

PERIODIC ORBITS OF AN HYBRID SYSTEM IN $\mathbb{R}^3$

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ABSTRACT. Roughly speaking hybrid systems are mixtures of continuous dynamics and discrete events. Hybrid systems can describe some natural phenomena more accurately, which purely continuous or discrete systems cannot achieve. As in the differential systems also in the hybrid systems the periodic orbits play a main role for understanding their dynamics. In this paper we classify analytically all the periodic orbits of an hybrid system in $\mathbb{R}^3$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Piecewise linear differential systems are widely used across multiple disciplines, including mechanical engineering, electrical circuit analysis, and notably in control theory, which provides a wealth of models for such systems, see for instance [1, 2, 3, 4].

A particular category of piecewise differential systems are the hybrid systems. The concept of a hybrid system varies depending on the field in which it is used. Broadly speaking, hybrid systems are dynamical systems characterized by the interplay between continuously evolving variables and variables that change discretely, often at specific points in time. A comprehensive overview, including formal definitions and a variety of applications, is provided in [5].

Hybrid systems offer a precise mathematical structure for representing and examining systems in which continuous and discrete behaviors are tightly linked. Understanding these systems involves expanding traditional dynamical systems theory to accommodate discontinuities, switching between modes, and logic-based control rules. As a result, hybrid systems play a key role in contemporary fields such as engineering, robotics, cyber-physical systems, and are gaining growing attention in applied mathematics research.

The periodic orbits of a differential system and also of an hybrid system play a main role for understanding the dynamics of these systems. In this paper we study the periodic orbits of one class of hybrid system in $\mathbb{R}^3$, and we illustrate how to study these periodic orbits when the differential systems that appear in

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the hybrid system are completely integrable, that is if the differential system is defined in $\mathbb{R}^n$ it has $n - 1$ independent first integrals.

The hybrid system here studied is determined by two continuous differential systems acting in the half spaces $y \leq 0$ and $y \geq 0$, which are linked by discrete reset rules that always enter in action when the trajectories cross the separation plane (or switching surface) $y = 0$ of the space $\mathbb{R}^3$. The continuous differential system that acts in the half-space $y > 0$ is

$$(1) \quad \begin{aligned} \dot{x} &= 0, \\ \dot{y} &= -ay - (a^2 + b^2)z, \\ \dot{z} &= y + az, \end{aligned}$$

with the following first integrals

$$H_1(x, y, z) = (az + y^2) + b^2z^2 \quad \text{and} \quad F_1(x, y, z) = x.$$

In the half-space $y < 0$ we have

$$(2) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -z, \\ \dot{z} &= y, \end{aligned}$$

with the following first integrals

$$H_2(x, y, z) = x - z \quad \text{and} \quad F_2(x, y, z) = y^2 + z^2.$$

The configuration of this hybrid system is different from the analogous continuous system on each side of the switching surface, which brings different qualitative behaviors and geometric structures of the trajectories. It is important to mention that there are two reset rules acting when the trajectories touch the plane $y = 0$, being

$$(3) \quad \begin{aligned} R_1 : \quad \{y = 0\} &\longrightarrow \{y = 0\}, \\ (x, 0, z) &\mapsto (s_1x, 0, s_2z), \end{aligned}$$

when the trajectories come from the half-space $y > 0$, determined by the system (1) with $0 < s_1 < 1$, $s_2 \in \mathbb{R}$, $s_2 \neq 0$, and

$$(4) \quad \begin{aligned} R_2 : \quad \{y = 0\} &\longrightarrow \{y = 0\}, \\ (x, 0, z) &\mapsto \left(r_1x, 0, \frac{z}{s_2}\right), \end{aligned}$$

when the trajectories come from the half-space $y < 0$ determined by the system (2) with $r_1 > 0$ and s_2 being the same as defined in (3).

Our objective is to search periodic orbits of this hybrid system, also referred to as *admissible periodic orbits* in this system. The main goal is to identify conditions under which such periodic orbits appear, determining when they create

limit cycles and understanding their dependence on the parameters of the reset rules and of the continuous part of the hybrid system.

The *admissible orbits* of the hybrid system are defined as follows. These orbits start at a point $A = (x_1, 0, z_1)$ with $x_1 > 0$ and $z_1 > 0$, and then follow in forward time the trajectory of the differential system (1) in the half-space $y \geq 0$ until reaching, for the first time, the switching plane $y = 0$ at the point $B = (x_2, 0, z_2)$ with $x_2 > 0$ and $z_2 < 0$. At this point, the reset rule is applied, producing the new point $C = (x_3, 0, z_3)$, where $x_3 = s_1 x_2 > 0$ and $z_3 = s_2 z_2 < 0$, as defined in (3). The orbit then continues following the flow in forward time of the differential system (2) in the half-space $y \leq 0$, starting at $C = (x_3, 0, z_3)$ and developing the trajectory until it reaches the switching plane $y = 0$ for the first time at the point $D = (x_4, 0, z_4)$ with $x_4 > 0$ and $z_4 > 0$. Applying the reset rule again yields the point $E = (x_5, 0, z_5)$, where $x_5 = r_1 x_4 > 0$ and $z_5 = \frac{z_4}{s_2} > 0$, as defined in (4).

This process repeats indefinitely, alternating between the two continuous dynamics and applying the discrete resets at each crossing of the switching plane $y = 0$. An admissible orbit is said to be *periodic* if it returns exactly to its initial point, that is, if $x_5 = x_1$ and $z_5 = z_1$ (i.e. $A = E$). Moreover, if such a periodic orbit is isolated within the set of all periodic admissible orbits, it is a *limit cycle*. See Figure 1 for an illustration.

As it will be explained in the proof of Theorem 1, for the conditions $0 < s_1 < 1$, $r_1 > 0$ and $\frac{(-1 + r_1 s_1) x_1}{2r_1 s_2} < 0$, the periodic orbit starts at the point $A = \left(x_1, 0, -\frac{(-1 + r_1 s_1) x_1}{2r_1 s_2}\right)$ of the plane $y = 0$ and continues following the orbit of the differential system (1) through the point A in the region $y > 0$ until it reaches in forward time for the first again the plane $y = 0$ at the point $B = \left(x_1, 0, \frac{(-1 + r_1 s_1) x_1}{2r_1 s_2}\right)$. After, the first reset map R_1 of (3) is applied to the point B and the orbit pass to the point $C = \left(s_1 x_1, 0, \frac{(-1 + r_1 s_1) x_1}{2r_1}\right)$. The orbit continues following in forward time the orbit of the second differential system (2) in the region $y < 0$ through the point C until it reaches for the first time the plane $y = 0$ at the point $D = \left(\frac{x_1}{r_1}, 0, -\frac{(-1 + r_1 s_1) x_1}{2r_1}\right)$. Then the second reset map R_2 from (4) sends the point D to the point $A = \left(x_1, 0, -\frac{(-1 + r_1 s_1) x_1}{2r_1 s_2}\right)$, completing the cycle of the periodic orbit.

In summary we have the following theorem.

Theorem 1. *For the conditions $\frac{(-1 + r_1 s_1) x_1}{2r_1 s_2} < 0$, $0 < s_1 < 1$ and $r_1 > 0$ the hybrid system defined by (1), (2), (3) and (4) admits a periodic orbit that*

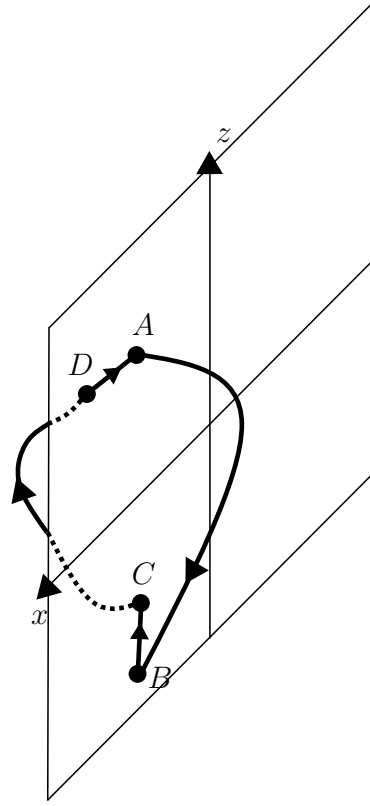


FIGURE 1. An example of a periodic orbit.

intersects the switching plane $y = 0$ at the points

$$A = \left(x_1, 0, -\frac{(-1 + r_1 s_1) x_1}{2r_1 s_2} \right), \quad B = \left(x_1, 0, \frac{(-1 + r_1 s_1) x_1}{2r_1 s_2} \right),$$

$$C = \left(s_1 x_1, 0, \frac{(-1 + r_1 s_1) x_1}{2r_1} \right), \quad D = \left(s_1 x_1, 0, -\frac{(-1 + r_1 s_1) x_1}{2r_1} \right),$$

as showed in Figure 1. Moreover, the one-parameter family of periodic orbits, parameterized by $x_1 > 0$ constitutes the whole set of the periodic orbits of the hybrid system.

Corollary 2. *The hybrid system composed by (1), (2), (3) and (4) has no limit cycles.*

2. PROOF OF THEOREM 1

If there exists a periodic orbit of the admissible orbits intersecting the plane $y = 0$ at the points $(x_i, 0, z_i)$ for $i = 1, 2, 3, 4, 5$, these points must satisfy

$$\begin{aligned}
 (5) \quad & H_1(x_1, 0, z_1) - H_1(x_2, 0, z_2) = 0, \\
 & F_1(x_1, 0, z_1) - F_1(x_2, 0, z_2) = 0, \\
 & (x_3, 0, z_3) = (s_1 x_2, 0, s_2 z_2), \\
 & H_2(x_3, 0, z_3) - H_2(x_4, 0, z_4) = 0, \\
 & F_2(x_3, 0, z_3) - F_2(x_4, 0, z_4) = 0, \\
 & (x_5, 0, z_5) = \left(r_1 x_4, 0, \frac{z_4}{s_2} \right), \\
 & x_5 - x_1 = 0, \quad z_5 - z_1 = 0.
 \end{aligned}$$

Of course, here we have used the H_1 and F_1 are first integrals of the differential system (1), and that H_2 and F_2 are first integrals of the differential system (2).

System (5) becomes

$$\begin{aligned}
 (6) \quad & (a^2 + b^2)(z_1 - z_2)(z_1 + z_2) = 0, \\
 & x_1 - x_2 = 0, \\
 & (x_3, 0, z_3) = (s_1 x_2, 0, s_2 z_2), \\
 & s_1 x_1 - x_4 + s_2 z_1 + z_4 = 0, \\
 & (s_2 z_1 - z_4)(s_2 z_1 + z_4) = 0, \\
 & (x_5, 0, z_5) = \left(r_1 x_4, 0, \frac{z_4}{s_2} \right), \\
 & x_5 - x_1 = 0, \quad z_5 - z_1 = 0.
 \end{aligned}$$

Since $x_5 - x_1 = r_1 x_4 - x_1 = 0$ and $z_5 - z_1 = \frac{z_4}{s_2} - z_1 = 0$, system (6) reduces to the system

$$\begin{aligned}
 & s_1 x_1 - x_4 + s_2 z_1 + z_4 = 0, \\
 & (s_2 z_1 - z_4)(s_2 z_1 + z_4) = 0, \\
 & r_1 x_4 - x_1 = 0, \\
 & \frac{z_4}{s_2} - z_1 = 0,
 \end{aligned}$$

whose solutions are $x_4 = \frac{x_1}{r_1}$, $z_1 = -\frac{(-1 + r_1 s_1)x_1}{2r_1 s_2}$ and $z_4 = -\frac{(-1 + r_1 s_1)x_1}{2r_1}$.

So considering $\frac{(-1 + r_1 s_1)x_1}{2r_1 s_2} < 0$ we obtain the periodic orbits for the hybrid system stated in Theorem 1.

Proof of Corollary 2. By Theorem 1, for each $x_1 > 0$, there exists a periodic orbit of the hybrid system composed by (1), (2), (3) and (4). These periodic orbits

form a one-parameter family continuously parameterized by x_1 . Therefore, every periodic orbit has arbitrarily close neighboring periodic orbits within this family.

Since a limit cycle is, by definition, an isolated periodic orbit in the set of all periodic orbits of the hybrid system, the existence of this continuum of periodic orbits implies that none of these hybrid systems has limit cycles. $\square$

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