

Article

New Tools for Finite-Time Fuzzy Control of Fractional-Order Systems: Stability Lemmas, Composite Disturbance Observers, and Singularity-Free Designs

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Abstract

This article presents a series of command-filter-based, finite-time adaptive fuzzy tracking control schemes for fractional-order nonlinear systems (FONSs) subject to input saturation and composite disturbances. A unified design framework is proposed to tackle these issues, combining adaptive fuzzy systems for online uncertainty estimation with novel fractional-order adaptive disturbance observers for robust perturbation approximation. Within this framework, several practical finite-time control schemes are developed, each offering a distinct approach to key design challenges, including finite-time fractional-order filter design, compensation strategies for filter errors, and measures to avoid singularity issues in virtual control law differentiation. The work also includes a review and theoretical comparison with existing methods. The practical finite-time stability of each closed-loop scheme is established via Lyapunov theory, supported by new technical lemmas introduced herein. Finally, numerical simulations are provided to illustrate the effectiveness and relative merits of the proposed control strategies.

Keywords: fractional-order nonlinear systems; adaptive fuzzy disturbance observer; multiple control schemes; practical finite-time stability; input saturation; singularity avoidance

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1. Introduction

Over the past several decades, there has been significant development in control approaches to meet the increasing performance requirements of uncertain nonlinear systems. Essentially, there are two primary categories of control methods designed to address uncertainties. The first category involves model-based approaches, such as adaptive control [1], sliding mode control (SMC) [2], or robust feedback control [3]. The second category utilizes model-free methods, including adaptive fuzzy logic (FL) controllers and adaptive neural network (NN) controllers, which have proven effective in controlling a wide range of uncertain nonlinear systems [4–7]. Additionally, the back-

stepping control concept has proven to be a highly efficient methodology ensuring the stability of closed-loop systems and has found widespread application in the design of adaptive FL/NN control strategies for uncertain nonlinear systems with non-matching conditions (which include pure-feedback nonlinear systems, strict-feedback nonlinear systems, non-strict nonlinear systems, and semi-strict-feedback systems) [8,9].

Fractional-order (FO) calculus, a concept dating back to the 17th century, is a generalization of integer-order calculus. Having resurfaced in the last three decades, it has established itself as a highly active research topic, increasingly attracting scholars [10–12]. Recognized for its distinctive features and appealing advantages, such as memory and heredity, fractional calculus provides a powerful mathematical framework for better understanding and accurately modeling complex phenomena in various scientific fields, including control engineering, chemistry, physics, chaotic financial systems, medicine and biology [10–15]. Additionally, there is evidence to support that integrating FO operators into traditional control methods can improve various aspects of closed-loop system performance, including transient and steady-state responses, and robustness against uncertainties and external disturbances [10,13]. Motivated by these factors, scholars have recently dedicated significant effort to investigating the control issues of uncertain fractional-order nonlinear systems (FONSs). Due to the complexity of FO derivatives, the control laws designed for integer-order nonlinear systems are not easily extendable to FONSs. Lately, different control strategies have been utilized for FONSs [10–24]. These methods commonly utilize techniques such as backstepping control, adaptive control, and robust control, as well as artificial intelligence-based methods like fuzzy logic systems (FLSs) and NNs. For instance, an adaptive backstepping control strategy is developed for fractional-order nonlinear systems (FONSs) in [13]. Unknown nonlinear functions are approximated by FLS, and FO adaptation laws are designed to update their parameters. The authors in [14] develop a new FO adaptive backstepping controller for a class of integer-order and FO systems subject to both model uncertainties and external disturbances. The stability of the closed-loop system is established using fractional Lyapunov theorems. In [15], the authors introduce an adaptive backstepping controller for a specific class of commensurate FONSs subject to both parametric uncertainty and external disturbances. Adaptive laws are developed to estimate unknown system parameters and disturbance bounds online.

Conventional backstepping control techniques are plagued by the well-known issue of “term explosion”, which arises from the repeated differentiation of virtual control (VC) laws. For FONSs, this problem significantly increases computational complexity. To confront this challenge, some scholars have utilized FLSs to approximate composite functions, which includes the system uncertainty and fractional derivative of the VCs [16–18]. In a different way, dynamic surface control (DSC) methods have also been employed to address the challenge of “complexity explosion” [12,19–21]. In these works, the DSC concept is integrated with adaptive NN and FLS techniques. Nevertheless, this method fails to address the error issue arising from the filtering process (i.e., from the first-order filter used in the DSC framework). On the other hand, as an alternative to addressing the drawbacks of DSC, a command-filtered backstepping control was recently proposed for FONSs [18,22,23]. In [18], the authors address the problem of adaptive fuzzy finite-time control for uncertain FONSs. They introduce a novel finite-time fractional-order command-filtering scheme for the adaptive backstepping method. This approach eliminates the need to analytically compute the fractional derivatives of the stabilizing functions. In [22], an FO command-filtered backstepping control scheme with a disturbance observer is proposed for a class of integer-order nonlinear systems. Using an indirect FO Lyapunov stability approach, the authors of [23] develop a command-filter-based FO-DSC technique for strict-feedback FONSs with unknown control

directions. Neuro-fuzzy networks are utilized to effectively model and approximate complex nonlinear relationships within the system. In [24], the authors solve the problem of sampled-data finite-time adaptive stabilization for a class of FONs without external disturbances, without compensating for approximation errors arising from the use of the fuzzy systems. In [25,26], the authors design adaptive backstepping control for FONs with actuator nonlinearities (dead zone and input saturation, respectively). In [27,28], the authors try to combine the FO control with a fractional-order disturbance observer (FODO). These observers enable precise estimation of disturbances, including both external perturbations and uncertain functions, which are subsequently integrated into the control scheme. While those control strategies can demonstrate satisfactory performance in specific scenarios, they are subject to several limitations. (1) The scope of the FONs analyzed is insufficiently broad; for example, [29] addresses only systems that satisfy matching conditions. (2) Many studies, such as [10–16,21,25,30,31], neglect external disturbances in their models. Others, like [23,24,26], consider disturbances but do not implement compensation strategies. While some works, including [14,15,19–21,29], have employed pseudo-term sliding mode control to mitigate disturbances, the use of disturbance observers for compensation in these systems is less common. Notable exceptions include [22,27,28]. (3) Most of these control approaches, as shown in [10,13–15,17,19–21,23,25–28], only guarantee asymptotic stability over an infinite time horizon. Therefore, further research is needed to address these limitations and improve performance.

In practical engineering applications, achieving control objectives within a finite time (FT) period is essential. In fact, FT stability represents a more intricate and progressive concept compared to asymptotic stability. Additionally, finite-time control (FTC) approaches offer more refined capabilities and performance enhancements, including faster convergence rates, robustness against uncertainties, and superior disturbance rejection [7]. Recently, some FTC schemes have been introduced for FONs [12,29–33]. An FT adaptive NN-DSC approach for a class of nonstrict-feedback FONs in the presence of input delay is addressed in [12]. Based on the fractional Lyapunov stability approach and the FT stability theory, an FO adaptive backstepping neuro-fuzzy SMC is designed for a class of FONs in [29]. A neuro-fuzzy network system is adopted to approximate unknown nonlinear function existing in FONs where approximation errors are ignored. The researchers of [30] introduce an FT adaptive neural resilient dynamic surface controller tailored for addressing sensor–actuator faults in large-scale FO systems. To achieve FT tracking control performance for a class of FONs subject to actuator faults, a command-filter-based fuzzy backstepping SMC technique is introduced in [31]. The FT stability analysis is established on the basis of the Lyapunov FO method. In [32], an event-triggered adaptive neural controller using the backstepping technique is proposed for non-strict feedback FONs. Note that, in all the previously mentioned papers, the FT stability of the closed-loop system is ensured through fractional Lyapunov stability analysis. In [33], a novel FT adaptive NN control scheme for a class of incommensurate FONs affected by external disturbances is designed based on the established criterion for practical FT stability. Unfortunately, most of the cited studies are hindered by one or more of the following limitations: (1) Some FT control schemes may suffer from the famous singularity issue generally associated with the (direct or indirect) time derivative of VCs [12,18,22,24,31,32]. (2) The practical finite-time stability (PFTS) analysis in [12,18,29,30,32,34,35] is not rigorously proven. This is because it relies on the following two FO derivative properties— $D_t^q f^{\mu_1}(t) = \frac{\Gamma(2-q)\Gamma(1+\mu_1)}{\Gamma(1+\mu_1-q)} f^{\mu_1-1}(t) D^q f(t)$ and $D_t^q f^{\mu_2}(t) = \frac{\Gamma(1+\mu_2)}{\Gamma(1+\mu_2-q)} f^{\mu_2-1}(t) D^q f(t)$, where $0 < q < 1$, $\mu_1 > 1, \mu_2 \in R$ —which have been questioned in the literature, particularly in [36–40].

Motivated by the foregoing discussions, this paper investigates a set of adaptive fuzzy disturbance observer (AFDO)-based FTC schemes for fractional-order nonlinear uncertain systems (FONUSs) in non-strict-feedback form. These systems are assumed to be subject simultaneously to actuator saturations, system uncertainties, and external disturbances. FLSs are employed to approximate unknown nonlinear functions online. The practical finite-time stability of each closed-loop control scheme is established via Lyapunov theory, supported by four new technical lemmas introduced in this work. To enhance the robustness of the proposed controllers, novel fractional-order disturbance observers (FODOs) are designed in conjunction with adaptive fuzzy systems. In comparison with existing studies [10–19,21–25,27–35], this paper offers the following contributions:

1. Unlike the approaches in [12,18,29,30,32,34,35], our proposed lemmas are derived in a rigorous manner, without relying on the FO derivative formulas used in those works, which have been questioned in recent studies [36–39].
2. In contrast to the methods in [12,18,22,24,31,32], the proposed FT fuzzy adaptive controllers are free from the singularity issue that commonly arises from the time derivative of virtual control (VC) laws. This is achieved by designing smooth VCs, and several alternative solutions are also presented to address this challenge.
3. Compared with previous works [10–19,21–25,27–32], the nonlinear systems considered here are characterized by a simultaneous combination of non-strict-feedback structure, FOs, uncertainties, input saturation, and unmatched external disturbances. This complex interplay adds considerable difficulty to the control design process.
4. Unlike the approaches in [11,12,16,18,24,29–33], this paper introduces new FODOs aimed at mitigating the adverse effects of unknown compound disturbances, which include fuzzy approximation errors and both matched and unmatched external disturbances. These observers are shown to contribute meaningfully to the robustness of the proposed control systems.

It is important to note that the present study is focused on the theoretical development and numerical validation of the proposed control schemes. While extensive simulations are provided to demonstrate the effectiveness of these approaches, experimental validation on physical hardware is beyond the scope of this work. The simulation results should therefore be interpreted as indicative rather than definitive with respect to practical implementation.

It is worth noting that Table A1 in the Appendix provides a list of abbreviations, whereas Table A2 summarizes the nomenclature, mathematical symbols, variables, and parameters used throughout this manuscript.

2. System Description and Preliminaries

For future reference, we present some lemmas and definitions related to fractional calculus, FT stability, and FLSs.

2.1. Fractional Calculus

The Caputo fractional derivative definition, commonly used in engineering for its well-understood physical meaning and interpretable initial conditions [41], will be exploited throughout this research.

Definition 1. [41] The definition of the q -th fractional integral of a function $f(t)$ is as follows:

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f(\tau) d\tau, \quad (1)$$

with $\Gamma(q) = \int_0^{+\infty} t^{q-1} e^{-t} dt$ being the so-called gamma function (also known as the Euler function).

Definition 2. [41] The q -th order Caputo derivative is defined as:

$$D_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t - \tau)^{m-q-1} f^{(m)}(\tau) d\tau, \quad (2)$$

with $m - 1 < q < m$, $m \in \mathbb{Z}^+$.

For convenience, we will only focus on the scenario where $0 < q < 1$ throughout the rest of this study. It should be noted that $J^q[D_t^q f(t)] = f(t) - f(0)$.

Definition 3. [41] The Mittag-Leffler function with two parameters (α, β) is defined as:

$$E_{\alpha, \beta}(s) = \sum_{m=0}^{+\infty} \frac{s^m}{\Gamma(\alpha m + \beta)}, \quad (3)$$

where $\alpha > 0, \beta > 0$, and s is a complex number. Obviously, one has $E_\alpha(s) = E_{\alpha,1}(s) = \sum_{m=0}^{+\infty} \frac{s^m}{\Gamma(\alpha m + 1)}$, $E_{0,1}(s) = \frac{1}{1-s}$, $E_{1,1}(s) = e^s$.

Property 1. [42,43] Consider a differentiable function vector $x(t) \in \mathbb{R}^n$. Then, $D_t^q x^T(t)x(t) \leq 2x^T(t)D_t^q x(t)$ remains valid at any time point $t \geq t_0 > 0$ and $0 < q \leq 1$, where t_0 is the initial time.

2.2. FT Stability

Lemma 1. [44] (Young's Inequality) For any real variables χ and ς , and strictly positive constants r_1, r_2 , and any $r_3 > 0$, the following inequality holds:

$$|\chi|^{r_1} |\varsigma|^{r_2} \leq \left(\frac{r_1}{r_1+r_2}\right) r_3 |\chi|^{r_1+r_2} + \left(\frac{r_2}{r_1+r_2}\right) r_3^{\frac{r_1}{r_2}} |\varsigma|^{r_1+r_2} \quad (4)$$

Lemma 2. [45] For $\Psi_i \in \mathbb{R}$, $i = 1, \dots, n$, $p_1 \in (0,1]$, one obtains:

$$(\sum_{i=1}^n |\Psi_i|)^{p_1} \leq \sum_{i=1}^n |\Psi_i|^{p_1} \leq n^{1-p_1} (\sum_{i=1}^n |\Psi_i|)^{p_1}. \quad (5)$$

Lemma 3. [46,47] For any $z_i \in \mathbb{R}$ and $\epsilon_i > 0$, the following inequality is satisfied: $0 \leq |z_i| - \frac{z_i^2}{\sqrt{z_i^2 + \epsilon_i^2}} \leq \epsilon_i$.

Lemma 4. [48] The inequality $0 \leq |z_i| - z_i \tanh(z_i/\epsilon_i) \leq \bar{\epsilon}_i$ holds for any $z_i \in \mathbb{R}$ and $\epsilon_i > 0$, with $\bar{\epsilon}_i = 0.2785\epsilon_i$.

Lemma 5. [49] For any $z_i \in \mathbb{R}$ and $\epsilon_i > 0$, the following inequality is satisfied: $\left(\frac{|z_i|}{2}\right)^{2q} - (z_i \tanh(z_i/\epsilon_i))^{2q} \leq (\bar{\epsilon}_i)^{2q}$, with $\bar{\epsilon}_i = 0.2785\epsilon_i$ and $0 < q < 1$.

Consider the following Caputo fractional-order nonlinear system (CFONS):

$$D_t^q \xi(t) = f(\xi(t)), \quad \text{with } f(0) = 0, \quad (6)$$

with $0 < q < 1$ and $\xi \in R^n$ being the pseudo-state vector. $f(\xi(t))$ is a continuous nonlinear function. Then, the following lemmas hold.

Lemma 6. [50] Let $V(\xi)$ be a differentiable and positive-definite function such that the following equation holds for $t \geq 0$: $D_t^q V(t) \leq -Y_1 V(t) - Y_2$, $t \geq 0$, where $Y_1 > 0$, $Y_2 > 0$, and $0 < q < 1$. It can be deduced that $V(t) \leq \frac{1}{q} \left(V(0) + \frac{Y_2}{Y_1} \right) \exp(-Y_1^{1/q} t) - \frac{Y_2}{Y_1}$. Then, we can judge that $V(t)$ converges to zero within the settling time (ST), T_s^* , which can be estimated as follows:

$$T_s^* \leq Y_1^{-1/q} \ln \left[\frac{Y_1 V(0) + Y_2}{q Y_2} \right]$$

Lemma 7. [51] Let $V(\xi)$ be a differentiable and positive-definite function such that $D_t^q V(\xi) \leq -YV(\xi) + \rho$, $t \geq 0$ with $0 < q < 1$, $\rho > 0$, and $\xi \geq 0$; then $V(t) \leq \left(V(0) - \frac{\rho}{Y} \right) E_q(-Yt^q) + \frac{\rho}{Y}$, $t \geq 0$.

Lemma 8. [38] For System (6), if there exists a continuous and non-negative function $V(t)$ satisfying:

$$D_t^q V(t) \leq -Y_1 V(t) - Y_2 V^\beta(t) + Y_3, \quad t \geq 0, \quad (7)$$

where $Y_1 > 0$, $Y_2 > 0$ and $Y_3 > 0$, and $0 < \beta < 1$, then the trajectory of $\xi(t)$ exhibits practical finite-time stability (PFTS) and converges to the following residual set:

$$\Omega = \left\{ \xi(t) | V(\xi(t)) \leq \min \left\{ \frac{Y_3}{Y_1(1-b)}, \left(\frac{Y_3}{Y_2(1-b)} \right)^{1/\beta} \right\} \right\}, \quad \text{for } t \geq T_s, \quad (8)$$

and the ST is:

$$T_s = \max \left\{ \left(\frac{t_{c1}^*}{bY_1 + Y_2 V^{\beta-1}(0)} \right)^{1/q}, \left(\frac{t_{c2}^*}{Y_1 + bY_2 V^{\beta-1}(0)} \right)^{1/q} \right\}, \quad (9)$$

where $0 < b < 1$, and t_{c1}^* and t_{c2}^* are the solutions of the following nonlinear equations:

$$E_q(-t_{c1}^*) - \frac{Y_3}{(1-b)Y_1 V(0)} = 0, \quad (10)$$

$$E_q(-t_{c2}^*) - \left(\frac{Y_3}{(1-b)Y_2 V^\beta(0)} \right)^{\frac{1}{\beta}} = 0. \quad (11)$$

Proof of Lemma 8. For any constant $0 < b < 1$, one has the following alternative forms for Equation (7):

$$D_t^q V(t) \leq -Y_1 b V(t) - Y_2 V^\beta(\xi) - Y_1(1-b)V(t) + Y_3, \quad (12)$$

or

$$D_t^q V(t) \leq -Y_1 V(t) - Y_2 b V^\beta(t) - Y_2(1-b)V^\beta(t) + Y_3. \quad (13)$$

Define the following sets: $\Omega_{\xi 1} = \left\{ \xi | V(\xi) \leq \frac{Y_3}{Y_1(1-b)} \right\}$, $\Omega_{\xi 2} = \left\{ \xi | V(\xi) \leq \left(\frac{Y_3}{Y_2(1-b)} \right)^{\frac{1}{\beta}} \right\}$ and $\Omega_{\xi 3} = \Omega_{\xi 1} \cup \Omega_{\xi 2}$.

Let us examine the following scenarios:

- **When $\xi(0) \notin \Omega_{\xi 1}$,** the inequality in Equation (12) becomes:

$$D_t^q V(t) \leq -Y_1 b V(t) - Y_2 V^\beta(t). \quad (14)$$

From Lemma 4 in [39] and Equation (14), it is clear that $V(t) \leq V(0)$. Then, it follows $V^{\beta-1}(t) \geq V^{\beta-1}(0)$ such that:

$$D_t^q V(t) \leq -\left(Y_1 b + Y_2 V^{\beta-1}(0)\right) V(t). \quad (15)$$

Using Lemma 7, we can obtain:

$$V(t) \leq V(0) E_q \left(-\left(Y_1 b + Y_2 V^{\beta-1}(0)\right) t^q \right). \quad (16)$$

From Equation (16), $V(t)$ decreases monotonically to the set Ω_{ξ_1} within an FT, T_{c1} . By setting the right hand of Equation (16) equal to $\frac{Y_3}{Y_1(1-b)}$, the ST can be determined as follows:

$$T_{c1} = \left(\frac{t_{c1}^*}{bY_1 + Y_2 V^{\beta-1}(0)} \right)^{1/q}, \quad (17)$$

where t_{c1}^* is the solution of the following nonlinear equation: $E_q(-t_{c1}^*) - \frac{Y_3}{(1-b)Y_1 V(0)} = 0$.

Remark 1. Assuming $\xi(0) \notin \Omega_{\xi_1}$, we obtain the inequality $1 > \frac{Y_3}{(1-b)Y_1 V(\xi)} \geq \frac{Y_3}{(1-b)Y_1 V(0)}$. Noting that $E_q(-t)$ is continuous and strictly decreasing from 1 (at $t = 0$) to 0 (as $t \rightarrow \infty$), the equation $E_q(-t) = \frac{Y_3}{(1-b)Y_1 V(0)}$ must have a unique solution because $0 < \frac{Y_3}{(1-b)Y_1 V(0)} < 1$ falls within the range of $E_q(-t)$.

- When $\xi(0) \notin \Omega_{\xi_2}$, the inequality in Equation (13) can be rewritten as:

$$D_t^q V(t) \leq -Y_1 V(t) - Y_2 b V^\beta(t). \quad (18)$$

Since this case is analogous to the preceding one, we will skip the detailed demonstration and present the convergence time directly:

$$T_{c2} = \left(\frac{t_{c2}^*}{Y_1 + bY_2 V^{\beta-1}(0)} \right)^{1/q}, \quad (19)$$

where t_{c2}^* is the solution of $E_q(-t_{c2}^*) - \left(\frac{Y_3}{(1-b)Y_2 V^\beta(0)} \right)^{\frac{1}{\beta}} = 0$.

Now, let us recapitulate the preceding outcomes from cases 1 and 2. There exists an ST, $T_s = \max(T_{c1}, T_{c2})$ such that $\xi \in \Omega_{\xi_3} = \Omega_{\xi_1} \cup \Omega_{\xi_2}$, for $t \geq T_c$. This ends the proof of Lemma 8. $\square$

2.3. New Criteria for Assessing PFTS

Four novel technical lemmas are established here (Lemmas 9–12). They can be used to investigate PFTS of a class of FO systems. Based on these novel lemmas, four FTFACs are obtained.

Lemma 9. Consider System (6). If there exists positive-definite C^2 function $V(\xi)$, K_∞ – functions Π_1 and Π_2 , constants $Y_1 > 0$, $Y_2 > 0$ and $Y_3 > 0$, and $0 < \beta < 1$, such that:

$$\begin{cases} \Pi_1(\|\xi(t)\|) \leq V(\xi(t)) \leq \Pi_2(\|\xi(t)\|), \\ D_t^q V(\xi) \leq -Y_1 V(\xi) - Y_2 V^\beta(\xi) + Y_3, \quad t \geq 0. \end{cases} \quad (20)$$

Then, one has:

$$V \leq \left(\frac{Y_3}{2^{\beta-1}Y_2} \right)^{\frac{1}{\beta}}, \text{ for } t \geq T_s \Rightarrow \|\xi(t)\| \leq \Pi_1^{-1} \left(\left(\frac{Y_3}{2^{\beta-1}Y_2} \right)^{\frac{1}{\beta}} \right), \text{ for } t \geq T_s \quad (21)$$

and the ST is:

$$T_s = \left[\frac{1}{(\gamma_1 + 2^{\beta-1}\gamma_2 W^{\beta-1}(0))} \right]^{1/q} \ln \left[\frac{(\gamma_1 + 2^{\beta-1}\gamma_2 W^{\beta-1}(0)) W(0) + \gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}}{q \gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}} \right], \quad (22)$$

where $W(0) = V(0) - \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{\frac{1}{\beta}}$.

Proof of Lemma 9. To define the following set $\Omega_{\xi 1} = \left\{ \xi | V(\xi) \leq \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta} \right\}$, one has two cases:

- **When $\xi(0) \notin \Omega_{\xi 1}$** (i.e., if $V(\xi) > \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}$):

By defining $W(\xi) = V(\xi) - \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}$ and using Lemma 2, we have:

$$V^\beta(\xi) = \left(W(\xi) + \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta} \right)^\beta \Rightarrow -V^\beta(\xi) \leq -2^{\beta-1} W^\beta(\xi) - \frac{\gamma_3}{\gamma_2}, \quad (23)$$

where $W(\xi) = V(\xi) - \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta} > 0$.

Considering Equations (23) and (20), we obtain:

$$D_t^q W(\xi) \leq -\gamma_1 W(\xi) - 2^{\beta-1} \gamma_2 W^\beta(\xi) - \gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{\frac{1}{\beta}}. \quad (24)$$

It is clear from Equation (24) that $W(t) \leq W(0)$ [39]. Then, for $0 < \beta < 1$, one has $W^{\beta-1}(t) \geq W^{\beta-1}(0)$. Hence:

$$D_t^q W(\xi) \leq -\left(\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0) \right) W(\xi) - \gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{\frac{1}{\beta}}. \quad (25)$$

According to Lemma 6, it is observed that:

$$W(\xi) \leq \frac{1}{q} \left(W(0) + \frac{\gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{\frac{1}{\beta}}}{\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0)} \right) \exp \left(-\left(\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0) \right)^{\frac{1}{q}} t \right) - \frac{\gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}}{\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0)}. \quad (26)$$

Then, through a straightforward computation, it can be determined that $(\xi) \leq \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{\frac{1}{\beta}}$, $\forall t \geq T_s$, where:

$$T_s = \left[\frac{1}{(\gamma_1 + 2^{\beta-1}\gamma_2 W^{\beta-1}(0))} \right]^{1/q} \ln \left[\frac{(\gamma_1 + 2^{\beta-1}\gamma_2 W^{\beta-1}(0)) W(0) + \gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}}{q \gamma_1 \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{1/\beta}} \right], \quad (27)$$

where $W(0) = V(0) - \left(\frac{\gamma_3}{2^{\beta-1}\gamma_2} \right)^{\frac{1}{\beta}}$.

- **When $\xi(0) \in \Omega_{\xi 1}$:** From the above analysis, $\xi(t)$ does not go beyond the set $\Omega_{\xi 1}$.

This ends the proof of Lemma 9. $\square$

Lemma 10. For System (6), if there exists a continuous and non-negative function $V(t)$ satisfying:

$$D_t^q V(\xi) \leq -\gamma_1 V(\xi) - \gamma_2 V^\beta(\xi) + \gamma_3, \quad t \geq 0, \quad (28)$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, and $0 < \beta < 1$, then the trajectory of $\xi(t)$ exhibits PFTS and converges to the following residual set:

$$\Omega = \left\{ \xi(t) | V(\xi) \leq \frac{\gamma_3}{\gamma_1} \right\}, \quad \text{for } t \geq T_s, \quad (29)$$

and the convergence time is:

$$T_s = \left(\frac{t_c^*}{\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0)} \right)^{1/q}, \quad (30)$$

where $W(0) = V(0) - \frac{\gamma_3}{\gamma_1}$, and t_c^* is the solution of the following nonlinear equation:

$$E_q(-t_c^*) - \frac{\frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}}{(\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0))W(0) + \frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}} = 0. \quad (31)$$

Proof of Lemma 10. To define the following set $\Omega = \{\xi(t) | V(\xi) \leq \frac{\gamma_3}{\gamma_1}\}$, one has two cases:

- **When $\xi(0) \notin \Omega$** (i.e., if $V(\xi) > \frac{\gamma_3}{\gamma_1}$). By defining $W(\xi) = V(\xi) - \frac{\gamma_3}{\gamma_1}$, we have:

$$V^\beta(\xi) = \left(W(\xi) + \frac{\gamma_3}{\gamma_1} \right)^\beta \Rightarrow -V^\beta(\xi) \leq -2^{\beta-1} W^\beta(\xi) - \frac{2^{\beta-1} \gamma_3}{\gamma_1}, \quad (32)$$

where $W(\xi) = V(\xi) - \frac{\gamma_3}{\gamma_1} > 0$.

Considering Equations (32) and (28), we obtain:

$$D_t^q W(\xi) \leq -\gamma_1 W(\xi) - 2^{\beta-1} \gamma_2 W^\beta(\xi) - 2^{\beta-1} \gamma_2 \gamma_3 \gamma_1^{-1}. \quad (33)$$

From Lemma 4 in [39] and Equation (33), it is clear that $W(\xi) \leq W(0)$. Then, it follows $W^{\beta-1}(\xi) \geq W^{\beta-1}(0)$ such that:

$$D_t^q W(\xi) \leq -\left(\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0) \right) W(\xi) - \frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}. \quad (34)$$

Using Lemma 7, we can obtain:

$$W(\xi) \leq \left(W(0) + \frac{\frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}}{\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0)} \right) E_q \left(-\left(\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0) \right) t^q \right) - \frac{\frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}}{\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0)}. \quad (35)$$

Then, through a straightforward computation, it can be determined that $V(\xi) \leq \frac{\gamma_3}{\gamma_1}, \forall t \geq T_s$, where:

$$T_s = \left(\frac{t_c^*}{\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0)} \right)^{1/q}, \quad (36)$$

where t_c^* is the solution of $E_q(-t_c^*) - \frac{\frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}}{(\gamma_1 + 2^{\beta-1} \gamma_2 W^{\beta-1}(0))W(0) + \frac{2^{\beta-1} \gamma_2 \gamma_3}{\gamma_1}} = 0$ and $W(0) = V(0) - \frac{\gamma_3}{\gamma_1}$.

- **When $\xi(0) \in \Omega$:** From the above analysis, $\xi(t)$ does not go beyond the set Ω .

This ends the proof of Lemma 10. $\square$

Lemma 11. For System (6), if there exists a continuous and non-negative function $V(t)$ satisfying:

$$D_t^q V(\xi) \leq -\gamma_1 V(\xi) - \gamma_2 V^\beta(\xi) + \gamma_3, \quad t \geq 0 \quad (37)$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, and $\gamma_3 > 0$, then the trajectory of $\xi(t)$ exhibits PFTS and converges to the following residual set:

$$\Omega = \{\xi(t) | V(\xi(t)) \leq \frac{\gamma_3}{\gamma_1}\}, \text{ for } t \geq T_s, \quad (38)$$

and the convergence time is:

$$T_s = \left(\frac{t_c^*}{\gamma_2 V^{\beta-1}(0)} \right)^{1/q}, \quad (39)$$

where t_c^* is the solution of the following nonlinear equation:

$$E_q(-t_c^*) - \frac{Y_3}{Y_1 V(0)} = 0. \quad (40)$$

Proof of Lemma 11. Define the following set $\Omega_\xi = \{\xi | V(\xi) \leq \frac{Y_3}{Y_1}\}$. When $\xi(0) \notin \Omega_\xi$, (37) can be rewritten as:

$$D_t^q V(\xi) \leq -Y_2 V^\beta(\xi), \quad (41)$$

From Lemma 4 in [39] and Equation (41), it is clear that $V(\xi) \leq V(0)$. Then, it follows $V^{\beta-1}(\xi) \geq V^{\beta-1}(0)$ such that:

$$D_t^q V(\xi) \leq -Y_2 V^{\beta-1}(0) V(\xi). \quad (42)$$

Using Lemma 7, we can obtain:

$$V(\xi) \leq V(0) E_q(-Y_2 V^{\beta-1}(0) t^q) \quad (43)$$

From Equation (43), $V(\xi)$ decreases monotonically to the set Ω_ξ within an FT, T_s . By setting the right hand of Equation (43) equal to $\frac{Y_3}{Y_1}$, the ST can be determined as follows:

$$T_s = \left(\frac{t_c^*}{Y_2 V^{\beta-1}(0)} \right)^{1/q}, \quad (44)$$

with t_c^* being the solution of $E_q(-t_c^*) - \frac{Y_3}{Y_1 V(0)} = 0$.

Therefore, there exists an ST, T_s such that $\xi(t) \in \Omega_\xi$, for all $t \geq T_s$. This completes the proof of Lemma 11. $\square$

Lemma 12. For System (6), if there exists a continuous and non-negative function $V(t)$ satisfying:

$$D_t^q V(\xi) \leq -Y_1 V(\xi) - Y_2 V^\beta(\xi) + Y_3, \quad t \geq 0, \quad (45)$$

where $Y_1 > 0$, $Y_2 > 0$, $Y_3 > 0$, and $0 < b < 1$, then the trajectory of $\xi(t)$ exhibits PFTS and converges to the following residual set:

$$\Omega = \left\{ \xi(t) | V(\xi(t)) \leq \frac{Y_3}{Y_1(1-b)} \right\}, \quad \text{for } t \geq T_s, \quad (46)$$

and the convergence time is:

$$T_s = \left(\frac{t_c^*}{b Y_1 + Y_2 V^{\beta-1}(0)} \right)^{1/q}, \quad (47)$$

where t_c^* is the solutions of the following nonlinear equation:

$$E_q(-t_c^*) = \frac{Y_3}{Y_1(1-b)V(0)}. \quad (48)$$

Proof of Lemma 12. Define the following set: $\Omega_\xi = \left\{ \xi | V(\xi) \leq \frac{Y_3}{Y_1(1-b)} \right\}$. When $\xi(0) \notin \Omega_\xi$, the inequality in Equation (45) becomes:

$$D_t^q V(\xi) \leq -Y_1 b V(\xi) - Y_2 V^\beta(\xi). \quad (49)$$

From Lemma 4 in [39] and Equation (49), it is clear that $V(\xi) \leq V(0)$. Then, it follows $V^{\beta-1}(\xi) \geq V^{\beta-1}(0)$ such that:

$$D_t^q V(\xi) \leq -\left(Y_1 b + Y_2 V^{\beta-1}(0) \right) V(\xi). \quad (50)$$

Using Lemma 7, we can obtain:

$$V(\xi) \leq V(0)E_q \left(- \left(Y_1 b + Y_2 V^{\beta-1}(0) \right) t^q \right), \quad (51)$$

From Equation (51), it is clear that $V(\xi)$ decreases monotonically to the set Ω_ξ within an FT, T_s . By setting the right hand of Equation (51) equal to $\frac{Y_3}{Y_1(1-b)}$, the ST can be determined as follows:

$$T_s = \left(\frac{t_c^*}{bY_1 + Y_2 V^{\beta-1}(0)} \right)^{1/q}, \quad (52)$$

where t_c^* is the solution of $E_q(-t_c^*) - \frac{Y_3}{(1-b)Y_1 V(0)} = 0$.

Therefore, there exists an ST, T_s such that $\xi(t) \in \Omega_\xi$, for all $t \geq T_s$. This completes the proof of Lemma 12. $\square$

Remark 2. The Lyapunov PFTS criteria considered in this paper, which are of the form $D_t^q V(\xi) \leq -Y_1 V(\xi) - Y_2 V^\beta(\xi) + Y_3$, where $0 < \beta < 1$, Y_1 , Y_2 and $Y_3 \geq 0$, can guarantee faster convergence rates for system states compared to the criteria $D_t^q V(\xi) \leq -Y_2 V^\beta(\xi) + Y_3$. Moreover, the convergence region can diminish in size. It is important to note that the rate of convergence is influenced by both the starting point $V(0)$ and the specific values of the parameters Y_1 and Y_2 . By employing FT control, which is characterized by its swift convergence, the system can react quickly and adjust effectively in the face of uncertainties and disturbances, demonstrating its resilience.

Remark 3. Lemmas 9–12 all address PFTS for fractional-order systems, yet they differ noticeably in their proofs, convergence-time expressions, and residual-set forms. They can be viewed as refinements of Lemma 8 from [38], and Table 1 lays out the main differences. The real trade-off comes down to three things: how simple the residual set is, whether the settling time is explicit or requires numerical solving, and whether the unknown constant b shows up (the latter making the calculation of this convergence time difficult). Lemmas 8 and 12 both suffer from that extra parameter and force one to solve nonlinear equations for the settling time—Lemma 8 even adds a min–max structure that makes it more conservative. Lemmas 10 and 11 drop b and keep the residual set simple, but they still rely on numerical resolution for the settling time. Lemma 9 comes out ahead on both counts: a clean residual set and an explicit settling time via Equation (22). Therefore, overall, Lemma 9 seems the most practical, while Lemmas 8 and 12 are the least convenient. That said, we are not making any claim about which estimate is more accurate—settling that would require a more thorough simulation study than what we have included here.

Table 1. Comparison of the existing lemma ([38]) and the proposed ones.

Existing Results	Residual Set	Settling Time (ST)	Remarks
Lemma 8 [38]	$\min \left\{ \frac{Y_3}{Y_1(1-b)}, \left(\frac{Y_3}{Y_2(1-b)} \right)^{1/\beta} \right\}$	$\max \left\{ \left(\frac{t_{c1}^*}{bY_1 + Y_2 V^{\beta-1}(0)} \right)^{1/q}, \left(\frac{t_{c2}^*}{Y_1 + bY_2 V^{\beta-1}(0)} \right)^{1/q} \right\}$	1- Residual set depends on unknown constant $b \in (0,1)$. 2- ST requires solving a non-linear equation. 3- Both are characterized by the min/max of two quantities.
Lemma 9	$\left(\frac{Y_3}{2^{\beta-1}Y_2} \right)^{\frac{1}{\beta}}$	Equation (22) (explicit)	1- Residual set has a simple expression. 2- ST is given explicitly; no non-linear equation solving required.
Lemma 10	$\frac{Y_3}{Y_1}$	$\left(\frac{t_c^*}{Y_1 + 2^{\beta-1}Y_2 V^{\beta-1}(0)} \right)^{1/q}$	1- Residual set is simple. 2- ST requires solving a non-linear equation.
Lemma 11	$\frac{Y_3}{Y_1}$	$\left(\frac{t_c^*}{Y_2 V^{\beta-1}(0)} \right)^{1/q}$	1- Residual set is simple. 2- ST requires solving a non-linear equation
Lemma 12	$\frac{Y_3}{Y_1(1-b)}$	$\left(\frac{t_c^*}{bY_1 + Y_2 V^{\beta-1}(0)} \right)^{1/q}$	1- Residual set depends on unknown $b \in (0,1)$.

2- ST requires solving a non-linear equation.

Remark 4. A natural question arises: Why do we propose these four new technical lemmas for the stability of fractional-order systems? It is well recognized that finite-time control of fractional-order nonlinear systems remains an open and challenging problem. To address this, numerous studies [12,18,29,30,32,34,35] have focused on establishing lemmas related to practical finite-time stability (PFTS) or finite-time stability (FTS), typically relying on inequalities of the form $D_t^q V(t) \leq -Y_1 V(t) - Y_2 V^\beta(t) + Y_3$, or its reduced variant, $D_t^q V(t) \leq -Y_2 V^\beta(t) + Y_3$. In most of these works, the following derivative formulas for power functions are employed:

1. In [12,34,35], for $0 < q < 1$ and $\mu_1 > 1$, $D_t^q f^{\mu_1}(t) = \frac{\Gamma(2-q)\Gamma(1+\mu_1)}{\Gamma(1+\mu_1-q)} f^{\mu_1-1}(t) D^q f(t)$.
2. In [18,29,30,32], for $0 < q < 1$ and $\mu_2 \in \mathbb{R}$, $D_t^q f^{\mu_2}(t) = \frac{\Gamma(1+\mu_2)}{\Gamma(1+\mu_2-q)} f^{\mu_2-1}(t) D^q f(t)$.

However, recent contributions [36–40] have rigorously demonstrated, via specific counterexamples, that these equalities are generally incorrect. This discrepancy highlights a critical gap in the existing theoretical framework. It is therefore both interesting and essential either to establish the correct formulations for such fractional derivatives or to simply avoid their use in the demonstration of finite-time stability lemmas. In response, this article proposes four new alternative lemmas adapted to the PFTS of fractional-order Caputo-type nonlinear systems (CFONS), without employing the formulations above. These lemmas can be viewed as an extension of the classical Lyapunov direct method. As a result, any Caputo FONS satisfying the proposed condition in Equation (7) is guaranteed to converge to a sufficiently small neighborhood within a finite time.

Remark 5. This remark provides counterexamples illustrating that the preceding chain-like formulations for the fractional-order derivative are not, in fact, generally applicable. In [39], the authors pointed out that the commonly used formula $D_t^q f^{\mu_2}(t) = \frac{\Gamma(1+\mu_2)}{\Gamma(1+\mu_2-q)} f^{\mu_2-1}(t) D^q f(t)$ for $0 < q < 1$, $\mu_2 \in \mathbb{R}$, is not universally valid. In particular, it fails for some specific parameter choices, such as $f(t) = t$, $q = 0.5$, and $\mu_2 = 1$. This observation likewise undermines the results of other works [18,29,30,32] that rely on the same identity. Moreover, the analogous expression $D_t^q f^{\mu_1}(t) = \frac{\Gamma(2-q)\Gamma(1+\mu_1)}{\Gamma(1+\mu_1-q)} f^{\mu_1-1}(t) D^q f(t)$ for $0 < q < 1$, $\mu_1 > 1$, employed in [12,34,35], is also generally incorrect. This formula appears to be a naive extrapolation of the classical integer-order chain rule; however, fractional derivatives do not satisfy such a rule in general. A straightforward and clear counterexample suffices to illustrate this fallacy: take $f(t) = t^2$, $q = 0.5$, and $\mu_1 = 2$. Under these conditions, the equality clearly breaks down. For a more comprehensive discussion, we refer the reader to [36–40].

2.4. Fuzzy Systems

The FLS is constructed of a knowledge base, fuzzifier, defuzzifier and fuzzy inference engine, wherein the knowledge base covers a series of fuzzy IF–THEN rules:

$$R^{(i)}: \text{if } z_1 \text{ is } A_1^i \text{ ... and ... and } z_n \text{ is } A_n^i, \text{ then } y \text{ is } Y^i, i = 1, \dots, N, \quad (53)$$

where $Z^T = [z_1, \dots, z_n]$ and y stand for the input and output of this FLS, respectively. N is the rule number.

When employing product-type inference, central average-type defuzzification, and singleton-type fuzzification, the outcome of the FLS can be expressed in the following manner:

$$y(Z) = \frac{\sum_{i=1}^N \mu_{A_j^i}(z_j)}{\sum_{i=1}^N \left(\prod_{j=1}^n \mu_{A_j^i}(z_j) \right)} = \theta^T \psi(Z), \quad (54)$$

with $\mu_{A_j^i}(z_j)$ being the memberships function of fuzzy set A_j^i . $\theta^T = [Y^1, \dots, Y^N]$ is the adaptive weight vector, and $\psi^T = [\psi^1, \dots, \psi^N]$ is the fuzzy basis function vector with $\psi^i(Z) = \prod_{j=1}^n \mu_{A_j^i}(z_j) / \sum_{i=1}^N \left(\prod_{j=1}^n \mu_{A_j^i}(z_j) \right)$.

According to [52], the continuous function $y(Z): R^n \rightarrow R$ can be ideally approximated by an FLS as:

$$y(Z) = \theta^{*T} \psi(Z) + \varepsilon_f(Z), \quad (55)$$

in which $\theta^{*T} \in R^N$ is an unknown constant vector. The latter represents the best parameter value. $\varepsilon_f(Z)$ is the minimum fuzzy approximation error being bounded.

Lemma 13. Suppose we have a continuous real function, $f(Z)$, defined on a bounded set Ω_Z in R^n . For any chosen positive constant $\bar{\varepsilon}$, there exists an optimal FLS, denoted as $\theta^{*T} \psi(Z)$, as described in reference [52]:

$$|f(Z) - \theta^{*T} \psi(Z)| \leq \bar{\varepsilon}. \quad (56)$$

2.5. System Description

Consider a CFONS in the following non-strict-feedback form:

$$\begin{cases} D_t^q x_i = f_i(x) + g_i x_{i+1} + d_i(x, t), & i = 1, \dots, n-1, \\ D_t^q x_n = f_n(x) + g_n u(v) + d_n(x, t), \\ y = x_1, \end{cases} \quad (57)$$

where $x = [x_1, \dots, x_n]^T \in R^n$, $y \in R$, and $v(t) \in R$ represent the system pseudo-state vector, the system output, and the control input, respectively. $f_i(x)$, where $1 \leq i \leq n$, stands for an uncertain nonlinear function. $d_i(x, t) \in R$ is the external disturbance, while g_i represents the known control gain. Furthermore, $u(v)$ is the actuator saturation output. The saturation model is given by:

$$u(v) = \text{sat}(v) = \begin{cases} \text{sign}(v) u_{\max}, & \text{for } |v(t)| \geq u_{\max}, \\ v, & \text{for } |v(t)| < u_{\max}. \end{cases} \quad (58)$$

where $v \in R$ is the saturation input, and $u_{\max}$ is a positive constant.

Based on [53], the saturation nonlinearity $u(v)$ can be modeled by a smooth function $h(v)$:

$$u(v) = h(v) + \rho(v), \quad (59)$$

where $\rho(v)$ is a bounded approximation error, and $\frac{\partial h(v)}{\partial v} > 0$ (i.e., $h(v)$ is strictly increasing).

The control objectives outlined in this article can be summarized as follows:

1. Achieving practical finite-time (PFT) tracking of the desired trajectory $y_d(t)$ by the control output $y(t)$.
2. Ensuring that all closed-loop signals remain bounded within an FT frame.

The following common assumptions are utilized throughout this paper.

Assumption 1. [53] The approximation error $\rho(v)$ must satisfy $|\rho(v)| \leq \bar{\rho}$, where $\bar{\rho}$ represents an uncertain constant.

Assumption 2. [53,54] The control gains g_i need to meet the following conditions: $g_i \neq 0$, for $1 \leq i \leq n$. Furthermore, $\text{sign}(g_i) = \text{sign}\left(g_i + \frac{\partial f_i(x)}{\partial x_{i+1}}\right)$, for $1 \leq i \leq n-1$.

Assumption 3. It is assumed that the desired signal y_d and its fractional-order time derivatives (FOTDs), namely $D_t^q y_d, D_t^q (D_t^q y_d)$, are continuous and bounded.

Assumption 4. [27,28,55] The external disturbance $d_i(x, t)$ and its FOTD should satisfy $|d_i| \leq \theta_{i,0}$ and $|D_t^q d_i| \leq \theta_{i,1}$, where $\theta_{i,0}$ and $\theta_{i,1} \geq 0$ are unknown constants.

Assumption 5. The FOTD of the augmented disturbance function P_i must fulfill $|D_t^q P_i| \leq \Delta_i$, $1 \leq i \leq n$, with Δ_i representing an unknown constant. The function P_i will be defined later.

Remark 6: The assumptions used in this work are standard and classical in adaptive backstepping control based on disturbance observers. We now clarify each one individually.

1. About Assumption 1: This assumption is realistic and widely adopted in the literature [53,54]. Several common choices for the function $h(v)$ support this, such as $h(v) = u_{\max} \tanh(v/\varepsilon_f)$, and $h(v) = \frac{2}{\pi} u_{\max} \arctan(v/\varepsilon_f)$. These functions are strictly increasing and yield uniformly bounded approximation errors ($\rho(v)$).
2. About Assumption 2: The condition $g_i \neq 0$ is a standard controllability requirement; if $g_i = 0$, the system becomes uncontrollable. The condition $\text{sign}(g_i) = \text{sign}(g_i + (\partial f_i(x)/\partial x_{i+1}))$ is known as the control direction condition. It ensures that the sign of the overall control direction is governed by g_i , and that the additional term does not reverse it [53,54]. Knowing the sign of the high-frequency gain is essential for control design, and this remains a classical assumption. When this sign is unknown, Nussbaum-type functions are typically employed to estimate it adaptively online.
3. About Assumption 3: In adaptive backstepping, the reference signal is not assumed to be smooth and differentiable for convenience, but because the recursive controller construction inherently requires it. Specifically, in this paper, we assume the existence of the first q and $2q$ derivatives of the desired reference y_d . If the reference is not sufficiently smooth, it should be pre-filtered to generate a differentiable trajectory.
4. About Assumption 4: The bounded fractional-order derivative assumption ensures that the disturbance cannot vary infinitely fast, allowing a finite-gain observer to maintain a bounded estimation error. Theoretically, this places the disturbance in the class of Lipschitz continuous functions and guarantees observer stability, with the variation rate directly determining the residual error magnitude. Without this standard assumption, no finite observer gain could prevent error divergence and loss of closed-loop stability [27,28,53–55].

Remark 7. For the special case where $q = 1$, System (57) degenerates into an integer-order system with a non-strict-feedback structure [53]. In this sense, our proposed technical lemmas and FT controllers can be interpreted as generalizations of existing results for integer-order nonlinear systems.

3. Disturbance Observer-Based FTFAC Schemes

In this section, a set of novel FTFAC schemes will be systematically designed within a backstepping framework. These schemes successfully circumvent the common singularity issue prevalent in numerous previous FT control designs.

3.1. First Control Scheme

Let us introduce the following coordinate transformation:

$$\begin{cases} z_i = x_i - \alpha_{fi-1} + l_i, & \text{for } i = 1, \dots, n-1, \\ z_n = x_n - \alpha_{fn-1} + l_n + \omega. \end{cases} \quad (60)$$

The variables l_i and ω will be defined later. $\alpha_{f0} = y_d$ and α_{fi} (for $i = 1, \dots, n-1$) are generated by the following fast finite-time filter (FFTF):

$$D_t^q \alpha_{fi} = -\frac{\tau_{1i}^2 e_{ai}^3}{\sqrt{\tau_{1i}^2 e_{ai}^4 + \epsilon_{a1i}^2}} - \frac{\tau_{2i}^2 (e_{ai}^{2p-1})^2 e_{ai}}{\sqrt{\tau_{2i}^2 (e_{ai}^p)^4 + \epsilon_{a2i}^2}} - \frac{\tau_{3i}^2 e_{ai}}{\sqrt{\tau_{3i}^2 e_{ai}^2 + \epsilon_{a3i}^2}}, \text{ with } \alpha_{fi}(0) = \alpha_i(0), \quad (61)$$

where $\frac{1}{2} < p < 1$, $\tau_{ji} > 0$ are design constants, and $\epsilon_{\alpha ji} > 0$ are small design constants. $e_{\alpha i} = \alpha_{fi} - \alpha_i$ are the filtering errors, and α_i are VC laws to be designed.

The compensation signal of the saturation effect ω can be generated by:

$$D_t^q \omega = -\tau_0 \omega + g_n(v - h(v)), \quad (62)$$

where $\tau_0 > 0$ is a design constant.

Theorem 1. When considering the FFTF (Equation (61)), the following two properties can be ensured:

- The signals $\alpha_{fi} - \alpha_i$ and $D_t^q \alpha_{fi}$ are bounded within an FT.
- The filtering errors $e_{\alpha i} = \alpha_{fi} - \alpha_i$ converge rapidly and practically, reaching an arbitrarily small neighborhood of the origin within an FT.

Proof of Theorem 1. Construct a quadratic Lyapunov function candidate (QLFC) of the form $V_\alpha = \sum_{i=1}^{n-1} \frac{1}{2} e_{\alpha i}^2$. Then, by invoking Property 1 and Lemma 3, the FOTD of V_α along the trajectories of System (61) can be bounded as follows:

$$D_t^q V_\alpha \leq -\sum_{i=1}^{n-1} \tau_{1i} e_{\alpha i}^2 - \sum_{i=1}^{n-1} \tau_{2i} e_{\alpha i}^{2p} - \sum_{i=1}^{n-1} \tau_{3i} |e_{\alpha i}| + \sum_{i=1}^{n-1} |e_{\alpha i}| |D_t^q \alpha_{fi}| + \sum_{i=1}^{n-1} (\epsilon_{\alpha 1i} + \epsilon_{\alpha 2i} + \epsilon_{\alpha 3i}). \quad (63)$$

Utilizing Lemmas 2 and choosing $\tau_{3i} \geq |D_t^q \alpha_{fi}|$, we can express Equation (63) as follows:

$$D_t^q V_\alpha \leq -\mu_{11} V_\alpha - \mu_{12} V_\alpha^p + \mu_{13}, \quad (64)$$

where $\mu_{11} = \min\{2\tau_{11}, \dots, 2\tau_{1n-1}\}$, $\mu_{12} = \min\{2^p \tau_{21}, \dots, 2^p \tau_{2n-1}\}$, and $\mu_{13} = \sum_{i=1}^{n-1} (\epsilon_{\alpha 1i} + \epsilon_{\alpha 2i} + \epsilon_{\alpha 3i})$.

According to Lemma 8 and our Lemmas 9–12, the filtering error $e_{\alpha i}$ rapidly approaches a small region near the origin within a finite duration $T_{\alpha 1}$. Note that Lemmas 8–12 estimate the convergence region and time. This completes the proof. $\square$

To compensate for the filtering errors, the following auxiliary system is designed:

$$\begin{cases} D_t^q l_i = -\frac{\gamma_{li}^2 l_i^3}{\sqrt{\gamma_{li}^2 l_i^4 + \epsilon_{l1i}^2}} - \frac{\kappa_{li}^2 (l_i^{2p-1})^2 l_i}{\sqrt{\kappa_{li}^2 (l_i^p)^4 + \epsilon_{l2i}^2}} + g_i(\alpha_i - \alpha_{fi}), \text{ for } i = 1, \dots, n-1, \\ D_t^q l_n = -\frac{\gamma_{ln}^2 l_n^3}{\sqrt{\gamma_{ln}^2 l_n^4 + \epsilon_{l1n}^2}} - \frac{\kappa_{ln}^2 (l_n^{2p-1})^2 l_n}{\sqrt{\kappa_{ln}^2 (l_n^p)^4 + \epsilon_{l2n}^2}}, \end{cases} \quad (65)$$

where ϵ_{l1i} and $\epsilon_{l2i} > 0$ are small design constants, and κ_{li} and $\gamma_{li} > 0$ are design constants.

Inserting Equation (59) into Equation (57), one has:

$$\begin{cases} D_t^q x_i = f_i(x) + g_i x_{i+1} + d_i(x, t), \quad i = 1, \dots, n-1, \\ D_t^q x_n = f_n(x) + g_n h(v) + g_n \rho(v) + d_n(x, t), \\ y = x_1. \end{cases} \quad (66)$$

To simplify the control design process and streamline the stability analysis, System (66) can be expressed as follows:

$$\begin{cases} D_t^q x_i = \bar{f}_i(\bar{x}_i) + \frac{\gamma_{li}^2 l_i^3}{\sqrt{\gamma_{li}^2 l_i^4 + \epsilon_{l1i}^2}} + \frac{\kappa_{li}^2 (l_i^{2p-1})^2 l_i}{\sqrt{\kappa_{li}^2 (l_i^p)^4 + \epsilon_{l2i}^2}} + g_i l_{i+1} + g_i x_{i+1} + d_i(x, t), \quad 1 \leq i \leq n-1 \\ D_t^q x_n = \bar{f}_n(\bar{x}_n) + \frac{\gamma_{ln}^2 l_n^3}{\sqrt{\gamma_{ln}^2 l_n^4 + \epsilon_{l1n}^2}} + \frac{\kappa_{ln}^2 (l_n^{2p-1})^2 l_n}{\sqrt{\kappa_{ln}^2 (l_n^p)^4 + \epsilon_{l2n}^2}} + g_n h(v) + d_n(x, t), \end{cases} \quad (67)$$

with $\bar{f}_i(\bar{x}_i) = f_i(x) - \frac{\gamma_{li}^2 l_i^3}{\sqrt{\gamma_{li}^2 l_i^4 + \epsilon_{li}^2}} - \frac{\kappa_{li}^2 (l_i^{2p-1})^2 l_i}{\sqrt{\kappa_{li}^2 (l_i^p)^4 + \epsilon_{li}^2}} - g_i l_{i+1}$, for $1 \leq i \leq n-1$, $\bar{f}_n(\bar{x}_n) = f_n(x) - \frac{\gamma_{ln}^2 l_n^3}{\sqrt{\gamma_{ln}^2 l_n^4 + \epsilon_{ln}^2}} - \frac{\kappa_{ln}^2 (l_n^{2p-1})^2 l_n}{\sqrt{\kappa_{ln}^2 (l_n^p)^4 + \epsilon_{ln}^2}} + g_n \rho(v)$, $\bar{x}_i = [x^T, l_i, l_{i+1}]^T$, $\bar{x}_n = [x^T, l_n, v]^T$, where l_i , for $1 \leq i \leq n$, represents auxiliary signals generated by Equation (65).

Substituting the ideal approximations of the unknown functions $\bar{f}_i(\bar{x}_i)$ obtained from Fuzzy System (55) into Equation (67) yields:

$$\begin{cases} D_t^q x_i = \theta_i^{*T} \psi_i(\bar{x}_i) + \frac{\gamma_{li}^2 l_i^3}{\sqrt{\gamma_{li}^2 l_i^4 + \epsilon_{li}^2}} + \frac{\kappa_{li}^2 (l_i^{2p-1})^2 l_i}{\sqrt{\kappa_{li}^2 (l_i^p)^4 + \epsilon_{li}^2}} + g_i l_{i+1} + g_i x_{i+1} + \varepsilon_i(\bar{x}_i) + d_i(x, t), \\ D_t^q x_n = \theta_n^{*T} \psi_n(\bar{x}_n) + \frac{\gamma_{ln}^2 l_n^3}{\sqrt{\gamma_{ln}^2 l_n^4 + \epsilon_{ln}^2}} + \frac{\kappa_{ln}^2 (l_n^{2p-1})^2 l_n}{\sqrt{\kappa_{ln}^2 (l_n^p)^4 + \epsilon_{ln}^2}} + g_n h(v) + \varepsilon_n(\bar{x}_n) + d_n(x, t) \end{cases} \quad (68)$$

To address the algebraic-loop issue caused by $\psi_i(\bar{x}_i)$, Equation (68) can be expressed as:

$$\begin{cases} D_t^q x_i = \theta_i^{*T} \psi_i(Z_i) + \frac{\gamma_{li}^2 l_i^3}{\sqrt{\gamma_{li}^2 l_i^4 + \epsilon_{li}^2}} + \frac{\kappa_{li}^2 (l_i^{2p-1})^2 l_i}{\sqrt{\kappa_{li}^2 (l_i^p)^4 + \epsilon_{li}^2}} + g_i l_{i+1} + g_i x_{i+1} + \theta_i^{*T} (\psi_i(\bar{x}_i) - \psi_i(Z_i)) + \varepsilon_i(\bar{x}_i) + d_i(x, t), \\ D_t^q x_n = \theta_n^{*T} \psi_n(Z_n) + \frac{\gamma_{ln}^2 l_n^3}{\sqrt{\gamma_{ln}^2 l_n^4 + \epsilon_{ln}^2}} + \frac{\kappa_{ln}^2 (l_n^{2p-1})^2 l_n}{\sqrt{\kappa_{ln}^2 (l_n^p)^4 + \epsilon_{ln}^2}} + g_n h(v) + \theta_n^{*T} (\psi_n(\bar{x}_n) - \psi_n(Z_n)) + \varepsilon_n(\bar{x}_n) + d_n(x, t), \end{cases} \quad (69)$$

with $Z_i = [x_1, x_2, \dots, x_i, l_i, l_{i+1}]^T \in R^{i+2}$, for $1 \leq i \leq n-1$, and $Z_n = [x^T, l_n]^T \in R^{n+1}$.

From Equations (60), (62), (65) and (69), we have:

$$\begin{cases} D_t^q Z_i = \theta_i^{*T} \psi_i(Z_i) + g_i z_{i+1} + g_i \alpha_i - D_t^q \alpha_{fi-1} + P_i, & 1 \leq i \leq n-1 \\ D_t^q Z_n = \theta_n^{*T} \psi_n(Z_n) + g_n v - \tau_0 \omega - D_t^q \alpha_{fn-1} + P_n, \end{cases} \quad (70)$$

where:

$$\begin{cases} P_i = \theta_i^{*T} (\psi_i(\bar{x}_i) - \psi_i(Z_i)) + \varepsilon_i(\bar{x}_i) + d_i(x, t), & 1 \leq i \leq n-1, \\ P_n = \theta_n^{*T} (\psi_n(\bar{x}_n) - \psi_n(Z_n)) + \varepsilon_n(\bar{x}_n) + d_n(x, t). \end{cases} \quad (71)$$

3.1.1. FODO Design

For System (70), let us design the following FODO:

$$\begin{cases} D_t^q \hat{P}_i = -\rho_i \hat{P}_i + \rho_i (D_t^q Z_i + D_t^q \alpha_{fi-1} - \hat{\theta}_i^T \psi_i(Z_i) - g_i z_{i+1} - g_i \alpha_i), & 1 \leq i \leq n-1 \\ D_t^q \hat{P}_n = -\rho_n \hat{P}_n + \rho_n (D_t^q Z_n + D_t^q \alpha_{fn-1} + \tau_0 \omega - \hat{\theta}_n^T \psi_n(Z_n) - g_n v), \end{cases} \quad (72)$$

Here, $\hat{\theta}_i$ represents an estimation of θ_i^* , and $\rho_i > 0$ denotes the observer gain.

Using Equations (70) and (72), we can formulate the dynamics of the observation error in the following manner:

$$D_t^q \tilde{P}_i = -\rho_i \tilde{P}_i - \rho_i \tilde{\theta}_i^T \psi_i(Z_i) - D_t^q P_i, \quad 1 \leq i \leq n \quad (73)$$

Here, $\tilde{\theta}_i = \hat{\theta}_i - \theta_i^*$ and $\tilde{P}_i = \hat{P}_i - P_i$ denote, respectively, the discrepancies in parameter estimation and disturbance estimation.

Remark 8. To simplify the practical application of the proposed FODO, Equation (72) can be rewritten as follows:

$$\begin{cases} D_t^q \hat{S}_i = -\rho_i \hat{S}_i + \rho_i (-\rho_i z_i - \rho_i \alpha_{fi-1} - \hat{\theta}_i^T \psi_i(Z_i) - g_i z_{i+1} - g_i \alpha_i), & 1 \leq i \leq n-1, \\ D_t^q \hat{S}_n = -\rho_n \hat{S}_n + \rho_n (-\rho_n z_n - \rho_n \alpha_{fn-1} + \tau_0 \omega - \hat{\theta}_n^T \psi_n(Z_n) - g_n v), \\ \hat{P}_i = \hat{S}_i + \rho_i z_i + \rho_i \alpha_{fi-1}, & 1 \leq i \leq n. \end{cases} \quad (74)$$

3.1.2. Design of FTFAC

Design the VCs α_i , the actual control law $\alpha_n = v$, and their respective adaptation laws as follows:

$$\begin{cases} \alpha_1 = \frac{1}{g_1} \left[-\gamma_1 z_1 - \frac{k_1^2 z_1^3}{\sqrt{k_1^2 z_1^4 + \epsilon_{z11}^2}} - \frac{s_1^2 (z_1^{2p-1})^2 z_1}{\sqrt{s_1^2 (z_1^p)^4 + \epsilon_{z21}^2}} - \hat{\theta}_1^T \psi_1(Z_1) - \hat{P}_1 + D_t^q \alpha_{f0} \right], \\ \alpha_i = \frac{1}{g_i} \left[-\gamma_i z_i - \frac{k_i^2 z_i^3}{\sqrt{k_i^2 z_i^4 + \epsilon_{z1i}^2}} - \frac{s_i^2 (z_i^{2p-1})^2 z_i}{\sqrt{s_i^2 (z_i^p)^4 + \epsilon_{z2i}^2}} - \hat{\theta}_i^T \psi_i(Z_i) - \hat{P}_i - g_{i-1} z_{i-1} + D_t^q \alpha_{fi-1} \right], 2 \leq i \leq n-1, \\ \alpha_n = \frac{1}{g_n} \left[-\gamma_n z_n - \frac{k_n^2 z_n^3}{\sqrt{k_n^2 z_n^4 + \epsilon_{z1n}^2}} - \frac{s_n^2 (z_n^{2p-1})^2 z_n}{\sqrt{s_n^2 (z_n^p)^4 + \epsilon_{z2n}^2}} - \hat{\theta}_n^T \psi_n(Z_n) - \hat{P}_n - g_{n-1} z_{n-1} + \tau_0 \omega + D_t^q \alpha_{fn-1} \right], \end{cases} \quad (75)$$

$$D_t^q \hat{\theta}_i = \Lambda_i [z_i \psi_i(Z_i) - \sigma_i \hat{\theta}_i], \text{ for } 1 \leq i \leq n, \quad (76)$$

where $\gamma_i, s_i, \Lambda_i, k_i, \epsilon_{z1i}, \epsilon_{z2i}$ and $\sigma_i > 0$ are design parameters, and $\frac{1}{2} < p < 1$.

Inserting Equation (75) into Equation (70) results in:

$$\begin{cases} D_t^q z_1 = -\gamma_1 z_1 - \frac{k_1^2 z_1^3}{\sqrt{k_1^2 z_1^4 + \epsilon_{z11}^2}} - \frac{s_1^2 (z_1^{2p-1})^2 z_1}{\sqrt{s_1^2 (z_1^p)^4 + \epsilon_{z21}^2}} - \tilde{\theta}_1^T \psi_1(Z_1) + g_1 z_2 - \tilde{P}_1, \\ D_t^q z_i = -\gamma_i z_i - \frac{k_i^2 z_i^3}{\sqrt{k_i^2 z_i^4 + \epsilon_{z1i}^2}} - \frac{s_i^2 (z_i^{2p-1})^2 z_i}{\sqrt{s_i^2 (z_i^p)^4 + \epsilon_{z2i}^2}} - g_{i-1} z_{i-1} - \tilde{\theta}_i^T \psi_i(Z_i) + g_i z_{i+1} - \tilde{P}_i, 2 \leq i \leq n-1, \\ D_t^q z_n = -\gamma_n z_n - \frac{k_n^2 z_n^3}{\sqrt{k_n^2 z_n^4 + \epsilon_{z1n}^2}} - \frac{s_n^2 (z_n^{2p-1})^2 z_n}{\sqrt{s_n^2 (z_n^p)^4 + \epsilon_{z2n}^2}} - \tilde{\theta}_n^T \psi_n(Z_n) - g_{n-1} z_{n-1} - \tilde{P}_n. \end{cases} \quad (77)$$

The backstepping framework is employed now to construct a Lyapunov function and analyze its time derivative to guarantee closed-loop stability.

Step 1. Select a QLFC as follows:

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\Lambda_1} \|\tilde{\theta}_1\|^2 + \frac{1}{2\gamma_1} \tilde{P}_1^2 \quad (78)$$

Then, by invoking Property 1, the first-order time derivative (FOTD) of V_1 is:

$$D_t^q V_1 \leq z_1 D_t^q z_1 + \frac{1}{\Lambda_1} \tilde{\theta}_1^T D_t^q \tilde{\theta}_1 + \frac{1}{\gamma_1} \tilde{P}_1 D_t^q \tilde{P}_1 \quad (79)$$

Using Lemma 3 and inserting Equations (73), (76) and (77) into Equation (79) yields:

$$D_t^q V_1 \leq -\gamma_1 z_1^2 - k_1 z_1^2 - s_1 z_1^{2p} + g_1 z_1 z_2 - \tilde{P}_1 z_1 + \epsilon_{z11} + \epsilon_{z21} - \sigma_1 \tilde{\theta}_1^T \tilde{\theta}_1 - \frac{\rho_1}{\gamma_1} \tilde{P}_1^2 - \quad (80)$$

$$\frac{\rho_1}{\gamma_1} \tilde{P}_1 \tilde{\theta}_1^T \psi_1(Z_1) - \frac{1}{\gamma_1} \tilde{P}_1 D_t^q P_1$$

Utilizing Young's inequality, we can derive:

$$\begin{cases} -\tilde{P}_1 z_1 \leq \gamma_1 z_1^2 + \frac{1}{4\gamma_1} \tilde{P}_1^2, \\ -\sigma_1 \tilde{\theta}_1^T \tilde{\theta}_1 \leq -\frac{1}{2} \sigma_1 \|\tilde{\theta}_1\|^2 + \frac{1}{2} \sigma_1 \|\theta_1^*\|^2, \\ -\frac{1}{\gamma_1} \tilde{P}_1 D_t^q P_1 \leq \frac{1}{4\gamma_1} \tilde{P}_1^2 + \frac{1}{\gamma_1} \Delta_1^2, \\ -\frac{\rho_1}{\gamma_1} \tilde{P}_1 \tilde{\theta}_1^T \psi_1(Z_1) \leq \frac{\rho_1}{4\gamma_1} \tilde{P}_1^2 + \frac{a_1 \rho_1}{\gamma_1} \|\tilde{\theta}_1\|^2, \end{cases} \quad (81)$$

where $a_1 \geq \|\psi_1(Z_1)\|^2$.

Applying Equation (81) to Equation (80) yields:

$$D_t^q V_1 \leq -k_1 z_1^2 - s_1 z_1^{2p} - \frac{3}{4\gamma_1} \left(\rho_1 - \frac{2}{3} \right) \tilde{P}_1^2 - \frac{1}{2} \left(\sigma_1 - \frac{2a_1 \rho_1}{\gamma_1} \right) \|\tilde{\theta}_1\|^2 + g_1 z_1 z_2 + \frac{1}{2} \sigma_1 \|\theta_1^*\|^2 + \epsilon_{z11} + \epsilon_{z21} + \frac{1}{\gamma_1} \Delta_1^2. \quad (82)$$

Step ($2 \leq i \leq n-1$). For the i th subsystem, we opt for the following QLFC:

$$V_i = V_{i-1} + \frac{1}{2} z_i^2 + \frac{1}{2\Lambda_i} \|\tilde{\theta}_i\|^2 + \frac{1}{2\gamma_i} \tilde{P}_i^2. \quad (83)$$

By invoking Property 1 and Lemma 3, the FOTD of Equation (83) along the solutions of Equations (73), (76), and (77) results in:

$$D_t^q V_i \leq D_t^q V_{i-1} - k_i z_i^2 - \gamma_i z_i^2 - s_i z_i^{2p} + g_i z_i z_{i+1} + \epsilon_{z1i} + \epsilon_{z2i} - \tilde{P}_i z_i - \sigma_i \tilde{\theta}_i^T \hat{\theta}_i - \frac{\rho_i}{\gamma_i} \tilde{P}_i^2 - \frac{\rho_i}{\gamma_i} \tilde{P}_i \tilde{\theta}_i^T \psi_i(Z_i) - \frac{1}{\gamma_i} \tilde{P}_i D_t^q P_i \quad (84)$$

Following a similar procedure to the one described above, the subsequent inequalities are valid:

$$\begin{cases} -\tilde{P}_i z_i \leq \gamma_i z_i^2 + \frac{1}{4\gamma_i} \tilde{P}_i^2, \\ -\sigma_i \tilde{\theta}_i^T \hat{\theta}_i \leq -\frac{1}{2} \sigma_i \|\tilde{\theta}_i\|^2 + \frac{1}{2} \sigma_i \|\theta_i^*\|^2, \\ -\frac{1}{\gamma_i} \tilde{P}_i D_t^q P_i \leq \frac{1}{4\gamma_i} \tilde{P}_i^2 + \frac{1}{\gamma_i} \Delta_i^2, \\ -\frac{\rho_i}{\gamma_i} \tilde{P}_i \tilde{\theta}_i^T \psi_i(Z_i) \leq \frac{\rho_i}{4\gamma_i} \tilde{P}_i^2 + \frac{a_i \rho_i}{\gamma_i} \|\tilde{\theta}_i\|^2, \end{cases} \quad (85)$$

where $a_i \geq \|\psi_i(Z_i)\|^2$.

Substituting Equation (85) into Equation (84) leads to:

$$\dot{V}_i \leq -\sum_{j=1}^i k_j z_j^2 - \sum_{j=1}^i s_j z_j^{2p} + g_i z_i z_{i+1} - \frac{1}{2} \sum_{j=1}^i \left(\sigma_j - \frac{2a_j \rho_j}{\gamma_j} \right) \|\tilde{\theta}_j\|^2 - \sum_{j=1}^i \frac{3}{4\gamma_j} \left(\rho_j - \frac{2}{3} \right) \tilde{P}_j^2 + \sum_{j=1}^i \left(\frac{1}{2} \sigma_j \|\theta_j^*\|^2 + \epsilon_{z1j} + \epsilon_{z2j} + \frac{1}{\gamma_j} \Delta_j^2 \right). \quad (86)$$

Step n . The QLFV for the n th subsystem can be chosen as:

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 + \frac{1}{2\lambda_n} \|\tilde{\theta}_n\|^2 + \frac{1}{2\gamma_n} \tilde{P}_n^2. \quad (87)$$

By invoking Property 1 and Lemma 3, the FOTD of Equation (87) along the solutions of Equations (73), (76), and (77) results in:

$$D_t^q V_n \leq D_t^q V_{n-1} - k_n z_n^2 - \gamma_n z_n^2 - s_n z_n^{2p} - \tilde{P}_n z_n + \epsilon_{z1n} + \epsilon_{z2n} - \sigma_n \tilde{\theta}_n^T \hat{\theta}_n - \frac{\rho_n}{\gamma_n} \tilde{P}_n^2 - \frac{\rho_n}{\gamma_n} \tilde{P}_n \tilde{\theta}_n^T \psi_n(Z_n) - \frac{1}{\gamma_n} \tilde{P}_n D_t^q P_n. \quad (88)$$

Using Young's inequality, one has:

$$\begin{cases} -\tilde{P}_n z_n \leq \gamma_n z_n^2 + \frac{1}{4\gamma_n} \tilde{P}_n^2, \\ -\sigma_n \tilde{\theta}_n^T \hat{\theta}_n \leq -\frac{1}{2} \sigma_n \|\tilde{\theta}_n\|^2 + \frac{1}{2} \sigma_n \|\theta_n^*\|^2, \\ -\frac{1}{\gamma_n} \tilde{P}_n D_t^q P_n \leq \frac{1}{4\gamma_n} \tilde{P}_n^2 + \frac{1}{\gamma_n} \Delta_n^2, \\ -\frac{\rho_n}{\gamma_n} \tilde{P}_n \tilde{\theta}_n^T \psi_n(Z_n) \leq \frac{\rho_n}{4\gamma_n} \tilde{P}_n^2 + \frac{a_n \rho_n}{\gamma_n} \|\tilde{\theta}_n\|^2, \end{cases} \quad (89)$$

where $a_n \geq \|\psi_n(Z_n)\|^2$.

Combining Equations (89), (86), and (88) yields:

$$D_t^q V_n \leq -\sum_{j=1}^n k_j z_j^2 - \sum_{j=1}^n s_j z_j^{2p} - \frac{1}{2} \sum_{j=1}^n \left(\sigma_j - \frac{2a_j \rho_j}{\gamma_j} \right) \|\tilde{\theta}_j\|^2 - \sum_{j=1}^n \frac{3}{4\gamma_j} \left(\rho_j - \frac{2}{3} \right) \tilde{P}_j^2 + \sum_{j=1}^n \left(\frac{\sigma_j}{2} \|\theta_j^*\|^2 + \epsilon_{z1j} + \epsilon_{z2j} + \frac{1}{\gamma_j} \Delta_j^2 \right) \quad (90)$$

3.1.3. Main Results and Stability Analysis

The following theorem encapsulates our primary findings.

Theorem 2. Consider the FONUS (Equation (57)) with saturated input and under Assumptions 1–5. If the actual and virtual controllers (Equation (75)), the update laws (Equation (76)), and the FODO (Equation (72)) are applied, then the following objectives are achieved:

- The FT boundedness of all closed-loop system signals is assured.

- The output tracking error e_i eventually converges to a small yet modifiable vicinity around the origin within an FT.

Proof of Theorem 2. To consider the influence of all intrinsic errors (namely, $z_i, \tilde{\theta}_i, \tilde{D}_i, l_i, e_{ai}$) on the stability of the closed-loop system, we formulate the subsequent QLFC as:

$$V = V_n + \sum_{i=1}^n \frac{1}{2} l_i^2 + \sum_{i=1}^{n-1} \frac{1}{2} e_{ai}^2$$

$$= \sum_{i=1}^n \frac{1}{2} z_i^2 + \sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 + \sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 + \sum_{i=1}^n \frac{1}{2} l_i^2 + \sum_{i=1}^{n-1} \frac{1}{2} e_{ai}^2$$

From Equations (63), (65) and (90), the FOTD can be expressed as follows:

$$D_t^q V \leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i^{2p} - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2a_i \rho_i}{\gamma_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{2}{3} \right) \tilde{P}_i^2 -$$

$$\sum_{i=1}^{n-1} \tau_{1i} e_{ai}^2 - \sum_{i=1}^{n-1} \tau_{2i} e_{ai}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} + \sum_{i=1}^n \left(\frac{\sigma_i}{2} \|\theta_i^*\|^2 + \frac{1}{\gamma_i} \Delta_i^2 + \right. \quad (91)$$

$$\left. \epsilon_{z1i} + \epsilon_{z2i} + \epsilon_{l1i} + \epsilon_{l2i} \right) + \sum_{i=1}^{n-1} (\epsilon_{\alpha 1i} + \epsilon_{\alpha 2i} + \epsilon_{\alpha 3i}).$$

Using Lemma 1, one gets:

$$\left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p + \left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p \leq \sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 + \sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 + 2(1-p)p^{\frac{p}{1-p}}. \quad (92)$$

Applying Lemma 2 and substituting Equation (92) into Equation (91) yields:

$$D_t^q V \leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i^{2p} - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2a_i \rho_i}{\gamma_i} - \frac{1}{\lambda_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{4}{3} \right) \tilde{P}_i^2 -$$

$$\sum_{i=1}^{n-1} \tau_{1i} e_{ai}^2 - \sum_{i=1}^{n-1} \tau_{2i} e_{ai}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} - \left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p -$$

$$\left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p + Y_3 \quad (93)$$

$$\leq -Y_1 V - Y_2 V^p + Y_3,$$

with:

$$Y_1 = \min \left\{ 2k_1, \dots, 2k_n, \lambda_1 \left(\sigma_1 - \frac{2a_1 \rho_1}{\gamma_1} - \frac{1}{\lambda_1} \right), \dots, \lambda_n \left(\sigma_n - \frac{2a_n \rho_n}{\gamma_n} - \frac{1}{\lambda_n} \right), \frac{3}{2} \left(\rho_1 - \frac{4}{3} \right), \dots, \frac{3}{2} \left(\rho_n - \frac{4}{3} \right), 2\tau_{11}, \dots, 2\tau_{1n-1}, 2\gamma_{l1}, \dots, 2\gamma_{ln} \right\},$$

$$Y_2 = \min \{ 2^p s_1, \dots, 2^p s_n, 2^p \tau_{21}, \dots, 2^p \tau_{2n-1}, 2^p \kappa_{l1}, \dots, 2^p \kappa_{ln}, 1 \}, \text{ and}$$

$$Y_3 = \sum_{i=1}^n \left(\frac{\sigma_i}{2} \|\theta_i^*\|^2 + \frac{1}{\gamma_i} \Delta_i^2 + \epsilon_{z1i} + \epsilon_{z2i} + \epsilon_{l1i} + \epsilon_{l2i} \right) + \sum_{i=1}^{n-1} (\epsilon_{\alpha 1i} + \epsilon_{\alpha 2i} + \epsilon_{\alpha 3i}) + 2(1-p)p^{\frac{p}{1-p}},$$

$$\text{where } \sigma_i > \frac{2a_i \rho_i}{\gamma_i} + \frac{1}{\lambda_i}.$$

From Lemmas 8–12, it is clear that all error variables, namely $z_i, \tilde{\theta}_i, \tilde{D}_i, l_i$, and e_{ai} , are bounded in FT. The error variable z_1 converges practically and in FT to a tiny neighborhood of zero. Because z_1 and l_1 converge practically in FT, so does the output tracking error ($e_1 = x_1 - \alpha_{f0} = x_1 - y_d$). $\square$

Remark 9: Rather than relying on an ad hoc trial-and-error approach, the design parameter tuning process can be structured into two stages. The first stage provides an analytical initialization based on conditions that guarantee closed-loop stability from the outset—specifically, the negative definiteness of Equation (93) and the positive definiteness of the Lyapunov function $V(t)$. The second stage refines these parameters through performance-oriented optimization—such as Bayesian optimization—of a cost functional that explicitly balances tracking accuracy against control effort. This structured approach yields reproducible tuning results that are largely independent of the designer's experience.

Remark 10. The design parameters exhibit systematic trade-offs that can be characterized through sensitivity analysis. These trade-offs are summarized as follows:

1. Tracking vs. control effort: Increasing k_i , s_i , and γ_i accelerates error convergence but proportionally increases control effort.
2. Adaptation speed vs. robustness: A larger Λ_i and smaller σ_i accelerate adaptation but increase sensitivity to measurement noise and unmodeled dynamics.
3. Filtering accuracy vs. noise sensitivity: Larger τ_{1j} , τ_{2j} , and τ_{3j} and smaller $\epsilon_{\alpha 1i}$, $\epsilon_{\alpha 2i}$, and $\epsilon_{\alpha 3i}$ reduce filtering errors but amplify high-frequency noise.
4. Observation speed vs. peaking phenomenon: A larger ρ_i accelerates observer convergence but risks the peaking phenomenon and excessive noise amplification—the well-known high-gain observer trade-off.
5. Compensation speed vs. noise sensitivity: Larger γ_{li} and κ_{li} accelerate the decay of the compensating signals but risk excessive noise amplification.

Remark 11. The boundedness of the disturbance's time derivative (whether integer- or fractional-order, as in our case) is a standard assumption in disturbance observer design [27,28,53–55]. This condition is not physically restrictive—real disturbances have finite energy and thus finite rates of change—but it is mathematically necessary to guarantee observer stability via Lyapunov analysis. It does not imply slow variation, as the bound may be arbitrarily large. The steady-state estimation error is proportional to this bound divided by the observer gain, which captures the inherent trade-off between estimation speed and noise sensitivity. Without this assumption, the estimation error could diverge, precluding any stability guarantee. Hence, the condition is both practically sound and theoretically indispensable.

3.2. Second Control Scheme

We will then present a new FTFAC strategy that addresses differently the singularity problem arising from the differentiation of VCs.

Let us introduce the following coordinate transformation:

$$\begin{aligned} z_i &= x_i - \alpha_{fi-1} + l_i, \text{ for } i = 1, \dots, n-1, \\ z_n &= x_n - \alpha_{fn-1} + l_n + \omega. \end{aligned} \quad (94)$$

The variables l_i and ω will be defined later. $\alpha_{f0} = y_d$ and α_{fi} (for $i = 1, \dots, n-1$) are generated by the following FTF:

$$D_t^q \alpha_{fi} = -\tau_{1i} e_{\alpha i} - \tau_{2i} \left(\tau_{2i} e_{\alpha i} \tanh \left(\frac{\tau_{2i} e_{\alpha i}}{\epsilon_{\alpha 1i}} \right) \right)^{2p-1} \tanh \left(\frac{\tau_{2i} e_{\alpha i}}{\epsilon_{\alpha 1i}} \right) - \tau_{3i} \tanh \left(\frac{\tau_{3i} e_{\alpha i}}{\epsilon_{\alpha 2i}} \right), \quad (95)$$

where $\frac{1}{2} < p < 1$ and $\tau_{ji} > 0$ are design constants, and $\epsilon_{\alpha ji} > 0$ is a small design constant. $e_{\alpha i} = \alpha_{fi} - \alpha_i$ is the filtering error, and α_i represents the VC laws to be designed.

The saturation effect compensation signal ω can be generated by:

$$D_t^q \omega = -\tau_0 \omega + g_n(v - h(v)), \quad (96)$$

where $\tau_0 > 0$ is a design constant.

Theorem 3. When considering the FTF (Equation (95)), two properties can be ensured:

- The signals $\alpha_{fi} - \alpha_i$ and $D_t^q \alpha_{fi}$ are bounded within an FT.
- The filtering errors $e_{\alpha i} = \alpha_{fi} - \alpha_i$ converge rapidly, reaching an arbitrarily small neighborhood of the origin within an FT.

Proof of Theorem 3. Construct a QLFC as $V_\alpha = \sum_{i=1}^{n-1} \left(\frac{1}{2} e_{\alpha i}^2 \right)$. Using Property 1 and Lemmas 4 and 5, its FOTD can be bounded as follows:

$$D_t^q V_\alpha \leq -\sum_{i=1}^{n-1} \tau_{1i} e_{\alpha i}^2 - \sum_{i=1}^{n-1} \tau_{2i} e_{\alpha i}^{2p} - \sum_{i=1}^{n-1} \tau_{3i} |e_{\alpha i}| + \sum_{i=1}^{n-1} |e_{\alpha i}| |D_t^q \alpha_i| + \sum_{i=1}^{n-1} ((\bar{\epsilon}_{\alpha 1i})^{2p} + \bar{\epsilon}_{\alpha 2i}), \quad (97)$$

If $\tau_{3i} \geq |D_t^q \alpha_i|$, Equation (97) becomes:

$$D_t^q V_\alpha \leq -\mu_{11} V_\alpha - \mu_{12} V_\alpha^p + \mu_{13}, \quad (98)$$

with $\mu_{11} = \min\{2\tau_{11}, \dots, 2\tau_{1n-1}\}$, $\mu_{12} = \min\{2^p\tau_{21}, \dots, 2^p\tau_{2n-1}\}$, and $\mu_{13} = \sum_{i=1}^{n-1} ((\bar{\epsilon}_{\alpha 1i})^{2p} + \bar{\epsilon}_{\alpha 2i})$.

According to Lemma 8 or our Lemmas 9–12, the filtering error $e_{\alpha i}$ rapidly approaches a small region near the origin within a finite duration $T_{\alpha 1}$. Lemmas 8 to 12 provide the necessary estimates for both the convergence region and the convergence time, thereby completing the proof of Theorem 3. $\square$

To address the filtering errors, we propose the following auxiliary system:

$$\begin{cases} D_t^q l_i = -\gamma_{li} l_i - \kappa_{li} \left(\kappa_{li} l_i \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) + g_i(\alpha_{i+1} - \alpha_{f(i+1)}), \\ D_t^q l_n = -\gamma_{ln} l_n - \kappa_{ln} \left(\kappa_{ln} l_n \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right), \end{cases} \quad (99)$$

where $i = 1, \dots, n-1$, and $\epsilon_{lii} > 0$ are small design constants, and κ_{li} and $\gamma_{li} > 0$ are design constants.

To facilitate the control design process and streamline the stability analysis, we rewrite System (66) as follows:

$$\begin{cases} D_t^q x_i = \bar{f}_i(\bar{x}_i) + \gamma_{li} l_i + \kappa_{li} \left(\kappa_{li} l_i \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) + g_i l_{i+1} + g_i x_{i+1} + d_i(x, t), \\ D_t^q x_n = \bar{f}_n(\bar{x}_n) + \gamma_{ln} l_n + \kappa_{ln} \left(\kappa_{ln} l_n \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) + g_n h(v) + d_n(x, t), \end{cases} \quad (100)$$

with $\bar{f}_i(\bar{x}_i) = f_i(x) - \gamma_{li} l_i - \kappa_{li} \left(\kappa_{li} l_i \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) - g_i l_{i+1}$, $\bar{f}_n(\bar{x}_n) = f_n(x) - \gamma_{ln} l_n - \kappa_{ln} \left(\kappa_{ln} l_n \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) + g_n \rho(v)$, and $\bar{x}_i = [x^T, l_i, l_{i+1}]^T$, $\bar{x}_n = [x^T, l_n, v]^T$.

Inserting Fuzzy Model (55) into Equation (100), one gets:

$$\begin{cases} D_t^q x_i = \theta_i^{*T} \psi_i(\bar{x}_i) + \gamma_{li} l_i + \kappa_{li} \left(\kappa_{li} l_i \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) + g_i l_{i+1} + g_i x_{i+1} + \varepsilon_i(\bar{x}_i) + d_i(x, t), 1 \leq i \leq n-1, \\ D_t^q x_n = \theta_n^{*T} \psi_n(\bar{x}_n) + \gamma_{ln} l_n + \kappa_{ln} \left(\kappa_{ln} l_n \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) + g_n h(v) + \varepsilon_n(\bar{x}_n) + d_n(x, t). \end{cases} \quad (101)$$

To deal with the algebraic-loop problem caused by the term $\psi_i(\bar{x}_i)$, one can rewrite Equation (101) as:

$$\begin{cases} D_t^q x_i = \theta_i^{*T} \psi_i(Z_i) + \gamma_{li} l_i + \kappa_{li} \left(\kappa_{li} l_i \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{li} l_i}{\epsilon_{lii}}\right) + g_i l_{i+1} + g_i x_{i+1} + \theta_i^{*T} (\psi_i(\bar{x}_i) - \psi_i(Z_i)) + \varepsilon_i(\bar{x}_i) + d_i(x, t), 1 \leq i \leq n-1, \\ D_t^q x_n = \theta_n^{*T} \psi_n(Z_n) + \gamma_{ln} l_n + \kappa_{ln} \left(\kappa_{ln} l_n \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) \right)^{2p-1} \tanh\left(\frac{\kappa_{ln} l_n}{\epsilon_{l1n}}\right) + g_n h(v) + \theta_n^{*T} (\psi_n(\bar{x}_n) - \psi_n(Z_n)) + \varepsilon_n(\bar{x}_n) + d_n(x, t), \end{cases} \quad (102)$$

with $Z_i = [x_1, x_2, \dots, x_i, l_i, l_{i+1}]^T \in R^{i+2}$, and $Z_n = [x^T, l_n]^T \in R^{n+1}$.

From Equations (94), (96), (99) and (102), we have:

$$\begin{cases} D_t^q z_i = \theta_i^{*T} \psi_i(Z_i) + g_i z_{i+1} + g_i \alpha_i - D_t^q \alpha_{f(i-1)} + P_i, 1 \leq i \leq n-1, \\ D_t^q z_n = \theta_n^{*T} \psi_n(Z_n) + g_n v - \tau_0 \omega - D_t^q \alpha_{f(n-1)} + P_n, \end{cases} \quad (103)$$

Where:

$$P_i = \theta_i^{*T} (\psi_i(\bar{x}_i) - \psi_i(Z_i)) + \varepsilon_i(\bar{x}_i) + d_i(x, t), 1 \leq i \leq n. \quad (104)$$

3.2.1. FODO Design

For System (103), let us construct the following FODO:

$$\begin{cases} D_t^q \tilde{P}_i = -\rho_i \tilde{P}_i + \rho_i (D_t^q z_i + D_t^q \alpha_{fi-1} - \hat{\theta}_i^T \psi_i(Z_i) - g_i z_{i+1} - g_i \alpha_i), 1 \leq i \leq n-1, \\ D_t^q \tilde{P}_n = -\rho_n \tilde{P}_n + \rho_n (D_t^q z_n + D_t^q \alpha_{fn-1} + \tau_0 \omega - \hat{\theta}_n^T \psi_n(Z_n) - g_n v), \end{cases} \quad (105)$$

where $\hat{\theta}_i$ is an estimation of θ_i^* , and $\rho_i > 0$ is the observer gain.

From Equations (105) and (103), one can obtain:

$$D_t^q \tilde{P}_i = -\rho_i \tilde{P}_i - \rho_i \tilde{\theta}_i^T \psi_i(Z_i) - D_t^q P_i, 1 \leq i \leq n, \quad (106)$$

with $\tilde{\theta}_i = \hat{\theta}_i - \theta_i^*$ and $\tilde{P}_i = \hat{P}_i - P_i$.

3.2.2. Design of the FTFAC

Let us design the VCs α_i , the actual control law $\alpha_n = v$, and their corresponding adaptation laws as follows:

$$\begin{cases} \alpha_1 = \frac{1}{g_1} \left[-\gamma_1 z_1 - k_1 z_1 - s_1 \left(s_1 z_1 \tanh \left(\frac{s_1 z_1}{\epsilon_{z11}} \right) \right)^{2p-1} \tanh \left(\frac{s_1 z_1}{\epsilon_{z11}} \right) - \hat{\theta}_1^T \psi_1(Z_1) \right. \\ \quad \left. - \hat{P}_1 + D_t^q \alpha_{f0} \right], \\ \alpha_i = \frac{1}{g_i} \left[-\gamma_i z_i - k_i z_i - s_i \left(s_i z_i \tanh \left(\frac{s_i z_i}{\epsilon_{z1i}} \right) \right)^{2p-1} \tanh \left(\frac{s_i z_i}{\epsilon_{z1i}} \right) \right. \\ \quad \left. - \hat{\theta}_i^T \psi_i(Z_i) - \hat{P}_i - g_{i-1} z_{i-1} + D_t^q \alpha_{fi-1} \right], 2 \leq i \leq n-1, \\ \alpha_n = \frac{1}{g_n} \left[-\gamma_n z_n - k_n z_n - s_n \left(s_n z_n \tanh \left(\frac{s_n z_n}{\epsilon_{z1n}} \right) \right)^{2p-1} \tanh \left(\frac{s_n z_n}{\epsilon_{z1n}} \right) - \hat{\theta}_n^T \psi_n(Z_n) \right. \\ \quad \left. - \hat{P}_n - g_{n-1} z_{n-1} + \tau_0 \omega + D_t^q \alpha_{fn-1} \right], \end{cases} \quad (107)$$

$$D_t^q \hat{\theta}_i = \Lambda_i [z_i \psi_i(Z_i) - \sigma_i \hat{\theta}_i], 1 \leq i \leq n, \quad (108)$$

where $\gamma_i, s_i, \Lambda_i, k_i, \epsilon_{z1i}$ and $\sigma_i > 0$ are design parameters, and $\frac{1}{2} < p < 1$.

Inserting Equation (107) into Equation (103) results in:

$$\begin{cases} D_t^q z_1 = -\gamma_1 z_1 - k_1 z_1 - s_1 \left(s_1 z_1 \tanh \left(\frac{s_1 z_1}{\epsilon_{z11}} \right) \right)^{2p-1} \tanh \left(\frac{s_1 z_1}{\epsilon_{z11}} \right) - \hat{\theta}_1^T \psi_1(Z_1) + g_1 z_2 - \tilde{P}_1, \\ D_t^q z_i = -\gamma_i z_i - k_i z_i - s_i \left(s_i z_i \tanh \left(\frac{s_i z_i}{\epsilon_{z1i}} \right) \right)^{2p-1} \tanh \left(\frac{s_i z_i}{\epsilon_{z1i}} \right) - g_{i-1} z_{i-1} - \hat{\theta}_i^T \psi_i(Z_i) + \\ \quad g_i z_{i+1} - \tilde{P}_i, 2 \leq i \leq n-1, \\ D_t^q z_n = -\gamma_n z_n - k_n z_n - s_n \left(s_n z_n \tanh \left(\frac{s_n z_n}{\epsilon_{z1n}} \right) \right)^{2p-1} \tanh \left(\frac{s_n z_n}{\epsilon_{z1n}} \right) - \hat{\theta}_n^T \psi_n(Z_n) - \\ \quad g_{n-1} z_{n-1} - \tilde{P}_n. \end{cases} \quad (109)$$

Since the design of VCs and the construction of the Lyapunov function using the backstepping approach are similar to those in Section 3.1, the details are omitted in this section.

3.2.3. Main Results and Stability Analysis

The following theorem encapsulates our main results.

Theorem 4. Consider the FONUS (Equation (57)) with saturated input and under Assumptions 1–5. If the actual and virtual controllers (Equation (107)), the update laws (Equation (108)), and the FODO (Equation (105)) are applied, then the following objectives are achieved:

- The FT boundedness of all closed-loop system signals is assured.
- The output tracking error e_i eventually converges to a small yet modifiable vicinity around the origin within an FT.

Proof of Theorem 4. Consider the following QLFC:

$$V = V_n + \sum_{i=1}^n \frac{1}{2} l_i^2 + \sum_{i=1}^{n-1} \frac{1}{2} e_{ai}^2 = \sum_{i=2}^n \frac{1}{2} z_i^2 + \sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 + \sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 + \sum_{i=1}^n \frac{1}{2} l_i^2 + \sum_{i=1}^{n-1} \frac{1}{2} e_{ai}^2. \quad (110)$$

From Equations (95), (99), (106), (108) and (109), its FOTD can be expressed as follows:

$$D_t^q V \leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i^{2p} - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2a_i \rho_i}{\gamma_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{2}{3} \right) \tilde{P}_i^2 - \sum_{j=1}^{n-1} \tau_{1j} e_{aj}^2 - \sum_{j=1}^{n-1} \tau_{2j} e_{aj}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} + \sum_{i=1}^n \left(\frac{\sigma_i}{2} \|\theta_i^*\|^2 + \frac{1}{\gamma_i} \Delta_i^2 + (\bar{\epsilon}_{z1i})^{2p} + (\bar{\epsilon}_{l1i})^{2p} \right) + \sum_{j=1}^{n-1} ((\bar{\epsilon}_{a1j})^{2p} + \bar{\epsilon}_{a2j}). \quad (111)$$

Using Lemma 1, one gets:

$$\left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p + \left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p \leq \sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 + \sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 + 2(1-p)p^{\frac{p}{1-p}}. \quad (112)$$

Applying Lemma 2 and substituting Equation (112) into Equation (111) yields:

$$D_t^q V \leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i^{2p} - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2a_i \rho_i}{\gamma_i} - \frac{1}{\lambda_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{4}{3} \right) \tilde{P}_i^2 - \sum_{j=1}^{n-1} \tau_{1j} e_{aj}^2 - \sum_{j=1}^{n-1} \tau_{2j} e_{aj}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} - \left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p - \left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p + Y_3, \quad (113)$$

$$\leq -Y_1 V - Y_2 V^p + Y_3,$$

with:

$$Y_1 = \min \left\{ 2k_1, \dots, 2k_n, \lambda_1 \left(\sigma_1 - \frac{2a_1 \rho_1}{\gamma_1} - \frac{1}{\lambda_1} \right), \dots, \lambda_n \left(\sigma_n - \frac{2a_n \rho_n}{\gamma_n} - \frac{1}{\lambda_n} \right), \frac{3}{2} \left(\rho_1 - \frac{4}{3} \right), \dots, \frac{3}{2} \left(\rho_n - \frac{4}{3} \right), 2\tau_{11}, \dots, 2\tau_{1n-1}, 2\gamma_{l1}, \dots, 2\gamma_{ln} \right\},$$

$$Y_2 = \min \{ 2^p s_1, \dots, 2^p s_n, 2^p \tau_{21}, \dots, 2^p \tau_{2n-1}, 2^p \kappa_{l1}, \dots, 2^p \kappa_{ln}, 1 \}, \text{ and}$$

$$Y_3 = \sum_{i=1}^n \left(\frac{\sigma_i}{2} \|\theta_i^*\|^2 + \frac{1}{\gamma_i} \Delta_i^2 + (\bar{\epsilon}_{z1i})^{2p} + (\bar{\epsilon}_{l1i})^{2p} \right) + \sum_{j=1}^{n-1} ((\bar{\epsilon}_{a1j})^{2p} + \bar{\epsilon}_{a2j}) + 2(1-p)p^{\frac{p}{1-p}}.$$

$$\text{where } \sigma_i > \frac{2a_i \rho_i}{\gamma_i} + \frac{1}{\lambda_i}.$$

Now, from Lemma 8–12 and Equation (113), it is clear that the error variables, namely $z_i, \tilde{\theta}_i, \tilde{D}_i, l_i$, and e_{ai} , are all bounded in FT. In particular, the error variables z_1 and $e_1 = x_1 - y_d$ converge practically and in FT towards a tiny neighborhood of the origin. $\square$

3.3. Third Control Scheme

Next, we propose a third FTFAC that addresses the singularity issue from differentiating VCs using a novel approach.

Let us introduce the following coordinate transformation:

$$\begin{cases} z_i = x_i - \alpha_{fi-1} + l_i, & i = 1, \dots, n-1, \\ z_n = x_n - \alpha_{fn-1} + l_n + \omega, \end{cases} \quad (114)$$

with $\alpha_{f0} = y_d$ and α_{fi} for $i = 1, \dots, n-1$, being generated by the following FFTF:

$$D_t^q \alpha_{fi} = -\tau_{1i} e_{ai} - \tau_{2i} e_{ai}^{2p-1} - \tau_{3i} \tanh \left(\frac{\tau_{3i} e_{ai}}{\epsilon_{a2i}} \right), \quad (115)$$

where $e_{ai} = \alpha_{fi} - \alpha_i$, and $\alpha_{fi}(0) = \alpha_i(0)$. $\tau_{ji} > 0$ represents design constants, and ϵ_{a2i} represents positive design constants.

The compensation signal ω to mitigate the saturation effect can be generated by:

$$D_t^q \omega = -\tau_0 \omega + g_n(v - h(v)), \quad (116)$$

where $\tau_0 > 0$ is a design constant.

Theorem 5. When considering the FFTF (Equation (115)), the following two properties are guaranteed:

- The signals $\alpha_{fi} - \alpha_i$ and $D_t^q \alpha_{fi}$ are bounded within an FT.
- The filtering errors $e_{ai} = \alpha_{fi} - \alpha_i$ converge rapidly and practically, reaching an arbitrarily small neighborhood of the origin within an FT.

Proof of Theorem 5. Because the proof for this theorem closely follows the same steps as the proof for Theorem 3, we will not include it here. $\square$

To compensate for the filtering errors, we design the following stable auxiliary system:

$$\begin{cases} D_t^q l_i = -\gamma_{li} l_i - \kappa_{li} l_i^{2p-1} + g_i(\alpha_i - \alpha_{fi}), i = 1, \dots, n-1, \\ D_t^q l_n = -\gamma_{ln} l_n - \kappa_{ln} l_n^{2p-1}, \end{cases} \quad (117)$$

where κ_{li} and $\gamma_{li} > 0$ are design constants.

To simplify the control design, we write System (66) as:

$$\begin{cases} D_t^q x_i = \bar{f}_i(\bar{x}_i) + \gamma_{li} l_i + \kappa_{li} l_i^{2p-1} + g_i l_{i+1} + g_i x_{i+1} + d_i(x, t), 1 \leq i \leq n-1, \\ D_t^q x_n = \bar{f}_n(\bar{x}_n) + \gamma_{ln} l_n + \kappa_{ln} l_n^{2p-1} + g_n h(v) + d_n(x, t), \end{cases} \quad (118)$$

where $\bar{f}_i(\bar{x}_i) = f_i(x) - \gamma_{li} l_i - \kappa_{li} l_i^{2p-1} - g_i l_{i+1}$, for $1 \leq i \leq n-1$, $\bar{f}_n(\bar{x}_n) = f_n(x) - \gamma_{ln} l_n - \kappa_{ln} l_n^{2p-1} + g_n \rho(v)$, $\bar{x}_i = [x^T, l_i, l_{i+1}]^T$, and $\bar{x}_n = [x^T, l_n, v]^T$.

Inserting Fuzzy Model (55) into Equation (118), one can obtain:

$$\begin{cases} D_t^q x_i = \theta_i^{*T} \psi_i(\bar{x}_i) + \gamma_{li} l_i + \kappa_{li} l_i^{2p-1} + g_i l_{i+1} + g_i x_{i+1} + \varepsilon_i(\bar{x}_i) + d_i(x, t), 1, \dots, n-1, \\ D_t^q x_n = \theta_n^{*T} \psi_n(\bar{x}_n) + \gamma_{ln} l_n + \kappa_{ln} l_n^{2p-1} + g_n h(v) + g_n \rho(v) + \varepsilon_n(\bar{x}_n) + d_n(x, t). \end{cases} \quad (119)$$

To deal with the algebraic-loop problem caused by $\psi_i(\bar{x}_i)$, one can rewrite Equation (119) as follows:

$$\begin{cases} D_t^q x_i = \theta_i^{*T} \psi_i(Z_i) + \gamma_{li} l_i + \kappa_{li} l_i^{2p-1} + g_i l_{i+1} + g_i x_{i+1} + \theta_i^{*T} (\psi_i(\bar{x}_i) - \psi_i(Z_i)) + \varepsilon_i(\bar{x}_i) + d_i(x, t), 1, \dots, n-1, \\ D_t^q x_n = \theta_n^{*T} \psi_n(Z_n) + \gamma_{ln} l_n + \kappa_{ln} l_n^{2p-1} + g_n h(v) + g_n \rho(v) + \theta_n^{*T} (\psi_n(\bar{x}_n) - \psi_n(Z_n)) + \varepsilon_n(\bar{x}_n) + d_n(x, t), \end{cases} \quad (120)$$

with $Z_i = [x_1, x_2, \dots, x_i, l_i, l_{i+1}]^T \in R^{i+2}$ for $1 \leq i \leq n-1$, and $Z_n = [x^T, l_n]^T \in R^{n+1}$.

From Equations (114)–(117) and (120), we have:

$$\begin{cases} D_t^q z_i = \theta_i^{*T} \psi_i(Z_i) + g_i z_{i+1} + g_i \alpha_i - D_t^q \alpha_{fi-1} + P_i, 1 \leq i \leq n-1, \\ D_t^q z_n = \theta_n^{*T} \psi_n(Z_n) + g_n v - \tau_0 \omega - D_t^q \alpha_{fn-1} + P_n, \end{cases} \quad (121)$$

where:

$$\begin{cases} P_i = \theta_i^{*T} (\psi_i(\bar{x}_i) - \psi_i(Z_i)) + \varepsilon_i(\bar{x}_i) + d_i(x, t), 1 \leq i \leq n-1 \\ P_n = \theta_n^{*T} (\psi_n(\bar{x}_n) - \psi_n(Z_n)) + \varepsilon_n(\bar{x}_n) + d_n(x, t) + g_n \rho(v). \end{cases} \quad (122)$$

3.3.1. FODO Design

For System (121), let us construct the following FODO:

$$\begin{cases} D_t^q \hat{P}_i = -\rho_i \hat{P}_i + \rho_i (D_t^q z_i + D_t^q \alpha_{fi-1} - \hat{\theta}_i^T \psi_i(Z_i) - g_i z_{i+1} - g_i \alpha_i), 1 \leq i \leq n-1, \\ D_t^q \hat{P}_n = -\rho_n \hat{P}_n + \rho_n (D_t^q z_n + D_t^q \alpha_{fn-1} + \tau_0 \omega - \hat{\theta}_n^T \psi_n(Z_n) - g_n v), \end{cases} \quad (123)$$

where $\hat{\theta}_i$ is an estimate of θ_i^* , and $\rho_i > 0$ is the observer gain.

From Equations (121) and (123), one can get:

$$D_t^q \tilde{P}_i = -\rho_i \tilde{P}_i - \rho_i \tilde{\theta}_i^T \psi_i(Z_i) - D_t^q P_i, 1 \leq i \leq n, \quad (124)$$

where $\tilde{\theta}_i = \hat{\theta}_i - \theta_i^*$ and $\tilde{P}_i = \hat{P}_i - P_i$.

3.3.2. Design of the FTFAC

Let us devise the VCs α_i , the actual control law $\alpha_n = v$, and their corresponding adaptation laws as follows:

$$\begin{cases} \alpha_1 = \frac{1}{g_1}[-\gamma_1 z_1 - k_1 z_1 - s_1 \Xi_1(z_1) - \hat{\theta}_1^T \psi_1(Z_1) - \hat{P}_1 + D_t^q \alpha_{f0}], \\ \alpha_i = \frac{1}{g_i}[-\gamma_i z_i - k_i z_i - s_i \Xi_i(z_i) - \hat{\theta}_i^T \psi_i(Z_i) - \hat{P}_i - g_{i-1} z_{i-1} + D_t^q \alpha_{fi-1}], 2 \leq i \leq n-1, \\ \alpha_n = \frac{1}{g_n}[-\gamma_n z_n - k_n z_n - s_n \Xi_n(z_n) - \hat{\theta}_n^T \psi_n(Z_n) - \hat{P}_n - g_{n-1} z_{n-1} + \tau_0 \omega + D_t^q \alpha_{fn-1}], \end{cases} \quad (125)$$

$$D_t^q \hat{\theta}_i = \Lambda_i [z_i \psi_i(Z_i) - \sigma_i \hat{\theta}_i], 1 \leq i \leq n, \quad (126)$$

where $\gamma_i, s_i, \Lambda_i, k_i$, and $\sigma_i > 0$ are design constants, and $\frac{1}{2} < p < 1$.

$\Xi_i(z)$ is a smooth switching function and can be designed as:

$$\Xi_i(z_i) = \begin{cases} z_i^{2p-1} = |z_i|^{2p-1} \text{sign}(z_i), & \text{for } |z_i| \geq \delta_i, \\ \varphi_i(z_i), & \text{for } |z_i| < \delta_i, \end{cases} \quad (127)$$

where δ_i is a small positive constant.

Design methodology for the function $\varphi_i(z_i)$: Since the function z_i^{2p-1} is odd, the function $\varphi_i(z_i)$ should also be odd. Employing the nonlinear filter in Equation (115), also known as a DSC technique, in the control design significantly reduces the complexity of designing the switching function $\varphi_i(z_i)$. In fact, this allows for simpler designs with either two or three parameters for this function. In these cases, $\varphi_i(z_i)$ should belong to the C^1 or C^2 class, respectively.

Case 1. Switching function with two unknown parameters: To identify the two parameters, one can use the following conditions of continuity and differentiability at the boundaries (i.e., when $z_i = \pm\delta_i$):

$$\begin{cases} \varphi_i(z_i) = z_i^{2p-1}, & \text{for } |z_i| = \delta_i, \\ \frac{d\varphi_i(z_i)}{dz_i} = (2p-1)|z_i|^{2p-2}, & \text{for } |z_i| = \delta_i. \end{cases} \quad (128)$$

Case 2. Switching function with three unknown parameters: To ensure the continuity and differentiability of $\Xi_i(z_i)$ at the boundaries, the unknown three parameters are determined using the following relations:

$$\begin{cases} \varphi_i(z_i) = z_i^{2p-1}, & \text{for } |z_i| = \delta_i, \\ \frac{d\varphi_i(z_i)}{dz_i} = (2p-1)|z_i|^{2p-2}, & \text{for } |z_i| = \delta_i, \\ \frac{d^2\varphi_i(z_i)}{dz_i^2} = (2p-1)(2p-2)|z_i|^{2p-3} \text{sign}(z_i), & \text{for } |z_i| = \delta_i. \end{cases} \quad (129)$$

Table 2 and Table 3 illustrate instances of functions $\varphi_i(z_i)$ characterized by two and three parameters, respectively.

Table 2. Examples of functions $\varphi_i(z_i)$ with two parameters.

$\varphi_i(z_i)$	Parameters
$v_{1i}z_i + v_{2i}z_i z_i $	$v_{1i} = (3-2p)\delta_i^{2p-2}, v_{2i} = 2(p-1)\delta_i^{2p-3}$
$v_{1i}z_i + v_{2i}z_i^3$	$v_{1i} = (3-2p)\delta_i^{2p-2}, v_{2i} = 2(p-1)\delta_i^{2p-3}$
$v_{1i}z_i + v_{2i}\tanh(z_i)$	$v_{1i} = (2p-1)\delta_i^{2p-2} - \frac{(4p-4)\delta_i^{2p-1}}{2-\sinh(2\delta_i)}, v_{2i} = \frac{(4p-4)\delta_i^{2p-1}\cosh^2(\delta_i)}{2-\sinh(2\delta_i)}$
$v_{1i}z_i + v_{2i}\sin(z_i)$	$v_{1i} = \frac{\delta_i^{2p-2}(\delta_i\cos(\delta_i)-(2p-1)\sinh(\delta_i))}{\delta_i\cos(\delta_i)-\sinh(\delta_i)}, v_{2i} = \frac{(2p-2)\delta_i^{2p-1}}{\delta_i\cos(\delta_i)-\sinh(\delta_i)}$

Table 3. Examples of $\varphi_i(z_i)$ functions with three parameters.

$\varphi_i(z_i)$	Parameters
$v_{1i}z_i + v_{2i}z_i z_i + v_{3i}z_i^3$	$v_{1i} = (p-2)(2p-3)\delta_i^{2p-2}, v_{2i} = 2(p-1)(4-2p)\delta_i^{2p-3},$ $v_{3i} = (p-1)(2p-3)\delta_i^{2p-4}$
$v_{1i}z_i + v_{2i}z_i^3 + v_{3i}z_i^5$	$v_{1i} = \frac{(p-2)(p-3)}{2}\delta_i^{2p-2}, v_{2i} = (p-1)(3-p)\delta_i^{2p-4},$ $v_{3i} = \frac{(p-1)(p-2)}{2}\delta_i^{2p-6}$
$v_{1i}z_i + v_{2i}z_i z_i + v_{3i}z_i^5$	$v_{1i} = \frac{(2p-3)(p-3)}{2}\delta_i^{2p-2}, v_{2i} = \frac{(2p-2)(12-4p)}{6}\delta_i^{2p-3},$

$$v_{3i} = \frac{(p-1)(2p-3)}{6} \delta_i^{2p-6}$$

Remark 12. The integer-order derivative (see Equations (128) and (129)) is adopted in the design of the $\varphi_i(z_i)$ functions for both theoretical and practical reasons. From a theoretical standpoint, the existence of the integer-order derivative of a given function is a sufficient condition for the existence of its fractional-order derivative, thereby ensuring a well-founded mathematical framework. From a practical perspective, the use of the integer-order derivative greatly reduces the computational complexity involved in determining the parameters of the $\varphi_i(z_i)$ functions, thus rendering the overall design process considerably more efficient.

3.3.3. Main Results and Stability Analysis

Substitution of Equation (125) into Equation (121) leads to:

$$\begin{cases} D_t^q z_1 = -\gamma_1 z_1 - k_1 z_1 - s_1 \Xi_1(z_1) - \tilde{\theta}_1^T \psi_1(Z_1) + g_1 z_2 - \tilde{P}_1, \\ D_t^q z_i = -\gamma_i z_i - k_i z_i - s_i \Xi_i(z_i) - g_{i-1} z_{i-1} - \tilde{\theta}_i^T \psi_i(Z_i) + g_i z_{i+1} - \tilde{P}_i, \quad 2 \leq i \leq n-1, \\ D_t^q z_n = -\gamma_n z_n - k_n z_n - s_n \Xi_n(z_n) - \tilde{\theta}_n^T \psi_n(Z_n) - g_{n-1} z_{n-1} - \tilde{P}_n. \end{cases} \quad (130)$$

The following theorem encapsulates our main results.

Theorem 6. Consider the FONUS (Equation (57)) with saturated input and under Assumptions 1–5. If the actual and virtual controllers (Equation (125)), the update laws (Equation (126)), and the FODO (Equation (123)) are applied, then the following objectives are achieved:

- The FT boundedness of all closed-loop system signals is assured.
- The output tracking error e_i eventually converges to a small yet modifiable vicinity around the origin within a finite period.

Proof of Theorem 6. Let us construct the following QLFC:

$$\begin{aligned} V &= V_n + \frac{1}{2} \sum_{i=1}^n l_i^2 + \frac{1}{2} \sum_{j=2}^n e_{\alpha j}^2, \\ &= \frac{1}{2} \sum_{i=1}^n z_i^2 + \sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 + \sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 + \frac{1}{2} \sum_{i=1}^n l_i^2 + \frac{1}{2} \sum_{j=1}^{n-1} e_{\alpha j}^2. \end{aligned} \quad (131)$$

From Equations (115), (117), (124), (126) and (130), the FOTD can be bounded as:

$$\begin{aligned} D_t^q V &\leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i \Xi_i(z_i) - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2\alpha_i \rho_i}{\gamma_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{2}{3} \right) \tilde{P}_i^2 - \sum_{j=1}^{n-1} \tau_{1j} e_{\alpha j}^2 - \\ &\quad \sum_{j=1}^{n-1} \tau_{2j} e_{\alpha j}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} + \sum_{i=1}^n \left(\frac{\sigma_i}{2} \|\theta_i^*\|^2 + \frac{1}{\gamma_i} \Delta_i^2 \right) + \\ &\quad \sum_{j=1}^{n-1} \bar{\epsilon}_{\alpha 2j}. \end{aligned} \quad (132)$$

Using Lemma 1, we have:

$$\left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p + \left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p \leq \sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 + \sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 + 2(1-p)p^{\frac{p}{1-p}}. \quad (133)$$

Using Lemma 2 and substituting Equation (133) into Equation (132) results in:

$$\begin{aligned} D_t^q V &\leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i \Xi_i(z_i) - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2\alpha_i \rho_i}{\gamma_i} - \frac{1}{\lambda_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{4}{3} \right) \tilde{P}_i^2 - \\ &\quad \sum_{j=1}^{n-1} \tau_{1j} e_{\alpha j}^2 - \sum_{j=1}^{n-1} \tau_{2j} e_{\alpha j}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} - \left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p - \left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p + \Upsilon_3. \end{aligned} \quad (134)$$

$$\text{where } \Upsilon_3 = \sum_{i=1}^n \left(\frac{\sigma_i}{2} \|\theta_i^*\|^2 + \frac{1}{\gamma_i} \Delta_i^2 \right) + \sum_{j=1}^{n-1} \bar{\epsilon}_{\alpha 2j} + 2(1-p)p^{\frac{p}{1-p}}.$$

Now, we have two scenarios to consider.

Case 1. If $|z_i| \geq \delta_i$, Equation (134) can be rearranged as:

$$\begin{aligned} D_t^q V &\leq -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i^2 - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2\alpha_i \rho_i}{\gamma_i} - \frac{1}{\lambda_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{4}{3} \right) \tilde{P}_i^2 - \sum_{j=1}^{n-1} \tau_{1j} e_{\alpha j}^2 - \\ &\quad \sum_{j=1}^{n-1} \tau_{2j} e_{\alpha j}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} - \left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{P}_i^2 \right)^p - \left(\sum_{i=1}^n \frac{1}{2\lambda_i} \|\tilde{\theta}_i\|^2 \right)^p + \Upsilon_3. \end{aligned} \quad (135)$$

Based on Lemma 2, Equation (135) can be rewritten as:

$$D_t^q V \leq -Y_1 V - Y_2 V^p + Y_3, \quad (136)$$

with:

$$Y_1 = \min \left\{ 2k_1, \dots, 2k_n, \Lambda_1 \left(\sigma_1 - \frac{2a_1 \rho_1}{\gamma_1} - \frac{1}{\Lambda_1} \right), \dots, \Lambda_n \left(\sigma_n - \frac{2a_n \rho_n}{\gamma_n} - \frac{1}{\Lambda_n} \right), \frac{3}{2} \left(\rho_1 - \frac{4}{3} \right), \dots, \frac{3}{2} \left(\rho_n - \frac{4}{3} \right), 2\tau_{11}, \dots, 2\tau_{1n-1}, 2\gamma_{l1}, \dots, 2\gamma_{ln} \right\},$$

$$Y_2 = \min \{ 2^p s_1, \dots, 2^p s_n, 2^p \tau_{21}, \dots, 2^p \tau_{2n-1}, 2^p \kappa_{l1}, \dots, 2^p \kappa_{ln}, 1 \}, \text{ where } \sigma_j > \frac{2a_j \rho_j}{\gamma_j} + \frac{1}{\Lambda_j}.$$

Case 2. If $|z_i| < \delta_i$, Equation (134) can be rearranged as:

$$\begin{aligned} D_t^q V \leq & -\sum_{i=1}^n k_i z_i^2 - \sum_{i=1}^n s_i z_i^{2p} - \frac{1}{2} \sum_{i=1}^n \left(\sigma_i - \frac{2a_i \rho_i}{\gamma_i} - \frac{1}{\Lambda_i} \right) \|\tilde{\theta}_i\|^2 - \sum_{i=1}^n \frac{3}{4\gamma_i} \left(\rho_i - \frac{4}{3} \right) \tilde{p}_i^2 - \\ & \sum_{j=1}^{n-1} \tau_{1j} e_{aj}^2 - \sum_{j=1}^{n-1} \tau_{2j} e_{aj}^{2p} - \sum_{i=1}^n \gamma_{li} l_i^2 - \sum_{i=1}^n \kappa_{li} l_i^{2p} - \left(\sum_{i=1}^n \frac{1}{2\gamma_i} \tilde{p}_i^2 \right)^p - \\ & \left(\sum_{i=1}^n \frac{1}{2\Lambda_i} \|\tilde{\theta}_i\|^2 \right)^p + Y_3 + \Delta, \end{aligned} \quad (137)$$

where $\Delta = \sum_{i=1}^n s_i z_i^{2p} - \sum_{i=1}^n s_i z_i \varphi_i(z_i)$. Since $|z_i| < \delta_i$, the term Δ has an upper bound $\bar{\Delta}$, where $|\sum_{i=1}^n s_i z_i^{2p} - \sum_{i=1}^n s_i z_i \varphi_i(z_i)| \leq \bar{\Delta}$.

As in case 1, according to the definition of V and according to Lemma 2, Equation (137) can be expressed as:

$$D_t^q V \leq -Y_1 V - Y_2 V^p + Y_3 + \Delta. \quad (138)$$

Therefore, for two cases, one can upper-bound $\dot{V}$ as:

$$D_t^q V \leq -Y_1 V - Y_2 V^p + Y_4, \quad (139)$$

where $Y_4 = \max(Y_3, Y_3 + \Delta)$.

Now, from Lemmas 8–12 and Equation (139), it is clear that all error variables, namely $z_i, \tilde{\theta}_i, \tilde{D}_i, l_i$, and e_{ai} , are bounded in FT. In particular, the error variable z_1 and $e_1 = x_1 - y_d$ converge practically in FT to a tiny neighborhood of zero. $\square$

3.4. Fourth Control Scheme

The coordinate transformation, FFTF, saturation compensation system, auxiliary system, and FODO are designed as in Section 3.3.

Design the VCs α_i . The actual control law $\alpha_n = v$ and the corresponding adaptation laws are as follows:

$$\begin{cases} \alpha_1 = \frac{1}{g_1} [-\gamma_1 z_1 - k_1 z_1 - s_1 \aleph_1(z_1) z_1^{2p-1} - \hat{\theta}_1^T \psi_1(Z_1) - \hat{P}_1 + D_t^q \alpha_{f0}], \\ \alpha_i = \frac{1}{g_i} [-\gamma_i z_i - k_i z_i - s_i \aleph_i(z_i) z_i^{2p-1} - \hat{\theta}_i^T \psi_i(Z_i) - \hat{P}_i - g_{i-1} z_{i-1} + D_t^q \alpha_{fi-1}], \\ \quad i = 2, \dots, n-1, \\ \alpha_n = \frac{1}{g_n} [-\gamma_n z_n - k_n z_n - s_n \aleph_n(z_n) z_n^{2p-1} - \hat{\theta}_n^T \psi_n(Z_n) - \hat{P}_n - g_{n-1} z_{n-1} + D_t^q \alpha_{fn-1}], \end{cases} \quad (140)$$

$$D_t^q \hat{\theta}_i = \Lambda_i [z_i \psi_i(Z_i) - \sigma_i \hat{\theta}_i], 1 \leq i \leq n \quad (141)$$

where $\gamma_i, s_i, \Lambda_i, k_i$, and $\sigma_i > 0$ are design constants, and $\frac{1}{2} < p < 1$. $\aleph_i(z_i)$ is a z_i -shifting function, which is a strictly increasing positive smooth function. It is of the class C^1 and can be designed as:

$$\aleph_i(z_i) = \begin{cases} 1, & \text{for } |z_i| \geq \delta_i, \\ \phi_i(z_i), & \text{for } |z_i| < \delta_i, \end{cases} \quad (142)$$

where δ_i is a small positive constant.

Design the methodology for the function $\phi_i(z_i)$. It can be designed using the following criteria:

$$\begin{cases} \phi_i(z_i) = 0, & \text{for } |z_i| = 0, \\ \phi_i(z_i) = 1, & \text{for } |z_i| = \delta_i, \\ \frac{d\phi_i(z_i)}{dz_i} = 0, & \text{for } |z_i| = \delta_i. \end{cases} \quad (143)$$

Table 4 shows examples of the $\phi_i(z_i)$ function.

Table 4. Examples of the $\phi_i(z_i)$ function.

$\phi_i(z_i)$	Parameters
$\phi_i(z_i) = \sin^n\left(\frac{\pi z_i^{2m}}{2\delta_i^{2m}}\right)$	$m \geq 1$ and $n \geq 2$ are positive design integers.
$\phi_i(z_i) = 1 - \cos^n\left(\frac{\pi z_i^{2m}}{2\delta_i^{2m}}\right)$	$m \geq 1$ and $n \geq 2$ are positive design integers.
$\phi_i(z_i) = v_{1i} \frac{z_i^{2m}}{\delta_i^{2m}} + v_{2i} \tanh\left(\frac{z_i^{2m}}{\delta_i^{2m}}\right)$	$v_{1i} = \frac{1}{1 - \sinh(1) \cosh(1)}$, $v_{2i} = \frac{-\cosh^2(1)}{1 - \sinh(1) \cosh(1)}$, and $m \geq 1$ is a positive design integer.
$\phi_i(z_i) = 1 - \left(1 - \frac{z_i^{2m}}{\delta_i^{2m}}\right)^{2n}$	$m \geq 1$ and $n \geq 1$ are positive design integers.

Substitution of Equation (140) into Equation (121) leads to the following closed-loop system dynamics:

$$\begin{cases} D_t^q z_1 = -\gamma_1 z_1 - k_1 z_1 - s_1 \aleph_1(z_1) z_1^{2p-1} - \tilde{\theta}_1^T \psi_1(Z_1) + g_1 z_2 - \tilde{P}_1, \\ D_t^q z_i = -\gamma_i z_i - k_i z_i - s_i \aleph_i(z_i) z_i^{2p-1} - g_{i-1} z_{i-1} - \tilde{\theta}_i^T \psi_i(Z_i) + g_i z_{i+1} - \tilde{P}_i, i = 2, \dots, n-1, \\ D_t^q z_n = -\gamma_n z_n - k_n z_n - s_n \aleph_n(z_n) z_n^{2p-1} - \tilde{\theta}_n^T \psi_n(Z_n) - g_{n-1} z_{n-1} - \tilde{P}_n. \end{cases} \quad (144)$$

The following theorem encapsulates our main results.

Theorem 7. Consider the FONUS (Equation (57)) with saturated input and under Assumptions 1–5. If the actual and virtual controllers (Equation (140)), the update laws (Equation (141)), and the FODO (Equation (123)) are applied, then the following objectives are achieved:

- The FT boundedness of all closed-loop system signals is assured.
- The output tracking error e_i eventually converges to a small yet modifiable vicinity around the origin within a finite period.

Proof of Theorem 7. Since the proof is similar to that of Theorem 6, it is omitted here. Note that in the expression of $D_t^q V$, one can write:

$$-s_i \aleph_i(z_i) z_i^{2p} = -s_i z_i^{2p} + s_i (1 - \aleph_i(z_i)) z_i^{2p} = -s_i z_i^{2p} + \Delta_i, \quad (145)$$

where $\Delta_i = s_i (1 - \aleph_i(z_i)) z_i^{2p}$, and $0 < (1 - \aleph_i(z_i)) \leq 1$. The stability analysis is also carried out in two scenarios:

Case 1. If $|z_i| \geq \delta_i$, one has $-s_i \aleph_i(z_i) z_i^{2p} = -s_i z_i^{2p}$.

Case 2. If $|z_i| < \delta_i$, $-s_i \aleph_i(z_i) z_i^{2p} = -s_i z_i^{2p} + \Delta_i$, where $|\Delta_i| \leq s_i \delta_i^{2p}$. $\square$

4. Comparison of the Proposed Control Schemes

Table 5 summarizes the key differences between the four control schemes proposed in this paper. Four strategies are proposed to solve the singularity problem. Table 6 shows the comparison of our control scheme with other neighboring control schemes proposed in the literature.

Table 5. Comparison of our proposed control schemes.

Schemes	The Key Contribution: How Is the Singularity Problem Solved?	Useful Remarks
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Scheme 1	1- The term $s_i z_i^{2p-1}$ is replaced by a function: $\frac{s_i^2(z_i^{2p-1})^2 z_i}{\sqrt{s_i^2(z_i^p)^4 + \epsilon_{z2i}^2}}$	1- Lemma 3 is used for stability analysis. 2- The use of the FFTF (Equation (61)) requires that VCs be at least class C^1 . 3- Lemma 3 is exploited also to smoothly approximate the sign function in Equation (61).
Scheme 2	1- The term $s_i z_i^{2p-1}$ is replaced by a function: $s_i \left(s_i z_i \tanh \left(\frac{s_i z_i}{\epsilon_{z1i}} \right) \right)^{2p-1} \tanh \left(\frac{s_i z_i}{\epsilon_{z1i}} \right)$	1- Lemma 5 is used for stability analysis. 2- The use of the FFTF (Equation (95)) requires that VCs be at least class C^1 . 3- This smooth approximation is used also in the auxiliary system and the nonlinear filter in order to ensure the finite-time convergence.
Scheme 3	1- The term $s_i z_i^{2p-1}$ is replaced by a function: $s_i \Xi_i(z_i).$	1- $\Xi_i(z)$ is a smooth switching function, being of the class C^1 . 2- The continuity and differentiability conditions at the boundaries (i.e., when $z_i = \pm \delta_i$) are exploited for designing this switching function. 3- Many design examples of this function are given in Tables 2 and 3. 4- The employment of the FFTF (Equation (115)) in the control design, known also as the DSC technique, evidently alleviates the demands on the design of the switching function.
Scheme 4	1- The term $s_i z_i^{2p-1}$ is replaced by a function: $s_i \aleph_i(z_i) z_i^{2p-1}.$	1- $\aleph_i(z_i)$ is a z_i -shifting function, being of the class C^1 . 2- $\aleph_i(z_i)$ is a strictly increasing positive smooth function. It takes values between 0 and 1. 3- To design this shifting function, the continuity and differentiability conditions are given in Equation (143). 4- Many design examples of this function are given in Table 4.

Table 6. Comparison with other neighboring control schemes proposed in the literature.

	Non-Strict-Feedback Model?	Questionable Derivative Properties (Section 1) Used in Stability Analysis?	Disturbances/Approximation on Error Compensation	Finite-Time Convergence	Singularity	Useful Remarks
Our paper	Yes	No	By designing a FODO.	Yes	No	1-The singularity issue has been solved by many methods. 2- The fuzzy approximation errors and external disturbances are effectively compensated for via designing a FODO. 3- Unmatched and matched disturbances are considered. 4- Our proposed lemmas avoid using the questionable derivative properties of Section 1.
[10]	No	No	Disturbances are not considered in the model.	No	No	1- Disturbances are not considered. 2- The system model is almost known. 3-The output tracking error asymptotically converges to zero. 4- The stability analysis does not rely on the questionable derivative properties from Section 1.
[11]	No	No	The fuzzy approximation errors are not compensated for.	Yes	No	1- Disturbances are not considered. 2- The system is with delayed input.
[12]	Yes	Yes	The fuzzy approximation errors are not compensated for.	Yes	Yes	1- Disturbances are not considered. 2- The system model is partially known. 3- A DSC concept is used, and the filtering error has not been compensated for. 4- The VC is not derivable, and it should beyond class C^1 . Therefore, the singularity problem can take place. 5- The system is with delayed input. 6- The output tracking error is PFT convergent. 7- The stability analysis relies on the questionable derivative properties from Section 1.

[13]	No	No	The fuzzy approximation errors are compensated for via an SMC term.	No	Yes	1- Only matched disturbances are considered. 2- VCs are designed as non-differentiable functions. Therefore, the singularity problem can take place. 3- The upper bounds of the fuzzy approximation errors are supposedly known. 4- The output tracking error is practically stable.
[14]	No	No	The matched disturbances are compensated for by an SMC term.	No	Yes	1- Only matched disturbances are considered. 2- VCs are designed as non-differentiable functions. Therefore, the singularity problem can take place. 3- The adaptive laws are questionable, since they depend on the unknown parametric estimation errors. 4- The output tracking error is finite-time convergent.
[15]	No	No	The matched disturbances are compensated for using a smooth SMC term.	No	No	1- Only matched disturbances are considered. 2- The system model is almost known. 3- The output tracking error is asymptotically stable.
[16]	No	No	The fuzzy approximation errors are not compensated for.	Yes	No	1- The external disturbances are not considered. 2- The designed control law suffers from an algebraic loop problem. 3- The proposed control law does not include terms with fractional power.
[17]	Yes	No	The fuzzy approximation errors are not compensated for.	No	No	1- The output tracking error is practically stable. 2- The upper bounds of the external disturbances and the uncertain functions are conjointly estimated via adaptive fuzzy systems. 3- An FO state observer is designed to estimate the pseudo-state variables.
[18]	No	Yes	By designing an adaptive pseudo-SMC.	Yes	Yes	1- The VC is not derivable, and it should be beyond class C^1 . 2- The fuzzy approximation errors and external disturbances are effectively compensated for via designing an adaptive pseudo-SMC. 3- The stability analysis relies on the questionable derivative properties from Section 1.
[19]	No	No	By designing an adaptive pseudo-SMC.	No	No	1- The output tracking error is practically stable. 2- The fuzzy approximation errors and external disturbances are effectively compensated for via designing an adaptive pseudo-SMC. 3- A DSC concept is used, and the filtering error has not been compensated for.
[20]	Yes	No	By designing an adaptive pseudo-SMC.	No	No	1- The output tracking error is practically stable. 2- The fuzzy approximation errors and external disturbances are effectively compensated for via designing an adaptive pseudo-SMC. 3- A DSC concept is used, and the filtering error has not been compensated for.
[21]	Yes	No	The fuzzy approximation errors are compensated for by designing a sliding-mode –like control.	No	No	1- The output tracking error is practically stable. 2- The external disturbances are not considered. 3- A DSC concept is used, and the filtering error has not been compensated for.
[22]	No	No	The disturbances are compensated for by designing a disturbance observer.	Yes	Yes	1- The system model is completely known. 2- The α_i VCs are not sufficiently differentiable. In fact, α_i does not go beyond class C^{i-1} . Therefore, the singularity problem can take place.
[23]	No	No	The fuzzy approximation errors and the external disturbances are not compensated for.	No	No	1- The system model is partially known. 2- There is unnecessary use of the Nussbaum function in the control laws, as control gains are known. 3- A DSC concept and FO command filter methodology are used in the control design.

[24]	No	No	The fuzzy approximation errors and the external disturbances are not compensated for.	Yes	Yes	1- The α_i VCs are not sufficiently differentiable. In fact, α_i does not go beyond class C^{i-1} . Note that all VCs are designed as functions of class C^1 . Therefore, the singularity problem can take place.
[25]	No	No	The fuzzy approximation errors are compensated for by a sliding-mode-like control.	No	No	1- The upper bound of the fuzzy approximation error is assumed to be known. 2- The external disturbances are not considered.
[26]	Yes	No	The fuzzy approximation errors and the external disturbances are not compensated for.	No	No	1- The designed control law suffers from an algebraic loop problem. 2- The feasibility condition naturally exists in the virtual controllers. 3- The output tracking error is only practically stable.
[27]	No	No	The mismatched disturbances are compensated for by using FODO.	No	No	1- The output tracking error does not converge in finite time. Only the sliding surface variable converges to zero in finite time. 2- The system model is completely known. 3- The control law depends on the fractional derivative of the tracking error. Note that the latter is not available for measurement.
[28]	No	No	The disturbances are compensated for using FODO.	No	No	1- The system model is known and has two pseudo-state variables. 2- An SMC based on FODO is designed. 3- Matched as well as mismatched disturbances are considered.
[29]	No	YES	The fuzzy approximation errors and the external disturbances are compensated for using an adaptive SMC.	Yes	No	1- The system under consideration satisfies the matching conditions. 2- The matching disturbances are considered. 3- The stability analysis relies on the questionable derivative properties from Section 1.
[30]	Yes	Yes	The fuzzy approximation errors are compensated for using an adaptive sliding- mode-like control.	Yes	No	1- The external disturbances are not considered. 2- The DSC concept is used, and the filtering error has not been compensated for. 3- The designed control law suffers from an algebraic loop problem. 4- The control law does not include fractional power terms. These terms are hidden in the adaptive neural networks. 5- The stability analysis relies on the questionable derivative properties from Section 1.
[31]	No	No	The fuzzy approximation errors are compensated for using an adaptive sliding mode term.	Yes	Yes	1- The control gain functions are assumed to be known. 2- The external disturbances are not considered. 3- VCs are designed as non-differentiable functions. Therefore, the singularity problem may occur.
[32]	No	Yes	The fuzzy approximation errors are not compensated for.	Yes	Yes	1- VCs are designed as non-differentiable functions. Therefore, the singularity problem can take place. 2- The external disturbances are not considered. 3- The stability analysis relies on the questionable derivative properties from Section 1.
[33]	Yes	No	The fuzzy approximation errors are not compensated for.	Yes	No	1- The designed control law suffers from an algebraic loop problem.

Remark 13. The proposed control architecture integrates five functional modules: (i) adaptive fuzzy systems with fractional-order integral adaptation laws incorporating sigma modification, (ii) a fractional-order adaptive fuzzy disturbance observer, (iii) a fractional-order finite-time filter for virtual control signals, (iv) a fractional-order auxiliary system to compensate for filtering errors, and (v) the virtual control signals ($n-1$ signals) and the actual control signal. Each module contributes to the overall computational load as detailed below.

- 1) The total computational cost per sampling instant is broken down as follows. Fractional-order operators dominate the cost, appearing in four distinct modules: the adaptive laws, the disturbance observer, the finite-time filter, and the auxiliary system. For a system of order n with history length L , the fractional operators across all modules require $O(n.L)$ operations per sample. For a typical configuration with $n = 3$ and $L = 200$, the fractional operators collectively require approximately 4500–5000 floating-point operations. The fuzzy inference systems—employed in both the adaptive laws and the disturbance observer—require $O(M.R)$ operations, where M is the number of inputs and R is the number of rules; with $M = 2$ and $R = 25$, this adds roughly 100 flops. The virtual control signals ($n - 1 = 2$ signals) and the actual control signal involve algebraic operations that add less than 100 flops. Thus, the total cost is approximately 4700–5200 flops per sample.
- 2) Memory requirements are dominated by the history buffers for the fractional operators across the four modules. Each fractional operator requires storing L past values per state variable. For $n = 3$ and $L = 200$, the total memory footprint for all fractional operators is approximately $3nL \times 8$ bytes ≈ 14.4 KB for double-precision storage. Additional memory for the fuzzy rule bases, precomputed coefficients, filter states, observer states, adaptive parameters, and stack/heap overhead brings the total to approximately 25–30 KB of SRAM. The code size, including the fuzzy rule bases and precomputed tables, requires approximately 60–100 KB of flash memory. These figures are well within the capabilities of modern ARM Cortex-M4 class microcontrollers with hardware floating-point units.
- 3) Regarding sampling-time feasibility, the minimum achievable sampling period is determined by the processor speed. On a 100 MHz ARM Cortex-M4 (peak ~ 50 MFLOPS), the minimum sampling period is approximately 100–110 μ s, corresponding to a maximum sampling frequency of ~ 9 kHz. Accounting for I/O overhead and RTOS jitter, sampling frequencies of 100–500 Hz are readily achievable on such platforms, while frequencies above 1 kHz require optimization (e.g., fixed-point arithmetic, lookup tables, reduced history length) or FPGA-based parallelization.

We acknowledge that shortening the history length L to reduce computational load inevitably degrades the accuracy of the fractional operator approximation across all modules, potentially inducing steady-state offset and degrading transient performance—a fundamental performance–complexity trade-off. Additionally, the sigma-modification terms in the adaptive laws, while essential for robustness, introduce small additional computational overhead. Real-time deployment therefore requires careful co-design between control performance requirements and hardware capabilities.

Remark 14. The primary sensitive tuning parameters are the finite-time exponents p (optimal range 0.5–0.7), control gains, observer gains, auxiliary compensation gains, and filter time constants—all of which present acute trade-offs between convergence speed, noise sensitivity, and robustness to unmodeled dynamics. The history length L creates a direct performance–complexity trade-off, as shortening it for real-time feasibility degrades transient response and induces steady-state offset. Tuning practically requires sequential offline optimization and hardware-in-the-loop validation to ensure robustness against parameter uncertainties and external disturbances. Ultimately, computational constraints and parameter sensitivity are tightly coupled—real-time compromises inevitably degrade theoretical performance, demanding careful co-design for successful deployment.

Remark 15. The theoretical contributions of this paper are detailed in Section 1 and are compared with related works in Table 1 and Table 6. In addition to the desirable practical properties of our control laws—namely, singularity-free behavior, oscillation-free operation, effective rejection of both matched and unmatched disturbances, and model-free implementation—the specific novelties of this work, with respect to [10–19,21–25,27–35,38], are as follows:

1. Four new alternative technical lemmas, extending Lemma 8 in [38], are developed to facilitate the analysis of practical finite-time stability for nonlinear fractional-order systems.
2. Several novel strategies are proposed to circumvent the singularity issue that typically arises from the time differentiation of virtual control laws.
3. A new design for a fractional-order disturbance observer (FODO) based on adaptive fuzzy systems is introduced.
4. Four new methodologies for practical finite-time fractional-order adaptive fuzzy control, using disturbance observers, are established.

Collectively, these contributions provide a unified and general framework for the design of fractional-order finite-time fuzzy adaptive control (FTFAC) based on FODOs.

5. Simulation Results

In this section, a simulation example is presented to verify the validity and the effectiveness of the suggested methods. The FONS considered is described as follows [13]:

$$\begin{cases} D_t^{0.95} x_1(t) = x_2(t) - 0.4 x_1^2(t) + d_1(x, t), \\ D_t^{0.95} x_2(t) = -0.1 x_2(t) + \frac{x_2(t) - 0.5 x_1^2(t)}{1 + x_1^2(t)} + d_2(x, t) + u(t), \\ y(t) = x_1(t), \end{cases} \quad (146)$$

where $f_1(x_1) = 0.4 x_1^2(t)$ and $f_2(x) = \frac{x_2(t) - 0.5 x_1^2(t)}{1 + x_1^2(t)}$ are the unknown functions.

The fuzzy systems $\theta_i^T \psi_i(Z_i)$ for $i = 1, 2$, have vectors $Z_1 = [x_1, x_2, l_1, l_2]^T$ and $Z_2 = [x_1, x_2, l_2]^T$ as input variables, respectively. For each input of these fuzzy systems, we define five sigmoidal membership functions, which are uniformly distributed on intervals $[-5, 5]$ for x_1 and x_2 , and $[-5, 5]$ for l_1 and l_2 .

The design parameter values are selected as follows:

$$\begin{aligned} p = 0.9, \tau_{11} = \tau_{21} = \tau_{31} = 3, \epsilon_{\alpha 11} = \epsilon_{\alpha 21} = \epsilon_{\alpha 31} = 4, \epsilon_{\alpha 12} = \epsilon_{\alpha 22} = \epsilon_{\alpha 32} = 3, \tau_0 = 7, \\ \epsilon_{l 11} = \epsilon_{l 21} = 2, \epsilon_{l 12} = \epsilon_{l 22} = 3, \kappa_{l 1} = \kappa_{l 2} = 2, \gamma_{l 1} = \gamma_{l 2} = 5, \rho_1 = \rho_2 = 2, \gamma_1 = \\ \gamma_2 = 4, s_1 = s_2 = 2, \Lambda_1 = \Lambda_2 = 3, k_1 = k_2 = 5, \epsilon_{z 11}, \epsilon_{z 12} = \epsilon_{z 21} = \epsilon_{z 22} = 2, \sigma_1 = 2, \\ \sigma_2 = 3. \end{aligned}$$

The functions $\varphi_i(z_i)$ in the third control strategy are selected as follows:

$$\varphi_i(z_i) = v_{1i} z_i + v_{2i} z_i |z_i|, \quad i = 1, 2, \quad (147)$$

with $v_{1i} = (3 - 2p)\delta_i^{2p-2}$ and $v_{2i} = 2(p - 1)\delta_i^{2p-3}$, where $\delta_i = 0.1$.

The functions $\varphi_i(z_i)$ in the fourth control strategy are chosen as:

$$\psi_i(z_i) = \sin^n\left(\frac{\pi z_i^{2m}}{2 \delta_i^{2m}}\right), \quad i = 1, 2, \quad (148)$$

where $\delta_i = 0.1$, $m = 1$ and $n = 2$.

In this section, two simulation scenarios are considered to comprehensively evaluate the performance of the proposed control schemes:

- *Scenario 1:* Evaluation of the performance of the four proposed control schemes, along with a numerical comparison with existing control methods.
- *Scenario 2:* Assessment of the robustness and adaptability of the proposed control schemes under various initial conditions, external disturbances, and reference signals.

1) Scenario 1: Performance Comparison:

In this scenario, the performance of the four proposed control schemes is evaluated and numerically compared with that of existing control methods.

The initial condition is $x(0) = [-3, 5]^T$, and $y_d(t) = \sin(t)$ is the output reference signal. Let $d_1(x, t) = 0.5 \cos(t)$ and $d_2(x, t) = \sin(t) + \cos(t)$.

The mathematical definitions of the adopted performance indices are given as follows:

- Overshoot $M_{po}(\%)$:

$$M_{po}(\%) = \max(0, (y_{max} - 1)) \times 100. \quad (149)$$

- Undershoot $M_{uo}(\%)$:

$$M_{uo}(\%) = (-y_{min} - 1) \times 100. \quad (150)$$

- Integral of Time-weighted Absolute Error (ITAE):

$$ITAE = \int_0^{15} |y(t) - y_d(t)| t dt. \quad (151)$$

- Maximum steady-state error:

$$e_{ss} = \max |y(t) - y_d(t)|, \quad (152)$$

where y_{min} and y_{max} denote the minimum and maximum values of the output y , respectively.

The simulation results are illustrated in Figures 1–4. The system output $y(t)$ and its reference signal $y_d(t)$ are depicted in Figure 1. The tracking error $y(t) - y_d(t)$ is shown in Figure 2. This figure highlights the tracking capabilities of the proposed controllers and indicates their effectiveness even in the presence of saturation nonlinearities and a completely unknown system model. The curves of the virtual control $\alpha_1(t)$ and the control input $u(t)$ are presented in Figure 3 and Figure 4, respectively. These results suggest that the four control strategies are able to ensure that all signals of the closed-loop system remain bounded.

To provide a more comprehensive qualitative and quantitative comparison between the four control strategies proposed in this paper and the existing methods reported in [13–15,19], additional simulation results based on the control schemes presented in [13–15,19] have been included (see Figures 5 and 6 along with Table 7).

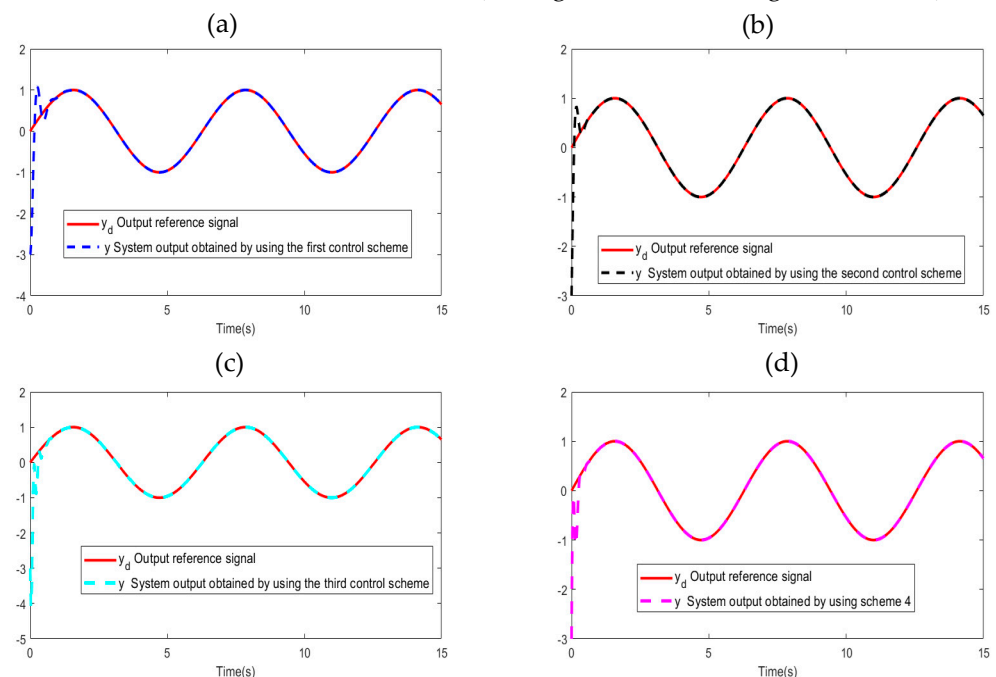


Figure 1. Tracking performance of the system output $y(t)$ with respect to the reference signal $y_d(t)$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

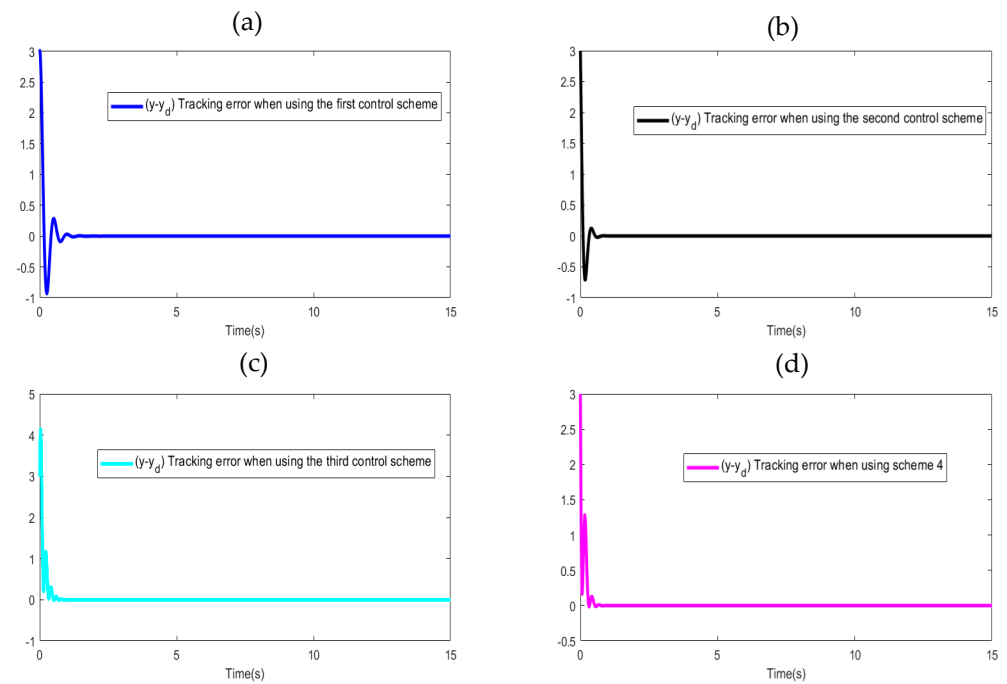


Figure 2. Evolution of the tracking error $y_d(t) - y(t)$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

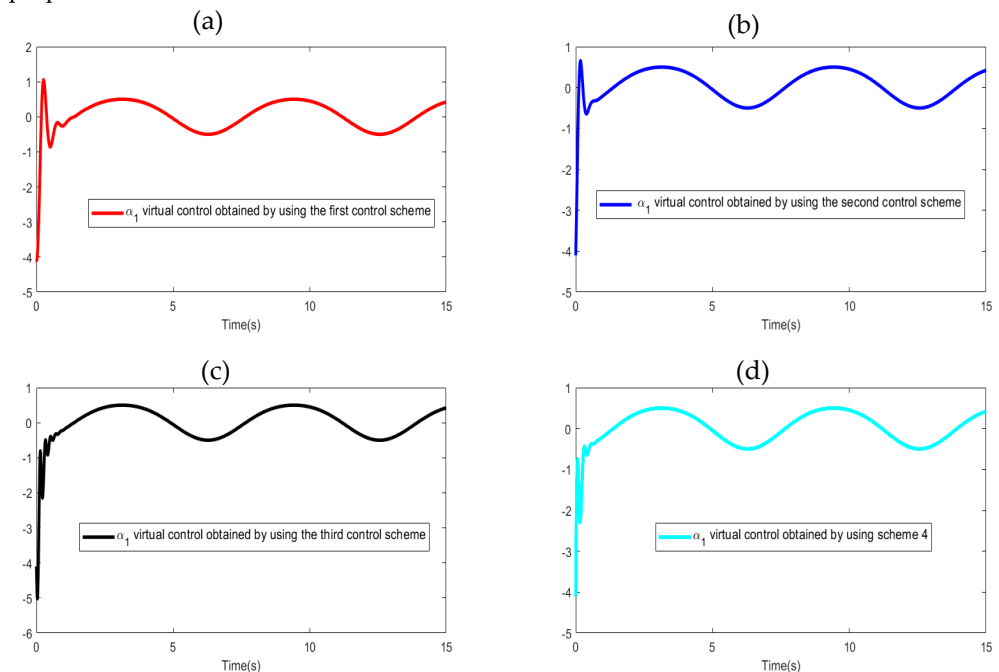


Figure 3. Virtual control signal $\alpha_1(t)$ obtained: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

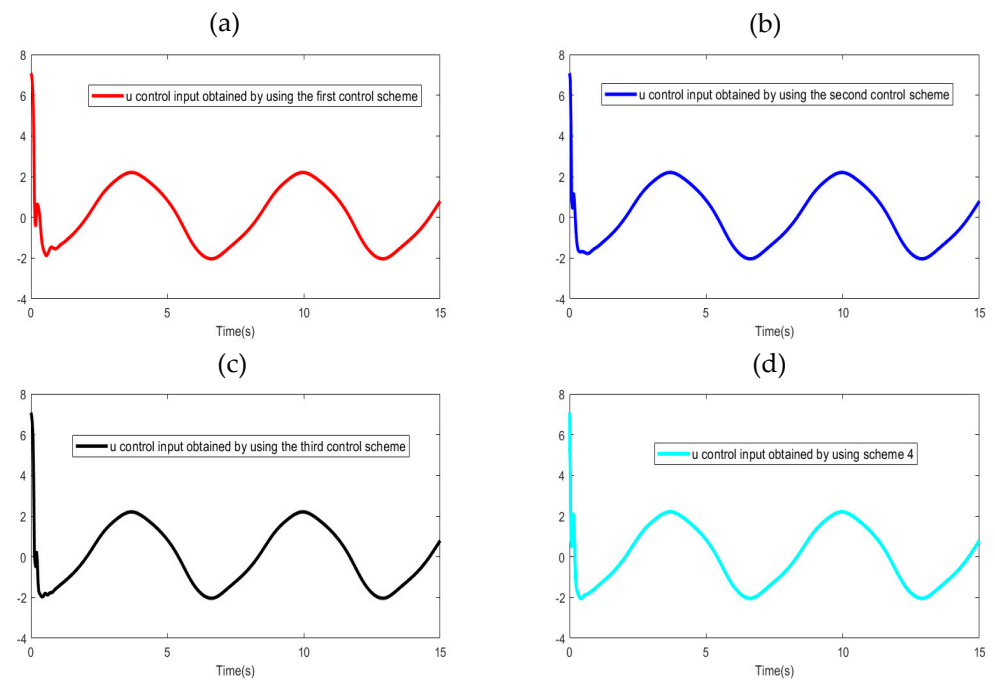


Figure 4. Evolution of control input $u(t)$ under: (a) the first proposed controller; (b) the second proposed controller; (c) the third proposed controller; (d) the fourth proposed controller.

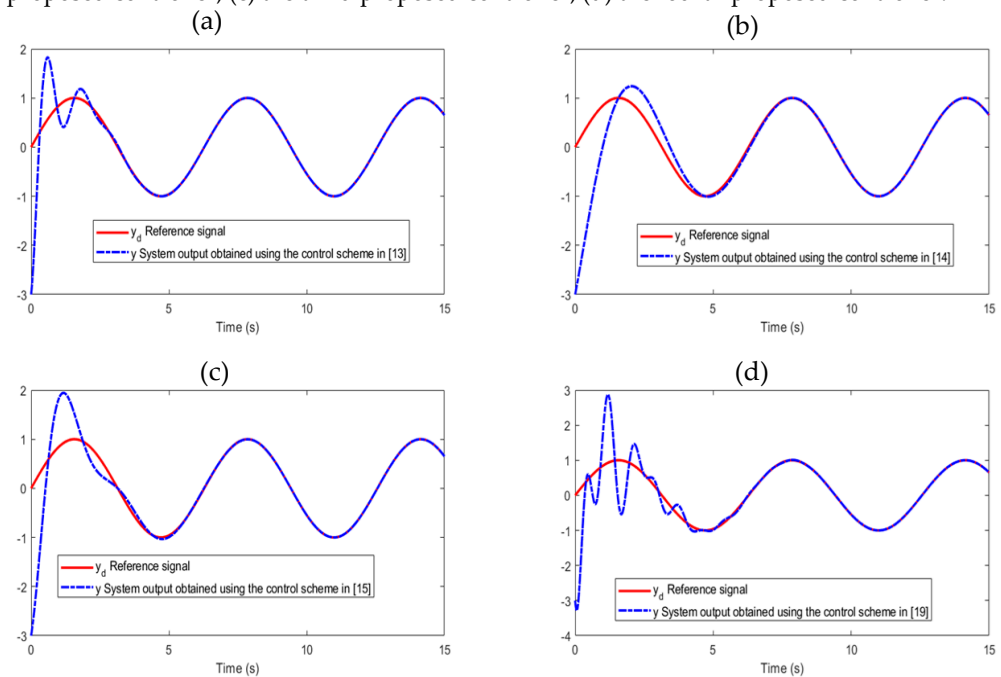


Figure 5. Tracking performance of the system output $y(t)$ and the reference signal $y_d(t)$ using: (a) the controller proposed in [13]; (b) the controller proposed in [14]; (c) the controller proposed in [15]; (d) the controller proposed in [19].

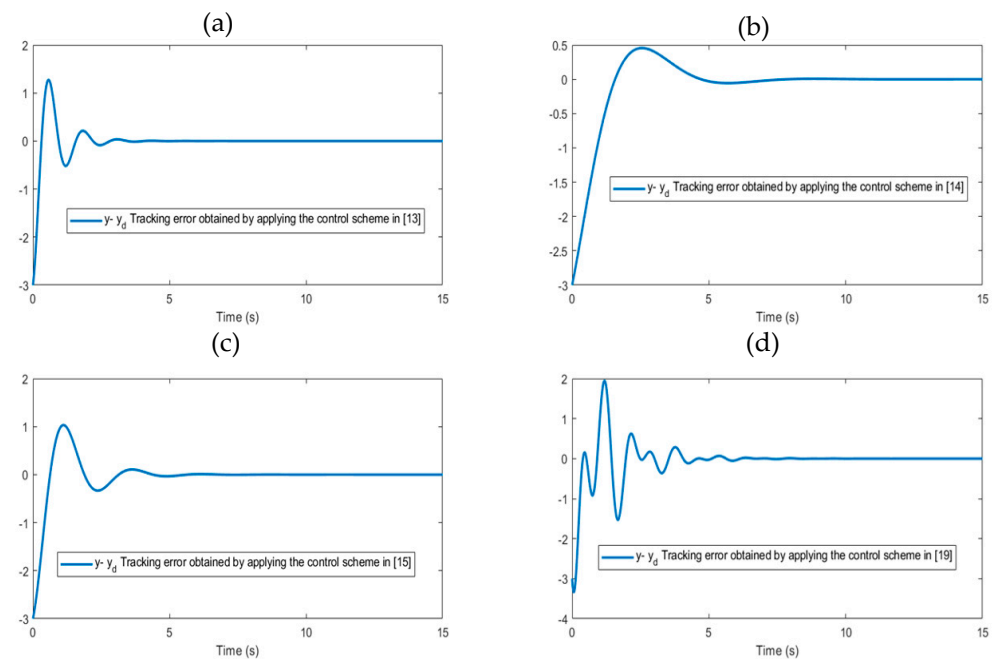


Figure 6. Evolution of the tracking error $y(t) - y_d(t)$ obtained using: (a) the controller proposed in [13]; (b) the controller proposed in [14]; (c) the controller proposed in [15]; (d) the controller proposed in [19].

Table 7 summarizes the widely accepted performance metrics used to objectively assess the transient and steady-state behavior of the controllers, including convergence time, undershoot, overshoot, Integral of Time-weighted Absolute Error (ITAE), and steady-state error. As shown in Table 7, the fourth proposed control strategy achieves the best overall performance, with the fastest convergence, smallest undershoot and overshoot, lowest ITAE value, and minimum steady-state error compared with the control schemes reported in [13–15, 19]. The other three proposed control strategies also outperform the corresponding existing methods in most of the considered metrics. These results demonstrate the superior tracking accuracy, convergence speed, control precision, robustness, and overall effectiveness of the proposed control strategies compared with the previously reported approaches.

Table 7. Quantitative performance comparison of the four proposed controllers with [13–15,19].

Control Scheme	Convergence Time (s)	Undershoot (%)	Overshoot (%)	ITAE	Maximum Steady-State Error
First proposed scheme	0.82	200	0.18	8.68t p [−]	3
Second proposed scheme	0.71	200	0	5.41nd [−]	3
Third proposed scheme	0.63	310	0	2.93d p [−]	4.2
Fourth proposed scheme	0.58	200	0	1.28th [−]	3
Scheme proposed in [13]	3.2	200	85	8.3×10^{-4}	3
Scheme proposed in [14]	5.1	200	20	6.5×10^{-4}	3
Scheme proposed in [15]	4.9	200	100	9.1×10^{-4}	3
Scheme proposed in [19]	5.3	210	198	3.4×10^{-3}	3.2

2) Scenario 2: Robustness and Adaptability Assessment:

To comprehensively and convincingly assess the effectiveness, robustness, and reliability of the four proposed control schemes, extensive numerical validation was performed under a wide range of operating conditions. In addition to actuator nonlinearities and model uncertainties, the controllers were systematically tested with different reference trajectories, varied initial conditions, and multiple disturbance profiles. These complementary scenarios were selected to challenge the controllers under diverse and potentially adverse conditions and thereby provide a more rigorous evaluation of their practical robustness. For each scheme, the system output and tracking error were carefully analyzed, with particular attention to tracking accuracy, convergence behavior, transient response, and stable performance under disturbances, uncertainties, and actuator constraints. The comparative results in Figures 7–18 consistently confirm the effectiveness and robustness of all four proposed schemes and highlight their superior tracking performance across a broad spectrum of operating conditions. Overall, these results provide strong numerical evidence that the proposed schemes are not limited to the nominal simulation case but can maintain accurate, stable, and reliable tracking under significant variations and uncertainties.

Figure 7. Tracking performance of the system output $y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(t)$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

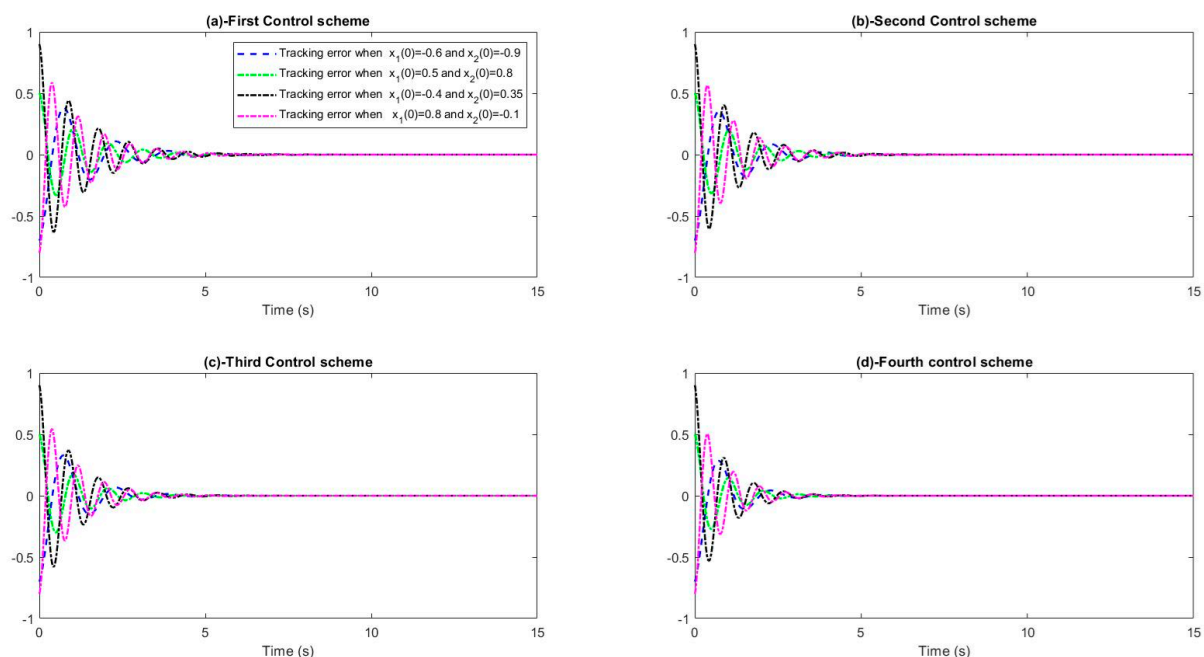


Figure 8. Evolution of the tracking error $y_d(t) - y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(t)$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

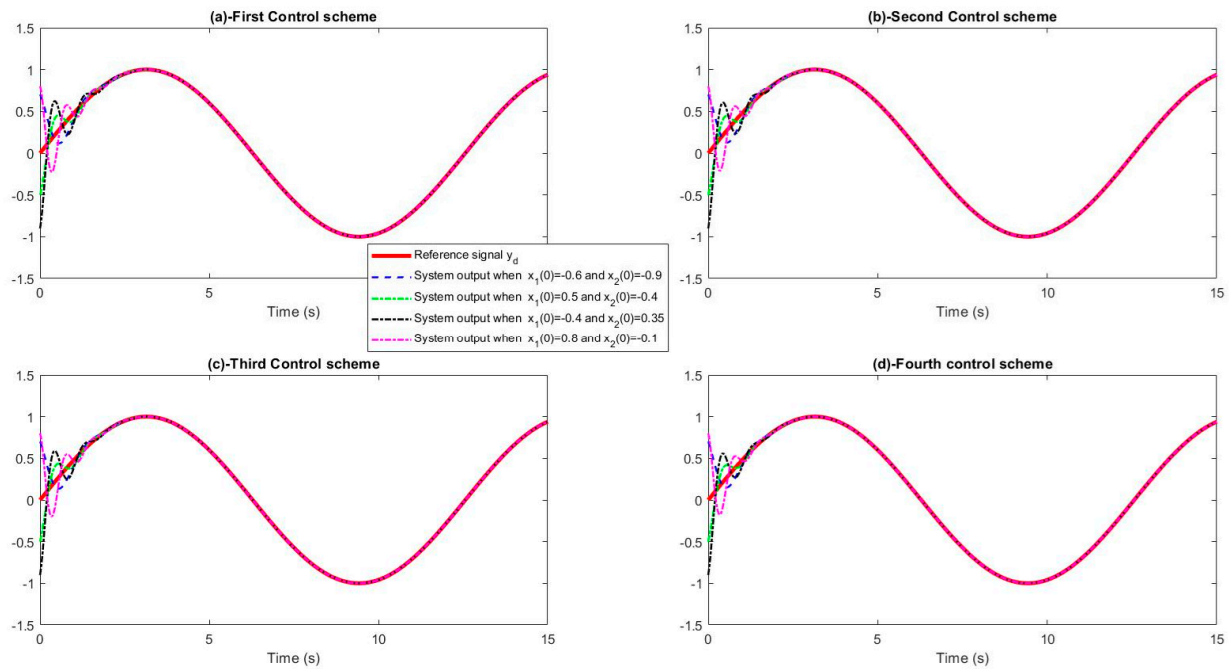


Figure 9. Tracking performance of the system output $y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

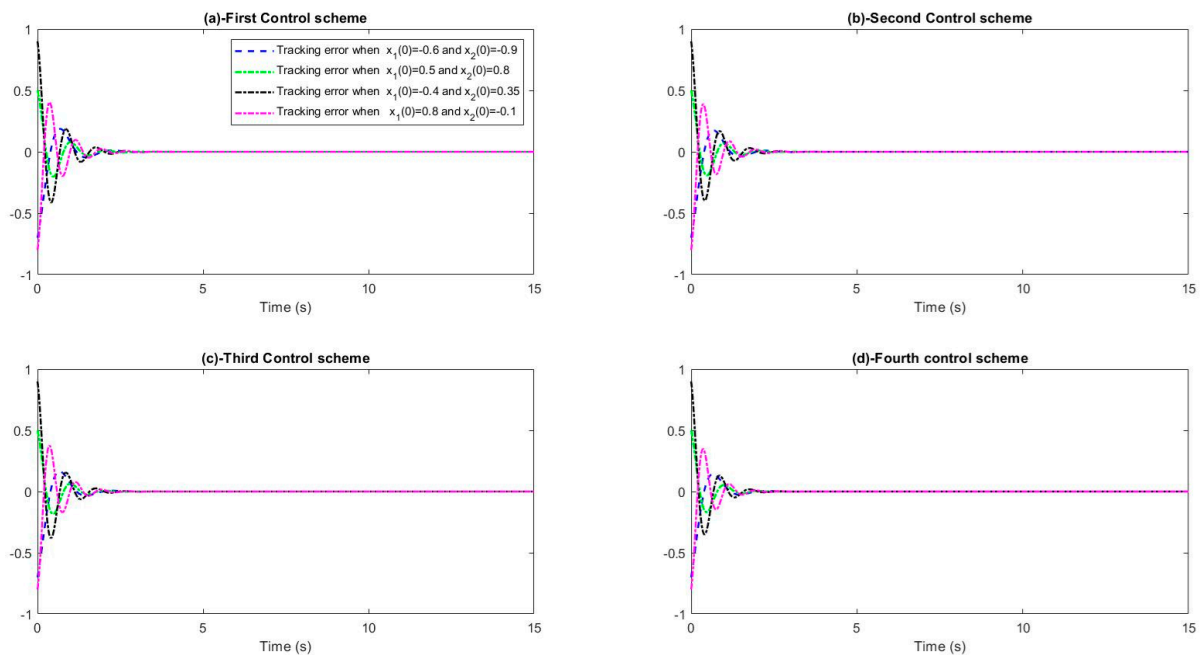


Figure 10. Evolution of the tracking error $y_d(t) - y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

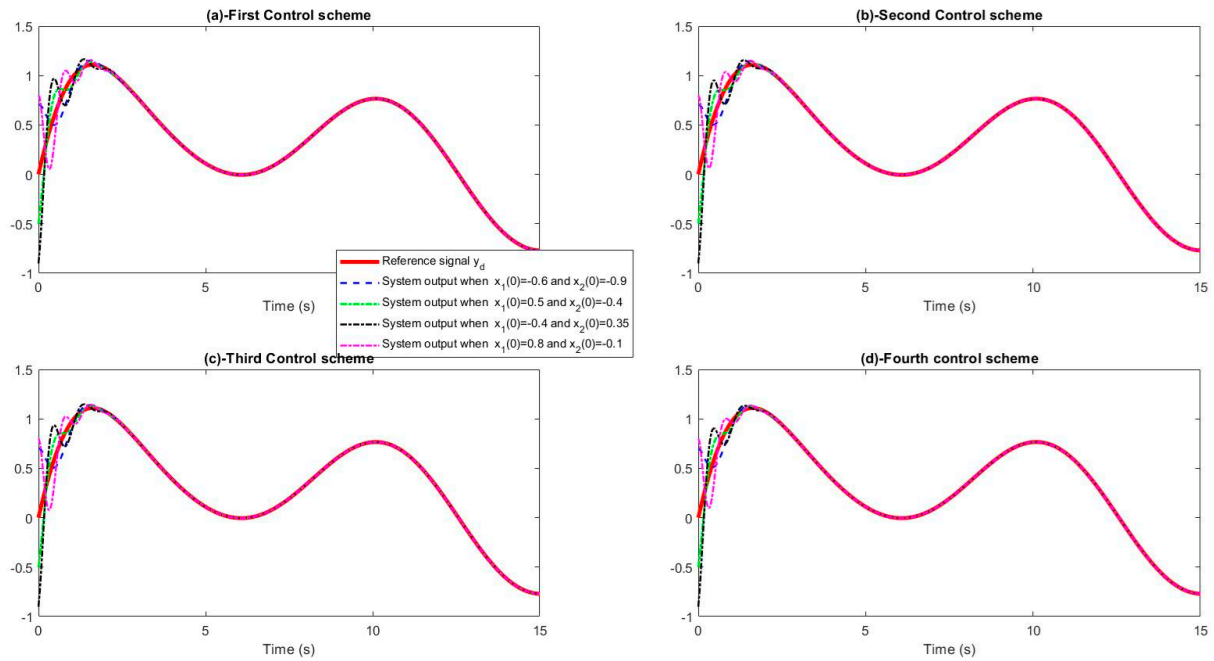


Figure 11. Tracking performance of the system output $y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})\cos(\frac{t}{4}) + \sin(t)\exp(-\frac{t}{2})$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

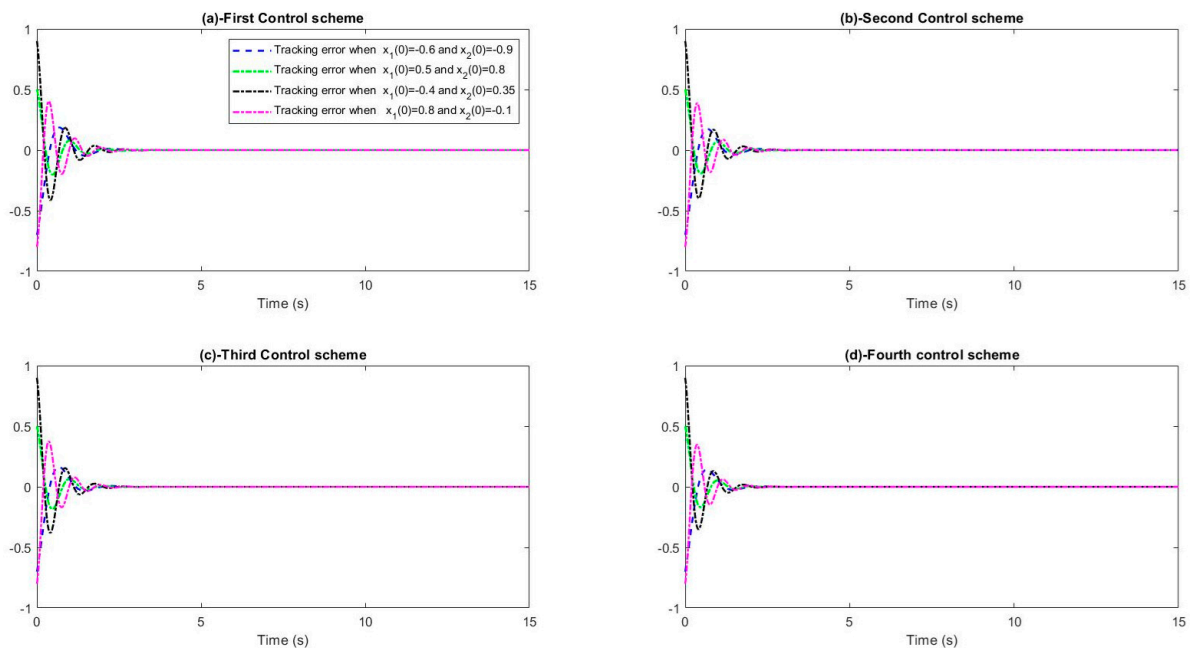


Figure 12. Evolution of the tracking error $y_d(t) - y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})\cos(\frac{t}{4}) + \sin(t)\exp(-\frac{t}{2})$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

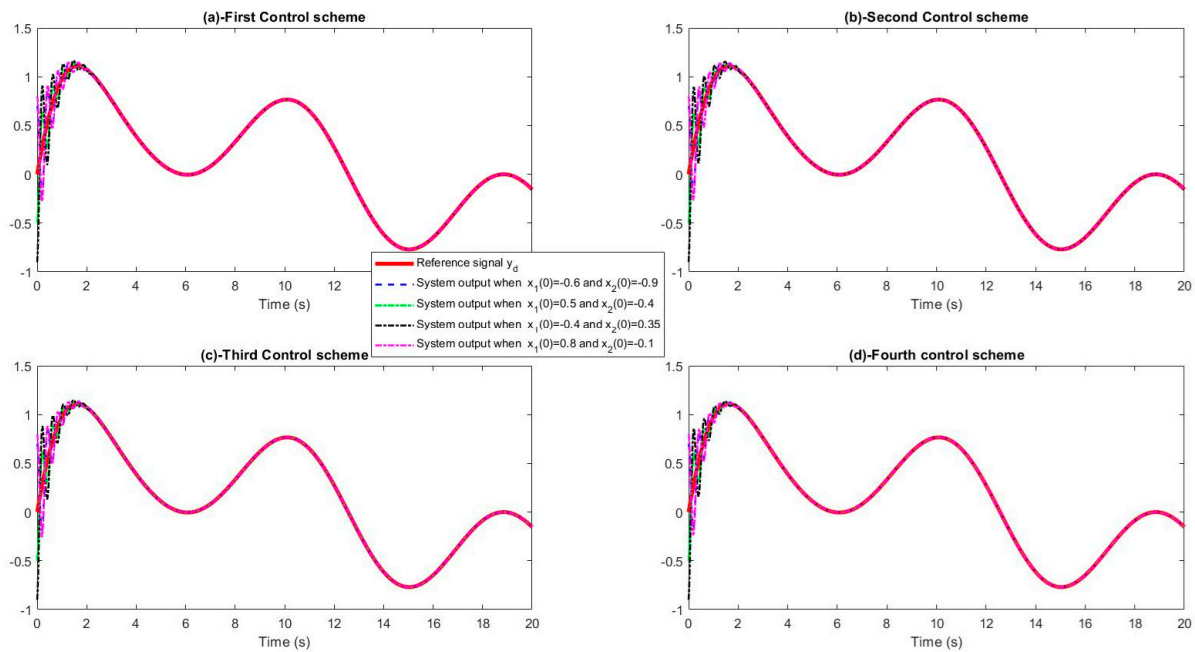


Figure 13. Tracking performance of the system output $y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})\cos(\frac{t}{4}) + \sin(t)\exp(-\frac{t}{2})$ and in the presence of disturbances $d_1(x, t) = \text{sign}(x_1) \cos(20t)$ and $d_2(x, t) = \text{sign}(x_2) \cos(30t)$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

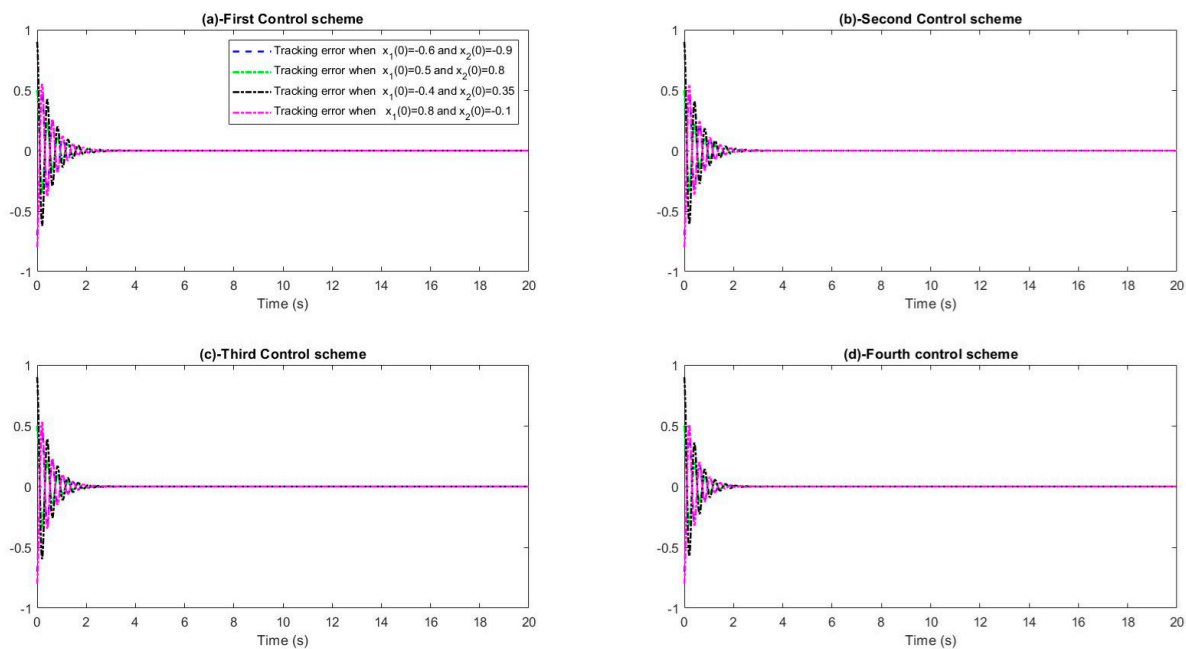


Figure 14. Evolution of the tracking error $y_d(t) - y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})\cos(\frac{t}{4}) + \sin(t)\exp(-\frac{t}{2})$ and in the presence of disturbances $d_1(x, t) = \text{sign}(x_1) \cos(20t)$ and $d_2(x, t) = \text{sign}(x_2) \cos(30t)$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

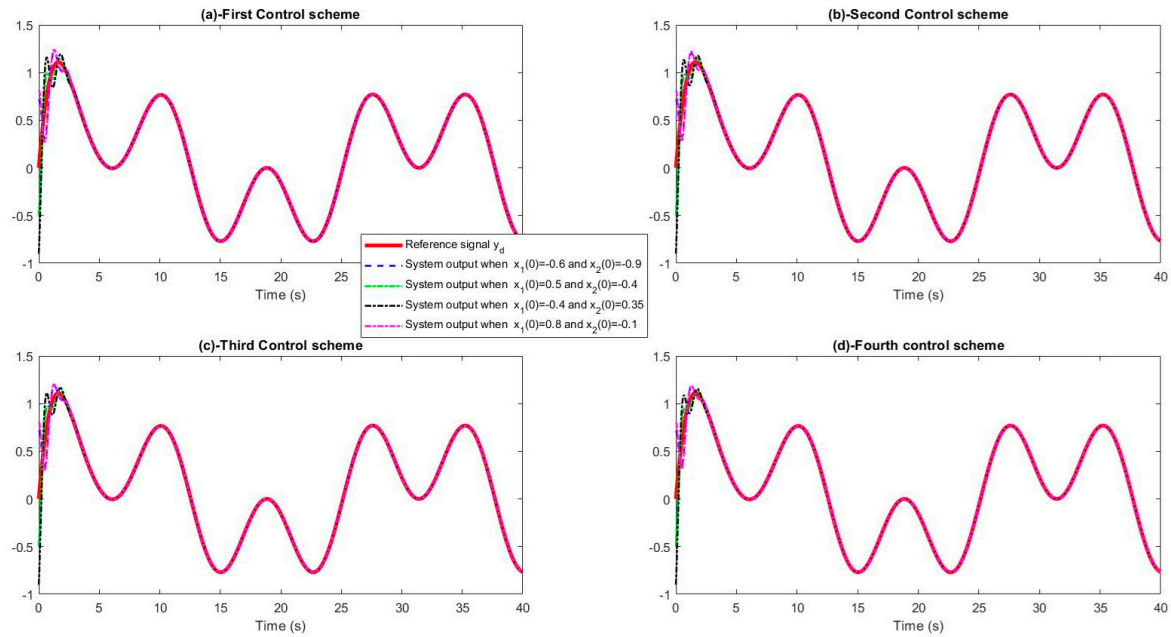


Figure 15. Tracking performance of the system output $y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})\cos(\frac{t}{4}) + \sin(t)\exp(-\frac{t}{2})$ and in the presence of disturbances $d_1(x, t) = \frac{\cos(20t)}{1+|x_2|}$ and $d_2(x, t) = \frac{\cos(30t)}{1+|x_1|}$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

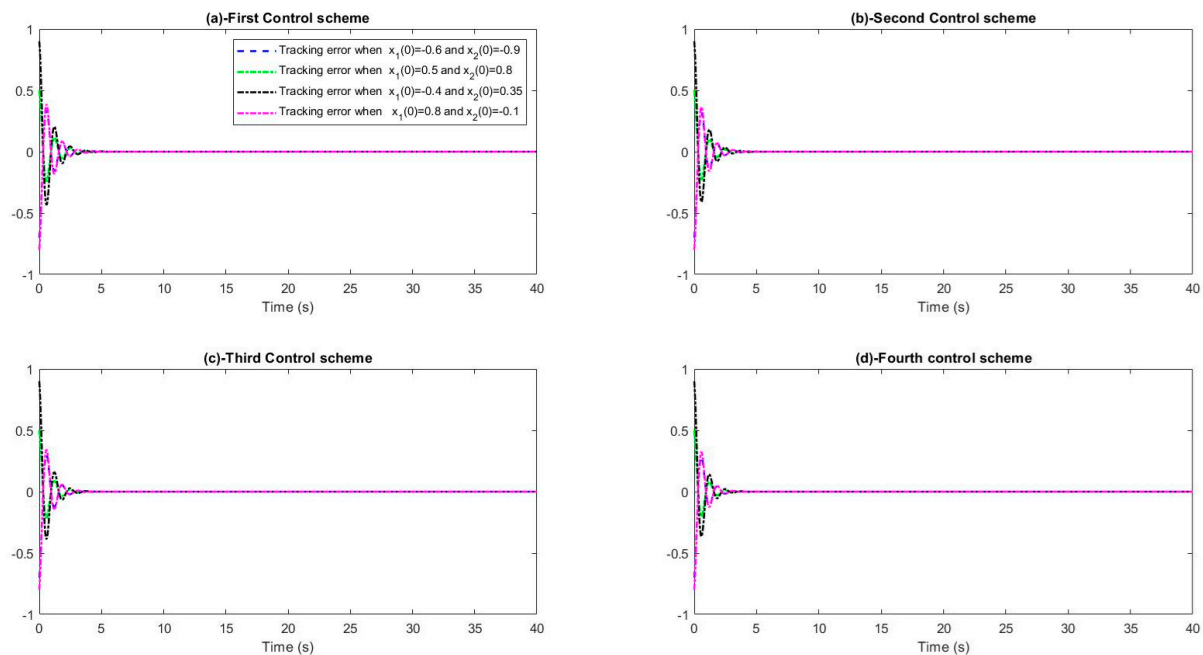


Figure 16. Evolution of the tracking error $y_d(t) - y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \sin(\frac{t}{2})\cos(\frac{t}{4}) + \sin(t)\exp(-\frac{t}{2})$ and in the presence of disturbances $d_1(x, t) = \frac{\cos(20t)}{1+|x_2|}$ and $d_2(x, t) = \frac{\cos(30t)}{1+|x_1|}$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

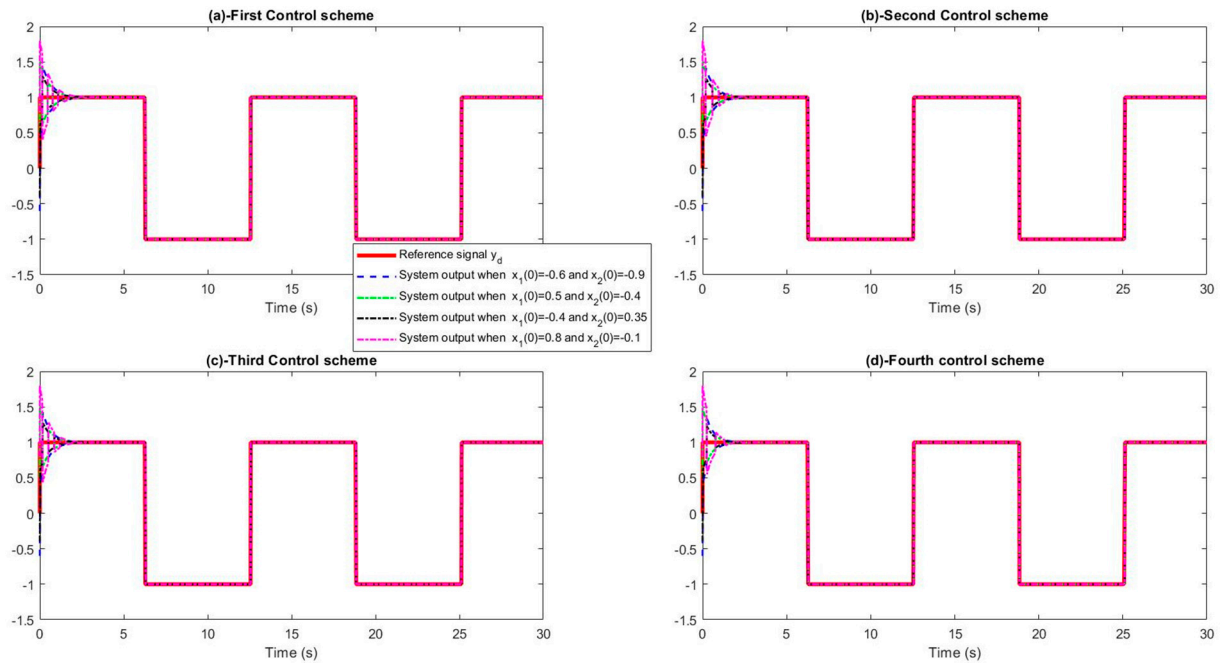


Figure 17. Tracking performance of the system output $y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \text{sign}(\sin(t/2))$ and in the presence of disturbances $d_1(x, t) = \text{sign}(\cos(20t))$ and $d_2(x, t) = \text{sign}(\cos(30t))$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

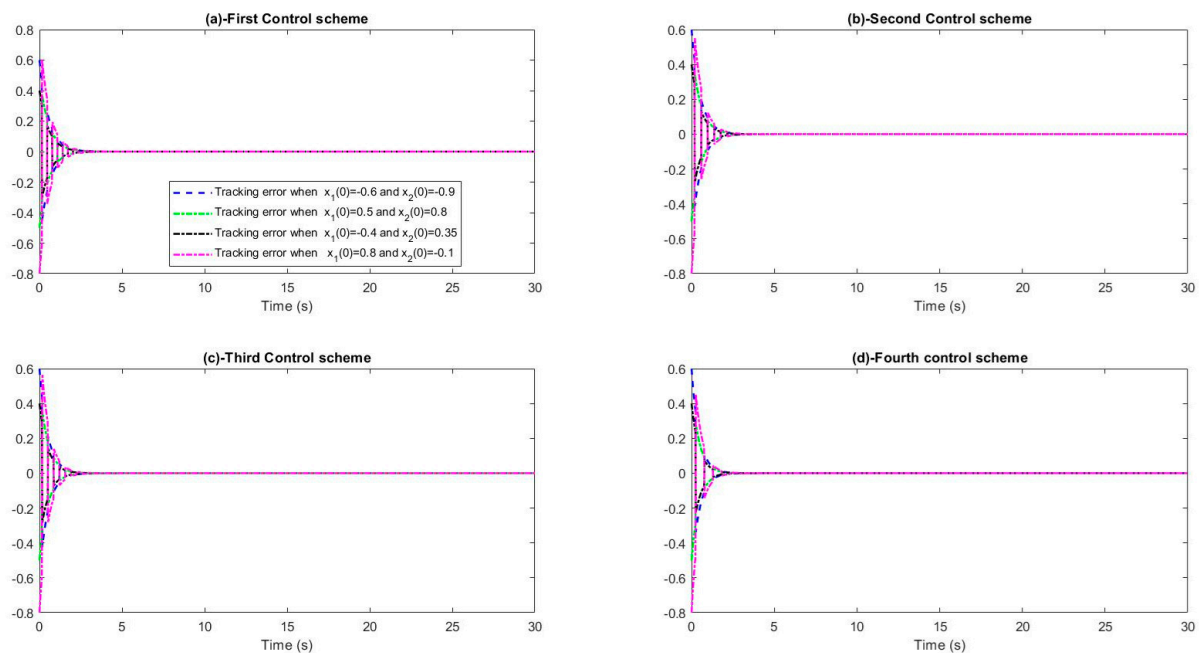


Figure 18. Evolution of the tracking error $y_d(t) - y(t)$ under different initial conditions with respect to the reference signal $y_d(t) = \text{sign}(\sin(t/2))$ and in the presence of disturbances $d_1(x, t) = \text{sign}(\cos(20t))$ and $d_2(x, t) = \text{sign}(\cos(30t))$: (a) using the first proposed controller; (b) using the second proposed controller; (c) using the third proposed controller; (d) using the fourth proposed controller.

6. Conclusions

In this paper, a set of FTFAC schemes has been developed for uncertain CFONs subject to actuator saturation and both matched and unmatched external disturbances. Adaptive FLSs are employed to estimate unknown nonlinear functions online, and FODOs are constructed to enhance control robustness. Notably, all proposed controllers are free from singularity, which is achieved by designing VCs as smooth functions and by employing FO filters. Through Lyapunov-based analysis, it is shown that all closed-loop system variables remain finite-time bounded, and that the tracking errors converge practically in finite time to a small neighborhood of the origin, which can be adjusted as needed. Simulation results confirm the effectiveness of the proposed control approaches. A qualitative and quantitative comparison with existing methods [13–15,19] further illustrates the relative advantages of the four control strategies, which exhibit improved performance across key indicators—particularly ITAE, MSSE, overshoot, and undershoot—for the critical controlled variables.

To advance this research direction, the following topics merit further investigation:

1. State estimation via the high-gain fractional-order observer introduced in [56], leveraging its demonstrated effectiveness in handling model uncertainties and measurement noise.
2. Extension of the proposed control frameworks to fractional-order dynamical models of electric vehicles and heat exchange systems [57] so as to broaden their practical relevance across diverse engineering applications.
3. Development of an FTFAC scheme for nonlinear systems with variable fractional order, which may offer promising prospects for achieving rapid convergence and robust performance under dynamic operating conditions.
4. A notable limitation of this work is the lack of experimental validation. Although the numerical simulations provide strong support for the theoretical findings and suggest practical potential, the transition to hardware implementation introduces additional challenges—such as time delays, quantization effects, and sensor imperfections—that are difficult to fully capture in a simulation environment. Experimental validation on a suitable testbed is therefore considered an important and necessary direction for future work.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

The following abbreviations are used in this manuscript:

Table A1. Abbreviations.

CFONs	Caputo (-type) fractional-order nonlinear system
DSC	Dynamic surface control

FFTF	Fast finite-time filter
FL	Fuzzy logic
FLS	Fuzzy logic system
FO	Fractional order
FODO	Fractional-order disturbance observer
FONS	Fractional-order nonlinear system
FONUS	Fractional-order nonlinear uncertain system
FTFAC	Finite-time fuzzy adaptive control
NN	Neural network
PFTS	Practical finite-time stability
FTS	Finite-time stability
QLFC	Quadratic Lyapunov function candidate
SMC	Sliding mode control
ST	Settling time
FOTD	Fractional-order time derivative
VC	Virtual control

The following nomenclature summarizes the mathematical symbols, variables, and parameters used throughout this manuscript.

Table A2. Summary of the primary variables and design parameters.

Parameter/Variable	Description	Selection Principle/Constraint
$E_q(\cdot)$	Mittag-Leffler function	$0 < q < 1$
$V(\cdot)$	Lyapunov candidate function	Symmetric, positive definite, and radially unbounded
D_t^q	Caputo fractional-order derivative	Real order, $0 < q < 1$
β	Fractional exponent	Real, $0 < \beta < 1$
p	Fractional exponent	Real, $\frac{1}{2} < p < 1$
x_i	State variable	$x_i \in R$
$x = [x_1, \dots, x_n]^T$	State vector	$x \in R^n$
$u(v)$	Actuator saturation output	$u \in R$
v	Saturation input	$v \in R$
$d_i(x, t)$	External disturbance	$d_i(x, t) \in R$, with d_i and $D_t^q d_i$ bounded
α_{fi-1}	Filter output	$\alpha_{fi-1} \in R$
α_{i-1}	Virtual control	$\alpha_{i-1} \in R$
ω	Compensation signal of the saturation effect	$\omega \in R$
y_d	Desired reference signal	$y_d \in R$, where y_d , $D_t^q y_d$, and $D_t^q(D_t^q y_d)$ are continuous and bounded
$\tau_{1j}, \tau_{2j}, \tau_{3j}$	Filter gains	Real positive
$\epsilon_{\alpha 1j}, \epsilon_{\alpha 2j}, \epsilon_{\alpha 3j}$	Small constants for filter performance improvement	Real positive
l_i	Auxiliary system state	$l_i \in R$
γ_{li}, κ_{li}	Auxiliary system gains	Real positive
$\epsilon_{l1i}, \epsilon_{l2i}$	Small constants for auxiliary system performance	Real positive
P_i	Augmented disturbance function	$P_i \in R$
$\hat{P}_i$	Estimate of the augmented disturbance	$\hat{P}_i \in R$
$\tilde{P}_i$	Disturbance observation errors	$\tilde{P}_i \in R$
ρ_i	Observer gain	Real positive, with $\rho_i > \frac{4}{3}$
θ_i^*	Unknown optimal fuzzy parameters	$\theta_i^* \in R^N$
$\psi_i(Z_i)$	Fuzzy basis function	$\psi_i(Z_i) \in R^N$
$\hat{\theta}_i$	Estimate of fuzzy parameters	$\hat{\theta}_i \in R^N$
$\tilde{\theta}_i = \hat{\theta}_i - \theta_i^*$	Parameter estimation error	$\tilde{\theta}_i \in R^N$
k_i, s_i, γ_i	Control gains	Real positive
$\epsilon_{z1i}, \epsilon_{z2i}$	Small constants for control performance improvement	Real positive
Λ_i	Adaptation law gain	Real positive
σ_i	σ -modification parameter	Small and real positive, with $\sigma_i > \frac{2a_i\rho_i}{\gamma_i} + \frac{1}{\Lambda_i}$

References

1. Xu, N.; Zhao, X.; Zong, G.; Wang, Y. Adaptive control design for uncertain switched nonstrict-feedback nonlinear systems to achieve asymptotic tracking performance. *Appl. Math. Comput.* **2021**, *408*, 126344. <https://doi.org/10.1016/j.amc.2021.126344>.
2. Wu, X.; Zhao, Y.; Xu, K. Nonlinear disturbance observer based sliding mode control for a benchmark system with uncertain disturbances. *ISA Trans.* **2021**, *110*, 63–70. <https://doi.org/10.1016/j.isatra.2020.10.032>.
3. Na, J.; Zhao, J.; Gao, G.; Li, Z. Output-feedback robust control of uncertain systems via online data-driven learning. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *32*, 2650–2662. <https://doi.org/10.1109/TNNLS.2020.3007414>.
4. Li, Y.; Yang, T.; Tong, S. Adaptive neural networks finite-time optimal control for a class of nonlinear systems. *IEEE Trans. Neural Netw. Learn. Syst.* **2020**, *31*, 4451–4460. <https://doi.org/10.1109/TNNLS.2019.2955438>.
5. Labdai, S.; Bounar, N.; Boukroune, A.; Hemici, B.; Nezli, L. Artificial neural network-based adaptive control for a DFIG-based WECS. *ISA Trans.* **2022**, *128*, 171–180. <https://doi.org/10.1016/j.isatra.2021.11.045>.
6. Zhang, J.; Chen, D.; Shen, G.; Sun, Z.; Xia, Y. Disturbance observer based adaptive fuzzy sliding mode control: A dynamic sliding surface approach. *Automatica* **2021**, *129*, 109606. <https://doi.org/10.1016/j.automatica.2021.109606>.
7. Bounar, N.; Boukroune, A.; Labdai, S.; Chrifi-Alaoui, L. Fixed-time fuzzy adaptive control scheme for doubly fed induction generator-based wind turbine. *Trans. Inst. Meas. Control* **2024**, *46*, 1579–1589. <https://doi.org/10.1177/01423312231197443>.
8. Su, X.; Liu, Z.; Lai, G.; Zhang, Y.; Chen, C.L.P. Event-triggered adaptive fuzzy control for uncertain strict-feedback nonlinear systems with guaranteed transient performance. *IEEE Trans. Fuzzy Syst.* **2019**, *27*, 2327–2337. <https://doi.org/10.1109/TFUZZ.2019.2898156>.
9. Jin, D.Y.; Niu, B.; Wang, H.Q.; Yang, D. A new adaptive DS-based finite-time neural tracking control scheme for non-strict-feedback nonlinear systems. *IEEE Trans. Syst. Man Cybern. Syst.* **2022**, *52*, 1014–1018. <https://doi.org/10.1109/TSMC.2020.3009405>.
10. Sheng, D.; Wei, Y.; Cheng, S.; Shuai, J. Adaptive backstepping control for fractional order systems with input saturation. *J. Frankl. Inst.* **2017**, *354*, 2245–2268. <https://doi.org/10.1016/j.jfranklin.2016.12.030>.
11. Lin, F.; Xue, G.; Qin, B.; Li, S.; Liu, H. Event-triggered finite-time fuzzy control approach for fractional-order nonlinear chaotic systems with input delay. *Chaos Solitons Fractals* **2013**, *175*, 114036. <https://doi.org/10.1016/j.chaos.2023.114036>.
12. Xing, Y.; Wang, Y. Finite-time adaptive NN backstepping dynamic surface control for input-delay fractional-order nonlinear systems. *IEEE Access* **2023**, *11*, 5206–5214. <https://doi.org/10.1109/ACCESS.2023.3234426>.
13. Liu, H.; Pan, Y.; Li, S.; Chen, Y. Adaptive fuzzy backstepping control of fractional-order nonlinear systems. *IEEE Trans. Syst. Man Cybern. Syst.* **2017**, *47*, 2209–2217. <https://doi.org/10.1109/TSMC.2016.2640950>.
14. Nikdel, N.; Badamchizadeh, M.A. Fractional-order adaptive backstepping control of a class of uncertain systems with external disturbances. *Int. J. Control* **2019**, *92*, 1344–1353. <https://doi.org/10.1080/00207179.2017.1393105>.
15. Li, X.; Wen, C.; Zou, Y. Adaptive backstepping control for fractional-order nonlinear systems with external disturbance and uncertain parameters using smooth control. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 7860–7869. <https://doi.org/10.1109/TSMC.2020.2987335>.
16. Xin, C.; Li, Y.; Niu, B. Event-triggered adaptive fuzzy finite time control of fractional-order non-strict feedback nonlinear systems. *J. Syst. Sci. Complex.* **2022**, *35*, 2166–2180. <https://doi.org/10.1007/s11424-022-1244-z>.
17. Yang, W.; Yu, W.; Lv, Y.; Zhu, L.; Hayat, T. Adaptive fuzzy tracking control design for a class of uncertain nonstrict-feedback fractional-order nonlinear SISO systems. *IEEE Trans. Cybern.* **2021**, *51*, 3039–3053. <https://doi.org/10.1109/TCYB.2019.2931401>.
18. You, X.; Dian, S.; Liu, K.; Guo, B.; Xiang, G.; Zhu, Y. Command filter-based adaptive fuzzy finite-time tracking control for uncertain fractional-order nonlinear systems. *IEEE Trans. Fuzzy Syst.* **2023**, *31*, 226–240. <https://doi.org/10.1109/TFUZZ.2022.3185453>.
19. Ma, Z.; Ma, H. Adaptive fuzzy backstepping dynamic surface control of strict-feedback fractional-order uncertain nonlinear systems. *IEEE Trans. Fuzzy Syst.* **2020**, *28*, 22–33. <https://doi.org/10.1109/tfuzz.2019.2900602>.
20. Sui, S.; Chen, C.L.P.; Tong, S. Neural-network-based adaptive DSC design for switched fractional-order nonlinear systems. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *32*, 4703–4712. <https://doi.org/10.1109/TNNLS.2020.3027339>.
21. Bi, W.; Wang, T.; Yu, X. Fuzzy adaptive decentralized control for nonstrict-feedback large-scale switched fractional-order nonlinear systems. *IEEE Trans. Cybern.* **2022**, *52*, 8887–8896. <https://doi.org/10.1109/TCYB.2021.3061136>.
22. Han, S.I. Fractional-order command filtered backstepping sliding mode control with fractional-order nonlinear disturbance observer for nonlinear systems. *J. Frankl. Inst.* **2020**, *357*, 6760–6776. <https://doi.org/10.1016/j.jfranklin.2020.04.055>.

23. Song, S.; Park, J.H.; Zhang, B.; Song, X.; Zhang, Z. Adaptive command filtered neuro-fuzzy control design for fractional-order nonlinear systems with unknown control directions and input quantization. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 7238–7249. <https://doi.org/10.1109/TSMC.2020.2967425>.
24. Zou, W.; Mao, J.; Xiang, Z. Adaptive fuzzy finite-time sampled-data control for a class of fractional-order nonlinear systems. *IEEE Trans. Fuzzy Syst.* **2024**, *32*, 3012–3025. <https://doi.org/10.1109/TFUZZ.2024.3366241>.
25. Ha, S.; Chen, L.; Liu, D.; Liu, H. Command filtered adaptive fuzzy control of fractional-order nonlinear systems with unknown dead zones. *IEEE Trans. Syst. Man Cybern. Syst.* **2024**, *54*, 3705–3016. <https://doi.org/10.1109/TSMC.2024.3370003>.
26. Wang, C.; Cui, L.; Liang, M.; Li, J.; Wang, Y. Adaptive neural network control for a class of fractional-order nonstrict-feedback nonlinear systems with full-state constraints and input saturation. *IEEE Trans. Neural Netw. Learn. Syst.* **2022**, *33*, 6677–6689. <https://doi.org/10.1109/TNNLS.2021.3082984>.
27. Wang, J.; Shao, C.; Chen, Y. Fractional order sliding mode control via disturbance observer for a class of fractional order systems with mismatched disturbance. *Mechatronics* **2018**, *53*, 8–19. <https://doi.org/10.1016/j.mechatronics.2018.05.006>.
28. Shi, S.L.; Li, J.X.; Fang, Y.M. Fractional-disturbance-observer-based sliding mode control for fractional order system with matched and mismatched disturbances. *Int. J. Control Autom. Syst.* **2019**, *17*, 1184–1190. <https://doi.org/10.1007/s12555-018-0654-0>.
29. Song, S.; Zhang, B.; Xia, J.; Zhang, Z. Adaptive backstepping hybrid fuzzy sliding mode control for uncertain fractional-order nonlinear systems based on finite-time scheme. *IEEE Trans. Syst. Man Cybern. Syst.* **2020**, *50*, 1559–1569. <https://doi.org/10.1109/TSMC.2018.2877042>.
30. Song, X.; Sun, P.; Song, S.; Stojanovic, V. Finite-time adaptive neural resilient DSC for fractional-order nonlinear large-scale systems against sensor-actuator faults. *Nonlinear Dyn.* **2023**, *111*, 12181–12196. <https://doi.org/10.1007/s11071-023-08456-0>.
31. Xue, G.; Lin, F.; Li, S. Adaptive fuzzy finite-time backstepping control of fractional-order nonlinear systems with actuator faults via command-filtering and sliding mode technique. *Inf. Sci.* **2022**, *600*, 189–208. <https://doi.org/10.1016/j.ins.2022.03.084>.
32. Li, Y.X.; Wei, M.; Tong, S. Event-triggered adaptive neural control for fractional-order nonlinear systems based on finite-time scheme. *IEEE Trans. Cybern.* **2022**, *52*, 9481–9489. <https://doi.org/10.1109/TCYB.2021.3056990>.
33. Cao, B.; Nie, X.; Cao, J.; Duan, P. Practical finite-time adaptive neural networks control for incommensurate fractional-order nonlinear systems. *Nonlinear Dyn.* **2023**, *111*, 4375–4393. <https://doi.org/10.1007/s11071-022-08096-w>.
34. Liu, Y.; Zhang, H.; Sun, J.; Wang, Y. Event-triggered adaptive finite-time containment control for fractional-order nonlinear multi agent systems. *IEEE Trans. Cybern.* **2024**, *54*, 1250–1260. <https://doi.org/10.1109/TCYB.2022.3208124>.
35. Liu, Y.; Zhang, H.; Shi, Z.; Gao, Z. Neural-network-based finite-time bipartite containment control for fractional-order multi-agent systems. *IEEE Trans. Neural Netw. Learn. Syst.* **2023**, *34*, 7418–7429. <https://doi.org/10.1109/TNNLS.2022.3143494>.
36. Du, F.; Lu, J.G. Novel methods of finite-time synchronization of fractional-order delayed memristor-based Cohen–Grossberg neural networks. *Nonlinear Dyn.* **2023**, *111*, 18985–19001. <https://doi.org/10.1007/s11071-023-08880-2>.
37. Du, F.; Lu, J.G. Finite-time synchronization of fractional-order delayed fuzzy cellular neural networks with parameter uncertainties. *IEEE Trans. Fuzzy Syst.* **2022**, *31*, 1769–1779. <https://doi.org/10.1109/TFUZZ.2022.3214070>.
38. Wei, C.; Wang, X.; Lai, J.; Zeng, Z. Practical finite-time synchronization of fractional-order complex dynamical networks with application to Lorenz’s circuit. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2024**, *72*, 4103–4114. <https://doi.org/10.1109/TCSI.2024.3468398>.
39. Yao, X.; Liu, Y.; Zhang, Z.; Wan, W. Synchronization rather than finite-time synchronization results of fractional-order multi-weighted complex networks. *IEEE Trans. Neural Netw. Learn. Syst.* **2021**, *33*, 7052–7063. <https://doi.org/10.1109/TNNLS.2021.3083886>.
40. You, X.; Dian, S.; Guo, B.; Xiao, Q.; Zhu, Y.; Liu, K. Novel finite-time adaptive fuzzy fault-tolerant control for fractional-order nonlinear systems and its applications. *IEEE/CAA J. Autom. Sin.* **2026**, *13*, 123–136. <https://doi.org/10.1109/JAS.2025.125852>.
41. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
42. Aguila, N.; Duarte, M.A.; Gallegos, J.A. Lyapunov functions for fractional-order systems. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2951–2957. <https://doi.org/10.1016/j.cnsns.2014.01.022>.
43. Zouari, F.; Boubellouta, A. Neural approximation-based adaptive control for pure-feedback fractional-order systems with output constraints and actuator nonlinearities. In *Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems*; IGI Global Scientific Publishing: Hershey, PA, USA, 2018. <https://doi.org/10.4018/978-1-5225-5418-9.ch015>.
44. Wang, F.; Chen, B.; Liu, X.; Lin, C. Finite-time adaptive fuzzy tracking control design for nonlinear systems. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 1207–1216. <https://doi.org/10.1109/TFUZZ.2017.2717804>.

45. Zuo, Z.; Tie, L. A new class of finite-time nonlinear consensus protocols for multi-agent systems. *Int. J. Control* **2014**, *87*, 363–370. <https://doi.org/10.1080/00207179.2013.834484>.
46. Li, Y.X.; Yang, G.H. Adaptive asymptotic tracking control of uncertain nonlinear systems with input quantization and actuator faults. *Automatica* **2016**, *72*, 177–185. <https://doi.org/10.1016/j.automatica.2016.06.008>.
47. Zouari, F.; Ibeas, A.; Boulkroune, A.; Cao, J. Finite-time adaptive event-triggered output feedback intelligent control for non-integer order nonstrict feedback systems with asymmetric time-varying pseudo-state constraints and nonsmooth input nonlinearities. *Commun. Nonlinear Sci. Numer. Simul.* **2024**, *136*, 108036. <https://doi.org/10.1016/j.cnsns.2024.108036>.
48. Polycarpou, M.; Ioannou, P.A. A robust adaptive nonlinear control design. *Automatica* **1996**, *32*, 423–427. [https://doi.org/10.1016/0005-1098\(95\)00147-6](https://doi.org/10.1016/0005-1098(95)00147-6).
49. Elikier, K.; Grouni, S.; Tadjine, M.; Zhang, W. Practical finite time adaptive robust flight control system for quad-copter UAVs. *Aerosp. Sci. Technol.* **2020**, *98*, 105708. <https://doi.org/10.1016/j.ast.2020.105708>.
50. Hui, M.; Wei, C.; Zhang, J.; Ho-Ching, H.; Luo, N.; Yao, R.; Bai, L. Finite-time projective synchronization of fractional-order memristive neural networks with mixed time-varying delays. *Complexity* **2020**, *4168705*, 1–27. <https://doi.org/10.1155/2020/4168705>.
51. Yan, H.; Qiao, Y.; Duan, L.; Miao, J. New results of quasi-projective synchronization for fractional-order complex-valued neural networks with leakage and discrete delays. *Chaos Solitons Fractals* **2022**, *159*, 112121. <https://doi.org/10.1016/j.chaos.2022.112121>.
52. Wang, L.X. Stable adaptive fuzzy control of nonlinear systems. *IEEE Trans. Fuzzy Syst.* **1993**, *1*, 146–155. <https://doi.org/10.1109/91.227383>.
53. Haddad, M.; Boulkroune, A.; Li, H. Practical finite-time fuzzy control of uncertain nonstrict-feedback systems with actuator saturation and output constraints. *Int. J. Adapt. Control Signal Process.* **2024**, *38*, 1829–1853. <https://doi.org/10.1002/acs.3783>.
54. Boulkroune, A.; Haddad, M.; Li, H. Adaptive fuzzy control design for nonlinear systems with actuation and state constraints: An approach with no feasibility condition. *ISA Trans.* **2023**, *142*, 1–11. <https://doi.org/10.1016/j.isatra.2023.07.040>.
55. Boudjemia, S.; Boulkroune, A.; Bounar, N. Enhanced robust backstepping control for wind energy systems with fixed-time disturbance observer. *Iran. J. Sci. Technol. Trans. Electr. Eng.* **2026**. <https://doi.org/10.1007/s40998-026-01091-6>.
56. Borja-Jaimes, V.; Adam-Medina, M.; García-Morales, J.; Cruz-Rojas, A.; Gil-Velasco, A.; Coronel-Escamilla, A. A Novel fractional multi-order high-gain observer design to estimate temperature in a heat exchange process. *Axioms* **2023**, *12*, 1107. <https://doi.org/10.3390/axioms12121107>.
57. Borja-Jaimes, V.; Valdez-Martínez, J.S.; Beltrán-Escobar, M.; Cruz-Rojas, A.; Gil-Velasco, A.; Coronel-Escamilla, A. A review of fractional order calculus applications in electric vehicle energy storage and management systems. *Mathematics* **2025**, *13*, 2920. <https://doi.org/10.3390/math13182920>.

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