \frac{1}{2}(a - c - d + e - \sqrt{\Delta}) \\ &= \frac{(a - c - d + e)(d - b) - (b + d)\sqrt{\Delta} - 2bd}{2d}. \end{aligned}$$

Since $\det(J(E_2)) > 0$, the sign of $\operatorname{tr}(J(E_2))$ determines the stability of E_2 . Thus, if $\operatorname{tr}(J(E_2)) < 0$, then E_2 is stable; it is a node if

$$\operatorname{tr}(J(E_2))^2 - 4\det(J(E_2)) \geq 0,$$

and it is a focus if

$$\operatorname{tr}(J(E_2))^2 - 4\det(J(E_2)) < 0.$$

Similarly, if $\operatorname{tr}(J(E_2)) > 0$, then E_2 is unstable; it is a node if

$$\operatorname{tr}(J(E_2))^2 - 4\det(J(E_2)) \geq 0,$$

and it is a focus if

$$\operatorname{tr}(J(E_2))^2 - 4\det(J(E_2)) < 0.$$

It remains to study the case

$$\text{tr}(J(E_2)) = 0.$$

Solving this equation with respect to b , we obtain

$$b = \frac{d(a - c - d + e - \sqrt{\Delta})}{a - c + d + e + \sqrt{\Delta}} := b^*.$$

Since $b = b^* > 0$, the eigenvalues of $J(E_2)$ are

$$\lambda_{1,2} = \pm i\sqrt{b\sqrt{\Delta}}.$$

Hence E_2 is either a center or a weak focus.

We now prove that E_2 is a first-order weak focus. We translate E_2 to the origin by taking

$$X = x - \frac{a + c + d + e - \sqrt{\Delta}}{2b}, \quad Y = y - \frac{a - c - d + e + \sqrt{\Delta}}{2d}.$$

After rewriting X, Y again as x, y , system (2) becomes

$$\begin{aligned} \frac{dx}{dt} &= bc_{10}x + c_{01}y - bxy, \\ \frac{dy}{dt} &= bd_{10}x + d_{01}y + bxy, \end{aligned} \tag{14}$$

where

$$c_{10} = -\frac{a - c + d + e + \sqrt{\Delta}}{2d}, \quad c_{01} = -\frac{a - c + d + e - \sqrt{\Delta}}{2},$$

and

$$d_{10} = \frac{a - c - d + e + \sqrt{\Delta}}{2d}, \quad d_{01} = -\frac{-a + c + d - e + \sqrt{\Delta}}{2}.$$

Substituting $b = b^*$ into system (14), and applying a linear change of variables together with a time rescaling, the linear part is put into its real Jordan normal form. More precisely, we obtain

$$\begin{aligned} \frac{du}{d\tau} &= -v, \\ \frac{dv}{d\tau} &= u + b_{20}u^2 + b_{11}uv + b_{02}v^2, \end{aligned} \tag{15}$$

where

$$b_{20} = \frac{d(-a + c + d - e + \sqrt{\Delta})}{2\sqrt{\Delta}(d + \sqrt{\Delta})}, \quad b_{02} = \frac{-a + c + d - e + \sqrt{\Delta}}{2(d + \sqrt{\Delta})},$$

and

$$b_{11} = \frac{(a - c + e)(-a + c + d - e + \sqrt{\Delta})}{\sqrt{2}(d + \sqrt{\Delta})\sqrt{(a - c - d - e)\sqrt{\Delta} - \Delta}}.$$

By Theorem 5, the first Liapunov constant of (15) is

$$W_1 = -(b_{20} + b_{02})b_{11}.$$

Since

$$-a + c + d - e + \sqrt{\Delta} > 0,$$

we have

$$b_{20} > 0, \quad b_{02} > 0.$$

Moreover, $b = b^* > 0$ implies $a \neq c - e$, and hence $b_{11} \neq 0$. Therefore

$$W_1 \neq 0.$$

Thus the origin of system (15) is a first-order weak focus, not a center.

Since

$$\text{sgn}(W_1) = -\text{sgn}(a - c + e),$$

the origin of system (15) is unstable when $a < c - e$, and stable when $a > c - e$. The time rescaling used to obtain (15) reverses the orientation of time. Therefore the stability of E_2 in the original system is the opposite one. Hence E_2 is a stable weak focus when $a < c - e$, and an unstable weak focus when $a > c - e$. ■

When $\Delta = 0$ system (2) has a unique finite equilibrium point, namely,

$$E_0 = \left(\frac{a + c + d + e}{2b}, \frac{a - c - d + e}{2d} \right).$$

Solving $\Delta = 0$ for a , one obtains either

$$a = a_1 := c + d - e + 2\sqrt{-de},$$

or

$$a = a_2 := c + d - e - 2\sqrt{-de}.$$

Obviously, $a_1 > 0$ and $a_2 \geq c + 2\sqrt{-de} - 2\sqrt{-de} \geq c > 0$.

If $a = a_1$, then

$$E_0 = E_0^+ := \left(\frac{c + d + \sqrt{-de}}{b}, \frac{\sqrt{-de}}{d} \right),$$

is an equilibrium point in the first quadrant.

The Jacobian matrix of system (2) at E_0^+ is

$$J(E_0^+) = \begin{pmatrix} -\frac{b}{d}(d + \sqrt{-de}) & -d - \sqrt{-de} \\ \frac{-be}{\sqrt{-de}} & \sqrt{-de} \end{pmatrix}.$$

We have $\det(J(E_0^+)) = 0$ and $\text{tr}(J(E_0^+)) = \sqrt{-de} - b(d + \sqrt{-de})/d$. If we take

$$b = b_1 := \frac{d\sqrt{-de}}{d + \sqrt{-de}}, \quad (16)$$

then $\text{tr}(J(E_0^+)) = 0$.

If $a = a_2$, then

$$E_0 = E_0^- := \left(\frac{c + d - \sqrt{-de}}{b}, -\frac{\sqrt{-de}}{d} \right),$$

is an equilibrium point in the half-plane $y \leq 0$. The Jacobian matrix of system (2) at E_0^- is

$$J(E_0^-) = \begin{pmatrix} -\frac{b}{d}(d - \sqrt{-de}) & -d + \sqrt{-de} \\ \frac{be}{\sqrt{-de}} & -\sqrt{-de} \end{pmatrix}.$$

We have $\det(J(E_0^-)) = 0$ and $\text{tr}(J(E_0^-)) = (b - d)\sqrt{-de}/d - b$. If we take

$$b = b_2 := \frac{d\sqrt{-de}}{\sqrt{-de} - d}, \quad (17)$$

then $\text{tr}(J(E_0^-)) = 0$. Since $b > 0$ we get that $-e > d > 0$. This implies that $0 < d < -e$. It is easy to obtain that if $0 < -e \leq d$, then $\text{tr}(J(E_0^-)) < 0$. So the eigenvalues of E_0 are zero and a negative real number.

The following two propositions give the details of the properties of the equilibrium points of system (2) when $e < 0$ and $\Delta = 0$.

Proposition 4. *When $e < 0$, $\Delta = 0$ and $a = a_1$, the unique equilibrium point E_0 of system (2) is in the first quadrant. Moreover,*

- (a) *if $b = b_1$, then E_0 is a nilpotent cusp;*
- (b) *if $b \neq b_1$, then E_0 is a semi-hyperbolic saddle-node.*

Proof. Substituting $a = a_1$ into system (2) and performing the translation

$$x_1 = x - \frac{c + d + \sqrt{-de}}{b}, \quad y_1 = y - \frac{\sqrt{-de}}{d},$$

we bring E_0 to the origin. Then system (2) becomes

$$\begin{aligned} \dot{x}_1 &= -\frac{b(d + \sqrt{-de})}{d}x_1 - (d + \sqrt{-de})y_1 - bx_1y_1, \\ \dot{y}_1 &= \frac{b\sqrt{-de}}{d}x_1 + \sqrt{-de}y_1 + bx_1y_1. \end{aligned} \tag{18}$$

(a) Assume first that $b = b_1$. Then (18) becomes

$$\begin{aligned} \dot{x}_1 &= -\sqrt{-de}x_1 - (d + \sqrt{-de})y_1 - \frac{d\sqrt{-de}}{d + \sqrt{-de}}x_1y_1, \\ \dot{y}_1 &= \frac{-de}{d + \sqrt{-de}}x_1 + \sqrt{-de}y_1 + \frac{d\sqrt{-de}}{d + \sqrt{-de}}x_1y_1. \end{aligned} \tag{19}$$

Taking

$$X = x_1, \quad Y = -\frac{b(d + \sqrt{-de})}{d}x_1 - (d + \sqrt{-de})y_1,$$

and using $b = b_1$, we obtain

$$\begin{aligned} \dot{X} &= Y + \alpha_1 XY + \alpha_2 X^2 := Y + P_3(X, Y), \\ \dot{Y} &= \beta_1 XY + \beta_2 X^2 := Q_3(X, Y), \end{aligned}$$

where

$$\alpha_1 = \frac{d\sqrt{-de}}{(d + \sqrt{-de})^2}, \quad \alpha_2 = \frac{-d^2e}{(d + \sqrt{-de})^2},$$

and

$$\beta_1 = \frac{d^2\sqrt{-de}}{(d + \sqrt{-de})^2}, \quad \beta_2 = \frac{-d^3e}{(d + \sqrt{-de})^2}.$$

From $Y + P_3(X, Y) = 0$, we obtain an analytic function $Y = \phi(X)$, where

$$\phi(X) = -\alpha_2 X^2 + O(X^3).$$

Moreover,

$$Q_3(X, \phi(X)) = \beta_2 X^2 + O(X^3),$$

and

$$\left(\frac{\partial P_3}{\partial X} + \frac{\partial Q_3}{\partial Y} \right) (X, \phi(X)) = (2\alpha_2 + \beta_1)X + O(X^2).$$

Since $\beta_2 > 0$ and

$$2\alpha_2 + \beta_1 = \frac{d\sqrt{-de}(d + 2\sqrt{-de})}{(d + \sqrt{-de})^2} > 0,$$

by Theorem 2, the origin is a nilpotent cusp. Therefore E_0 is a nilpotent cusp.

(b) Now assume that $b \neq b_1$. Since $\Delta = 0$, the determinant of the Jacobian matrix at E_0 is zero. On the other hand, $b \neq b_1$ implies

$$\text{tr}(J_+(E_0)) \neq 0.$$

Thus E_0 has one zero eigenvalue and one non-zero eigenvalue. More precisely, if $b > b_1$, then

$$\text{tr}(J_+(E_0)) < 0,$$

and the non-zero eigenvalue is negative; if $0 < b < b_1$, then

$$\text{tr}(J_+(E_0)) > 0,$$

and the non-zero eigenvalue is positive. Hence E_0 is a semi-hyperbolic singular point.

To determine its local phase portrait, we use (18) and perform the linear change of variables

$$x_1 = \frac{-de}{be - de - b\sqrt{-de}}u + \frac{b(e - \sqrt{-de})}{be - de - b\sqrt{-de}}v, \quad y_1 = \frac{be}{be - de - b\sqrt{-de}}(u - v).$$

This transformation is well defined because

$$be - de - b\sqrt{-de} \neq 0$$

when $b \neq b_1$. In the new coordinates, system (18) becomes

$$\begin{aligned} \frac{du}{dt} &= \frac{b^2 de \sqrt{-de}}{(be - de - b\sqrt{-de})^2} u^2 - \frac{b^2 e (bd + (b + d)\sqrt{-de})}{(be - de - b\sqrt{-de})^2} uv \\ &\quad + \frac{b^3 e (d + \sqrt{-de})}{(be - de - b\sqrt{-de})^2} v^2, \\ \frac{dv}{dt} &= \left(\frac{d - b}{d} \sqrt{-de} - b \right) v + \frac{bde^2(b - d)}{(be - de - b\sqrt{-de})^2} u^2 \\ &\quad - \frac{be(b - d)(de + be - b\sqrt{-de})}{(be - de - b\sqrt{-de})^2} uv + \frac{b^2 e (b - d)(e - \sqrt{-de})}{(be - de - b\sqrt{-de})^2} v^2. \end{aligned} \tag{20}$$

Since

$$\lambda_1 := \frac{d - b}{d} \sqrt{-de} - b \neq 0,$$

we introduce the constant change of time

$$d\tau = \lambda_1 dt. \tag{21}$$

Then (20) becomes

$$\begin{aligned} \frac{du}{d\tau} &= \gamma_1 u^2 + \gamma_2 uv + \gamma_3 v^2, \\ \frac{dv}{d\tau} &= v + \delta_1 u^2 + \delta_2 uv + \delta_3 v^2, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \gamma_1 &= bd\sqrt{-de}K, & \gamma_2 &= -b(bd + (b + d)\sqrt{-de})K, & \gamma_3 &= -b^2(d + \sqrt{-de})K, \\ \delta_1 &= de(b - d)K, & \delta_2 &= (b - d)(-de - be + b\sqrt{-de})K, & \delta_3 &= b(e - \sqrt{-de})(b - d)K, \end{aligned}$$

and

$$K = \frac{-bde}{(bd + (b-d)\sqrt{-de})(be - de - b\sqrt{-de})^2}.$$

Since $e < 0$, we have $-bde > 0$. Hence the sign of K is determined by the sign of $bd + (b-d)\sqrt{-de}$. Therefore

$$K < 0 \quad \text{if } 0 < b < b_1, \quad K > 0 \quad \text{if } b > b_1.$$

Consequently,

$$\gamma_1 < 0 \quad \text{if } 0 < b < b_1, \quad \gamma_1 > 0 \quad \text{if } b > b_1.$$

Assume first that $b \neq d$. From the second equation of (22), there exists an analytic function $v = f_1(u)$ such that

$$v = f_1(u) = -\delta_1 u^2 + \left(\delta_1 \delta_2 - \frac{3}{2} \delta_1^2 \delta_3 \right) u^3 + O(u^4).$$

Substituting this expression into the first equation of (22), we obtain the reduced differential equation

$$\frac{du}{d\tau} = \gamma_1 u^2 + O(u^3).$$

Since $\gamma_1 \neq 0$, it follows from Subsection 2.2 that E_0 is a semi-hyperbolic saddle-node. Moreover, in the (u, v) -coordinates, the parabolic side lies on the side $u > 0$ if $b > b_1$, and on the side $u < 0$ if $0 < b < b_1$. If $\lambda_1 < 0$, equivalently $b > b_1$, the time change (21) reverses the orientation of the orbits, but it does not change the position of the parabolic and hyperbolic sectors.

It remains to consider the case $b = d$. Since $b_1 < d$, this case belongs to $b > b_1$. In this case, (22) reduces to

$$\begin{aligned} \frac{du}{d\tau} &= \sqrt{-de} u^2 - (d + 2\sqrt{-de})uv - (d + \sqrt{-de})v^2, \\ \frac{dv}{d\tau} &= v. \end{aligned} \tag{23}$$

Thus the associated reduced differential equation is

$$\frac{du}{d\tau} = \sqrt{-de} u^2.$$

Since $\sqrt{-de} > 0$, the origin is again a semi-hyperbolic saddle-node, and the parabolic side lies on the side $u > 0$. Since in this case $d\tau = -d dt$, the orientation of the orbits is reversed in the original time t . This completes the proof. $\blacksquare$

When $a = a_2$ we have the similar results as in Proposition 4.

Proposition 5. *When $e < 0$, $\Delta = 0$ and $a = a_2$, the unique equilibrium point E_0 of system (2) is in the third or fourth quadrant. Moreover,*

- (a) *if $b = b_2 > 0$, then E_0 is a nilpotent cusp;*
- (b) *if $b \neq b_2$, then E_0 is a semi-hyperbolic saddle-node.*

Proof. Substituting $a = a_2$ into system (2) and performing the translation

$$x_1 = x - \frac{c + d - \sqrt{-de}}{b}, \quad y_1 = y + \frac{\sqrt{-de}}{d},$$

we bring E_0 to the origin. Then system (2) becomes

$$\begin{aligned} \dot{x}_1 &= \frac{b(\sqrt{-de} - d)}{d} x_1 + (\sqrt{-de} - d) y_1 - b x_1 y_1, \\ \dot{y}_1 &= \frac{be}{\sqrt{-de}} x_1 - \sqrt{-de} y_1 + b x_1 y_1. \end{aligned} \tag{24}$$

(a) Assume first that $b = b_2 > 0$. Then $b_2 > 0$ implies $-e > d$, and hence $\sqrt{-de} - d > 0$. Taking

$$X = x_1, \quad Y = \frac{b(\sqrt{-de} - d)}{d}x_1 + (\sqrt{-de} - d)y_1,$$

and using $b = b_2$, system (24) is transformed into

$$\begin{aligned} \dot{X} &= Y + \tilde{\alpha}_1 XY + \tilde{\alpha}_2 X^2 := Y + \tilde{P}_3(X, Y), \\ \dot{Y} &= \tilde{\beta}_1 XY + \tilde{\beta}_2 X^2 := \tilde{Q}_3(X, Y), \end{aligned}$$

where

$$\tilde{\alpha}_1 = \sqrt{-de} h, \quad \tilde{\alpha}_2 = de h, \quad \tilde{\beta}_1 = d\sqrt{-de} h, \quad \tilde{\beta}_2 = d^2 e h, \quad h = -\frac{1}{d - e - 2\sqrt{-de}}.$$

Since

$$d - e - 2\sqrt{-de} = (\sqrt{d} - \sqrt{-e})^2 > 0,$$

we have $h < 0$, and therefore

$$\tilde{\beta}_2 = d^2 e h > 0.$$

From $Y + \tilde{P}_3(X, Y) = 0$, we obtain an analytic function $Y = \psi(X)$ satisfying

$$\psi(X) = -\tilde{\alpha}_2 X^2 + O(X^3).$$

Hence

$$\tilde{Q}_3(X, \psi(X)) = \tilde{\beta}_2 X^2 + O(X^3), \quad \tilde{\beta}_2 > 0.$$

Moreover,

$$\left(\frac{\partial \tilde{P}_3}{\partial X} + \frac{\partial \tilde{Q}_3}{\partial Y} \right) (X, \psi(X)) = (2\tilde{\alpha}_2 + \tilde{\beta}_1)X + O(X^2).$$

Since

$$2\tilde{\alpha}_2 + \tilde{\beta}_1 = dh(2e + \sqrt{-de}),$$

and $-e > d > 0$ implies $2e + \sqrt{-de} < 0$, we get

$$2\tilde{\alpha}_2 + \tilde{\beta}_1 > 0.$$

Thus the hypotheses of Theorem 2 are satisfied. Therefore the origin is a nilpotent cusp, and consequently E_0 is a nilpotent cusp.

(b) Now assume that $b \neq b_2$. Since $\Delta = 0$, the determinant of the Jacobian matrix at E_0 is zero. Moreover, under the present assumption one has

$$\text{tr}(J_-(E_0)) \neq 0.$$

Thus E_0 has one zero eigenvalue and one non-zero eigenvalue, and hence E_0 is a semi-hyperbolic singular point.

To determine its local phase portrait, we use (24) and perform the linear change of variables

$$x_1 = \frac{-de}{be - de + b\sqrt{-de}}u + \frac{b(e + \sqrt{-de})}{be - de + b\sqrt{-de}}v, \quad y_1 = \frac{be}{be - de + b\sqrt{-de}}(u - v).$$

This transformation is well defined because

$$be - de + b\sqrt{-de} \neq 0$$

when $b \neq b_2$. In the new coordinates, system (24) becomes

$$\begin{aligned}\frac{du}{dt} &= bH \left(-d\sqrt{-de}u^2 - (bd - (b+d)\sqrt{-de})uv + b(d - \sqrt{-de})v^2 \right), \\ \frac{dv}{dt} &= - \left(b + \frac{d-b}{d}\sqrt{-de} \right) v \\ &\quad + (b-d)H \left(de u^2 - (be + de + b\sqrt{-de})uv + b(e + \sqrt{-de})v^2 \right),\end{aligned}\tag{25}$$

where

$$H = \frac{be}{(be - de + b\sqrt{-de})^2} \neq 0.$$

Since $b \neq b_2$, we have

$$\lambda_2 := - \left(b + \frac{d-b}{d}\sqrt{-de} \right) = - \left(b + \left(1 - \frac{b}{d} \right) \sqrt{-de} \right) \neq 0.$$

We introduce the constant change of time

$$d\tau = \lambda_2 dt.\tag{26}$$

Then (25) becomes

$$\begin{aligned}\frac{du}{d\tau} &= \mu_1 u^2 + \mu_2 uv + \mu_3 v^2, \\ \frac{dv}{d\tau} &= v + \eta_1 u^2 + \eta_2 uv + \eta_3 v^2,\end{aligned}\tag{27}$$

where

$$\begin{aligned}\mu_1 &= bd^2 eL, & \mu_2 &= -bd(be + de + b\sqrt{-de})L, & \mu_3 &= b^2 d(e + \sqrt{-de})L, \\ \eta_1 &= -(-de)^{3/2}(b-d)L, & \eta_2 &= \sqrt{-de}(b-d)(be - de - b\sqrt{-de})L, \\ \eta_3 &= -be(b-d)(d - \sqrt{-de})L, & L &= \frac{-be}{(be - de + b\sqrt{-de})^3}.\end{aligned}$$

We now determine the sign of the leading coefficient μ_1 . If $0 < -e \leq d$, then

$$be - de + b\sqrt{-de} = -de + b\sqrt{-e}(\sqrt{d} - \sqrt{-e}) > 0.$$

Thus $L > 0$, and since $e < 0$, we have

$$\mu_1 < 0.$$

If $0 < d < -e$, then the sign of $be - de + b\sqrt{-de}$ changes at $b = b_2$. More precisely,

$$L > 0 \quad \text{if } b < b_2, \quad L < 0 \quad \text{if } b > b_2.$$

Consequently,

$$\mu_1 < 0 \quad \text{if } 0 < -e \leq d \quad \text{or} \quad 0 < d < -e, \quad b < b_2,$$

whereas

$$\mu_1 > 0 \quad \text{if } 0 < d < -e, \quad b > b_2.$$

Assume first that $b \neq d$. From the second equation of (27), there exists an analytic function $v = f_2(u)$ such that

$$v = f_2(u) = -\eta_1 u^2 + O(u^3).$$

Substituting this expression into the first equation of (27), we obtain the reduced differential equation

$$\frac{du}{d\tau} = \mu_1 u^2 + O(u^3).$$

Since $\mu_1 \neq 0$, it follows from Subsection 2.2 that E_0 is a semi-hyperbolic saddle-node. Moreover, in the (u, v) -coordinates, the parabolic side lies on the side $u < 0$ if

$$0 < -e \leq d \quad \text{or} \quad 0 < d < -e, \quad b < b_2,$$

and it lies on the side $u > 0$ if

$$0 < d < -e, \quad b > b_2.$$

If $\lambda_2 < 0$, equivalently if

$$0 < -e \leq d \quad \text{or} \quad 0 < d < -e, \quad b < b_2,$$

then the time change (26) reverses the orientation of the orbits, but it does not change the position of the parabolic and hyperbolic sectors.

It remains to consider the case $b = d$. In this case, $\lambda_2 = -d < 0$, and (27) reduces to

$$\begin{aligned} \frac{du}{d\tau} &= \frac{e}{\sqrt{-de}} u^2 - \left(\frac{2e}{\sqrt{-de}} + 1 \right) uv + \left(\frac{e}{\sqrt{-de}} + 1 \right) v^2, \\ \frac{dv}{d\tau} &= v. \end{aligned} \tag{28}$$

Thus the associated reduced differential equation is

$$\frac{du}{d\tau} = \frac{e}{\sqrt{-de}} u^2.$$

Since $e < 0$, we have $e/\sqrt{-de} < 0$. Hence the origin is again a semi-hyperbolic saddle-node, and the parabolic side lies on the side $u < 0$. Since $d\tau = -d dt$, the orientation of the orbits is reversed in the original time t . This completes the proof. ■

4. Proof of Theorem 1

We determine the global phase portraits by combining the local phase portraits of the finite equilibrium points, the local phase portraits of the infinite equilibrium points described in Proposition 1, and the possible connections between separatrices. The information on periodic orbits obtained below will be used together with the Poincaré–Bendixson theorem to determine the possible α - and ω -limit sets of the separatrices.

On the x -axis we have

$$\dot{y}|_{y=0} = e.$$

Hence the vector field is directed upward on the x -axis when $e > 0$, and downward when $e < 0$. In particular, the half-plane $y > 0$ is positively invariant for $e > 0$, whereas the half-plane $y < 0$ is positively invariant for $e < 0$.

On the y -axis we have

$$\dot{x}|_{x=0} = a + cy.$$

Let $y_0 = -a/c$. Then $\dot{x} > 0$ for $y > y_0$, and $\dot{x} < 0$ for $y < y_0$. Thus, on the y -axis, the vector field is directed to the right for $y > y_0$, and to the left for $y < y_0$.

We shall repeatedly use the following standard separatrix-continuation argument. Consider a one-parameter family of vector fields X_μ , where the local phase portraits of the singular points involved do not change for $\mu \in [\mu_1, \mu_2]$. Let $\Gamma_1(\mu)$ and $\Gamma_2(\mu)$ be two separatrix branches depending continuously on μ . Choose a small transverse section Σ , with a coordinate s , which is crossed by both separatrices for all $\mu \in [\mu_1, \mu_2]$. Define

$$H(\mu) = s(\Gamma_1(\mu) \cap \Sigma) - s(\Gamma_2(\mu) \cap \Sigma).$$

By the continuous dependence of solutions on parameters, H is continuous. If $H(\mu_1)H(\mu_2) < 0$, then there exists $\mu_0 \in (\mu_1, \mu_2)$ such that $H(\mu_0) = 0$. Hence the two separatrices meet on Σ , and by uniqueness of solutions they coincide. This gives the required separatrix connection.

4.1. Phase Portrait of System (2) with $e > 0$

From Theorems 3, 4 and the local phase portraits of the equilibrium points of system (2) when $e > 0$, the following lemma shows that there are no limit cycles.

Lemma 1. *System (2) has no limit cycles when $e > 0$.*

Proof. Since

$$\dot{y}|_{y=0} = e > 0,$$

a periodic orbit cannot intersect the x -axis. Hence any periodic orbit must be entirely contained either in $y > 0$ or in $y < 0$.

In the half-plane $y < 0$, the only finite equilibrium point is the saddle E_1 . Hence no periodic orbit can lie in $y < 0$, because by Theorem 3 the interior of a closed path must contain a focus or a center.

It remains to consider the half-plane $y > 0$. Taking the Dulac function

$$B(x, y) = \frac{1}{y},$$

we obtain

$$\frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = -\frac{b}{y} - b - \frac{e}{y^2} < 0, \quad y > 0.$$

Therefore, by the Bendixson–Dulac criterion, there is no periodic orbit in $y > 0$. Thus system (2) has no limit cycles when $e > 0$. ■

Since the half-plane $y > 0$ is positively invariant and there are no limit cycles, every non-stationary orbit in $y > 0$, except the separatrices already specified, has ω -limit E_2 . In particular, the unstable separatrix of D_3 entering this region has ω -limit E_2 . Thus the unstable separatrix for D_3 ends at E_2 . The unstable separatrix of D'_2 (see Fig. 3) due to the fact that the half-plane $y > 0$ is positively invariant ends at E_2 . Since there are no closed paths surrounding E_2 , the saddle E_1 cannot have homoclinic loop, because inside a homoclinic loop it must be a focus (see Theorem 3). So one unstable separatrix of E_1 must go to E_2 , and the other unstable separatrix can only go to D_1 . Finally the stable separatrices of E_1 can only start at D'_3 and D_2 , respectively.

Taking into account the local phase portraits of the finite and infinite equilibrium points we obtain the phase portrait of system (2) in the Poincaré disc described in Fig. 3.

4.2. Phase Portraits of System (2) with $e < 0$

The following lemma gives the information about the limit cycles of system (2) when $e < 0$.

Lemma 2. *For the case $\Delta \leq 0$ system (2) with $e < 0$ has no limit cycles and for the case $\Delta > 0$ it has at most one limit cycle.*

Proof. When $e < 0$ system (2) has no limit cycles for the case $\Delta < 0$, because there are no finite equilibrium points, see Theorem 3.

For $\Delta = 0$, due to Propositions 4 and 5, the unique finite equilibrium point is saddle-node or cusp. According to Theorem 3, the equilibrium point in the interior of a limit cycle must be either a focus or a center. Thus there are no limit cycles for $\Delta = 0$.

For $\Delta > 0$, since the highest degree terms of $P(x, y)$ and $Q(x, y)$ are proportional because they are $-bxy$ and bxy , respectively, due to Theorem 4, system (2) has at most one limit cycle with a non-zero characteristic exponent, that is, when this limit cycle exists, then it is stable or unstable. ■

In the following we will consider the phase portraits of system (2) with $e < 0$ in the three cases $\Delta < 0$, $\Delta = 0$ and $\Delta > 0$.

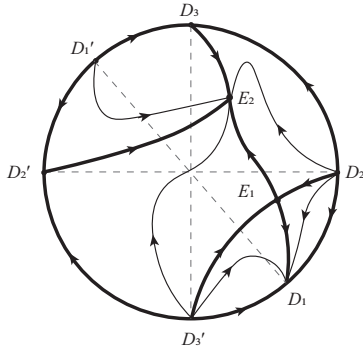


Fig. 3. Phase portrait in the Poincaré disc of system (2) for $e > 0$. The finite equilibrium point E_1 is a hyperbolic saddle in the fourth quadrant, while E_2 is a stable node or focus in the first quadrant. The labelled points on the boundary are the infinite equilibrium points described in Proposition 1. Arrows indicate the positive time direction of the orbits, and the solid curves represent the corresponding separatrices. No limit cycle occurs in this case.

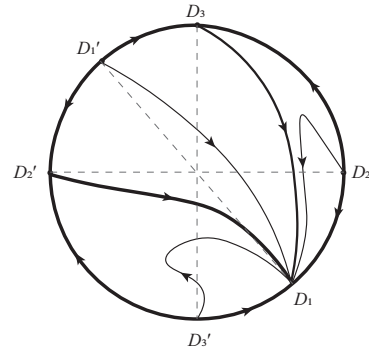


Fig. 4. Phase portrait in the Poincaré disc of system (2) for $e < 0$ and $\Delta < 0$. In this case, system (2) has no finite equilibrium points and no limit cycles. Hence the global configuration is completely determined by the infinite equilibrium points and their separatrices. Arrows indicate the positive time direction of the flow.

When $\Delta < 0$, system (2) has no finite equilibrium points and, by Lemma 2, no limit cycles. Hence, using the Poincaré–Bendixson theorem in the Poincaré disc and the local phase portraits at infinity described in Proposition 1, the non-stationary orbits in the canonical regions determined by the separatrices have ω -limit D_1 . In particular, the separatrix branches from D_2' and D_3 also have ω -limit D_1 . Therefore the phase portrait in the Poincaré disc is the one shown in Fig. 4.

When $\Delta = 0$, the unique finite equilibrium point E_0 may lie either in the first quadrant or in the half-plane $y < 0$, and it can be either a cusp or a saddle-node. So from Lemma 2 there are no limit cycles. Recall that when $\Delta = 0$, we have $a = a_1$ or $a = a_2$.

We first consider the case that $a = a_1$. Now E_0 is in the first quadrant. According to the local phase portraits of the equilibrium points and the behaviour of the vector field on y -axis, the unstable separatrix of D_2' due to the fact that the half-plane $y < 0$ is positively invariant ends at D_1 . Thus, in the corresponding region, the non-separatrix orbits have α -limit D_3' and ω -limit D_1 .

When $b \neq b_1$ then E_0 is a saddle-node. According to the local phase portraits of the equilibrium points, the possible ω -limit sets of the non-stationary orbits in the corresponding regions are D_1 and the stable set of E_0 . Thus the unstable separatrix of E_0 must end at D_1 .

For the case $b > b_1$, E_0 is a saddle-node with its stable parabolic sector on the right and its unique unstable separatrix on the left. Moreover, its unique unstable separatrix ends at D_1 . And its stable separatrices may start at D_2, D_3 or D_1' .

We claim that one of the stable separatrices of E_0 must start at D_2 . Consider the straight line

$$L_1 : y = y_{E_0^+} := \sqrt{-de}/d,$$

that connects E_0 and the infinite equilibrium points D_2 and D_2' . According to Theorem 6, the flow goes in different directions on each half-line of L_1 separated by E_0^+ . Consequently, it is impossible that two stable separatrices of E_0 start at D_1' or one of them starts at D_1' and another starts at D_3 . The claim is proved. Thus, the other stable separatrix of E_0 may start at D_2, D_3 or D_1' .

If the other stable separatrix of E_0 starts at D_1' , then the unstable separatrix of D_3 ends at E_0 . The behaviour of the remaining trajectories in this case is easy to determine by the local phase portraits of the equilibrium points. We describe the unique possible phase portrait in the Poincaré disc in Fig. 5(a). Taking the parameters $(a, b, c, d, e) = (5, 0.8, 1, 1, -1)$ in system (2) we obtain the phase portrait of Fig. 5(a).

If the other stable separatrix of E_0 starts at D_2 , then the unique unstable separatrix of D_3 must end at D_2 , as well as the other trajectories starting at D_2 and the trajectories starting at D_1' . The behaviour of the

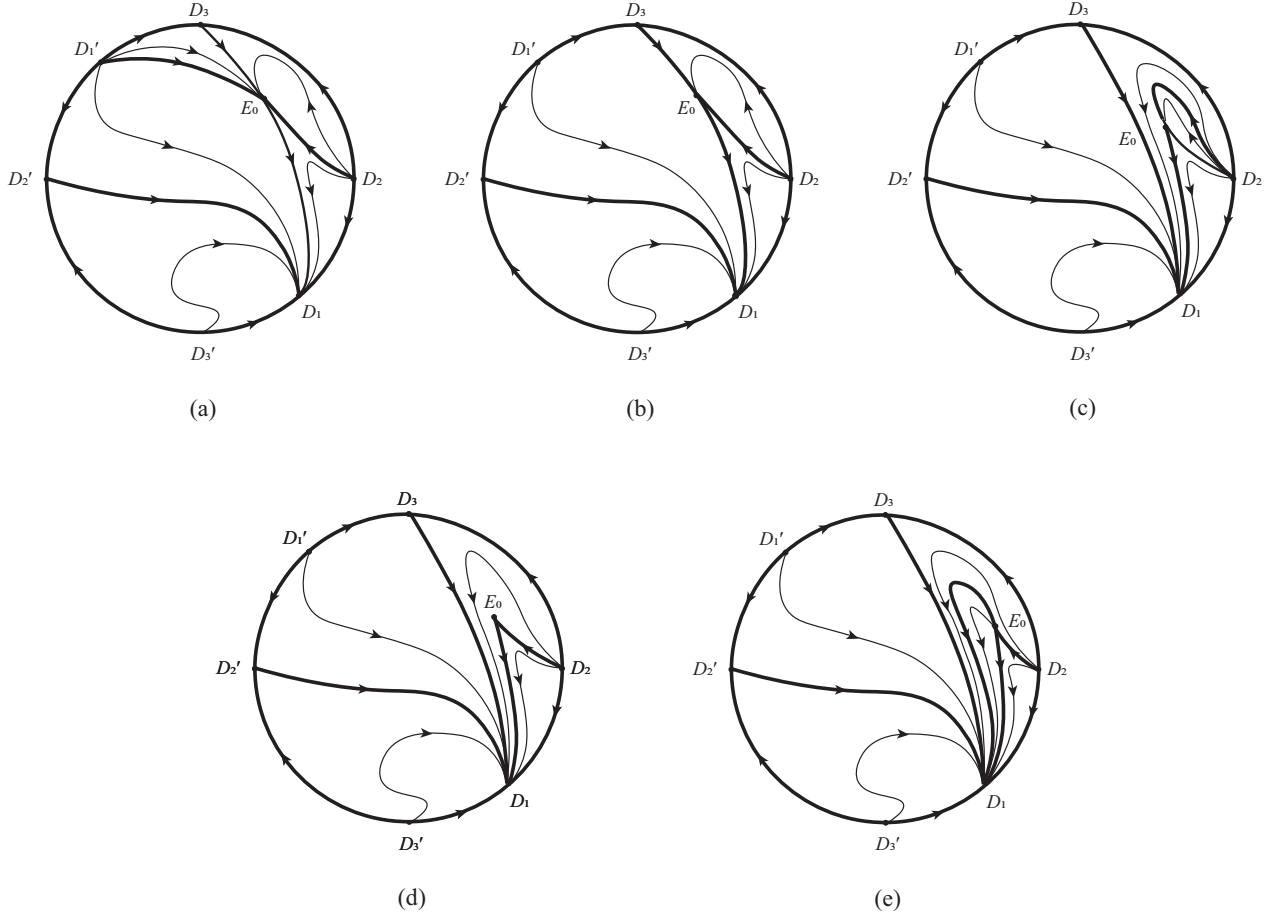


Fig. 5. Phase portraits in the Poincaré disc of system (2) for $e < 0$, $\Delta = 0$ and $a = a_1$. The unique finite equilibrium point E_0 lies in the first quadrant. The portraits (a)–(c) correspond to $b > b_1$, with (b) representing the intermediate separatrix connection; (d) corresponds to the nilpotent cusp case $b = b_1$; and (e) corresponds to $0 < b < b_1$. Arrows indicate the positive time direction and the solid curves represent separatrices.

remaining trajectories in this case is easy to determine by taking into account the local phase portraits of the equilibrium points. This case is shown in Fig. 5(c). Taking the parameters $(a, b, c, d, e) = (5, 0.7, 1, 1, -1)$ in system (2), we can obtain the phase portrait in the Poincaré disc in this case.

The phase portraits in Fig. 5(a) and Fig. 5(c) are realized, for instance, by the parameter values

$$(a, b, c, d, e) = (5, 0.8, 1, 1, -1) \quad \text{and} \quad (a, b, c, d, e) = (5, 0.7, 1, 1, -1),$$

respectively. We now let b vary in the interval $[0.7, 0.8]$, keeping the other parameters fixed. Let $\gamma_s(b)$ denote the corresponding stable separatrix of E_0 , traced backward, and let $\gamma_3(b)$ denote the unstable separatrix of D_3 . Choose a small transverse section Σ in the region between the two configurations. In Fig. 5(c), $\gamma_s(b)$ lies on one side of $\gamma_3(b)$ on Σ , whereas in Fig. 5(a), it lies on the other side. Therefore the signed distance between these two separatrices on Σ changes sign. By the argument above, there exists $b_0 \in (0.7, 0.8)$ such that $\gamma_s(b_0)$ coincides with $\gamma_3(b_0)$. This gives the separatrix connection shown in Fig. 5(b).

For the case $b < b_1$ we have that E_0 is a saddle-node with its unstable parabolic sector on the left and its unique stable separatrix on the right. Thus the unstable separatrices of E_0 , and the non-separatrix orbits in the corresponding unstable sector, have ω -limit D_1 . For the stable separatrix of E_0 , it may start at D_2 , D_3 or D_1' . We claim that it can only start at D_2 . Otherwise, if the stable separatrix of E_0 starts at D_3 or D_1' , then the flow on the straight line L_1 goes in the same direction, specifically upward and this is a contradiction with Theorem 6. In this case the unstable separatrix of D_3 ends at D_1 . The phase portrait in the Poincaré disc of this case is shown in Fig. 5(e).

When $b = b_1$, E_0 is a cusp and has two separatrices, one stable and another unstable. The unstable one must end at D_1 , while the stable one may start at D_2, D_3 or D'_1 . For the same reason as in the case $b < b_1$, the stable separatrix must start at D_2 . Thus the unstable separatrix of D_3 ends at D_1 . See in Fig. 5(d) for the phase portrait in the Poincaré disc of this case. The phase portrait can be realized, for instance, for

$$(a, b, c, d, e) = (5, 0.5, 1, 1, -1).$$

Next we consider the case $a = a_2$. In this case E_0 is in the half-plane $y < 0$ and it can be a saddle-node or a cusp. According to the local phase portraits of the equilibrium points, the non-stationary orbits in the corresponding canonical regions have ω -limit either D_1 or a point on the stable set of E_0 .

Consider the straight line

$$L_2 : x = x_{E_0^-} := (c + d - \sqrt{-de})/b,$$

that connects E_0 with the infinite equilibrium points D_3 and D'_3 , and the straight line

$$L_3 : y = y_{E_0^-} := -\sqrt{-de}/d,$$

that connects E_0 with the infinite equilibrium points D_2 and D'_2 . Note that

$$\dot{x}|_{L_2} = -e - \sqrt{-de} - (d - \sqrt{-de})y, \quad \dot{y}|_{L_2} = -e - \sqrt{-de}y,$$

and

$$\dot{x}|_{L_3} = c + d - e - \frac{c + 2d}{d}\sqrt{-de} + \frac{b\sqrt{-de} - bd}{d}x, \quad \dot{y}|_{L_3} = e + \frac{c + d}{d}\sqrt{-de} - \frac{b\sqrt{-de}}{d}x.$$

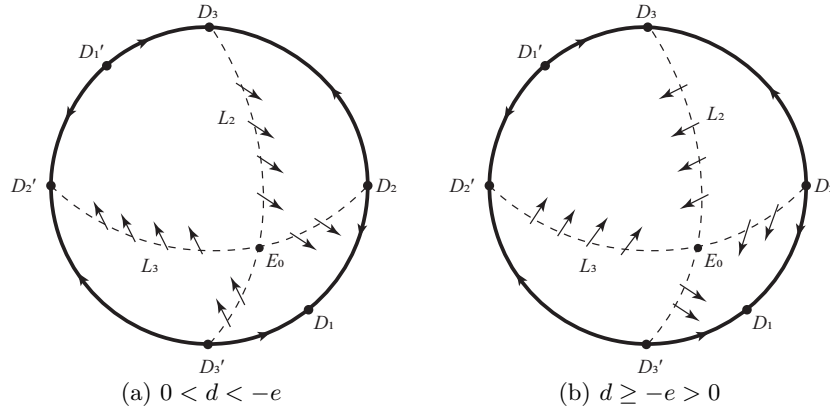


Fig. 6. Direction of the vector field on the straight lines L_2 and L_3 for system (2) in the case $e < 0$, $\Delta = 0$ and $a = a_2$. The lines L_2 and L_3 pass through the degenerate finite equilibrium point E_0 and connect it with pairs of infinite equilibrium points. The arrows on these lines are used to determine the possible separatrix connections in the proof of the phase portraits shown in Fig. 7. Panel (a) corresponds to $0 < d < -e$, while panel (b) corresponds to $d \geq -e > 0$.

When $y > y_{E_0^-}$ then we can obtain $\dot{y}|_{L_2} < 0$, $\dot{x}|_{L_2} > 0$ for $0 < d < -e$, and $\dot{x}|_{L_2} \leq 0$ for $0 < -e \leq d$ with equality if and only if $d = -e$. When $x > x_{E_0^-}$ then we can obtain $\dot{y}|_{L_3} < 0$, $\dot{x}|_{L_3} > 0$ for $0 < d < -e$, and $\dot{x}|_{L_3} \leq 0$ for $0 < -e \leq d$ with equality if and only if $d = -e$. According to Theorem 6 the flow goes in different directions on each half-line of L_2 or L_3 separated by E_0 . So we can obtain the direction of vector field on the line L_2 and L_3 except at the point E_0 , that is shown in Fig. 6.

For $0 < d < -e$ the vector field on the lines L_2 and L_3 is shown in Fig. 6(a). So the unstable separatrix of D'_2 must end at E_0 or D_1 passing through L_2^+ .

When $0 < d < -e, b < b_2$ and $d \geq -e > 0$, E_0 is a saddle-node with its stable parabolic sector on the right and its unique unstable separatrix on the left.

For the case $0 < d < -e$ and $b < b_2$ we claim that the unstable separatrix of D_3 ends at D_1 . Otherwise, it ends at E_0 . Then this unstable separatrix of D_3 is located on the left side of line L_2 . Then there are no trajectories passing through L_2^+ . This is a contradiction.

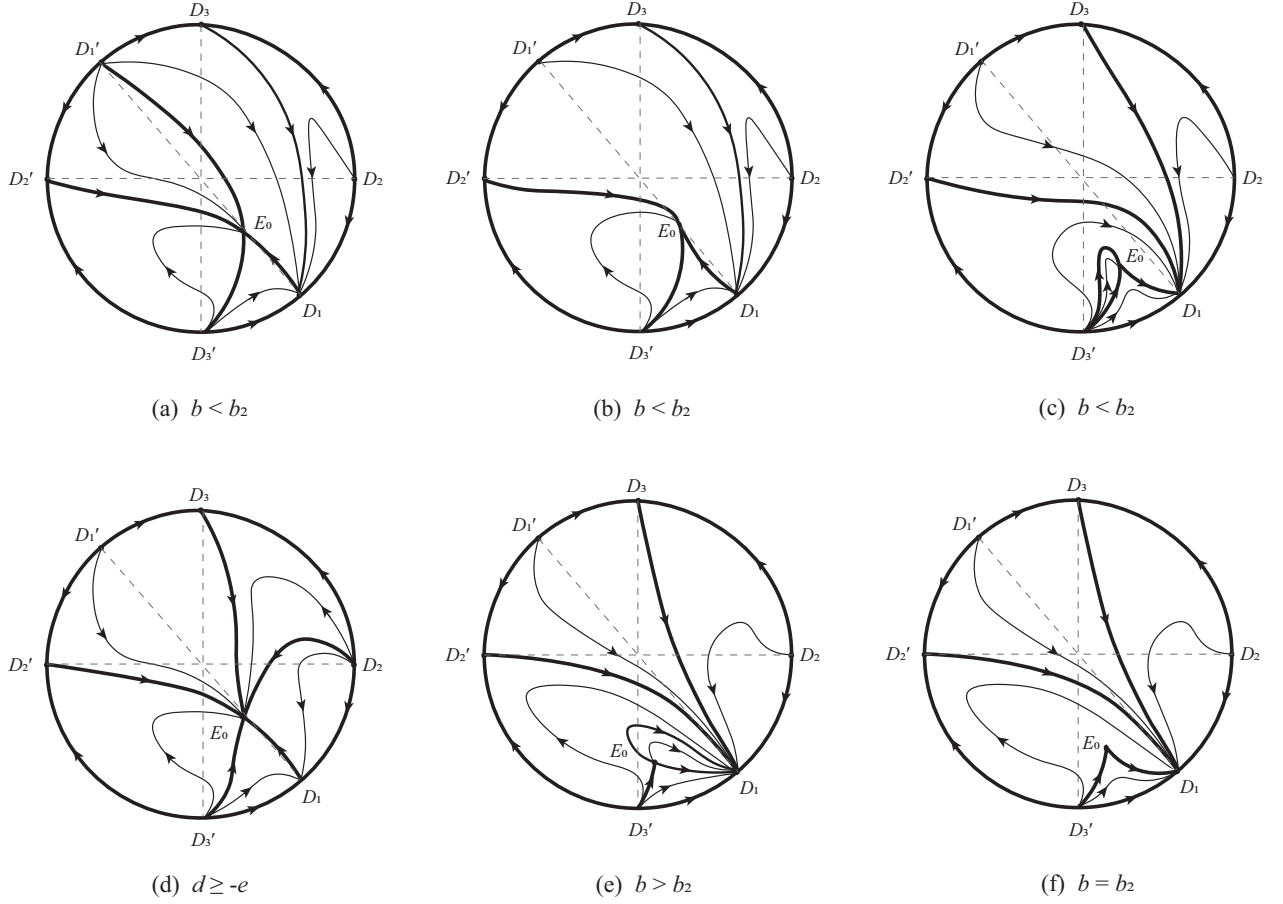


Fig. 7. Phase portraits in the Poincaré disc of system (2) for $e < 0$, $\Delta = 0$ and $a = a_2$. In this case the unique finite equilibrium point E_0 lies in the half-plane $y < 0$. The portraits (a), (b) and (c) correspond to $0 < d < -e$ and $b < b_2$, with (b) representing the intermediate separatrix connection. The portrait (d) corresponds to $d \geq -e > 0$. The portrait (e) corresponds to $0 < d < -e$ and $b > b_2$, while (f) corresponds to the nilpotent cusp case $0 < d < -e$ and $b = b_2$. Arrows indicate the positive time direction; solid curves represent separatrices, and dashed straight lines indicate the auxiliary lines used to determine the separatrix connections.

If the unstable separatrix of D'_2 ends at D_1 passing over E_0 , then the orbits belonging to the corresponding stable sector of E_0 have α -limit D'_3 , particularly its stable separatrix. In this case we obtain the phase portrait in the Poincaré disc shown in Fig. 7(c). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (1.01, 7, 1, 1, -1.21).$$

If the unstable separatrix of D'_2 ends at E_0 , then one of the stable separatrices of E_0 must start at D'_3 . The other stable separatrix of E_0 must start at D'_1 or coincide with the unstable separatrix of D'_2 . If this separatrix starts at D'_1 , we obtain the phase portrait in the Poincaré disc of system (2) shown in Fig. 7(a). This configuration is realized, for instance, for

$$(a, b, c, d, e) = (1.01, 2, 1, 1, -1.21).$$

The phase portraits in Fig. 7(a) and Fig. 7(c) are realized by the parameter values

$$(a, b, c, d, e) = (1.01, 2, 1, 1, -1.21) \quad \text{and} \quad (a, b, c, d, e) = (1.01, 7, 1, 1, -1.21),$$

respectively. Let b vary in the interval $[2, 7]$, with the other parameters fixed. Denote by $\gamma_u(b)$ the unstable separatrix of D'_2 , and by $\gamma_s(b)$ the relevant stable separatrix of E_0 . We choose a transverse section Σ crossed by these two separatrix branches. In the two endpoint configurations, the intersections of $\gamma_u(b)$ and $\gamma_s(b)$ with Σ have opposite order. Equivalently, the signed distance between them changes sign. Hence there

exists $b_0 \in (2, 7)$ for which this signed distance is zero. Therefore $\gamma_u(b_0)$ and $\gamma_s(b_0)$ coincide, giving the separatrix connection shown in Fig. 7(b).

For the case $d \geq -e > 0$, E_0 is in the fourth quadrant and the vector field on the lines L_2 and L_3 is shown in Fig. 6(b). Hence the unstable separatrices of D'_2 and D_3 end at E_0 . Since the unstable separatrix of E_0 ends at D_1 , the non-separatrix orbits in the corresponding regions issuing from D'_3 have ω -limit either E_0 or D_1 . Particularly, there exists an orbit starting at D'_3 being one of the stable separatrices of E_0 . According to the vector field on L_2^+ , there exist trajectories crossing through L_2 and ending at E_0 . According to the local phase portraits of all the equilibrium points, these orbits must start at D_2 . So the other stable separatrix of E_0 starts at D_2 . The remaining orbits in the corresponding region issuing from D_2 have ω -limit D_1 . The phase portrait in the Poincaré disc of this case is shown in Fig. 7(d). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (2, 1, 1, 4, -1).$$

When $0 < d < -e$ and $b > b_2$, E_0 is a saddle-node with its unstable parabolic sector on the right and its unique stable separatrix on the left. We claim that the unstable separatrix of D_3 must end at D_1 . Otherwise, it ends at E_0 . Then the unstable separatrix of D'_2 should end at D_1 passing below E_0 , and this is a contradiction. Thus the stable separatrix of E_0 can start at D_3 or D'_2 .

If the stable separatrix of E_0 starts at D'_2 , then due to the continuous dependence of the orbits with respect to the initial conditions and parameters, there exists a suitable set of parameters (a, b, c, d, e) to ensure that the stable separatrix of E_0 starts at D'_1 . In this case the unstable separatrix of D'_2 will end at D_1 passing below E_0 , this is a contradiction. So the stable separatrix of E_0 starts at D_3 . Thus the remaining orbits in the corresponding canonical regions have ω -limit D_1 , including the unstable separatrix of E_0 and the unstable separatrix of D'_2 . The phase portrait in the Poincaré disc of this case is shown in Fig. 7(e). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (2, 3, 1, 1, -4).$$

When $0 < d < -e$ and $b = b_2$, E_0 is a cusp and has two separatrices, one stable and another unstable. Its unstable separatrix ends at D_1 . We claim that its stable separatrix must start at D'_3 . If it starts at D_3 or D'_1 , then the unstable separatrix of D'_2 will end at D_1 passing below E_0 , this is a contradiction with the vector field on L_2 . If it starts at D'_2 the trajectories starting at D'_3 will end at D_1 . Then some of these trajectories will pass through L_2 from left to right. This is a contradiction with the vector field on L_2^- . Thus the stable separatrix of E_0 starts at D'_3 . Therefore the unstable separatrix of D'_2 ends at D_1 . The phase portrait in the Poincaré disc in this case is shown in Fig. 7(f). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (2, 2, 1, 1, -4).$$

We have described the phase portraits of system (2) in the Poincaré disc for the case $\Delta = 0$ and $a = a_2$ in Fig. 7.

Finally we consider the case $\Delta > 0$. First we introduce a lemma for this case that under certain condition the system can exhibit a Hopf bifurcation.

Lemma 3. *When $\Delta > 0$, the equilibrium point E_2 of system (2) with $e < 0$ can exhibit a degenerate Hopf bifurcation.*

Proof. We first translate E_2 to the origin and obtain system (14). The characteristic polynomial of the linear part of system (14) at the origin is

$$p(\lambda) = \lambda^2 - \text{tr}(J(E_2))\lambda + \det(J(E_2)).$$

Since

$$\det(J(E_2)) = b\sqrt{\Delta} > 0,$$

we have

$$p(\lambda) = \lambda^2 - \text{tr}(J(E_2))\lambda + b\sqrt{\Delta}.$$

For a Hopf bifurcation, we impose

$$\text{tr}(J(E_2)) = 0.$$

Then

$$p(\lambda) = \lambda^2 + \omega^2, \quad \omega = \sqrt{b^* \sqrt{\Delta}},$$

where

$$b = b^* = \frac{d(a - c - d + e - \sqrt{\Delta})}{a - c + d + e + \sqrt{\Delta}} = -\frac{d(-a + c + d - e + \sqrt{\Delta})}{a - c + d + e + \sqrt{\Delta}}.$$

In particular, for the parameter values considered below we assume $b^* > 0$.

Let

$$b = b^* + \varepsilon\beta_1 + \varepsilon^2\beta_2,$$

where $\varepsilon > 0$ is sufficiently small. Then system (14) becomes

$$\begin{aligned} \dot{x} &= c_{10}(b^* + \varepsilon\beta_1 + \varepsilon^2\beta_2)x + c_{01}y - (b^* + \varepsilon\beta_1 + \varepsilon^2\beta_2)xy, \\ \dot{y} &= d_{10}(b^* + \varepsilon\beta_1 + \varepsilon^2\beta_2)x + d_{01}y + (b^* + \varepsilon\beta_1 + \varepsilon^2\beta_2)xy, \end{aligned} \tag{29}$$

where

$$c_{10} = -\frac{a - c + d + e + \sqrt{\Delta}}{2d}, \quad c_{01} = -\frac{a - c + d + e - \sqrt{\Delta}}{2},$$

and

$$d_{10} = \frac{a - c - d + e + \sqrt{\Delta}}{2d}, \quad d_{01} = -\frac{-a + c + d - e + \sqrt{\Delta}}{2}.$$

Notice that

$$b^* = -\frac{d_{01}}{c_{10}} > 0,$$

and hence

$$\omega = \sqrt{-\frac{d_{01}}{c_{10}} \sqrt{\Delta}}.$$

Making the rescaling of the variables

$$(x, y) = (\varepsilon X, \varepsilon Y),$$

system (29) in the new variables (X, Y) becomes

$$\begin{aligned} \dot{X} &= b^*c_{10}X + c_{01}Y + \varepsilon(c_{10}(\beta_1 + \varepsilon\beta_2)X - b^*XY - \varepsilon(\beta_1 + \varepsilon\beta_2)XY), \\ \dot{Y} &= b^*d_{10}X + d_{01}Y + \varepsilon(d_{10}(\beta_1 + \varepsilon\beta_2)X + b^*XY + \varepsilon(\beta_1 + \varepsilon\beta_2)XY). \end{aligned} \tag{30}$$

We now put the linear part of system (30) at $\varepsilon = 0$ into the real Jordan normal form

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

For this purpose, we use the linear change of variables

$$X = u, \quad Y = \frac{d_{01}}{c_{01}}u - \frac{\omega}{c_{01}}v.$$

In the new variables (u, v) , system (30) becomes

$$\begin{aligned}
\dot{u} = & -\omega v + \varepsilon \frac{c_{10}}{a-c} u \left((a-c)\beta_1 + \left(\frac{\omega^2}{2} + b^{*2}c_{10} \right) u + b^*\omega v \right) \\
& + \frac{\varepsilon^2 u}{c_{01}} ((a-c)\beta_2 - d_{01}\beta_1 u + \beta_1\omega v) + \varepsilon^3 \beta_2 u (-d_{01}u + \omega v), \\
\dot{v} = & \omega u - \varepsilon du \left(\frac{\beta_1\omega}{b^*d^2} - \frac{e+\omega^2}{(a-c)\omega} - \frac{e_{11}}{(a-c)c_{10}} v \right) \\
& + \varepsilon^2 c_{10}^2 u \left(\beta_2 \frac{b^*}{\omega} - \beta_1 \frac{b^*d}{(a-c)\omega} u - \frac{d\beta_1}{(a-c)c_{10}} v \right) \\
& + \varepsilon^3 \beta_2 du \left(\frac{e_{11}}{(a-c)\omega} u - \frac{c_{10}}{a-c} v \right),
\end{aligned} \tag{31}$$

where

$$e_{11} = \frac{-a+c+d+e+\sqrt{\Delta}}{2}.$$

Passing system (31) to polar coordinates

$$u = r \cos \theta, \quad v = r \sin \theta,$$

and taking θ as the new independent variable, we obtain

$$\begin{aligned}
\frac{dr}{d\theta} = & \varepsilon c_{10} \left(\frac{\beta_1}{\omega} r \cos^2 \theta - \beta_1 \frac{e_{11}}{c_{10}d_{01}^2} r \cos \theta \sin \theta - \frac{b^*d_{01}}{(a-c)\omega} r^2 \cos^3 \theta \right. \\
& \left. + \frac{b^*-d}{a-c} r^2 \cos^2 \theta \sin \theta - \frac{d^2d_{01}}{(a-c)\omega} r^2 \cos \theta \sin \theta \right) + O(\varepsilon^2) \\
= & \varepsilon F_1(r, \theta) + O(\varepsilon^2).
\end{aligned} \tag{32}$$

The first-order averaging function is

$$f_1(r) = \frac{\beta_1 c_{01}}{2\omega} r.$$

The only zero of f_1 is $r = 0$, which corresponds to the equilibrium point. Therefore we take $\beta_1 = 0$, so that $f_1 \equiv 0$, and apply the averaging theory of second order.

For $\beta_1 = 0$, system (32) takes the form

$$\begin{aligned}
\frac{dr}{d\theta} = & \varepsilon \frac{c_{10}}{a-c} r^2 \cos \theta \left(-\frac{b^*d_{01}}{\omega} \cos^2 \theta + \frac{b^*-d}{2} \sin(2\theta) - \frac{d^2d_{01}}{\omega} \sin \theta \right) \\
& + \varepsilon^2 \frac{c_{10}\omega^3}{b^*} r \cos \theta \left(\frac{\beta_2}{b^*} \left(\cos \theta - \frac{\omega}{d_{01}} \sin \theta \right) \right. \\
& - \frac{c_{10}\omega}{(a-c)^2} r^2 \cos \theta \left(-\omega^2 + (d_{01}d + \omega^2) \cos^2 \theta + \frac{(d_{01}-d)\omega}{2} \sin(2\theta) \right) \\
& \left. \cdot \left(c_{10} \cos^2 \theta - \frac{d}{b^*} \sin^2 \theta + \frac{b^*(d_{01}d + \omega^2)}{2\omega\Delta} \sin(2\theta) \right) \right) + O(\varepsilon^3) \\
= & \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + O(\varepsilon^3).
\end{aligned} \tag{33}$$

The second-order averaging function is

$$f_2(r) = \frac{r}{16(a-c)d^3c_{10}^4\omega} (K_2 r^2 + K_0),$$

where

$$\begin{aligned}
K_2 = & (-a+c-e)d^3 \left(((-a+c+d)^3 + 2de(-a+c+d) + de^2) \Delta \right. \\
& \left. + ((a-c-d)^4 + e(a-c+3d)(a-c-d)^2 - de^2(a-c-3d-e)) \sqrt{\Delta} \right),
\end{aligned}$$

and

$$\begin{aligned} K_0 = & 4\beta_2 (d^5 - d^4(2a - 2c - 5e) + d^3((a - c)(a - c - 3e) + 10e^2) + (a - c + e)^3e^2 \\ & + d^2e^2(3a - 3c + 10e) + de^2(a - c + e)(2a - 2c + 5e) \\ & + (d^4 + 6d^2e^2 - d^3(a - c - 4e) + de^2(3a - 3c + 4e) + e^2(a - c + e)^2) \sqrt{\Delta}). \end{aligned}$$

Recall that

$$\Delta = (-a + c + d + e)^2 + 4de.$$

When $d \rightarrow +\infty$, the signs of K_2 and K_0 are determined by $-a + c - e$ and β_2 , respectively. Hence, choosing

$$\operatorname{sgn}(\beta_2) = -\operatorname{sgn}(-a + c - e),$$

and taking d sufficiently large, we obtain

$$K_2 K_0 < 0.$$

Therefore, the equation $f_2(r) = 0$ has a positive solution

$$r^* = \sqrt{-\frac{K_0}{K_2}}.$$

Moreover,

$$f'_2(r)|_{r=r^*} = \frac{3K_2 r^{*2}}{16(a - c)d^3c_{10}^4\omega} = -\frac{3K_0}{16(a - c)d^3c_{10}^4\omega} \neq 0.$$

By the second-order averaging theorem, system (2) has a periodic orbit bifurcating from the equilibrium point E_2 . Moreover, the limit cycle is stable when

$$(a - c)K_0 > 0,$$

and unstable when

$$(a - c)K_0 < 0.$$

Thus the equilibrium point E_2 can exhibit a Hopf bifurcation. $\blacksquare$

Since $\Delta > 0$ we obtain $a > a_1$ or $a < a_2$. Note that $y_{E_2} > y_{E_1}$ and $x_{E_1} > x_{E_2} > y_{E_1}$. Thus E_1 lies to the lower right of E_2 . When $a > a_1$ we have $a - c - d + e > 0$. Since $\Delta < (a - c - d + e)^2$ we have $y_{E_1} > 0$. Thus E_1 and E_2 are both in the first quadrant. When $a < a_2$ we have $a - c - d + e < 0$. Since $\Delta < (a - c - d + e)^2$ we have $y_{E_2} < 0$. Thus E_1 and E_2 are both in the half-plane $y < 0$.

We first consider the case that $a > a_1$. According to the direction of the vector field on the coordinate axes and the local phase portraits at infinity, the unstable separatrix of D'_2 ends at D_1 . The other orbits in the same canonical region, in particular those issuing from the corresponding boundary arc near D'_3 , also have D_1 as their ω -limit. In the following we only need to discuss the separatrix configurations associated with the points E_1, E_2, D_1, D_2, D_3 and D'_1 .

Assume first that E_2 is an unstable node or focus. Then, by the Poincaré–Bendixson theorem, Lemma 2 and the local phase portrait at infinity, each remaining non-stationary orbit in the corresponding canonical regions has ω -limit either D_1 or a point on the stable set of E_1 . Thus the two unstable separatrices of E_1 must end at D_1 . Thus one of the stable separatrices of E_1 starts at E_2 , and the remaining orbits starting at E_2 end at D_1 . Another stable separatrix of E_1 starts at D_2 . Therefore the rest of the orbits starting at D_2, D_3, D'_1 and D'_3 will go to D_1 . The phase portrait in the Poincaré disc in this case is shown in Fig. 8(a). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (12, 0.2, 1, 1, -5).$$

When E_2 is a stable node or focus, from Lemma 3 if the values of the parameters (a, b, c, d, e) change, the stability of E_2 may change and the system can exhibit a Hopf bifurcation. When the stability of E_2 changes from unstable to stable, an unstable limit cycle bifurcates from E_2 . Since this limit cycle is unstable,

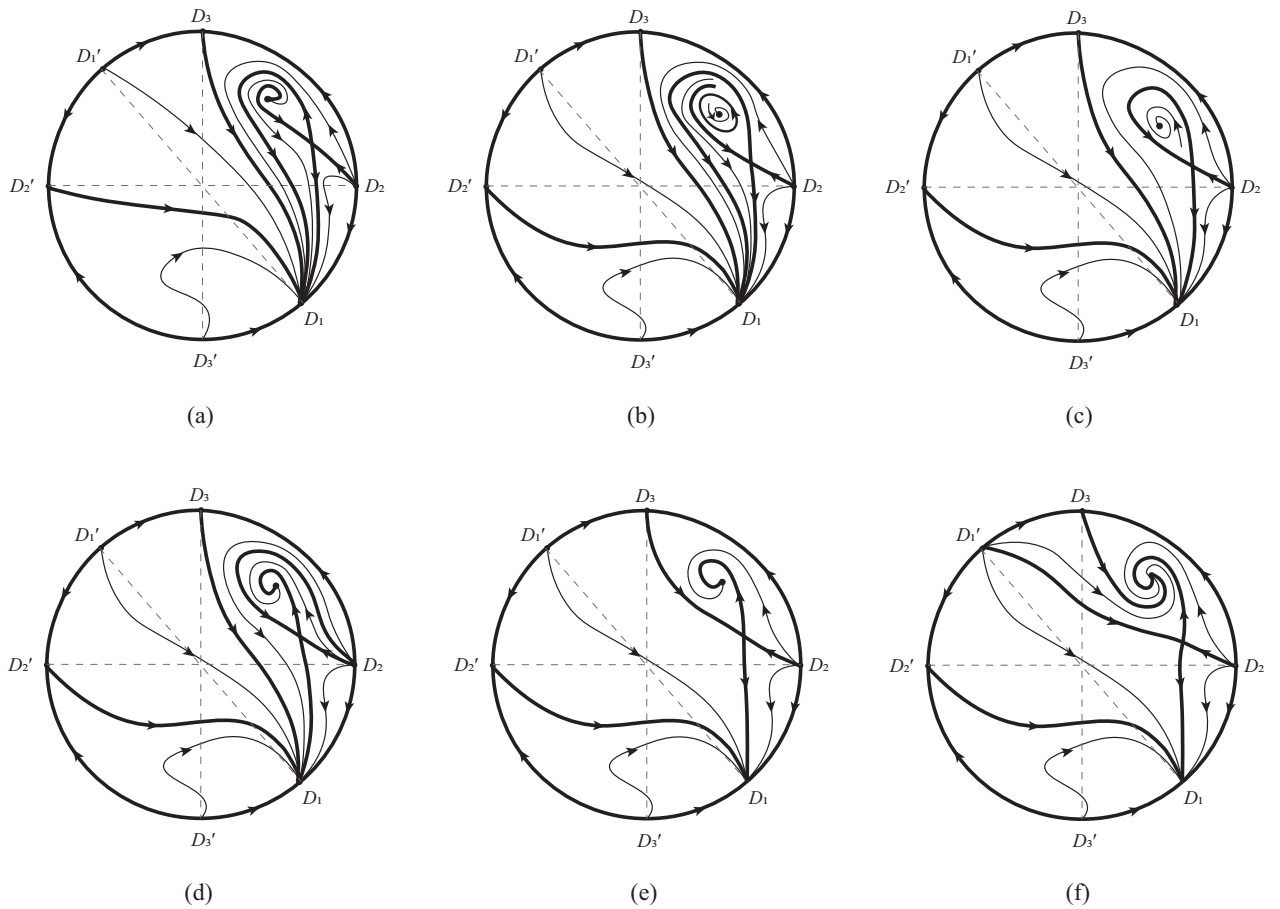


Fig. 8. Phase portraits in the Poincaré disc of system (2) for $e < 0$, $\Delta > 0$ and $a > a_1$. In this case the two finite equilibrium points E_1 and E_2 lie in the first quadrant, where E_1 is a hyperbolic saddle and E_2 is a node or focus. Panel (a) corresponds to the case where E_2 is unstable and no limit cycle is present. Panel (b) contains the unstable limit cycle bifurcating from E_2 . Panel (c) contains a homoclinic loop based at the saddle E_1 . Panels (d), (e) and (f) show the three possible separatrix configurations after the breaking of the homoclinic loop. Arrows indicate the positive time direction, solid curves represent separatrices, and closed curves denote limit cycles or homoclinic loops when they occur.

the non-stationary trajectories inside the region bounded by it tend to E_2 , whereas the trajectories just outside the cycle move away from it.

The analysis of the remaining trajectories is similar to that for the case when E_2 is unstable. Such phase portrait can be realized, for instance, for

$$(a, b, c, d, e) = (12, 0.3, 1, 1, -5).$$

This phase portrait in the Poincaré disc is shown in Fig. 8(b).

The limit cycle born from E_2 belongs to a one-parameter family of limit cycles. By Lemma 2, system (2) has at most one limit cycle. Hence, along this family, the limit cycle cannot disappear by coalescing with another periodic orbit. By the Wintner–Perko termination principle, the maximal family of limit cycles must terminate either at a singular point or at a separatrix cycle. Since this family bifurcates from E_2 , the termination at a singular point corresponds to the Hopf bifurcation point. Away from this endpoint, the family cannot terminate at another finite singular point. Moreover, the local phase portrait at infinity described in Proposition 1, together with the direction of the vector field on the coordinate axes, excludes a separatrix cycle involving infinite singularities. Therefore, in the present configuration, the only possible separatrix cycle is the homoclinic loop of the saddle E_1 . Consequently, for an intermediate parameter value, the two separatrices of E_1 form the homoclinic loop shown in Fig. 8(c).

When the parameters vary, this homoclinic loop breaks, and the unstable separatrix of the saddle E_1 constituting the loop approaches to E_2 . After the stable separatrix of E_1 forming the homoclinic loop, denoted as l_s^1 , breaks and it may start at D_2, D_3 or D'_1 .

If the stable separatrix l_s^1 starts at D_2 , then another stable separatrix of E_1 should start at D_2 . Thus the non-separatrix orbits in the region bounded by these two separatrices have α -limit D_1 and ω -limit E_2 . Outside this region, the remaining orbits issuing from the corresponding boundary arc near D_2 have ω -limit D_1 . Hence the remaining unstable separatrix of E_1 , the unstable separatrix of D_3 , and the orbits in the corresponding region issuing from D'_1 have ω -limit D_1 . In this case we have the phase portrait in the Poincaré disc shown in Fig. 8(d). This configuration is realized, for instance, for

$$(a, b, c, d, e) = (12, 0.36, 1, 1, -5).$$

If the stable separatrix l_s^1 starts at D'_1 , then another stable separatrix of E_1 should start at D_2 and the unstable separatrix of D_3 must end at E_2 . The remaining unstable separatrix of E_1 ends at D_1 . According to the local phase portraits of the equilibrium points of system (2) we obtain the phase portrait of Poincaré disc of Fig. 8(f). This configuration is realized, for instance, for

$$(a, b, c, d, e) = (12, 0.38, 1, 1, -5).$$

Since the phase portraits in Fig. 8(d) and Fig. 8(f) are realized for $b = 0.36$ and $b = 0.38$, respectively, the relevant stable separatrix $l_s^1(b)$ of E_1 lies on different sides of the unstable separatrix of D_3 on a common transverse section. By the separatrix-continuation argument stated above, there exists $b_0 \in (0.36, 0.38)$ such that $l_s^1(b_0)$ coincides with the unstable separatrix of D_3 . This gives the phase portrait in Fig. 8(e).

We describe the phase portraits of system (2) with $e < 0$ in the Poincaré disc when $\Delta > 0$ for the case $a > a_1$ in Fig. 8.

Next we consider the case that $a < a_2$. Now from the values of the coordinates of E_1 and E_2 , it follows that they are either both situated in the third quadrant, both in the fourth quadrant, or E_2 is in the third quadrant while E_1 is in the fourth quadrant. In the following we consider the case that both E_1 and E_2 are in the third quadrant. The other cases can be analyzed analogously, and by topological equivalence, they yield the same phase portraits.

According to the direction of the vector field on the coordinate axes and the local phase portraits at infinity, the separatrix of D_3 entering the corresponding region cannot cross the relevant separatrices and cannot end at E_1 or E_2 . Since periodic orbits are excluded in this configuration, its ω -limit set must be D_1 . Hence the orbits in the same canonical region issuing from the corresponding boundary arc near D_2 also have ω -limit D_1 . In the following we only need to discuss the separatrix configurations associated with $E_1, E_2, D'_1, D'_2, D'_3$ and D_1 .

Assume first that E_2 is an unstable node or focus. Then, by the Poincaré–Bendixson theorem, Lemma 2 and the local phase portrait at infinity, each remaining non-stationary orbit in the corresponding canonical regions has ω -limit either D_1 or a point on the stable set of E_1 . Thus the two unstable separatrices of E_1 must end at D_1 . The region enclosed by them must contain E_2 , otherwise there would be no source for the stable separatrix of E_1 situated in this region. Since E_2 is located in this region the stable separatrix of E_1 situated in it starts at E_2 , while the other trajectories starting at E_2 end at D_1 .

We claim that the other stable separatrix of E_1 can only start at D'_3 , otherwise it starts at D'_2 or D'_1 . According to Theorem 6 on the straight line connecting E_2 with D_3 and D'_3 , denoted by L_2 , the flow goes in different directions on either side of E_2 . If the other stable separatrix of E_1 starts at D'_1 , then the remaining trajectories starting at D'_1 end at D_1 , passing through L_2 in the same direction. This is a contradiction with Theorem 6 because these trajectories pass through L_2 in the two sides of E_2 . Similarly, we can prove that the other stable separatrix of E_1 cannot start at D'_2 . Indeed, the unstable separatrix of D'_2 ends at D_1 , passing through the line L_2 above E_2 . The behavior of the remaining trajectories in the corresponding canonical regions follows from the local phase portraits of the equilibrium points. We have shown the phase portrait in the Poincaré disc of this case in Fig. 9(f). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (4.9, 1.2, 0.5, 0.5, -8).$$

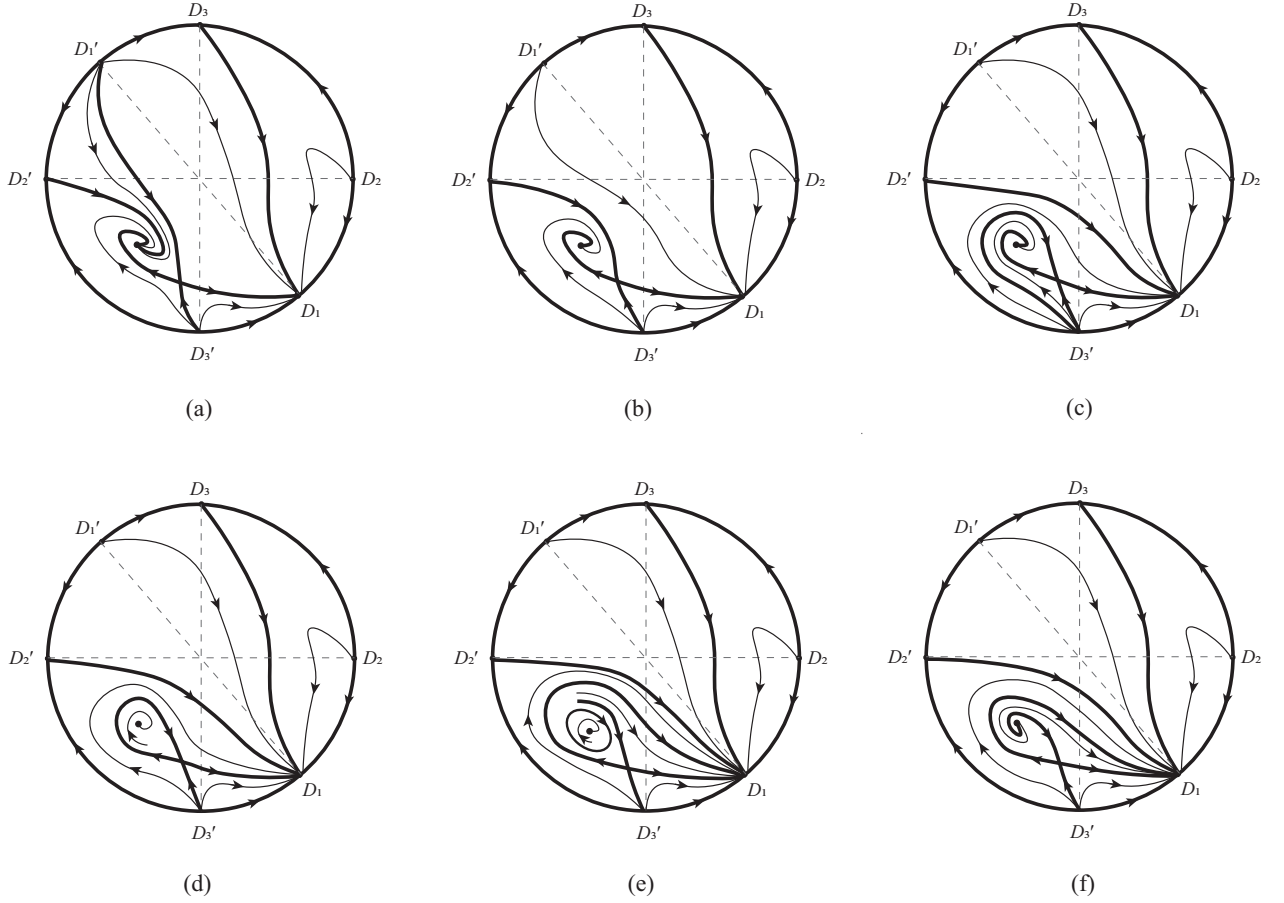


Fig. 9. Phase portraits in the Poincaré disc of system (2) for $e < 0$, $\Delta > 0$ and $a < a_2$. In this case the two finite equilibrium points E_1 and E_2 lie in the half-plane $y < 0$. These six portraits occur in a different parameter region from those in Fig. 8, but they are topologically equivalent to the corresponding portraits in Fig. 8, as stated in Remark 1.1. Panel (f) corresponds to the case where E_2 is unstable and no limit cycle is present. Panel (e) contains the limit cycle bifurcating from E_2 . Panel (d) contains a homoclinic loop based at the saddle E_1 . Panels (a), (b) and (c) show the possible separatrix configurations after the breaking of this homoclinic loop, with (b) representing the intermediate separatrix connection. Arrows indicate the positive time direction, solid curves represent separatrices, and closed curves denote limit cycles or homoclinic loops when they occur.

When E_2 is a stable node or focus, from Lemma 3, the system can exhibit a Hopf bifurcation. When the stability of E_2 changes from unstable to stable, an unstable limit cycle will bifurcate from E_2 . Since this limit cycle is unstable, the non-stationary trajectories inside the region bounded by it tend to E_2 , whereas the trajectories just outside the cycle move away from it. The analysis of the remaining trajectories is similar to that of the case when E_2 is unstable. The phase portrait in the Poincaré disc in this case can be realized, for instance, for

$$(a, b, c, d, e) = (4.9, 1.1, 0.5, 0.5, -8).$$

The phase portrait in the Poincaré disc is shown in Fig. 9(e).

As in the case $a > a_1$, the limit cycle born from E_2 belongs to a one-parameter family of limit cycles. Since system (2) has at most one limit cycle, this family cannot terminate at another periodic orbit. By the Wintner–Perko termination principle, the maximal family must terminate either at a singular point or at a separatrix cycle. The termination at a singular point corresponds to the Hopf bifurcation point at E_2 , and the local phase portraits of the finite and infinite singular points exclude any separatrix cycle involving infinity. Hence the only possible separatrix cycle in this configuration is the homoclinic loop of the saddle E_1 . Thus, for an intermediate parameter value, the homoclinic loop shown in Fig. 9(d) occurs.

When this homoclinic loop breaks, the stable branch of E_1 involved in the loop, denoted by l_s^2 , may have its α -limit at D_3' , D_2' , or D_1' .

If the stable separatrix l_s^2 starts at D'_3 , then the other stable separatrix of E_1 also starts at D'_3 . Thus the non-separatrix orbits in the region bounded by these two separatrices have α -limit D'_3 and ω -limit E_2 . Outside this region, the remaining orbits issuing from the corresponding boundary arc near D'_3 have ω -limit D_1 , as do the remaining unstable separatrix of E_1 , the unstable separatrix of D'_3 , and the orbits in the corresponding region issuing from D'_1 . This phase portrait in the Poincaré disc is shown in Fig. 9(c). This configuration is realized, for instance, for

$$(a, b, c, d, e) = (4.9, 1, 0.5, 0.5, -8).$$

If the stable separatrix l_s^2 starts at D'_1 , then the other stable separatrix of E_1 may start at D'_3 or D'_2 . Using Theorem 6 it is easy to see that it cannot start at D'_2 . So it starts at D'_3 . Therefore the unstable separatrix of D'_2 ends at E_2 , and the remaining unstable separatrix ends at D_1 . From the local phase portraits of the equilibrium points of system (2) we obtain the phase portrait in the Poincaré disc of Fig. 9(a). This phase portrait is realized, for instance, for

$$(a, b, c, d, e) = (4.9, 0.9, 0.5, 0.5, -8).$$

We now fix the remaining parameters and vary b continuously from the value corresponding to Fig. 9(a) to the value corresponding to Fig. 9(c). Let $l_s^2(b)$ be the relevant stable separatrix of E_1 , and let $\gamma_2(b)$ be the unstable separatrix of D'_2 . Choose a transverse section Σ , endowed with a local coordinate, which is crossed by both separatrices during this variation of b . In Fig. 9(a) and Fig. 9(c), the intersections of $l_s^2(b)$ and $\gamma_2(b)$ with Σ occur in opposite orders. Hence the signed distance between these two intersections changes sign. Therefore, by the separatrix-continuation argument, there exists an intermediate value b_0 such that the two separatrices meet on Σ . By uniqueness of solutions, $l_s^2(b_0)$ coincides with the unstable separatrix of D'_2 . This gives the phase portrait shown in Fig. 9(b).

We describe the phase portraits of system (2) with $e < 0$ in the Poincaré disc when $\Delta > 0$ for the case $a < a_2$ in Fig. 9.

It remains to justify the topological equivalences stated in Remark 1.1.

Consider the separatrix skeleton of a phase portrait, namely the union of all finite and infinite equilibrium points, separatrices, limit cycles and homoclinic loops. For each corresponding pair of portraits in Fig. 8 and Fig. 9 listed in Remark 1.1, the two separatrix skeletons are isomorphic as oriented embedded graphs in the closed Poincaré disc. More precisely, the corresponding finite and infinite singular points have the same local topological type, the corresponding separatrix branches have the same endpoints, the cyclic order of the separatrices on the boundary of each canonical region is the same, and the presence or absence of a limit cycle or a homoclinic loop is preserved.

Therefore, we first define a homeomorphism between the two separatrix skeletons by sending each equilibrium point, separatrix branch, limit cycle and homoclinic loop of one portrait onto the corresponding object of the other portrait, preserving the orientation of the orbits. The complement of the separatrix skeleton is a finite union of canonical regions. For each corresponding pair of canonical regions, the boundaries have the same cyclic order and the orbits inside the region have the same limiting sets on the boundary. Hence the boundary homeomorphism extends to a homeomorphism of the whole canonical region, mapping oriented orbits onto oriented orbits. Gluing these extensions over all canonical regions gives a homeomorphism of the closed Poincaré disc which maps oriented orbits of one portrait onto oriented orbits of the other. Thus the corresponding phase portraits in Fig. 8 and Fig. 9 are topologically equivalent.

In particular, the phase portraits represented by Fig. 8(c) and Fig. 9(d) are topologically equivalent. Both contain a homoclinic loop based at the saddle E_1 , and the remaining equilibrium points and separatrix connections have the same cyclic configuration in the Poincaré disc.

On the other hand, no other phase portraits drawn in Figs. 3–9 are topologically equivalent. Indeed, the number and local topological type of finite equilibria, the existence of nilpotent cusps or semi-hyperbolic saddle-nodes, the number and arrangement of separatrix connections, and the presence or absence of a limit cycle or a homoclinic loop are invariants under topological equivalence. Any two portraits not listed as corresponding pairs in Remark 1.1 differ in at least one of these invariants. Therefore, after identifying the six equivalent pairs between Fig. 8 and Fig. 9, system (2) has precisely nineteen topologically distinct phase portraits in the Poincaré disc.

This completes the proof of Theorem 1.

Biological interpretation and conclusion

In this paper we have described the global phase portraits in the Poincaré disc of the SIS model with importation or removal of infected individuals. Although the compactified system is studied in the whole plane in order to obtain a complete mathematical classification, the epidemiologically meaningful region is the first quadrant, where both the susceptible and infected populations are non-negative.

The sign of the parameter e plays an essential role in the dynamics. When $e > 0$, infected individuals are continuously imported into the population. In this case the first quadrant is positively invariant, and the system has a unique biologically meaningful equilibrium point in this region. This equilibrium is stable, and no limit cycles occur. Therefore, every trajectory with initial condition in the biologically meaningful region, except the equilibrium itself, tends to a steady endemic state. From the epidemiological point of view, the continuous importation of infectives prevents the disappearance of the disease and leads to a stable long-term level of infection.

When $e < 0$, infected individuals are removed from the population at a constant rate. In this case the first quadrant is no longer positively invariant for the polynomial system considered in the whole plane, because the vector field on the axis $y = 0$ points toward the half-plane $y < 0$. Thus, trajectories that cross this axis should be interpreted as situations in which the constant removal mechanism has driven the infected population to extinction in the biological model. The continuation of these trajectories to the half-plane $y < 0$ has no direct epidemiological meaning, but it is necessary for the complete topological classification of the polynomial vector field.

For $e < 0$, the global dynamics is much richer. Depending on the sign of Δ , the system may have no finite equilibrium point, one degenerate equilibrium point, or two finite equilibrium points. The case $\Delta = 0$ corresponds to a transition between these different configurations and gives rise to degenerate singularities such as saddle-nodes or cusps. When the corresponding equilibrium lies in the first quadrant, this transition can be interpreted as the appearance or disappearance of an endemic steady state.

In the case $\Delta > 0$, the system may have two finite equilibria. When both equilibria are in the first quadrant, one of them is a saddle and the other one represents an endemic equilibrium whose stability depends on the parameters. Moreover, the system can exhibit a Hopf bifurcation. Hence, for suitable parameter values, periodic oscillations of the infected population may occur. The limit cycles and homoclinic loops appearing in the global phase portraits describe changes in the basin structure and in the qualitative organization of the orbits.

Consequently, the removal of infected individuals does not only affect the number and position of equilibria, but may also change the global structure of the dynamics. In particular, it may produce degenerate equilibria, changes of stability, Hopf bifurcations, limit cycles and homoclinic connections. The classification obtained in this work therefore gives a complete qualitative description of the model, including both the biologically meaningful dynamics in the first quadrant and the global behavior of the associated polynomial vector field in the Poincaré disc.

Acknowledgments

The authors are grateful to the editor and the anonymous reviewers for their careful reading of the manuscript and for their constructive comments and suggestions, which have greatly helped us improve the presentation and clarity of the paper. Jaume Llibre is partially supported by the Agència Estatal de Investigació of Spain grant PID2022-136613NB-I00, and by the Reial Acadèmia de Ciències i Arts de Barcelona. Hui Zeng is supported by International Program for Candidates of Sun Yat-Sen University.

References

Artés, J. C., Llibre, J. & Rezende, A. C. [2018] *Structurally unstable quadratic vector fields of codimension one*, Springer.

- Artés, J. C., Llibre, J., Schlomiuk, D. & Vulpe, N. [2024] “Global analysis of Riccati quadratic differential systems,” *Int. J. Bifurcation and Chaos* **34**, 2450004.
- Brauer, F. & van den Driessche, P. [2001] “Models for transmission of disease with immigration of infectives,” *Math. Biosci.* **171**(2), 143–154.
- Coppel, W. A. [1966] “A survey of quadratic systems,” *J. Differ. Equ.* **2**(3), 293–304.
- Coppel, W. A. [1989] “Some quadratic systems with at most one limit cycle,” *Dynamics reported: A series in dynamical systems and their applications*, 61–88.
- Cui, Q., Du, Q., & Wang, L. [2020] “Global dynamics of a generalized sirs epidemic model with constant immigration,” *Math. Probl. Eng.* **2020**(1), 7845390.
- Dumortier, F., Llibre, J. & Artés, J. C. [2006] *Qualitative theory of planar differential systems*, Springer.
- Henshaw, S. & McCluskey, C. C. [2015] “Global stability of a vaccination model with immigration,” *Electron. J. Differ. Equ.* **92**, 1–10.
- Ichida, Y. [2022] “On global behavior of a some SIR epidemic model based on the Poincaré compactification,” *JSIAM Lett.* **14**, 65–68.
- Kermack, W. O. & McKendrick, A. G. [1927] “A contribution to the mathematical theory of epidemics,” *Proc. R. Soc. Lond.* **115**, 700–721.
- Llibre, J. & Pantazi, C. [2023] “Global phase portraits of the quadratic systems having a singular and irreducible invariant curve of degree 3,” *Int. J. Bifurcation Chaos* **33**, 2350003.
- Llibre, J. & Salhi, T. [2024] “Phase portraits of an SIR epidemic model,” *Appl. Anal.* **103**(6), 1165–1175.
- Llibre, J. & Valls, C. [2008] “Integrability of a SIS model,” *J. Math. Anal. Appl.* **344**, 574–578.
- McCluskey, C. [2021] “Lyapunov functions for disease models with immigration of infected hosts,” *Discrete Contin. Dyn. Syst. Ser. B* **26**(8), 4479–4491.
- Novaes, D. D., Llibre, J. & Teixeira, M. A. [2014] “Higher order averaging theory for finding periodic solutions via Brouwer degree,” *Nonlinearity* **27**(3), 563–583.
- Nucci, M. C. & Leach, P. G. L. [2004] “An integrable SIS model,” *J. Math. Anal. Appl.* **290**, 506–518.
- Oliveira, R. D. S. & Rezende, A. C. [2013] “Global phase portraits of a SIS model”, *Appl. Math. Comput.* **219**(9), 4924–4930.
- Perko, L. [2001] *Differential equations and dynamical systems*, 3rd ed., Springer.
- Shim, E. [2006] “A note on epidemic models with infective immigrants and vaccination,” *Math. Biosci. Eng.* **3**(3), 557–566.
- Sanders, J. A., Verhulst, F., & Murdock, J. [2007] *Averaging methods in nonlinear dynamical systems*, Springer.
- Sigdel, R. P. & McCluskey, C. C. [2014] “Global stability for an SEI model of infectious disease with immigration,” *Appl. Math. Comput.* **243**, 684–689.
- Uggenti, C. & McCluskey, C. [2018] “Global stability for infectious disease models that include immigration of infected individuals and delay in the incidence,” *Electron. J. Differ. Equ.* **2018**(64), 1–14.
- Verhulst, F. [1990] *Nonlinear differential equations and dynamical systems*, Springer-Verlag.
- Wang, L. & Wang, X. [2012] “Influence of temporary migration on the transmission of infectious diseases in a migrants’ home village,” *J. Theor. Biol.* **300**, 100–109.
- Yaacob, N., Mohd Siam, F. & Nasir, H. [2025] “Dynamics of an SIS model with an influx of infective visitors or immigrants,” *Differ. Equ. Dyn. Syst.* **33**, 513–533.
- Zhang, Z. [1992] *Qualitative theory of differential equations*, American Mathematical Society.