

Characterization of isochronous centers of polynomial quadratic-like Hamiltonian systems

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Abstract

We characterize isochronous centers of real planar polynomial quadratic-like Hamiltonian systems, i.e., planar differential systems $\dot{x} = -H_y(x, y)$, $\dot{y} = H_x(x, y)$ where $H(x, y) = A(x) + B(x)y + C(x)y^2$ with $A(x)$, $B(x)$, $C(x) \in \mathbb{R}[x]$. Given a polynomial quadratic-like Hamiltonian system with a center at the origin we can determine whether it is isochronous or not. Conversely, we can construct most of the polynomial quadratic-like Hamiltonian systems with an isochronous center, up to the determination of a particular Belyi function. Our results show the strong algebraic properties that are forced by the isochronicity of the center. © 2026 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

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1. Introduction

In this paper we study isochronous centers of real polynomial Hamiltonian systems. The problem of characterizing isochronous centers, either for polynomial or analytic planar systems, has attracted the attention of several authors. However there are very few families of polynomial dif-

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ferential systems in which a complete classification of the isochronous centers has been found. Quadratic systems were classified by Loud [10] and cubic systems with homogeneous nonlinearities by Pleshkan [14]. Kukles' systems were classified by Christopher and Devlin [2]. Concerning Hamiltonian systems, i.e., planar differential systems of the form

$$\begin{cases} \dot{x} = H_y(x, y), \\ \dot{y} = -H_x(x, y), \end{cases} \quad (1)$$

there are also very few results. It is well known, see for instance [1], that the only polynomial potential system having an isochronous center is the linear one. Several authors proved independently (see [2,4,15]) that there are no Hamiltonian systems with homogeneous nonlinearities having an isochronous center at the origin. Cubic Hamiltonian systems, i.e., differential systems of the form (1) with $H(x, y)$ being a fourth degree polynomial, were classified in [3]. Apart from few other special cases, the knowledge of polynomial systems with isochronous centers is slight. For some other results on isochronicity we refer the reader to [2,5,11,12] and references therein.

For any center p of a planar differential system, the largest punctured neighborhood of p which is entirely covered by periodic orbits is called the *period annulus* of p and we will denote it by $\mathcal{P}$. A center is said to be *global* when its period annulus is $\mathbb{R}^2 \setminus \{p\}$. The function which associates to each periodic orbit in $\mathcal{P}$ its period is called the *period function*. The center is called *isochronous* when the period function is constant. By a result that goes back to Poincaré, it is well known that a center of an analytic differential system is isochronous if and only if it is *linearizable*, i.e., there exists an analytic local change of coordinates transforming the initial system to the linear center $\dot{x} = -\omega y$, $\dot{y} = \omega x$, with $\omega \in \mathbb{R} \setminus \{0\}$. In particular, only non-degenerate centers can be isochronous, see for instance [2]. Moreover it is also well-known that the period annulus of an isochronous center is unbounded, see [2] again. When the differential system is polynomial one can compactify the vector field to $\mathbb{S}^2$ and in doing so the boundary of the period annulus has two connected components; the center itself and a polycycle. If the center is isochronous then the polycycle can only have critical points at infinity.

Our aim with the present study is to go further in the knowledge of isochronicity in the so-called *quadratic-like* Hamiltonian systems, namely the planar differential systems (1) associated to a Hamiltonian of the form $H(x, y) = A(x) + B(x)y + C(x)y^2$. These Hamiltonian systems were previously studied in [3], where the authors obtained several results for the case in which the functions A , B and C are analytic or polynomial. In the present paper we focus on the latter case. Thus, we are interested in the isochronous centers of the differential system

$$\begin{cases} \dot{x} = B(x) + 2C(x)y, \\ \dot{y} = -A'(x) - B'(x)y - C'(x)y^2, \end{cases} \quad (2)$$

where $A, B, C \in \mathbb{R}[x]$. Clearly there is no loss of generality in assuming that the center is at the origin and, since it must be non-degenerate, we can perform a rotation and a constant rescaling of time so that

$$A(x) = x^2/2 + o(x^2), \quad B(x) = o(x) \quad \text{and} \quad C(0) = 1/2.$$

Here, the notation $o(\cdot)$ is one of the Landau symbols to symbolically express the asymptotic behavior of a given function and throughout the paper we use in a neighborhood of $x = 0$. Given two analytic functions $f(x)$ and $g(x)$ in a neighborhood of $x = 0$ we say that $f(x) = o(g(x))$ if $\lim_{x \rightarrow 0} f(x)/g(x) = 0$.

Let us define

$$G := 4AC - B^2$$

and note that then $G(x) = x^2 + o(x^2)$. Isochronous centers can be characterized in terms of the polynomials $C(x)$ and $G(x)$ alone as follows. We define the following two algebraic functions

$$M(x) := \frac{G(x)}{C(x)} \text{ and } N(x) := \frac{\sqrt{G(x)}}{C(x)M'(x)}, \quad (3)$$

where the square root is taken such that $\sqrt{G(x)} = x + o(x)$. Then $M(x) = 2x^2 + o(x)$ and $N(0) = 1/2$. Since M has a local minimum at $x = 0$, there exists a unique analytic involution σ with $\sigma(x) = -x + o(x)$ such that $M = M \circ \sigma$ in a neighborhood of $x = 0$. (Recall that an involution is a function $\sigma \neq \text{Id}$ such that $\sigma^2 = \text{Id}$, where Id denotes the identity.) Note that σ is an algebraic function because M is rational. With this notation (see Proposition 2.2) the center is isochronous if, and only if, for all $x \approx 0$

$$N(x) + N(\sigma(x)) = 1, \text{ where } \sigma(x) = -x + o(x) \text{ verifies } M(x) = M(\sigma(x)). \quad (4)$$

Observe, see (3), that the above condition depends on G and C only. This fact leads us to the following definition:

Definition 1.1. We say that (G, C) is an *isochronous pair* if G and C are real polynomials with $G(x) = x^2 + o(x^2)$ and $C(0) = 1/2$ for which the functions M and N given in (3) verify (4). $\square$

Our first main result shows that G needs to be a squared polynomial.

Theorem A. *If (G, C) is an isochronous pair then $G = G_0^2$ where $G_0(x) = x + o(x)$ is a real polynomial.*

The function M as introduced in (3) is rational by definition. The previous result shows that this is also the case for N when (G, C) is an isochronous pair. It also enables us to suppose without loss of generality that any isochronous pair is of the form (G_0^2, C) with $G_0(x) = x + o(x)$ being a real polynomial. Moreover, as a byproduct of Theorem A, we have that any quadratic-like Hamiltonian polynomial differential system with an isochronous center has odd degree, see Corollary 5.8. This reinforces the conjecture posed in [8] about the degree of the Hamiltonian polynomial differential systems having an isochronous center.

Our second main result states that if $N(x) \not\equiv 1/2$ then the study of isochronous pairs can be reduced (essentially) to the simplest involution, namely, the Möbius transformation.

Theorem B. *Let (G_0^2, C) be an isochronous pair with $N(x) \not\equiv 1/2$. Then there exist an isochronous pair $(\hat{G}_0^2, \hat{C})$ and a real polynomial t with $t(x) = x + o(x)$ which satisfy*

$$G_0(x) = \frac{\hat{G}_0(t(x))}{t'(x)} \text{ and } C(x) = \frac{\hat{C}(t(x))}{(t'(x))^2} \quad (5)$$

and such that the involution $\hat{\sigma}$ defined by the rational function $\hat{M} = \frac{\hat{G}_0^2}{\hat{C}}$ (i.e., verifying the identity $\hat{M} = \hat{M} \circ \hat{\sigma}$) is $\hat{\sigma}(x) = \frac{-x}{1+\kappa x}$ with $\kappa \in \mathbb{R}$.

Although Theorem B narrows the isochronicity problem in the quadratic-like Hamiltonian differential systems, the determination of all the isochronous pairs with σ being a Möbius transformation is still an open problem. The properties of the trio $\hat{G}_0$, $\hat{C}$ and t giving rise to polynomials G_0 and C by means of the equalities in (5) are described in Proposition 4.1. In particular this result enables us to construct isochronous pairs by composition. In this regard it is also worth to note, following the notation in the statement of Theorem B, that $t \circ \sigma = \hat{\sigma} \circ t$.

It is obvious but nevertheless important to remark, that the conditions in (4) for the pair (G, C) to be isochronous depend on M and N , which in turn are only defined by means of the polynomials G and C , see (3). That being said, we point out that in Theorem B we do not suppose that (G_0^2, C) comes from a quadratic-like Hamiltonian differential system (2) for which $4AC - B^2 = G_0^2$. This is not the case of our next result and this fact naturally leads to the following definition.

Definition 1.2. We say that (G, C) is a *liftable pair* if $G, C \in \mathbb{R}[x]$ with $G(x) = x^2 + o(x^2)$ and $C(0) = 1/2$ for which there exist $A, B \in \mathbb{R}[x]$ with $A(x) = \frac{1}{2}x^2 + o(x^2)$ and $B(x) = o(x)$ satisfying that $G = 4AC - B^2$. $\square$

We study liftable pairs in Proposition 3.1 and Lemma 4.2. In Example 5.7 we give an isochronous pair (G_0^2, C) with $N(x) \not\equiv 1/2$ that is not liftable. Moreover, in Example 5.9 we construct an isochronous liftable pair that leads to an isochronous center with a specific type of period annulus whose existence was posed as an open question in [3, p. 408].

There are examples of isochronous pairs (G_0^2, C) with $N(x) \equiv 1/2$ for which the conclusion of Theorem B is not true, see Example 5.3. Our next result deals precisely with the case $N(x) \equiv 1/2$. Observe that in this case the condition (4) is trivially verified, so that (G_0^2, C) is an isochronous pair. In this regard one can readily verify from (3) that

$$N(x) \equiv 1/2 \Leftrightarrow G_0 C' = 2C(G_0' - 1) \quad (6)$$

In this case the characterization of isochronicity entails the determination of a Belyi function. (Recall that a Belyi function is a rational function with at most three critical values.) The following is our third main result and in its statement $\bar{\alpha}$ denotes the complex conjugate of a given $\alpha \in \mathbb{C}$. Moreover, here and in the rest of the paper, $\mathbb{N}$ denotes the set of natural numbers greater or equal than 1.

Theorem C. Let (G_0^2, C) be a liftable pair. Then $N \equiv 1/2$ if, and only if, either $G_0(x) = x$ and $C(x) = 1/2$, or there exist $k_1, k_2, \dots, k_n \in \mathbb{N} \cup \{-2, -1\}$ with $\sum_{i=1}^n k_i = 1$ and pairwise distinct $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C} \setminus \mathbb{R}$ such that

$$G_0(x) = x \prod_{i=1}^n (1 - \alpha_i x)(1 - \bar{\alpha}_i x) \text{ and } C(x) = \frac{1}{2} \prod_{i=1}^n ((1 - \alpha_i x)(1 - \bar{\alpha}_i x))^{k_i+2} \quad (7)$$

with $G_0'(1/\alpha_i) = -2/k_i$ for $i = 1, 2, \dots, n$. Moreover in this case $M = \frac{G_0^2}{C}$ is a Belyi function.

With regard to this result let us note that there are isochronous pairs (G_0^2, C) with $N \equiv 1/2$ that are not liftable (see Example 4.4).

The paper is organized in the following way. In Section 2 we show some auxiliary results needed for the proofs of our main theorems. We begin by proving, see Proposition 2.2, that the center at the origin of (2) is isochronous if, and only if, (G, C) is a liftable isochronous pair (cf. Definitions 1.1 and 1.2). We also state L uroth's Theorem, which is a key tool to study the generating function of the involution σ verifying $M = M \circ \sigma$. Section 3 is devoted to prove a number of results about the liftability of polynomial pairs (G, C) . In Section 4 we are concerned with the construction of isochronous pairs by composition. In particular, the properties of $\hat{G}_0$, $\hat{C}$ and t which, by means of the equalities in (5), may give rise to polynomials G_0 and C . Finally, the proofs of the main theorems are carried out in Section 5.

2. Auxiliary results

Lemma 2.1. *Let us consider the planar differential system*

$$\begin{cases} \dot{x} = -y/f(x), \\ \dot{y} = x/f(x), \end{cases}$$

where f is an analytic function in a neighborhood of $x = 0$ with $f(0) = 1$. Then the origin is an isochronous center if and only if $f(x) + f(-x) \equiv 2$ for all $x \approx 0$.

Proof. The solution of the differential system under consideration with initial condition $(x_0, 0)$ is $t \mapsto (x(t), y(t)) = (x_0 \cos(\theta(t)), x_0 \sin(\theta(t)))$ where θ verifies

$$\theta'(t) = \frac{1}{f(x_0 \cos(\theta(t)))} \text{ and } \theta(0) = 0.$$

On the other hand, since $H(x, y) = x^2 + y^2$ is a first integral, all the solutions in a punctured neighborhood of the origin are periodic. The period of the one passing through the point $(x_0, 0)$ is given by

$$T(x_0) = \int_0^{2\pi} f(x_0 \cos(\theta)) d\theta.$$

By using the Taylor expansion of f at $x = 0$ one can readily show that $T(x_0) \equiv 2\pi$ if and only if the function $x \mapsto f(x) - 1$ is odd, i.e., $f(x) + f(-x) \equiv 2$. $\square$

Proposition 2.2. *Consider the differential system (2) and, setting $G = 4AC - B^2$, let M and N be the functions defined in (3). Then the center at the origin is isochronous if, and only if, the identity $N(x) + N(\sigma(x)) = 1$ holds for all $x \approx 0$, where $\sigma(x) = -x + o(x)$ is the unique analytic involution such that $M = M \circ \sigma$.*

Proof. One can verify that the local change of variables given by

$$\left\{ x = x, y = \frac{v}{\sqrt{2C(x)}} - \frac{B(x)}{2C(x)} \right\}$$

and the constant rescaling of time $t \mapsto -t$ brings the differential system (2) to

$$\begin{cases} \dot{x} = -v\sqrt{2C(x)} \\ \dot{v} = \frac{1}{4}\sqrt{2C(x)}M'(x), \end{cases}$$

where M is the function defined in (3) taking $G = 4AC - B^2$. Next, we perform an additional change of variables given by $\{u = g(x), v = v\}$, where $g(x) = \frac{x}{2}\sqrt{\frac{M(x)}{x^2}} = x + o(x)$. It brings the above differential system to

$$\begin{cases} \dot{u} = -v/f(u), \\ \dot{v} = u/f(u), \end{cases} \quad \text{with } f(u) = 2N(g^{-1}(u)),$$

where N is the function defined in (3). Hence, by applying Lemma 2.1, the differential system (2) has an isochronous center at the origin if and only if $N(g^{-1}(u)) + N(g^{-1}(-u)) = 1$, i.e.,

$$N(u) + N(g^{-1}(-g(u))) = 1.$$

Since $M(g^{-1}(x)) = M(g^{-1}(-x)) = 4x^2$, setting $\sigma(x) = g^{-1}(-g(x))$, we get $M = M \circ \sigma$. $\square$

It is to be pointed out that the previous result can also be obtained from [3, Theorem C]. We give here a different proof for the sake of completeness.

The following lemma will be used to prove Theorem A. Even though it is quite classical, we prove it here for reader's convenience. In its statement k is any field of characteristic zero but in our application we shall take $k = \mathbb{R}$.

Lemma 2.3. *If φ is an algebraic function over $k(x)$ such that $\varphi'(x) = \sqrt{\chi(x)}$ for some $\chi \in k(x)$, then there exists $q \in k(x)$ and $\eta \in k$ such that $\varphi = q\sqrt{\chi} + \eta$.*

Proof. We denote by K the normal closure of $k(\varphi(x), \sqrt{\chi(x)}, x)$, that is K is the minimal extension field of $k(x)$ which is normal. As usual, $\text{Gal}(K/k(x))$ denotes the Galois group of K over $k(x)$. We denote by H the subgroup of $\text{Gal}(K/k(x))$ which fixes $k(\sqrt{\chi(x)}, x)$. We define

$$\psi(x) := \sum_{\varsigma \in H} \varsigma(\varphi(x)).$$

We note that ψ is the field trace of $\varphi(x)$ and, thus, $\psi(x)$ belongs to the base field $k(\sqrt{\chi(x)}, x)$. Hence, there exist two rational functions $r, s \in k(x)$ such that $\psi(x) = r(x) + s(x)\sqrt{\chi(x)}$. We derive with respect to x and we get

$$\psi'(x) = r'(x) + \left(s'(x) + s(x) \frac{\chi'(x)}{2\chi(x)} \right) \sqrt{\chi(x)}. \quad (8)$$

On the other hand, we recall that if $\varsigma \in \text{Gal}(K/k(x))$ and φ is an algebraic function, then

$$\varsigma(\varphi(x))' = \varsigma(\varphi'(x)). \quad (9)$$

Indeed, if $P(x, y) \in k(x)[y]$ is such that $P(x, \varphi(x)) \equiv 0$, then $\varsigma(\varphi(x))$ is another y -root of $P(x, y)$ because the extension $K/k(x)$ is normal, that is $P(x, \varsigma(\varphi(x))) \equiv 0$. We derive the last two identities with respect to x and we get

$$\begin{aligned} \frac{\partial P}{\partial x}(x, \varphi(x)) + \frac{\partial P}{\partial y}(x, \varphi(x)) \varphi'(x) &\equiv 0 \quad \text{and} \\ \frac{\partial P}{\partial x}(x, \varsigma(\varphi(x))) + \frac{\partial P}{\partial y}(x, \varsigma(\varphi(x))) \varsigma(\varphi(x))' &\equiv 0. \end{aligned} \quad (10)$$

The coefficients of $P(x, y)$ are in $k(x)$ and, therefore, they are fixed by ς . We apply ς to the first identity of (10) and we have

$$\frac{\partial P}{\partial x}(x, \varsigma(\varphi(x))) + \frac{\partial P}{\partial y}(x, \varsigma(\varphi(x))) \varsigma(\varphi'(x)) \equiv 0.$$

This last expression together with the second identity of (10) lead to (9).

We derive the definition $\psi(x) := \sum_{\varsigma \in H} \varsigma(\varphi(x))$ with respect to x and we get

$$\psi'(x) = \sum_{\varsigma \in H} \varsigma(\varphi(x))' = \sum_{\varsigma \in H} \varsigma(\varphi(x)') = \sum_{\varsigma \in H} \varsigma(\sqrt{\chi(x)}),$$

where we have used (9) and the hypothesis $\varphi'(x) = \sqrt{\chi(x)}$. Since ς is an automorphism which fixes $\sqrt{\chi(x)}$, we deduce that $\psi'(x) = |H| \sqrt{\chi(x)}$, where $|H|$ denotes the cardinal of the subgroup H . Equating with (8), we obtain that $r'(x) \equiv 0$ and $s'(x) + s(x) \chi'(x)/(2\chi(x)) = |H|$. We can integrate the last expression to get that $s(x) = (c + |H|\varphi(x))/\sqrt{\chi(x)}$ for a certain constant $c \in k$. Hence, equating $\varphi(x)$ we get

$$\varphi(x) = -\frac{c}{|H|} + \frac{s(x)}{|H|} \sqrt{\chi(x)}.$$

We denote $s(x)/|H|$ by $q(x)$ and $-c/|H|$ by η and we have shown that $\varphi = q\sqrt{\chi} + \eta$ for $q \in \mathbb{C}(x)$ and $\eta \in k$. $\square$

Next we state Lüroth's Theorem (see [16, p. 218]), that will be used to study the generating function of the involution.

Theorem 2.4 (Lüroth). *Let $k(t)$ be a transcendental extension of a field k . Any subfield $K \subset k(t)$, such that $k \subsetneq K$, is of the form $K = k(r)$ for some $r \in k(t)$.*

The following result gives a property of a field generated by a polynomial.

Lemma 2.5. *Let $t(x)$ and $p(x)$ be two nonconstant real polynomials such that $p(x) \in \mathbb{R}(t(x))$. Then, $p(x) = q(t(x))$ with $q(x)$ a real polynomial.*

Proof. Since $p(x) \in \mathbb{R}(t(x))$, we have that $p(x) = q(t(x))$ with $q(x) \in \mathbb{R}(x)$. Let us write $q(x) = q_1(x)/q_2(x)$ with $q_1(x)$ and $q_2(x)$ two coprime real polynomials. If $q_2(x)$ is not a constant, there exists $c \in \mathbb{C}$ such that $q_2(c) = 0$. Let us consider the equation $t(x) = c$, which has at

least one solution $\alpha \in \mathbb{C}$. Since $p(\alpha) \in \mathbb{C}$, we deduce that $q_1(t(\alpha)) = q_1(c) = 0$, which is in contradiction with q_1 and q_2 being coprime. The fact that $q(x)$ is indeed a polynomial follows. $\square$

At this point we recall the definition of critical point and critical value of a rational function. So let us take $\phi \in \mathbb{C}(x)$ and consider $\phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, i.e., the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In order to study the points of $\hat{\mathbb{C}}$ close to ∞ , we take a local chart. Let $\mathcal{U}_0$ and $\mathcal{U}_\infty$ be, respectively, neighborhoods of 0 and ∞ in $\hat{\mathbb{C}}$. A local chart in $\mathcal{U}_\infty$ is $\varphi : \mathcal{U}_\infty \rightarrow \mathcal{U}_0$ with $\varphi(x) := 1/x$. Using this local chart, we can define $\phi'(\infty)$ as $\hat{\phi}'(0)$ with $\hat{\phi}(x) := \phi(1/x)$ if $\phi(\infty) \in \mathbb{C}$ or $\hat{\phi}(x) := 1/\phi(1/x)$ if $\phi(\infty) = \infty$. Analogously, if $x_0 \in \mathbb{C}$ is such that $\phi(x_0) = \infty$ (a pole), then $\phi'(x_0)$ is defined by $\hat{\phi}'(x_0)$ with $\hat{\phi}(x) := 1/\phi(x)$. We say that $x_0 \in \hat{\mathbb{C}}$ is a *critical point* of ϕ if $\phi'(x_0) = 0$. In such a case, the value $c_0 \in \hat{\mathbb{C}}$ with $c_0 := \phi(x_0)$ is a *critical value* of ϕ . The *multiplicity* of a critical point coincides with its multiplicity as a zero of $\phi(x) - c_0$; namely, it is equal to $m \geq 2$ if the consecutive derivatives of ϕ satisfy

$$\phi'(x_0) = 0, \phi''(x_0) = 0, \dots, \phi^{(m-1)}(x_0) = 0 \quad \text{and} \quad \phi^{(m)}(x_0) \neq 0.$$

For a rational function, it is clear that the set of critical points and the set of critical values are finite. Note that if x_0 is a critical point, then its multiplicity is at least 2. For instance, any polynomial of degree $k \geq 2$ has ∞ as a critical point of multiplicity k with critical value ∞ . A rational function without critical points needs to be a Möbius transformation, i.e., of the form $\phi(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma \neq 0$ (see [9, Chapter 1] for instance).

We recall as well the Riemann–Hurwitz formula (called also Hurwitz’s theorem) in the case of a rational function defined on the Riemann sphere. Let $\phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a nonconstant rational function and we denote by v the degree of the covering induced by ϕ , that is, if $\phi(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are two coprime polynomials, then $v := \max\{\deg p, \deg q\}$. In this context, critical points are also called *ramification points* and critical values are called *branch points*. Given $\alpha \in \mathbb{C}$ with $\phi(\alpha) \in \mathbb{C}$, we define its multiplicity m_α as the maximum integer number such that $p(x) - \phi(\alpha)q(x) = (x - \alpha)^{m_\alpha}r(x)$ where $r(x)$ is a polynomial such that $r(\alpha) \neq 0$. If $\alpha \in \mathbb{C}$ is a pole of ϕ , that is $\phi(\alpha) = \infty$, then m_α is the multiplicity of α for the rational function $1/\phi(x)$. For $\alpha = \infty$, we define m_α as the multiplicity of 0 for the rational function $\phi(1/x)$ if $\phi(\alpha) \in \mathbb{C}$ and for the rational function $1/\phi(1/x)$ if $\phi(\alpha) = \infty$. Note that $m_\alpha \geq 1$ and that this value coincides with the multiplicity of α in case that α is a critical point of ϕ , as we have previously defined. For maps between genus zero surfaces, the Riemann–Hurwitz formula reads as

$$2 = 2v - \sum_{\alpha \in \hat{\mathbb{C}}} (m_\alpha - 1), \quad (11)$$

see [9, p. 23] or [7, p. 301] for instance. Note that the sum runs all over $\alpha \in \hat{\mathbb{C}}$ but the summand is different from zero only if α is a critical point of ϕ . We know that the number of critical points is finite because a critical point is a zero of $\phi'(x)$ and a rational function has a finite number of zeroes.

Given a nonconstant rational function ϕ , let $\{\alpha_1, \dots, \alpha_r\}$ be the set of its critical points and let $\{c_1, \dots, c_b\}$ be the set of its critical values. Since the value $\phi(\alpha_i)$ can coincide with the value $\phi(\alpha_j)$ when $i \neq j$, it is clear that $b \leq r$. For each critical value c_j , we define r_j as the cardinal of the set $\{\phi^{-1}(c_j)\}$, that is, r_j is the number of points $\alpha \in \hat{\mathbb{C}}$, either critical or not, such that $\phi(\alpha) = c_j$. Note that, since ϕ is not constant $r_j \geq 1$.

Lemma 2.6. *If ϕ be a nonconstant rational function then $2 - \sum_{j=1}^b r_j = v(2 - b)$.*

Proof. In the Riemann–Hurwitz formula (11) we can run the sum over the finite set formed by the union of the sets $\{\phi^{-1}(c_j)\}$, for $j = 1, 2, \dots, b$ because in its complement set the sum is zero. Therefore we can write that

$$\sum_{\alpha \in \hat{\mathbb{C}}} (m_\alpha - 1) = \sum_{j=1}^b \sum_{\alpha \in \{\phi^{-1}(c_j)\}} (m_\alpha - 1) = \left(\sum_{j=1}^b \sum_{\alpha \in \{\phi^{-1}(c_j)\}} m_\alpha \right) - \sum_{j=1}^b r_j. \quad (12)$$

Assume that we have a critical value $c \in \mathbb{C}$ and that the set $\{\phi^{-1}(c)\}$ only contains finite points. Then, from the factorization of the polynomial $p(x) - cq(x)$ and the definition of v we have that

$$\sum_{\alpha \in \{\phi^{-1}(c)\}} m_\alpha = v.$$

The cases where either c or one element of $\{\phi^{-1}(c_j)\}$ are ∞ satisfy the same identity by taking local charts as we have explained before. Therefore we have that

$$\sum_{j=1}^b \sum_{\alpha \in \{\phi^{-1}(c_j)\}} m_\alpha = \sum_{j=1}^b v = vb.$$

We substitute the latter expression in (12) and we have that $\sum_{\alpha \in \hat{\mathbb{C}}} (m_\alpha - 1) = vb - \sum_{j=1}^b r_j$. We substitute in the Riemann–Hurwitz formula (11) and we get the result. $\square$

The following result is Lemma 1.1 in [13] written in our notation.

Lemma 2.7. *Consider a nonconstant rational function ϕ with exactly two critical points, α_1 and α_2 . Let h be a Möbius transformation such that $h(\infty) = \alpha_1$ and $h(0) = \alpha_2$. Then,*

$$\phi(h(x)) = \tilde{h}(x^v)$$

where v is the degree of the covering induced by ϕ and $\tilde{h}$ is a Möbius transformation.

Next result determines the form of those rational functions of $\mathbb{C}(x)$ with at most two critical values. It is a classical result but we include here a proof for the sake of completeness.

Lemma 2.8. *Let ϕ be a nonconstant rational function with induced covering degree equal to v . Assume that ϕ has at most two critical values on $\hat{\mathbb{C}}$. Then ϕ has exactly two critical values or it has none. In the first case $v \geq 2$ and there exist two Möbius transformations h_1 and h_2 such that $\phi = h_1 \circ (h_2)^v$. In the second case ϕ is a Möbius transformation, i.e., $v = 1$.*

Proof. Let b denote the number of critical values of ϕ . Then

1. If $b = 0$ then, by Lemma 2.6, $2 = 2v$ and thus $v = 1$. Any rational function with $v = 1$ is a Möbius transformation.
2. If $b = 1$ then, by Lemma 2.6 again, $2 - r_1 = v$. Since a Möbius transformation has no critical points, $v \geq 2$. This yields $2 - r_1 \geq 2$ and, hence, $r_1 \leq 0$. This is impossible because by definition $r_1 \geq 1$.
3. If $b = 2$ then we have, as before, $v \geq 2$. Moreover, by Lemma 2.6, we deduce that $r_1 + r_2 = 2$. Since r_1 and r_2 are positive integers with $r_1 \geq 1$ and $r_2 \geq 1$, we have that $r_1 = r_2 = 1$. We are now under the hypothesis of Lemma 2.7, which gives the result.

This exhausts all possibilities, and so the proof follows. $\square$

3. Liftability

The following result characterizes liftable pairs, see Definition 1.2, of the form (G_0^2, C) . As usual, given two polynomials $p(x)$ and $q(x)$, we write $p(x)|q(x)$ to indicate that $p(x)$ divides $q(x)$, i.e., that there exists another polynomial $r(x)$ such that $q(x) = r(x)p(x)$.

Proposition 3.1. *Given $G_0, C \in \mathbb{R}[x]$ with $G_0(x) = x + o(x)$ and $C(0) = 1/2$, the pair (G_0^2, C) is liftable if, and only if,*

$$(x - \alpha)^k | C(x) \text{ with } \alpha \in \mathbb{R} \Rightarrow (x - \alpha)^{\lceil k/2 \rceil} | G_0(x). \quad (13)$$

Proof. The necessity of (13) follows easily from the identity $4AC = G_0^2 + B^2$ and the fact that all the polynomials are real. We are going to show that (13) is also sufficient. We decompose C as

$$C(x) = c \underbrace{\prod_{i=1}^{\ell} (x - \alpha_i)^{k_i}}_{R(x)} \underbrace{\prod_{i=1}^k (x - w_i)^{r_i}}_{L(x)} \prod_{i=1}^k (x - \bar{w}_i)^{r_i},$$

where $c \neq 0$, $\alpha_i \in \mathbb{R}$ for $i = 1, 2, \dots, \ell$ and $w_i \in \mathbb{C} \setminus \mathbb{R}$ has positive imaginary part for $i = 1, 2, \dots, k$. Here, as usual, $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. Accordingly $C = RL\bar{L}$. On the other hand, by the assumption (13),

$$G_0^2(x) = x^2 \prod_{i=1}^{\ell} (x - \alpha_i)^{2\lceil k_i/2 \rceil} \tilde{G}_0^2(x),$$

where $\tilde{G}_0 \in \mathbb{R}[x]$. We also define the real polynomial

$$\tilde{R}(x) := \left(\prod_{i=1}^{\ell} (x - \alpha_i)^{2\lceil k_i/2 \rceil} \right) / R(x),$$

so that $G_0^2(x) = x^2 R(x) \tilde{R}(x) \tilde{G}_0^2(x)$. We then consider the real polynomials

$$A(x) := x^2 \tilde{R}(x) \tilde{A}(x) \quad \text{and} \quad B(x) := x \prod_{i=1}^{\ell} (x - \alpha_i)^{[k_i/2]} \tilde{B}(x),$$

for certain $\tilde{A}, \tilde{B} \in \mathbb{R}[x]$ to be chosen such that the equation $4AC = G_0^2 + B^2$ is verified. On account of the previous definitions the equation writes as

$$4x^2 \tilde{R}(x) \tilde{A}(x) R(x) L(x) \overline{L(x)} = x^2 R(x) \tilde{R}(x) \tilde{G}_0^2(x) + x^2 R(x) \tilde{R}(x) \tilde{B}^2(x),$$

which implies

$$4\tilde{A}L\bar{L} = \tilde{G}_0^2 + \tilde{B}^2. \quad (14)$$

We claim that there exist $K \in \mathbb{C}[x]$ and $\tilde{B} \in \mathbb{R}[x]$ satisfying that, cf. (14),

$$\tilde{G}_0 + \mathbf{i}\tilde{B} = 2KL \quad (15)$$

where, as usual, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit. Once we prove the claim the result will follow because then, setting $\tilde{A} = K\bar{K}$,

$$4\tilde{A}L\bar{L} = (2KL)(2\bar{K}\bar{L}) = (\tilde{G}_0 + \mathbf{i}\tilde{B})(\tilde{G}_0 - \mathbf{i}\tilde{B}) = \tilde{G}_0^2 + \tilde{B}^2,$$

as desired. In order to prove the claim we set $d := \deg L$ and $n := \max\{d, \deg \tilde{G}_0 - d\}$. We shall find K with $\deg K \leq n$. Note that in this case, due to $\deg \tilde{G}_0 \leq d + n$, from (15) we must have $\deg \tilde{B} \leq d + n$. We identify the unknown coefficients of $K \in \mathbb{C}[x]$ and $\tilde{B} \in \mathbb{R}[x]$ as vectors in $\mathbb{R}^{2(n+1)}$ and $\mathbb{R}^{d+n+1}$, respectively, and define the linear map

$$\begin{aligned} T_L : \mathbb{R}^{2(n+1)} \times \mathbb{R}^{d+n+1} &\longrightarrow \mathbb{R}^{2(n+d+1)} \\ (K, \tilde{B}) &\longmapsto 2K(x)L(x) - \mathbf{i}\tilde{B}(x). \end{aligned}$$

Let us denote the image and kernel of T_L by $\text{im}(T_L)$ and $\ker(T_L)$, respectively. We aim to determine the dimension of $\text{im } T_L$ so that we can guarantee that the coefficients of the given real polynomial $\tilde{G}_0$ belong to it. We have that

$$\ker(T_L) = \left\{ (K, \tilde{B}) : 2K(x)L(x) - \mathbf{i}\tilde{B}(x) \equiv 0 \right\}.$$

If $2K(x)L(x) - \mathbf{i}\tilde{B}(x) \equiv 0$ then, due to $\tilde{B} \in \mathbb{R}[x]$, we deduce that $2\bar{K}(x)\bar{L}(x) + \mathbf{i}\tilde{B}(x) \equiv 0$. Hence, both complex polynomials, $L(x)$ and $\bar{L}(x)$, divide the real polynomial $\tilde{B}(x)$ and we get that $\tilde{B} = L\bar{L}D$ for some $D \in \mathbb{R}[x]$ with $\deg D \leq n - d$. The substitution of this expression for $\tilde{B}$ in the identity $2K(x)L(x) - \mathbf{i}\tilde{B}(x) \equiv 0$ yields $K = \frac{1}{2}\mathbf{i}\bar{L}D$. Accordingly

$$\ker(T_L) = \left\{ \left(\frac{1}{2}\mathbf{i}\bar{L}D, L\bar{L}D \right) : \text{for any } D \in \mathbb{R}[x] \text{ with } \deg D \leq n - d \right\}$$

and so we can assert that $\dim(\ker(T_L)) = n - d + 1$. Therefore, by applying Grassmann's formula,

$$\dim(\operatorname{im}(T_L)) = 2n + 2 + d + n + 1 - (n - d + 1) = 2(n + d + 1)$$

and we deduce that T_L is surjective. Hence, given $L \in \mathbb{C}[x]$ with $\deg L = d$ and $\tilde{G}_0 \in \mathbb{R}[x]$, one can always find $\tilde{B} \in \mathbb{R}[x]$ and $K \in \mathbb{C}[x]$ such that (15) is satisfied. This shows the validity of the claim and so the result follows. $\square$

The proof of Proposition 3.1 is constructive, i.e., given $G_0, C \in \mathbb{R}[x]$ satisfying the assumptions, a method to find $A, B \in \mathbb{R}[x]$ verifying $A(x) = x^2/2 + o(x^2)$ and $B(x) = o(x)$ together with $4AC = G_0^2 + B^2$ is provided. We note however that the solution of this problem is not unique. Indeed, taking any $p \in \mathbb{R}[x]$ with $p(x) = o(x^2)$, the polynomials $\tilde{A} := A + Bp + Cp^2$ and $\tilde{B} := \pm B \pm 2Cp$ also satisfy that $\tilde{A}(x) = x^2/2 + o(x^2)$, $\tilde{B}(x) = o(x)$ and $4\tilde{A}C = G_0^2 + \tilde{B}^2$.

In the next result we determine the set of $A, B \in \mathbb{R}[x]$ that verify $4AC - B^2 = G$ for a given $C, G \in \mathbb{R}[x]$.

Lemma 3.2. *Consider $G, C \in \mathbb{R}[x]$ with $C \not\equiv 0$ and assume that there exist $A_1, B_1, A_2, B_2 \in \mathbb{R}[x]$ such that*

$$4A_1C - B_1^2 = G \text{ and } 4A_2C - B_2^2 = G.$$

Then, either $A_1 = A_2$ and $B_1 = \pm B_2$, or there exist $p_1, p_2, C_1, C_2 \in \mathbb{R}[x]$ such that

$$B_1 = C_2p_2 - C_1p_1, \quad B_2 = C_1p_1 + C_2p_2, \quad A_2 = A_1 + p_1p_2 \text{ and } C = C_1C_2.$$

Proof. From $4A_1C - B_1^2 = 4A_2C - B_2^2$ we deduce that

$$4(A_2 - A_1)C = (B_2 - B_1)(B_2 + B_1). \quad (16)$$

If either $B_2 \equiv B_1$ or $B_2 \equiv -B_1$, since C is not identically null, we deduce that $A_2 \equiv A_1$. Otherwise, let C_1 be the polynomial formed by the factors of C which divide the polynomial $B_2 - B_1$ and let C_2 be the polynomial formed by the factors of C which divide the polynomial $B_2 + B_1$. It is clear that $C = C_1C_2$ and that $B_2 - B_1 = 2C_1p_1$ for a certain real polynomial p_1 and $B_2 + B_1 = 2C_2p_2$ for a certain real polynomial p_2 . From (16) we deduce that $A_2 - A_1 = p_1p_2$ and equating B_1 and B_2 from the identities $B_2 - B_1 = 2C_1p_1$ and $B_2 + B_1 = 2C_2p_2$, we have the desired result. Note that if p_1, p_2, C_1 and C_2 are real polynomials such that

$$B_1 = C_2p_2 - C_1p_1, \quad B_2 = C_1p_1 + C_2p_2, \quad A_2 = A_1 + p_1p_2$$

and $C = C_1C_2$, then easy computations show that $4A_2C - B_2^2 = 4A_1C - B_1^2$. $\square$

4. Composition via canonical changes

By a canonical change we mean a change of variables $(u, v) = \Phi(x, y)$, analytic in a neighborhood of the origin, that preserves area (i.e., such that the Jacobian of Φ is identically one). It is well-known (see for instance [6, Lemma 8]) that if we perform a canonical coordinate change $(u, v) = \Phi(x, y)$ to a Hamiltonian differential system with Hamiltonian function $H(x, y)$, then we get a differential system which is also Hamiltonian with Hamiltonian function

$(H \circ \Psi)(u, v)$ where $\Psi = \Phi^{-1}$. In this paper we study quadratic-like Hamiltonians and to preserve this structure we consider the particular canonical changes given by $\Psi(u, v) = \left(t(u), \frac{v}{t'(u)}\right)$ where $t(u) = u + o(u)$ is a rational function. This rational change of variables brings the Hamiltonian $\hat{H}(x, y) = \hat{A}(x) + \hat{B}(x)y + \hat{C}(x)y^2$ to the Hamiltonian $H(u, v) = A(u) + B(u)v + C(u)v^2$, where

$$A(u) = \hat{A}(t(u)), \quad B(u) = \frac{\hat{B}(t(u))}{t'(u)} \quad \text{and} \quad C(u) = \frac{\hat{C}(t(u))}{(t'(u))^2}.$$

Moreover, setting $\hat{G} = 4\hat{A}\hat{C} - \hat{B}^2$ and following the obvious notation for $\hat{M}$ and $\hat{N}$, cf. (3), one can verify that

$$M(u) = \hat{M}(t(u)) \quad \text{and} \quad N(u) = \hat{N}(t(u)).$$

The isochronicity is preserved by the coordinate change but not necessarily the polynomiality of the Hamiltonian system. In our next results we study when this is also preserved.

Proposition 4.1. *Consider real rational functions $\hat{G}_0$, $\hat{C}$ and t such that $\hat{G}_0(x) = x + o(x)$, $\hat{C}(0) = 1/2$ and $t(x) = x + o(x)$. Define the rational functions*

$$G_0(x) := \frac{\hat{G}_0(t(x))}{t'(x)} \quad \text{and} \quad C(x) := \frac{\hat{C}(t(x))}{(t'(x))^2}. \quad (17)$$

Then the following statements hold:

- (a) $(\hat{G}_0^2, \hat{C})$ satisfies (4) for $\hat{M}$ and $\hat{N}$ if and only if (G_0^2, C) satisfies (4) for M and N .
- (b) If $G_0, C \in \mathbb{R}[x]$ then there exist $\tilde{G}_0, \tilde{C}, \tilde{t} \in \mathbb{R}[x]$ such that

$$G_0(x) = \frac{\tilde{G}_0(\tilde{t}(x))}{\tilde{t}'(x)} \quad \text{and} \quad C(x) = \frac{\tilde{C}(\tilde{t}(x))}{(\tilde{t}'(x))^2}.$$

Moreover, $\tilde{G}_0 = G_0 \circ h^{-1}$, $\tilde{C} = C \circ h^{-1}$ and $\tilde{t} = h \circ t$ with $h(x) = \frac{x}{1+\gamma x}$ for some $\gamma \in \mathbb{R}$.

- (c) Assume that $\hat{G}_0, \hat{C} \in \mathbb{R}[x]$. Then $G_0, C \in \mathbb{R}[x]$ if and only if t is a polynomial with only simple roots and whose critical values are roots of $\hat{G}_0$ and $\hat{C}$. Moreover, if x_0 is a critical point of t of multiplicity strictly greater than 2, then $t(x_0)$ is a root of multiplicity at least two of $\hat{C}$.
- (d) Assume that $G_0, C, \hat{G}_0, \hat{C}, t \in \mathbb{R}[x]$, $\deg \hat{C} > 2$ and $\deg t \geq 2$. Then $\deg G_0 > \deg \hat{G}_0$ and $\deg C > \deg \hat{C}$.

Proof. Since $t(x) = x + o(x)$, $\hat{G}_0(x) = x + o(x)$ and $\hat{C}(0) = 1/2$, by (17) we have that $G_0(x) = x + o(x)$ and $C(0) = 1/2$. Following (3) we define the rational functions

$$\hat{M} := \frac{\hat{G}_0^2}{\hat{C}}, \quad \hat{N} := \frac{\hat{G}_0}{\hat{C}\hat{M}'}, \quad M := \frac{G_0^2}{C}, \quad N := \frac{G_0}{CM'},$$

and from (17) it is clear that the following identities hold:

$$M(x) = \hat{M}(t(x)) \quad \text{and} \quad N(x) = \hat{N}(t(x)).$$

We denote by $\sigma(x) = -x + o(x)$ the involution defined by $M(x) = M(\sigma(x))$ and we denote by $\hat{\sigma}(x) = -x + o(x)$ the involution defined by $\hat{M}(x) = \hat{M}(\hat{\sigma}(x))$. We have that the involution $\sigma(x) = -x + o(x)$ defined by $M(x) = M(\sigma(x))$ satisfies that $\hat{M}(t(x)) = \hat{M}(t(\sigma(x)))$ which implies that $t(\sigma(x)) = \hat{\sigma}(t(x))$. Hence,

$$N(x) + N(\sigma(x)) = \hat{N}(t(x)) + \hat{N}(t(\sigma(x))) = \hat{N}(t(x)) + \hat{N}(\hat{\sigma}(t(x))) = \hat{N}(y) + \hat{N}(\hat{\sigma}(y)),$$

where we have denoted $y = t(x)$ which is a change of variable since $t(x) = x + o(x)$ is a diffeomorphism. By Proposition 2.2 we have that statement (a) is proved.

We prove now statement (b). Assume first that $t(x)$ is a polynomial. Then the polynomial $G_0(x)t'(x) \in \mathbb{R}(t(x))$, which by Lemma 2.5 implies that $G_0(x)t'(x) = \hat{G}_0(t(x))$ with $\hat{G}_0(x)$ a real polynomial. In an analogous way we have that $C(x)(t'(x))^2 = \hat{C}(t(x))$ with $\hat{C}(x)$ a real polynomial.

Assume now that $t(x)$ is not a polynomial. We write the rational function $t(x)$ as the quotient of two coprime polynomials $t(x) = t_1(x)/t_2(x)$ with $t_1(x) = x + o(x)$ and $t_2(0) = 1$. In the same way, we write the rational functions $\hat{G}_0(x)$ and $\hat{C}(x)$ as the quotient of two coprime polynomials: $\hat{G}_0(x) = \hat{G}_1(x)/\hat{G}_2(x)$ and $\hat{C}(x) = \hat{C}_1(x)/\hat{C}_2(x)$ with $\hat{G}_1(x) = x + o(x)$, $\hat{G}_2(0) = 1$, $\hat{C}_1(0) = 1/2$ and $\hat{C}_2(0) = 1$. We also define the two-variable homogeneous polynomials coming from the homogenization of the polynomials $\hat{G}_i$ and $\hat{C}_i$, $i = 1, 2$. That is, consider the two real variables x and y and

$$G_{Hi}(x, y) := \hat{G}_i(x/y) y^{\delta_{\hat{G}_i}},$$

where $\delta_{\hat{G}_i} = \deg \hat{G}_i$ for $i = 1, 2$. The homogeneous polynomials $C_{Hi}(x, y)$ are defined analogously for $i = 1, 2$. On account of (17) we have that

$$\begin{aligned} G_0(x) &= \frac{\hat{G}_1\left(\frac{t_1(x)}{t_2(x)}\right)}{\hat{G}_2\left(\frac{t_1(x)}{t_2(x)}\right)} \frac{t_2^2(x)}{t_1'(x)t_2(x) - t_1(x)t_2'(x)} \\ &= \frac{G_{H1}(t_1(x), t_2(x))}{G_{H2}(t_1(x), t_2(x))} \frac{t_2(x)^{\delta_{\hat{G}_2} - \delta_{\hat{G}_1} + 2}}{t_1'(x)t_2(x) - t_1(x)t_2'(x)}, \end{aligned} \quad (18)$$

and

$$\begin{aligned} C(x) &= \frac{\hat{C}_1\left(\frac{t_1(x)}{t_2(x)}\right)}{\hat{C}_2\left(\frac{t_1(x)}{t_2(x)}\right)} \frac{t_2^4(x)}{(t_1'(x)t_2(x) - t_1(x)t_2'(x))^2} \\ &= \frac{C_{H1}(t_1(x), t_2(x))}{C_{H2}(t_1(x), t_2(x))} \frac{t_2(x)^{\delta_{\hat{C}_2} - \delta_{\hat{C}_1} + 4}}{(t_1'(x)t_2(x) - t_1(x)t_2'(x))^2}. \end{aligned} \quad (19)$$

Let us assume first that $\delta_{\hat{G}_2} - \delta_{\hat{G}_1} + 2 < 0$ and let $\alpha \in \mathbb{C}$ be such that $t_2(\alpha) = 0$. Then, in order that $G_0(\alpha) \in \mathbb{C}$, from (18), we need that $G_{H1}(t_1(\alpha), t_2(\alpha)) = 0$. By construction of $G_{H1}(x, y)$ this implies that $t_1(\alpha) = 0$ which is not possible because t_1 and t_2 are assumed to be coprime. Thus, we deduce that $\delta_{\hat{G}_2} - \delta_{\hat{G}_1} + 2 \geq 0$ and, analogously, that $\delta_{\hat{C}_2} - \delta_{\hat{C}_1} + 4 \geq 0$. We consider now two cases, either there is a value $c \in \mathbb{C}$ such that $\hat{G}_2(c) = 0$ or $\hat{G}_2(x)$ is a constant, that is $\hat{G}_0(x)$ is a polynomial. If there is a value $c \in \mathbb{C}$ such that $\hat{G}_2(c) = 0$, then this value cannot be zero because $\hat{G}_2(0) = 1$ by definition. We have a value $c \in \mathbb{C}$, $c \neq 0$, such that $\hat{G}_2(c) = 0$ and we consider the equation $t_1(x) - ct_2(x) = 0$. If such equation has a complex solution $\alpha \in \mathbb{C}$, we have that $G_{H2}(t_1(\alpha), t_2(\alpha)) = 0$. From (18) and the fact that $G_0(\alpha)$ takes a complex value, we have that either $G_{H1}(t_1(\alpha), t_2(\alpha)) = 0$, which is not possible because $\hat{G}_1$ and $\hat{G}_2$ are coprime, or $t_2(\alpha) = 0$, which is neither possible because then $t_1(\alpha) = 0$ and this is a contradiction with the fact that t_1 and t_2 are coprime. Thus, the equation $t_1(x) - ct_2(x) = 0$ cannot have any complex solution, which implies that $t_1(x) - ct_2(x) = k$, with $k \in \mathbb{C}$ and $k \neq 0$ because t_1 and t_2 are coprime. From $t_1(x) = x + o(x)$ and $t_2(0) = 1$ we can write that $t_2(x) = 1 + \gamma t_1(x)$ with $\gamma \in \mathbb{C}$. Hence $t'_1 t_2 - t_1 t'_2 = t'_1$ and from (18) we deduce that the polynomial $G_0 t'_1 \in \mathbb{R}(t_1)$. The fact that G_0 and t_1 are real polynomials implies that the base field is $\mathbb{R}$ and $\gamma \in \mathbb{R}$. Then, by Lemma 2.5 we have that there exists a real polynomial $\hat{G}_0(x)$ such that $G_0 t'_1 = \hat{G}_0(t_1)$. And we are as in the previous case because $t_1(x)$ is a polynomial.

We can discard the case that $\hat{C}(x)$ is not a polynomial using analogous reasonings to the previous paragraph. If we assume the existence of a value $c \in \mathbb{C}$ such that $\hat{C}_2(c) = 0$, we can construct a polynomial $t_1(x)$ out from the rational function $t(x)$ and we can show the existence of a polynomial $\hat{C}(x)$ such that $C(t'_1)^2 = \hat{C}(t_1)$.

We need to analyze the case that $\hat{G}_0$ and $\hat{C}$ are polynomials. Analogously to (18) and (19) we can write

$$G_0(x) = \hat{G}_0 \left(\frac{t_1(x)}{t_2(x)} \right) \frac{t_2^2(x)}{t'_1(x)t_2(x) - t_1(x)t'_2(x)} = \frac{G_{H0}(t_1(x), t_2(x)) t_2(x)^{2-\delta_{\hat{G}_0}}}{t'_1(x)t_2(x) - t_1(x)t'_2(x)} \quad (20)$$

and

$$C(x) = \hat{C} \left(\frac{t_1(x)}{t_2(x)} \right) \frac{t_2^4(x)}{(t'_1(x)t_2(x) - t_1(x)t'_2(x))^2} = \frac{C_H(t_1(x), t_2(x)) t_2(x)^{4-\delta_{\hat{C}}}}{(t'_1(x)t_2(x) - t_1(x)t'_2(x))^2}, \quad (21)$$

where $\delta_{\hat{C}} = \deg C$ and $C_H(x, y)$ is the homogenization of the polynomial $\hat{C}$. Following the same reasoning as in the previous paragraph it is clear that $2 - \delta_{\hat{G}_0} \geq 0$ and $4 - \delta_{\hat{C}} \geq 0$. We consider now the number of critical values of the rational function $t(x) = t_1(x)/t_2(x)$. If this rational function has no critical values, then by Lemma 2.8, it is a Möbius transformation and since $t(x) = x + o(x)$ we deduce that $t(x) = x/(1 + \gamma x)$ with $\gamma \in \mathbb{R}$. Then $t'_1 t_2 - t_1 t'_2 = 1$ and $t(x)$ only changes the position of the roots of the polynomials G_0 and C with respect of the roots of the polynomials $\hat{G}_0$ and $\hat{C}$ and the statement is true taking the Möbius transformation $x/(1 + \gamma x)$. On the other hand, assume that there exists $\alpha \in \mathbb{C}$ such that $(t'_1 t_2 - t_1 t'_2)(\alpha) = 0$, that is α is a critical point of the rational function $t(x)$. Note first that $\alpha \neq 0$ because $t(x) = x + o(x)$. Let $c \in \mathbb{C}$ be the critical value corresponding to α , that is $t(\alpha) = c$. Since $\delta_{\hat{G}_0} \leq 2$ and $\hat{G}_0(x) = x + o(x)$, we have that there exists $c_0 \in \mathbb{R}$ such that $\hat{G}_0(x) = x(1 + c_0 x)$. From (20) and (21), we have that c needs to be a root of both polynomials $\hat{G}_0(x)$ and $\hat{C}(x)$. Since $C(0) = 1/2$, we have that $c \neq 0$.

We have that $t(x)$ has at most two critical values, one is $t(\alpha) = c$ and the other one is $t(\infty)$. This is because the critical values of $t(x)$ are roots of $\hat{G}_0(x) = x(1 + c_0x)$ or $t(\infty)$ and 0 cannot be a critical value. By Lemma 2.8, $t(x)$ takes the form

$$t(x) = \frac{a_1(b_0 + b_1x)^n + a_0(b_2 + b_3x)^n}{a_3(b_0 + b_1x)^n + a_2(b_2 + b_3x)^n},$$

where $a_i, b_i \in \mathbb{C}$, for $i = 0, 1, 2, 3$, $(b_0b_3 - b_1b_2)(a_0a_3 - a_1a_2) \neq 0$ and $n \in \mathbb{N}$ with $n \geq 2$. Then

$$t'(x) = \frac{n(b_0b_3 - b_1b_2)(a_0a_3 - a_1a_2)(b_1x + b_0)^{n-1}(b_3x + b_2)^{n-1}}{(a_3(b_0 + b_1x)^n + a_2(b_2 + b_3x)^n)^2}.$$

Since α and ∞ are critical points, we have that $\alpha = -b_0/b_1$ and $b_3 = 0$. We can write

$$t(x) = \frac{\mu_1(x - \alpha)^n + \mu_0}{\mu_3(x - \alpha)^n + \mu_2},$$

with $\mu_i \in \mathbb{C}$, $i = 0, 1, 2, 3$ and $\mu_0\mu_3 - \mu_1\mu_2 \neq 0$. Note also that $\mu_1 \neq 0$ because $t(x) = x + o(x)$. The numerator of t is $t_1(x) = \mu_1(x - \alpha)^n + \mu_0$ and the denominator is $t_2(x) = \mu_3(x - \alpha)^n + \mu_2$. We can take $\ell_0 = \mu_2 - \mu_0\mu_3/\mu_1$ and $\ell_1 = \mu_3/\mu_1$ and we write $t_2(x) = \ell_0 + \ell_1 t_1(x)$. Since $t_1(x) = x + o(x)$ and $t_2(0) = 1$, we deduce that $t_2(x) = 1 + \gamma t_1(x)$ for $\gamma \in \mathbb{R}$. And the statement follows reasoning as in the previous paragraph.

We provide now the proof of statement (c). The proof of the “if” part of statement (c) is straightforward. Assume that $\hat{G}_0$ and $\hat{C}$ are polynomial and $t(x)$ is a polynomial with only simple roots and whose critical values are roots of $\hat{G}_0$ and of $\hat{C}$, with multiplicity at least 2 in $\hat{C}$ if the corresponding critical point has multiplicity > 2 in $t(x)$. In order to show that G_0 and C are polynomial, we will prove that they take a complex value at any root of $t'(x)$. Let $\alpha \in \mathbb{C}$ be such a root, that is, $t'(\alpha) = 0$. Thus, α is a critical point of $t(x)$. Assume that it has multiplicity m and recall that $m \geq 2$. Denote by c the corresponding critical value. We have that $t(x) - c = (x - \alpha)^m \tilde{t}(x)$ with $\tilde{t}(\alpha) \neq 0$. Hence $t'(x) = (x - \alpha)^{m-1} (m\tilde{t}(x) + (x - \alpha)\tilde{t}'(x))$. By assumption $\hat{G}_0(x) = (x - c)\tilde{G}_0(x)$ for some polynomial $\tilde{G}_0(x)$. Thus

$$G_0(x) = \frac{\hat{G}_0(t(x))}{t'(x)} = \frac{(t(x) - c)\tilde{G}_0(t(x))}{t'(x)} = \frac{(x - \alpha)^m \tilde{t}(x) \tilde{G}_0(t(x))}{(x - \alpha)^{m-1} (m\tilde{t}(x) + (x - \alpha)\tilde{t}'(x))}. \quad (22)$$

Note that α is a root of G_0 of multiplicity ≥ 1 . Analogously, by assumption $\hat{C}(x) = (x - c)^\ell \tilde{C}(x)$ for some polynomial $\tilde{C}$ and $\ell \geq 2$ if $m > 2$. Then

$$C(x) = \frac{\hat{C}(t(x))}{(t'(x))^2} = \frac{(t(x) - c)^\ell \tilde{C}(x)}{(t'(x))^2} = \frac{(x - \alpha)^{m\ell} (\tilde{t}(x))^\ell \tilde{C}(x)}{(x - \alpha)^{2m-2} (m\tilde{t}(x) + (x - \alpha)\tilde{t}'(x))^2}. \quad (23)$$

We see that if $m = 2$ and $\ell \geq 1$, then $C(x)$ takes a complex value at α . If $m > 2$ and $\ell \geq 2$ then, $m\ell \geq 2m$ which is greater than $2m - 2$. Thus, $G_0(x)$ and $C(x)$ are polynomials.

To prove the “only if” part, assume that there is $\alpha \in \mathbb{C}$ such that $t(\alpha) = t'(\alpha) = 0$. The polynomial $C(x)$ takes a complex value in α and since $\hat{C}(0) = 1/2$ we get a contradiction. Thus, all the roots of the polynomial $t(x)$ are simple. Let us consider now a critical point $\alpha \in \mathbb{C}$ of $t(x)$,

that is, $t'(\alpha) = 0$. Let $c = t(\alpha)$ be its critical value. It is clear that $\hat{G}_0(c)$ and $\hat{C}(c)$ need to be zero in order that the polynomials $G_0(x)$ and $C(x)$ evaluate at a complex value when $x = \alpha$. If α is a critical point of multiplicity m , then $t(x) - c = (x - \alpha)^m \tilde{t}(x)$ with $\tilde{t}(x)$ a polynomial such that $\tilde{t}(\alpha) \neq 0$. We write $\hat{G}_0(x) = (x - c)\tilde{G}_0(x)$ with $\tilde{G}_0(x)$ a polynomial. Then identity (22) shows that $G_0(\alpha) = 0$. With analogous computations and using identity (23) it is clear that c needs to be a root of the polynomial $\hat{C}(x)$, of multiplicity at least 2 if $m > 2$, in order to have that the polynomial C takes a complex value at $x = \alpha$. This proves the statement in (c).

Let us turn finally to the proof of (d), so assume that $\deg t \geq 2$. We consider polynomial pairs $(\hat{G}_0^2, \hat{C})$ and, from them, the polynomial pair (G_0^2, C) defined by (17). We aim at showing that the degrees of the polynomials (G_0^2, C) are strictly greater than the degrees of the polynomials $(\hat{G}_0^2, \hat{C})$. We first show that $\deg \hat{G}_0 > 1$. If $\deg \hat{G}_0 = 1$ then $\hat{G}_0(x) = x$ and $G_0(x) = t(x)/t'(x)$ which is not a polynomial because $t(x)$ is a polynomial with only simple roots. Accordingly,

$$\deg G_0 = \deg \hat{G}_0 \cdot \deg t - \deg t + 1 = \deg t (\deg \hat{G}_0 - 1) + 1 \geq 2 \deg \hat{G}_0 - 1 > \deg \hat{G}_0$$

where we take $\deg t \geq 2$ and $\deg \hat{G}_0 > 1$ into account. Thus $\deg G_0 > \deg \hat{G}_0$. On the other hand, by hypothesis we have that $\deg \hat{C} > 2$. By (17) we have that

$$\deg C = \deg \hat{C} \cdot \deg t - 2 \deg t + 2 = \deg t (\deg \hat{C} - 2) + 2 \geq 2 \deg \hat{C} - 2 > \deg \hat{C}$$

where we use that $\deg t \geq 2$ and $\deg \hat{C} > 2$. Therefore, $\deg C > \deg \hat{C}$. This shows (d) and concludes the proof of the result. $\square$

Assertion (b) in Proposition 4.1 shows that the search for polynomials G_0 and C satisfying (17) can be restricted to $\hat{G}_0$, $\hat{C}$ and t that are polynomial as well. Note moreover that in the statement of Proposition 4.1 we do not assume that (G_0^2, C) is a liftable pair, i.e., the existence of $A, B \in \mathbb{R}[x]$ such that $G_0^2 = 4AC - B^2$. We relate the existence of such polynomials A and B for the pair (G_0^2, C) with the existence of $\hat{A}, \hat{B} \in \mathbb{R}[x]$ such that $\hat{G}_0^2 = 4\hat{A}\hat{C} - \hat{B}^2$ in the following result.

Lemma 4.2. *Let $G_0(x) = x + o(x)$, $\hat{G}_0(x) = x + o(x)$, $C(x)$ and $\hat{C}(x)$ with $C(0) = \hat{C}(0) = 1/2$ and $t(x) = x + o(x)$ be real polynomials verifying the equalities in (17). If $(\hat{G}_0^2, \hat{C})$ is a liftable pair then so it is (G_0^2, C) .*

Proof. By hypothesis there exist real polynomials $\hat{A}(x) = x^2/2 + o(x^2)$ and $\hat{B}(x) = o(x)$ such that $4\hat{A}\hat{C} = \hat{G}_0^2 + \hat{B}^2$. Then we define

$$A(x) := \hat{A}(t(x)) \text{ and } B(x) := \frac{\hat{B}(t(x))}{t'(x)}. \quad (24)$$

Since $t(x) = x + o(x)$, $\hat{A}(x) = x^2/2 + o(x^2)$ and $\hat{B}(x) = o(x)$ it is clear that (24) implies that the rational functions $A(x)$ and $B(x)$ satisfy that $A(x) = x^2 + o(x^2)$ and $B(x) = o(x)$. We substitute x by $t(x)$ in identity $4\hat{A}\hat{C} = \hat{G}_0^2 + \hat{B}^2$ and we divide by $(t'(x))^2$ to get that

$$4\hat{A}(t(x)) \frac{\hat{C}(t(x))}{(t'(x))^2} = \frac{\hat{G}_0^2(t(x))}{(t'(x))^2} + \frac{\hat{B}^2(t(x))}{(t'(x))^2}.$$

Thus, by (17) and (24) we deduce that $4AC = G_0^2 + B^2$. It is clear that $A(x) = \hat{A}(t(x))$ is a real polynomial and since $4AC = G_0^2 + B^2$ we deduce that the real rational function $B(x)$ is indeed a real polynomial. $\square$

The converse of Lemma 4.2 is not true as the following example exhibits.

Example 4.3. Consider $A(x) = x^2(1 + 5x + 7x^2)/2$, $B(x) = \sqrt{3}x^3$, $C(x) = \frac{1}{2}(1 + x + x^2)$ and $G_0(x) = x(1 + x)(1 + 2x)$, which verify the identity $4AC = G_0^2 + B^2$. Let us also take $t(x) = x + x^2$ together with $\hat{C}(x) = (1 + x)(1 + 4x)/2$ and $\hat{G}_0(x) = x(1 + 4x)$. Then it is easy to check that the relations (17) are satisfied. However, by Proposition 3.1, there do not exist real polynomials $\hat{A}(x)$ and $\hat{B}(x)$ such that $4\hat{A}\hat{C} = \hat{G}_0^2 + \hat{B}^2$. $\square$

The previous example is not isochronous and the existence of such an example in the isochronous case is an open problem.

Example 4.4. One can easily check that $G(x) = x^2(1 + ax^2)^2$ and $C(x) = \frac{1}{2}(1 + ax^2)^3$ verify $N(x) \equiv 1/2$ for all $a \in \mathbb{R}$, see (3). However Proposition 3.1 shows that if $a < 0$ then (G, C) is not a liftable pair. The same happens with the following pair of real polynomials

$$G_0(x) = \frac{x}{72} \left(\sqrt{2}\zeta + x \right) \left(\sqrt{2}\zeta + 2x \right) \left(3\sqrt{2}\zeta - 2\sqrt{3}\zeta + 6x \right) \left(3\sqrt{2}\zeta + 2\sqrt{3}\zeta + 6x \right),$$

and

$$C(x) = \frac{1}{2^8 3^7 \sqrt{6}} \left(3\sqrt{2}\zeta - 2\sqrt{3}\zeta + 6x \right)^5 \left(3\sqrt{2}\zeta + 2\sqrt{3}\zeta + 6x \right)^5,$$

with $\zeta = \sqrt[4]{6}$. $\square$

5. Proofs of the main results

Let us recall that the rational function M defined in (3) in terms of G and C defines an involution σ verifying $M = M \circ \sigma$. In what follows we consider the field

$$\mathcal{F} := \{\phi \in \mathbb{R}(x) : \phi = \phi \circ \sigma\}.$$

Note that $\mathbb{R} \subsetneq \mathcal{F}$ because M is not constant and belongs to $\mathcal{F}$. Moreover $\mathcal{F} \subsetneq \mathbb{R}(x)$ because $x \notin \mathcal{F}$ due to $\sigma(x) = -x + o(x)$. Therefore, by Lüroth's Theorem (see Theorem 2.4), there exists a non-constant rational function R so that $\mathcal{F} = \mathbb{R}(R)$. We aim at knowing properties of the generator $R(x)$.

Lemma 5.1. Consider $G(x)$ and $C(x)$ two real polynomials such that $G(x) = x^2 + o(x)$ and $C(0) = 1/2$. The generator $R \in \mathbb{R}(x)$ of the field $\mathcal{F}$ satisfies the following statements.

- (a) There are two coprime polynomials $r(x)$ and $s(x)$ such that $r(x) = x^2 + o(x^2)$, $s(0) = 1$ and $R(x) = r(x)/s(x)$.
 (b) The polynomial r divides the polynomial G .

Proof. Note that if $\alpha, \beta, \delta, \gamma \in \mathbb{R}$ with $\alpha\delta - \beta\gamma \neq 0$, then

$$\mathbb{R}(R) = \mathbb{R}\left(\frac{\alpha R + \beta}{\gamma R + \delta}\right).$$

Thus, by choosing α, β, γ and δ conveniently, we can assume without loss of generality that the Taylor development of R at $x = 0$ is $R(x) = x^n + o(x^n)$ with $n \geq 1$. Since $R(x) = R(\sigma(x))$ and $\sigma(x) = -x + o(x)$, we deduce that n is even. The function M belongs to the field $\mathcal{F} = \mathbb{R}(R)$, therefore there exists a rational function ϕ such that $M = \phi(R)$. Since $M(0) = 0$ and $R(0) = 0$, we have that the Taylor development of ϕ at $x = 0$ is

$$\phi(x) = \alpha x^k + o(x^k) \quad \text{with } k \geq 1 \text{ and } \alpha \neq 0.$$

Recall that $M(x) = 2x^2 + o(x^2)$. Equating the lowest order terms in the Taylor developments at $x = 0$ in identity $M = \phi(R)$, we get that $\alpha = 2$, $k = 1$ and $n = 2$. Hence the generator verifies $R(x) = x^2 + o(x^2)$. Let us fix that the generator is $R = r/s$, where r and s are relatively prime polynomials verifying that $r(x) = x^2 + o(x^2)$ and $s(0) = 1$. We have proved statement (a) and that $M = \phi(R)$, with $\phi(x) = 2x + o(x)$.

In order to prove statement (b), we can write the function ϕ as the quotient of two coprime polynomials $\phi = P/Q$. We define the homogeneous and coprime polynomials

$$P_H(x, y) := y^{\delta_P} P\left(\frac{x}{y}\right) \quad \text{and} \quad Q_H(x, y) := y^{\delta_Q} Q\left(\frac{x}{y}\right),$$

where δ_P denotes the degree of the polynomial P . Thus, putting $R = r/s$ and $M = G/C$ we get

$$\frac{G}{C} = \frac{s^{\delta_Q} P_H(r, s)}{s^{\delta_P} Q_H(r, s)}.$$

Thus $s^{\delta_P} Q_H(r, s) G = s^{\delta_Q} P_H(r, s) C$. We evaluate at $x = 0$ and we recall that $s(0) = 1$, $r(0) = 0$, $G(0) = 0$ and $C(0) = 1/2$, to get that $0 = P_H(0, 1)/2$. Therefore, $P_H(x, y) = x \tilde{P}(x, y)$ for some homogeneous polynomial $\tilde{P}(x, y)$. Hence,

$$s^{\delta_P} Q_H(r, s) G = s^{\delta_Q} r \tilde{P}(r, s) C.$$

The polynomials r and s are coprime and the polynomials P_H and Q_H are coprime, therefore r divides G . $\square$

Proof of Theorem A. Since $G(x) = x^2 + o(x^2)$, $C(0) = 1/2$ and $M(x) = 2x^2 + o(x^2)$, from the definition of N given in (3) we get that $N(0) = 1/2$. We deal with two cases: either $N(x) \equiv 1/2$ (i.e., N is a constant function) or $N(x) = 1/2 + \omega x^{2k-1} + o(x^{2k-1})$ with $\omega \neq 0$ and $k \geq 1$. The fact that the Taylor development of $N(x) - 1/2$ at $x = 0$ has a lower term of odd degree (if N is not constant) comes from the identity $N(x) + N(\sigma(x)) = 1$ and that $\sigma(x) = -x + o(x)$.

The case $N(x) \equiv 1/2$ is trivial because, from (3), we have that

$$G(x) = \left(\frac{C(x)M'(x)}{2} \right)^2.$$

Thus, $G(x)$ is a squared polynomial, that is, $G = G_0^2$ with $G_0(x) = x + o(x)$.

Assume now that

$$N(x) = 1/2 + \omega x^{2k-1} + o(x^{2k-1}) \quad \text{with } \omega \neq 0 \text{ and } k \geq 1. \quad (25)$$

By (3) we can write $G(x) = (N(x)C(x)M'(x))^2$ and we aim at proving that $N(x)$ is a rational function in order to deduce that G is a squared polynomial. Since $N(x) + N(\sigma(x)) = 1$, we get $N'(x) + N'(\sigma(x))\sigma'(x) = 0$. Moreover the generator R of $\mathcal{F}$ satisfies $R(x) = R(\sigma(x))$ and we have that $R'(x) = R'(\sigma(x))\sigma'(x)$. Hence,

$$\frac{N'}{R'}(x) = -\frac{N'}{R'}(\sigma(x)).$$

Note also, see (3), that $(N')^2 \in \mathbb{R}(x)$. Therefore $(N'/R')^2 \in \mathcal{F}$ and there exists $\psi_1 \in \mathbb{R}(x)$ such that $N'/R' = \sqrt{\psi_1 \circ R}$. This implies that

$$N'(x) = \sqrt{\psi_1(R(x))} R'(x). \quad (26)$$

Note that $R(x) = x^2 + o(x^2)$ by Lemma 5.1. We set $\rho(x) := x\sqrt{R(x)/x^2}$, which is a local diffeomorphism, and take the change $x \mapsto \rho^{-1}(x)$. Since $\rho^2(x) = R(x)$ we have $x^2 = R(\rho^{-1}(x))$. We derive the former expression with respect to x to get $2x = R'(\rho^{-1}(x))(\rho^{-1})'(x)$. Then (26) yields

$$N'(\rho^{-1}(x)) = \sqrt{\psi_1(R(\rho^{-1}(x)))} R'(\rho^{-1}(x)).$$

Thus, $N'(\rho^{-1}(x))(\rho^{-1})'(x) = 2x\sqrt{\psi_1(x^2)}$. We deduce that

$$(N \circ \rho^{-1})'(x) = 2x\sqrt{\psi_1(x^2)}. \quad (27)$$

By Lemma 2.3 there exist $q(x) \in \mathbb{R}(x)$ and $\eta \in \mathbb{R}$ such that

$$(N \circ \rho^{-1})(x) = xq(x)\sqrt{\psi_1(x^2)} + \eta.$$

Note, also, that from (27) we deduce that $q(x)$ is an odd function. Thus,

$$(N \circ \rho^{-1})(x) = \sqrt{\psi_2(x^2)} + \eta \quad (28)$$

for a rational function ψ_2 and a constant η . We substitute x by $\rho(x)$ in (28) to get

$$N(x) = \sqrt{\psi_2(R(x))} + \eta, \quad (29)$$

where we have used that $\rho^2 = R$. Recall that $R(x) = x^2 + o(x^2)$ and we evaluate (29) at $x = 0$ to get $1/2 = \sqrt{\psi_2(0)} + \eta$. Moreover, since we are under the hypothesis (25), from (29) we deduce that the Taylor development of $\psi_2(x)$ at $x = 0$ is

$$\psi_2(x) = b_0 + b_\ell x^\ell + o(x^\ell), \quad (30)$$

with $b_0, b_\ell \in \mathbb{R}$ and $b_\ell \neq 0$ (because N is not constant), $\ell \geq 1$. From (25) and (29), we deduce that

$$\sqrt{\psi_2(R(x))} = \frac{1}{2} - \eta + \omega x^{2k-1} + o(x^{2k-1}), \quad (31)$$

with $\omega \neq 0$ and $k \geq 1$. We recall that $R(x) = x^2 + o(x)$ and we take the Taylor development of $\sqrt{\psi_2(R(x))}$ at $x = 0$. Consider (30) and we have that

- if $b_0 \neq 0$, the Taylor development of $\sqrt{\psi_2(R(x))}$ at $x = 0$ is

$$\sqrt{\psi_2(R(x))} = \sqrt{b_0} + \frac{b_\ell}{2\sqrt{b_0}} x^{2\ell} + o(x^{2\ell});$$

- if $b_0 = 0$, the Taylor development of $\sqrt{\psi_2(R(x))}$ at $x = 0$ is

$$\sqrt{\psi_2(R(x))} = \sqrt{b_\ell} x^\ell + o(x^\ell).$$

Equating the lowest order terms in (31), we deduce that $b_0 = 0$, $\sqrt{b_\ell} = \omega$, $\eta = 1/2$ and $\ell = 2k - 1$. In short

$$N(x) = \sqrt{\psi_2(R(x))} + 1/2. \quad (32)$$

Therefore,

$$\left(N - \frac{1}{2}\right)^2 = N^2 - N + \frac{1}{4} = \psi_2 \circ R.$$

Since $N^2 \in \mathbb{R}(x)$ and $\psi_2 \circ R \in \mathbb{R}(x)$, we deduce that $N(x)$ is a rational function. We conclude that G is a squared polynomial. $\square$

At this point we focus on the case that (G, C) is an isochronous pair with $N(x) \neq 1/2$, see (3). The case $N(x) \equiv 1/2$ will be studied later on. In either case let us note that, thanks to Theorem A, $G = G_0^2$ with $G_0 \in \mathbb{R}[x]$. That being said, we study next the case of an isochronous pair (G_0^2, C) with $N(x) \neq 1/2$ and we will give a more detailed expression of the generator R of the field $\mathcal{F}$.

Lemma 5.2. Assume that (G_0^2, C) is an isochronous pair with $N \neq 1/2$ and consider the field $\mathcal{F}$. Then there exist a polynomial $t(x) = x + o(x)$, whose roots are all simple, and a constant $\kappa \in \mathbb{R}$ such that

$$\mathcal{F} = \mathbb{R} \left(\left(\frac{t(x)}{1 + \kappa t(x)} \right)^2 \right).$$

Proof. By Lemma 5.1, $R(x) = r(x)/s(x)$ with $(r, s) = 1$ (i.e., r and s are coprime polynomials), $r(x) = x^2 + o(x^2)$ and $s(0) = 1$. From the proof of Theorem A, $N(x) = 1/2 + \omega x^{2k-1} + o(x^{2k-1})$ with $\omega \in \mathbb{R} \setminus \{0\}$. Moreover, see (32), we have that $N(x) - 1/2 = \sqrt{\psi_2(R(x))}$, where $\psi_2(x)$ is a rational function such that $\psi_2(x) = \omega^2 x^{2k-1} + o(x^{2k-1})$. We write the rational function $\psi_2(x)$ as the quotient of two coprime polynomials and its numerator and denominator in linear factors. Thus we can write

$$\psi_2(R(x)) = \prod_{i=1}^m \frac{\alpha_i r(x) + \beta_i s(x)}{\alpha_{m+i} r(x) + \beta_{m+i} s(x)},$$

where $m \in \mathbb{N}$ is the maximum of the degrees of the numerator and denominator of $\psi_2(x)$ and $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, \dots, 2m$. We remark that the number of factors m in the numerator of the previous expression coincides with the number of factors in the denominator m . So the total number of factors is $2m$.

As a consequence of $\sqrt{\psi_2(R(x))} \in \mathbb{R}(x)$, we have that any factor in the above decomposition repeated an odd number of times must be a square. Thus, $r(x)$ needs to be a square, because ψ_2 has an odd order $2k-1$ at $x=0$. This implies that there is at least another factor with odd multiplicity. Therefore, $\alpha_j r + \beta_j s = u^2$ for some α_j, β_j , with $\beta_j \neq 0$. We can choose $\alpha_j, \beta_j \in \mathbb{R}$ and $u(0) = 1$, because $r(x)$ is a real polynomial. We take

$$\frac{R}{\alpha_j R + \beta_j} = \frac{r}{\alpha_j r + \beta_j s} = \frac{t^2}{u^2}$$

as a generator of the field $\mathcal{F}$. Since $r(x) = x^2 + o(x^2)$ and $s(0) = 1$, we can choose $t(x) = x + o(x)$ and $u(0) = 1$. Note that

$$\mathcal{F} = \mathbb{R} \left(\frac{t(x)^2}{u(x)^2} \right) \subsetneq \mathbb{R} \left(\frac{t(x)}{u(x)} \right).$$

We know that $M \in \mathbb{R}(t^2/u^2)$ since $M \in \mathcal{F}$. Let us show next that $N \in \mathbb{R}(t/u)$. To this end we note first that

$$\frac{t(x)}{u(x)} = -\frac{t(\sigma(x))}{u(\sigma(x))}.$$

Moreover, on account of $N(x) + N(\sigma(x)) = 1$, $N(x) = 1/2 + \sqrt{\psi_2(R(x))}$ and $N(x) \in \mathbb{R}(x)$, we deduce that

$$(N - 1/2)(x) = -(N - 1/2)(\sigma(x)).$$

Thus, $(t/u)(N - 1/2) \in \mathcal{F}$. And, therefore,

$$N = \frac{1}{2} + \frac{t}{u} \psi_3 \left(\frac{t^2}{u^2} \right), \quad \text{with } \psi_3 \in \mathbb{R}(x). \quad (33)$$

On the other hand, recall that $M = G_0^2/C$ and $N = G_0/(C M')$. So, $G_0 = M/(N M')$. Since $M \in \mathbb{R}(R)$, there exists $\phi \in \mathbb{R}(x)$ such that $M = \phi(R)$. We have that $R = t^2/u^2$. Thus,

$$R' = 2 \left(\frac{t}{u} \right) \frac{t'u - tu'}{u^2}.$$

From (33), we get

$$\begin{aligned} G_0 = \frac{1}{N} \frac{M}{M'} &= \frac{1}{\frac{1}{2} + \frac{t}{u} \psi_3(R)} \frac{\phi(R)}{\phi'(R)} \frac{1}{2 \left(\frac{t}{u} \right) \frac{t'u - tu'}{u^2}} \\ &= \frac{1}{\frac{1}{2} + \frac{t}{u} \psi_3(R)} \frac{\phi(R)}{\phi'(R)} \frac{\left(\frac{t}{u} \right)}{2 R \frac{t'u - tu'}{u^2}}. \end{aligned}$$

Therefore,

$$G_0 = \varphi \left(\frac{t}{u} \right) \frac{t u}{t'u - tu'}, \text{ for some } \varphi \in \mathbb{R}(x). \quad (34)$$

Recall that we have proved that $\mathcal{F} = \mathbb{R}(R)$ with $R = t^2/u^2$ and $t(x)$ and $u(x)$ two coprime real polynomials such that $t(x) = x + o(x)$ and $u(0) = 1$. By Lemma 5.1 we have that t^2 divides G_0^2 . Thus, t divides G_0 , i.e., there exists a polynomial G_1 , with $G_1(0) = 1$, such that $G_0 = t G_1$. From (34) we obtain

$$G_1 = \varphi \left(\frac{t}{u} \right) \frac{u}{t'u - tu'}.$$

Since $G_1(0) = 1$, $t(x) = x + o(x)$ and $u(0) = 1$, we get that $\varphi(0) = 1$. We write the rational function φ as a quotient of two coprime polynomials and we expand these polynomials in linear factors. Hence, we can write

$$\varphi \left(\frac{t}{u} \right) = \prod_{i=1}^m \frac{\delta_i t + \gamma_i u}{\delta_{m+i} t + \gamma_{m+i} u},$$

where $m \in \mathbb{N}$ is the maximum of the degrees of the numerator and the denominator of φ and $\delta_i, \gamma_i \in \mathbb{C}$. Since $\varphi(0) = 1$ we have that $\gamma_i \neq 0$ for $i = 1, 2, \dots, 2m$. Hence,

$$G_1 = \prod_{i=1}^m \frac{\delta_i t + \gamma_i u}{\delta_{m+i} t + \gamma_{m+i} u} \frac{u}{t'u - tu'}.$$

Since G_1 is a polynomial, we get that any factor in the denominator of the previous expression is either a nonzero constant or it cancels with a factor in the numerator. In the first case, there are constants δ, γ , with $\gamma \neq 0$, such that $\delta t + \gamma u$ is a nonzero constant and, thus, we deduce that $u = 1 + \kappa t$ for some $\kappa \in \mathbb{R}$. In the second case, all the factors in the denominator need to cancel with the numerator u and, thus, either $m = 0$ or $m = 1$. That is,

$$G_1 = \frac{u}{t'u - tu'} \text{ if } (m = 0); \quad \text{or} \quad G_1 = \frac{\delta t + \gamma u}{t'u - tu'} \text{ if } (m = 1).$$

Since $G_1(0) = 1$, $t(x) = x + o(x)$ and $u(0) = 1$, we have $\gamma = 1$. We denote by $\tilde{u} = u$ in the case $(m = 0)$ and $\tilde{u} = u + \delta t$ in case $(m = 1)$. We have $\tilde{u}(0) = 1$ and $G_1 = \tilde{u}/(t'\tilde{u} - t\tilde{u}')$. We consider the rational function $\phi = t/\tilde{u}$. If $x_0 \in \mathbb{C}$ is a critical point of ϕ , since G_1 is a polynomial, then $\tilde{u}(x_0) = 0$. Hence, ϕ has at most two critical values: $\phi(x_0) = \infty$ and $\phi(\infty)$. By Lemma 2.8, either $\phi(x)$ is a Möbius transformation or there exist two Möbius transformations, $h_i(x) = (\alpha_i x + \beta_i)/(\gamma_i x + \delta_i)$, $i = 1, 2$, and $d \geq 2$, such that

$$\phi(x) = \frac{t(x)}{\tilde{u}(x)} = \frac{\alpha_2 \left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1} \right)^d + \beta_2}{\gamma_2 \left(\frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1} \right)^d + \delta_2}.$$

If $\phi(x)$ is a Möbius transformation, then $t(x) = x$ and $\tilde{u}(x) = 1 + \tilde{\kappa}x$ for $\tilde{\kappa}$ constant. In the second case, let x_0 be a finite critical point of ϕ with critical value ∞ . Let $\phi(\infty)$ be the other critical value. Recall that $d \geq 2$ and we have

$$\begin{aligned} \phi(x) &= \frac{\alpha_2(\alpha_1 x + \beta_1)^d + \beta_2(\gamma_1 x + \delta_1)^d}{\gamma_2(\alpha_1 x + \beta_1)^d + \delta_2(\gamma_1 x + \delta_1)^d}, \\ \phi'(x) &= \frac{d(\beta_1 \gamma_1 - \alpha_1 \delta_1)(\beta_2 \gamma_2 - \alpha_2 \delta_2)(\alpha_1 x + \beta_1)^{d-1}(\gamma_1 x + \delta_1)^{d-1}}{(\gamma_2(\alpha_1 x + \beta_1)^d + \delta_2(\gamma_1 x + \delta_1)^d)^2}. \end{aligned}$$

Then $x_0 = -\beta_1/\alpha_1$, $\delta_2 = 0$ and $\gamma_1 = 0$. We can write

$$\phi(x) = \frac{(x - x_0)^d + \mu_0}{\mu_1(x - x_0)^d},$$

for constants μ_i , $i = 0, 1$, $\mu_0 \mu_1 \neq 0$. We have that $\tilde{u}(x) = 1 + \tilde{\kappa} t(x)$ for $\tilde{\kappa}$ constant. Since $\tilde{u} = u + \delta t$, we deduce that there exists $\kappa \in \mathbb{R}$ such that $u = 1 + \kappa t$.

We aim at proving that all the roots of t are simple. We have $u = 1 + \kappa t$,

$$G_1 = \varphi\left(\frac{t}{u}\right) \frac{u}{t'u - tu'} = \varphi\left(\frac{t}{1 + \kappa t}\right) \frac{1 + \kappa t}{t'}$$

and that $\varphi(0) = 1$. Since $G_1(x)$ is a polynomial we deduce that if x^* is such that $t'(x^*) = 0$, then $\varphi(t(x^*)/(1 + \kappa t(x^*))) (1 + \kappa t(x^*)) = 0$. Hence $t(x^*) = 0$ would contradict $\varphi(0) = 1$. $\square$

We are in position of proving Theorem B.

Proof of Theorem B. We are under the hypothesis of Lemma 5.2 and from its proof we have that $G_0 = t G_1$, with t and G_1 polynomials and

$$G_1 = \varphi\left(\frac{t}{1 + \kappa t}\right) \frac{1 + \kappa t}{t'},$$

with φ a rational function. Thus, $G_0 t' \in \mathbb{R}(t)$. As a consequence of Lemma 2.5 we have that there exists a real polynomial, we denote it by $\hat{G}_0$, such that

$$G_0(x) = \frac{\hat{G}_0(t(x))}{t'(x)}.$$

On the other hand, since we have that the rational function $M = G_0^2/C$ belongs to the field $\mathcal{F}$ and, using Lemma 5.2,

$$\mathcal{F} = \mathbb{R} \left(\left(\frac{t(x)}{1 + \kappa t(x)} \right)^2 \right) \subsetneq \mathbb{R}(t(x))$$

we deduce that $(t')^2 C \in \mathbb{R}(t)$. Again, from Lemma 2.5, there exists $\hat{C} \in \mathbb{R}[x]$ such that

$$C(x) = \frac{\hat{C}(t(x))}{(t'(x))^2}.$$

Since $t(x) = x + o(x)$, $G_0(x) = x + o(x)$ and $C(0) = 1/2$, we have that $\hat{G}_0(x) = x + o(x)$ and $\hat{C}(0) = 1/2$. By (a) in Proposition 4.1, $(\hat{G}_0^2, \hat{C})$ is an isochronous pair because so it is (G_0^2, C) .

The involution $\sigma(x) = -x + o(x)$ defined by $M(x) = M(\sigma(x))$ satisfies that

$$\frac{t(x)}{1 + \kappa t(x)} = -\frac{t(\sigma(x))}{1 + \kappa t(\sigma(x))}. \quad (35)$$

We consider the rational function $\hat{M} := \hat{G}_0^2/\hat{C}$, which defines an involution $\hat{\sigma}(x) = -x + o(x)$ that satisfies $\hat{M}(x) = \hat{M}(\hat{\sigma}(x))$. From the proof of statement (a) of Proposition 4.1 we have that $t(\sigma(x)) = \hat{\sigma}(t(x))$. We substitute this identity in (35) to deduce that

$$\hat{\sigma}(t(x)) = -\frac{t(x)}{1 + 2\kappa t(x)}.$$

The fact that $t(x) = x + o(x)$ is a diffeomorphism provides that $\hat{\sigma}(x) = -x/(1 + 2\kappa x)$ as desired. This concludes the proof of the result. $\square$

Example 5.3. There are isochronous pairs (G_0^2, C) verifying $N(x) \equiv 1/2$ for which the conclusion of Theorem B is not true. Indeed, consider for instance

$$G_0(x) = x(x+1)(3x+1) \text{ and } C(x) = \frac{1}{2}(x+1)(3x+1)^5.$$

One can readily verify, see (3), that

$$M(x) = \frac{2x^2(x+1)}{(3x+1)^3} \text{ and } N(x) \equiv \frac{1}{2}.$$

If there exist polynomials $\hat{G}_0(x) = x + o(x)$ and $t(x) = x + o(x)$ such that $G_0 = \frac{\hat{G}_0(t)}{t'}$, see (5), then either $\{\deg(\hat{G}_0) = 2, \deg t = 2\}$ or $\{\deg(\hat{G}_0) = 3, \deg t = 1\}$. In the first case, taking the coefficients of $\hat{G}_0$ and t as unknowns, it is easy to conclude that there is not any choice so that $G_0 = \frac{\hat{G}_0(t)}{t'}$ holds. In the second case $t(x) = x$ and consequently, according to Theorem B, the involution associated to $M(x)$ should be $\sigma(x) = -\frac{x}{1+\kappa x}$ for some $\kappa \in \mathbb{R}$. However the equation $M(x) = M(z)$ yields to $9z^2x + z^2 + 10zx + z + 9zx^2 + z^2 + x = 0$ and some computations show that this cannot be verified with $z = -\frac{x}{1+\kappa x}$ for any choice of κ . $\square$

In Theorem B we prove that isochronous pairs with $N(x) \not\equiv 1/2$ can be reduced to an isochronous pair whose corresponding involution is $\sigma(x) = -\frac{x}{1+\kappa x}$ where $\kappa \in \mathbb{R}$. In the case that $\kappa = 0$ we have that $\sigma(x) = -x$. The following result (that was previously proved in [3, Theorem C]) and the next example deal with this case.

Proposition 5.4. Assume $G(x)$ and $C(x)$ are real even polynomials with $G(x) = x^2 + o(x^2)$ and $C(0) = 1/2$. Then (G, C) is an isochronous pair if, and only if, $N(x) \equiv 1/2$.

Proof. Since G and C are even functions then, see (3), so they are M and N . Hence $\sigma = -\text{Id}$. Then, if (G, C) is an isochronous pair, from (4) we get that $N(x) \equiv 1/2$. This proves the necessity. The sufficiency of this condition is obvious. $\square$

The following example shows that there are isochronous pairs (G, C) where G and C are not even functions but the associated involution is $\sigma = -\text{Id}$.

Example 5.5. Let $c \in \mathbb{R}$ and $c \neq 0$. Define the real polynomials

$$G_0(x) = x(1 + cx) \quad C(x) = \frac{1}{2}(1 - cx)(1 + cx)^3.$$

In this case $M(x) = \frac{2x^2}{1-cx^2}$ is an even function and $N(x) = \frac{1}{2}(1 - cx)$. Since $N(x) + N(-x) \equiv 1$, (G_0^2, C) is an isochronous pair. Note also that (G_0^2, C) is not a liftable pair by Proposition 3.1. $\square$

As we already mentioned, the isochronicity problem for quadratic-like Hamiltonian differential systems was studied previously in [3] using tools from qualitative theory. Using our notation, Theorem D in that paper asserts the following:

Proposition 5.6. If (G, C) is a liftable isochronous pair then, either $G(x) = x^2$ and $C(x) \equiv 1/2$, or $\deg G = \deg C \geq 4$.

As the following example shows, the liftability assumption in Proposition 5.6 is essential.

Example 5.7. One can readily check that $(G(x), C(x)) = (x^2, (1 + kx)/2)$ is an isochronous pair for all $k \in \mathbb{R}$. However, by applying Proposition 3.1, it is not liftable when $k \neq 0$. The isochronous pair given by

$$(G(x), C(x)) = \left(x^2(1 + kx + k^2x^2/2)^2(1 + kx)^2, (1/2)(1 + kx + k^2x^2/2)(1 + kx)^4\right)$$

is another example of this situation when $k \neq 0$. $\square$

Corollary 5.8. *If the polynomial Hamiltonian differential system (2) has an isochronous center then $\deg A$ and $\deg C$ are both even and $\deg A + \deg C = 2 \deg B$. In particular the Hamiltonian has even degree.*

Proof. Recall that $H(x, y) = A(x) + B(x)y + C(x)y^2$ so that the degree of this polynomial is given by $\deg H = \max \{\deg A, \deg B + 1, \deg C + 2\}$. By Theorem A, $G = G_0^2$ with $G_0 \in \mathbb{R}[x]$ and, consequently, $\deg C$ is even because either $C(x) \equiv 1/2$ or $\deg C = 2 \deg G_0$ by Proposition 5.6.

- If $G(x) = x^2$ and $C(x) \equiv 1/2$, by $G = 4AC - B^2$, we have that $2A = x^2 + B^2$, which implies that $\deg A$ is even. Recall that $B(x) = o(x)$ by hypothesis. The fact that $2A = x^2 + B^2$ also implies that if $\deg B \leq 1$ (that is $B(x) \equiv 0$) then $\deg A = 2$ (because $A(x) = x^2/2 + o(x^2)$ by hypothesis) and $\deg H = 2$. If $\deg B > 1$ then the identity $2A = x^2 + B^2$ implies that $\deg A = 2 \deg B$. This also gives that $\deg B + 1 = \deg A/2 + 1 \leq \deg A$ because $\deg A \geq 2$. Therefore $\deg H = \deg A$, which is even.
- If either $G(x) \neq x^2$ or $C(x) \not\equiv 1/2$ then $\deg C = 2 \deg G_0$ and we argue as follows. Observe first that $\deg A$ is even because $G_0^2 + B^2 = 4AC$ and $\deg C$ is even. On account of this it suffices to verify that

$$\deg B + 1 \leq \max \{\deg A, \deg C + 2\}. \quad (36)$$

To show this observe that $2 \max(\deg G_0, \deg B) = \deg A + \deg C = \deg A + 2 \deg G_0$, which easily implies that $\deg G_0 < \deg B$ due to $A(x) = \frac{1}{2}x^2 + o(x^2)$. Therefore we obtain that $2 \deg B = 2 \deg G_0 + \deg A$, which in turn yields $2 \deg B + 2 = 2 \deg G_0 + \deg A + 2$. By using this equality one can readily prove by contradiction that (36) holds.

This exhausts all possibilities, and so the result follows. $\square$

Example 5.9. Some easy computations show that the only liftable isochronous pair (G_0^2, C) with $\deg C = 2 \deg G_0 = 6$ (cf. Proposition 5.6) and associated involution $\sigma(x) = \frac{-x}{1+2x}$ is given by

$$G_0(x) = x(1 + 2x + 2x^2) \text{ and } C(x) = \frac{1}{2}(1 + 2x + 2x^2)^3.$$

In this case one can choose for instance $A(x) = \frac{1}{2}x^2(1 - 2x + 2x^2)$ and $B(x) = 2x^3(1 + 2x + 2x^2)$ so that $4AC - B^2 = G_0^2$ holds (cf. Proposition 3.1). Thus the Hamiltonian differential system with $H(x, y) = A(x) + B(x)y + C(x)y^2$ has an isochronous center at the origin. The particularity of this isochronous center is that the projection of its period annulus on the x -axis is not the whole real line. The existence of such isochronous centers with $A, B, C \in \mathbb{R}[x]$ was posed as an open question in [3, p. 408]. $\square$

Let us consider next the case $N(x) \equiv 1/2$. This, recall (6), is equivalent to require that

$$G_0 C' = 2C(G_0' - 1). \quad (37)$$

The next result gives some properties of the real polynomials G_0 and C satisfying this condition. Recall that given $k \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\alpha \neq 0$, and a polynomial $q(x)$ we write $(1 - \alpha x)^k || q(x)$ to

indicate that the polynomial $(1 - \alpha x)^k$ divides $q(x)$ and k is the maximum multiplicity that divides $q(x)$, i.e., that there exists another polynomial $r(x)$ such that $q(x) = r(x)(1 - \alpha x)^k$ and $r(1/\alpha) \neq 0$.

Proposition 5.10. *Let $G_0(x) = x + o(x)$ and $C(x) = 1/2 + o(1)$ be real polynomials satisfying (37), i.e., so that $N(x) \equiv 1/2$. Then, either $G_0(x) = x$ and $C(x) \equiv 1/2$, or the following hold:*

- (a) $\deg C = 2 \deg G_0$.
- (b) All the roots of G_0 are simple.
- (c) If $C(\alpha) = 0$ with $\alpha \in \mathbb{C}$ then $G_0(\alpha) = 0$.
- (d) Let $\alpha \in \mathbb{C} \setminus \{0\}$ be such that $G_0(1/\alpha) = 0$ and $(1 - \alpha x)^{k+2} \mid C(x)$ with $k \in \mathbb{Z}_{\geq -2}$. Then $k \neq 0$ and $G'_0(1/\alpha) = -2/k$.
- (e) The factorization of G_0 and C over $\mathbb{C}$ is

$$G_0(x) = x \prod_{i=1}^{\tilde{n}} (1 - \alpha_i x) \text{ and } C(x) = \frac{1}{2} \prod_{i=1}^{\tilde{n}} (1 - \alpha_i x)^{k_i+2}, \quad (38)$$

respectively, where $\alpha_i \in \mathbb{C} \setminus \{0\}$ and $k_i \in \mathbb{N} \cup \{-2, -1\}$ for $i = 1, 2, \dots, \tilde{n}$. Moreover $\sum_{i=1}^{\tilde{n}} k_i = 2$.

- (f) If (G_0^2, C) is a liftable pair then all the roots of G_0 but $x = 0$ are in $\mathbb{C} \setminus \mathbb{R}$.

Proof. It is clear that if $G_0(x) \equiv x$ and $C(x) \equiv 1/2$, then identity (37) is satisfied. Moreover if $G_0(x) \equiv x$, then identity (37) implies that $C(x) \equiv 1/2$ and the other way round. We discard this trivial case for the rest of the proof.

- (a) Assume that $\ell := \deg G_0 > 1$ and $s := \deg C > 0$. Let us write $G_0(x) = g_\ell x^\ell + \dots$ and $C(x) = c_s x^s + \dots$, where the dots denote terms of lower order and $g_\ell, c_s \in \mathbb{R} \setminus \{0\}$. By equating the higher order terms in (37) we get $g_\ell s c_s = 2 c_s \ell g_\ell$. This implies that $s = 2\ell$ which is $\deg C = 2 \deg G_0$.
- (b) We have that $x = 0$ is a simple root of G_0 . Let $1/\alpha \in \mathbb{C}$ be a root of $G_0(x)$ of multiplicity m , with $m \in \mathbb{N}$, $m \geq 2$. That is $G_0(x) = (1 - \alpha x)^m \tilde{G}_0(x)$ where $\tilde{G}_0(x)$ is a real polynomial such that $\tilde{G}_0(1/\alpha) \neq 0$. If $C(1/\alpha) \neq 0$, then the multiplicity of the root $1/\alpha$ in the left hand side of (37) is at least m , whereas the right hand side of (37) does not vanish at $1/\alpha$, this gives a contradiction. If $C(1/\alpha) = 0$, we write $C(x) = (1 - \alpha x)^\mu \tilde{C}(x)$ with $\tilde{C}(x)$ a real polynomial such that $\tilde{C}(1/\alpha) \neq 0$. Note that $\mu \in \mathbb{Z}$ is such that $\mu \geq 1$. The multiplicity of the root $1/\alpha$ in the left hand side of (37) is $m + \mu - 1$ and its multiplicity in the right hand side is μ (because $G'_0(1/\alpha) = 0$ when $m \geq 2$). Therefore $m + \mu - 1 = \mu$ which gives $m = 1$, a contradiction. We deduce that all the roots of G_0 are simple.
- (c) We write $C(x) = (x - \alpha)^\mu \tilde{C}(x)$ with $\mu \geq 1$ and $\tilde{C} \in \mathbb{R}[x]$ such that $\tilde{C}(\alpha) \neq 0$. Since the multiplicity of α in $C'(x)$ is exactly $\mu - 1$, from (37) it follows that $G_0(\alpha) = 0$.
- (d) Note first that for the root $x = 0$ we have that $G_0(0) = 0$, $G'_0(0) = 1$ and $C(0) = 1/2$. By abuse of notation, we can relate to the root $x = 0$ the value of $k = -2$ and the statement is also satisfied. Now we are concerned with the rest of roots of G_0 . Consider $\alpha \in \mathbb{C}$, $\alpha \neq 0$, such that $G_0(1/\alpha) = 0$. By statement (b), this root is simple in G_0 , that is, we can write $G_0(x) = (1 - \alpha x) \tilde{G}_0(x)$ with $\tilde{G}_0(x)$ a real polynomial such that $\tilde{G}_0(1/\alpha) \neq 0$. Following

the notation of the statement, we can write $C(x) = (1 - \alpha x)^{k+2} \tilde{C}(x)$ with $\tilde{C}(x)$ a real polynomial such that $\tilde{C}(1/\alpha) \neq 0$. If $C(1/\alpha) \neq 0$, then the value of k corresponding to this root is $k = -2$. Moreover evaluating (37) at $x = 1/\alpha$, we get that $G'_0(1/\alpha) = 1$ and the statement is satisfied. If $C(1/\alpha) = 0$, then $k \geq -1$ and (37) reads for

$$(1 - \alpha x) \tilde{G}_0(x) \left((k+2)(1 - \alpha x)^{k+1} (-\alpha) \tilde{C}(x) + (1 - \alpha x)^{k+2} \tilde{C}'(x) \right) = 2(1 - \alpha x)^{k+2} \tilde{C}(x) \left((-\alpha) \tilde{G}_0(x) + (1 - \alpha x) \tilde{G}'_0(x) - 1 \right).$$

We divide the previous identity by $(1 - \alpha x)^{k+2}$ and we get

$$\begin{aligned} \tilde{G}_0(x) \left((k+2)(-\alpha) \tilde{C}(x) + (1 - \alpha x) \tilde{C}'(x) \right) \\ = 2\tilde{C}(x) \left((-\alpha) \tilde{G}_0(x) + (1 - \alpha x) \tilde{G}'_0(x) - 1 \right). \end{aligned}$$

We evaluate in $x = 1/\alpha$ and we have that

$$\tilde{G}_0(1/\alpha)(k+2)(-\alpha) \tilde{C}(1/\alpha) = 2\tilde{C}(1/\alpha) \left((-\alpha) \tilde{G}_0(1/\alpha) - 1 \right).$$

We sum $2\alpha \tilde{G}_0(1/\alpha) \tilde{C}(1/\alpha)$ in both sides of the previous expression to get that

$$-\alpha \tilde{G}_0(1/\alpha) \tilde{C}(1/\alpha) k = -2\tilde{C}(1/\alpha).$$

We divide by $\tilde{C}(1/\alpha)$ which, by assumption, is different from zero and we have that

$$-\alpha \tilde{G}_0(1/\alpha) k = -2.$$

This implies that $k \neq 0$ and that

$$-\alpha \tilde{G}_0(1/\alpha) = -2/k.$$

Note that since $G_0(x) = (1 - \alpha x) \tilde{G}_0(x)$, we have that $G'_0(1/\alpha) = -\alpha \tilde{G}_0(1/\alpha)$ and we are done.

- (e) The decompositions of the statement are a consequence of the previous statements. Note that $\deg G_0 = \tilde{n} + 1$ and $\deg C = \tilde{n} + \sum_{i=1}^{\tilde{n}} k_i$. By statement (a), we have that $\deg C = 2 \deg G_0$ and, thus, $\sum_{i=1}^{\tilde{n}} k_i = 2$.
- (f) Since (G_0^2, C) is a liftable pair by hypothesis we can apply Proposition 3.1. In doing so, since all the roots of G_0 are simple by assertion (b), if $G_0(\alpha) = 0$ with $\alpha \in \mathbb{R} \setminus \{0\}$ then either $C(\alpha) \neq 0$ or $x = \alpha$ is a simple root of the polynomial C . Following the notation in (d) we have that its corresponding value of k is either -2 or -1 , and in both cases we have that $G'_0(\alpha) > 0$. Note moreover that $G'_0(0) = 1$. Thus G_0 is increasing at all its real roots, which leads to a contradiction if there are other ones apart from $x = 0$.

This concludes the proof of the result. $\square$

By (b) in Proposition 5.10, if (G_0, C) is an isochronous pair with $N(x) \equiv 1/2$ then all the roots of G_0 are simple. This is not always the case when $N(x) \not\equiv 1/2$ as the following example shows.

Example 5.11. Let us take $G_0(x) = x(1+x)^2$ and $C(x) = \frac{1}{2}(1+x)^6$. Then the functions in (3) are

$$M(x) = \frac{2x^2}{(1+x)^2} \quad \text{and} \quad N(x) = \frac{1}{2(1+x)}.$$

It is easy to see that the involution defined by $M(x) = M(\sigma(x))$ is $\sigma(x) = -\frac{x}{1+2x}$ and that $N(x) + N(\sigma(x)) = 1$. Thus, (G_0^2, C) is an isochronous pair and the roots of G_0 are not all simple. Note that (G_0^2, C) is not a liftable pair, see Proposition 3.1. $\square$

The following result will allow us to prove the sufficiency conditions in Theorem C.

Proposition 5.12. Assume that $G_0, C \in \mathbb{C}[x]$ factorize as in (38), where $\alpha_1, \dots, \alpha_{\tilde{n}} \in \mathbb{C} \setminus \{0\}$ are pairwise distinct, $k_i \in \mathbb{N} \cup \{-2, -1\}$ and $G'_0(1/\alpha_i) = -2/k_i$ for $i = 1, \dots, \tilde{n}$, and $\sum_{i=1}^{\tilde{n}} k_i = 2$. Then the equality in (37) holds, i.e., $N(x) \equiv 1/2$.

Proof. Let us consider the polynomial $V(x) := G_0(x)C'(x) - 2C(x)(G'_0(x) - 1)$. We will show that $V(x) \equiv 0$, i.e., that (37) holds. From (38) we have that $\deg G_0 = \tilde{n} + 1$ and

$$\deg C = \sum_{i=1}^{\tilde{n}} (k_i + 2) = \sum_{i=1}^{\tilde{n}} k_i + 2\tilde{n} = 2 + 2\tilde{n},$$

where we have used the assumption that $\sum_{i=1}^{\tilde{n}} k_i = 2$. Therefore $\deg V \leq 3\tilde{n} + 2$.

Note that, since $G_0(0) = 0$ and $G'_0(0) = 0$, we have $V(0) = 0$. Let us fix $i \in \{1, 2, \dots, \tilde{n}\}$. From (38) we can write that

$$G_0(x) = (1 - \alpha_i x) \tilde{G}_0(x) \quad \text{and} \quad C(x) = (1 - \alpha_i x)^{k_i+2} \tilde{C}(x),$$

where $\tilde{G}_0(x)$ and $\tilde{C}(x)$ are polynomials. We can write that

$$V(x) = (1 - \alpha_i x)^{k_i+2} \left(\tilde{G}_0(x) \left((k_i + 2)(-\alpha_i) \tilde{C}(x) + (1 - \alpha_i x) \tilde{C}'(x) \right) - 2\tilde{C}(x) (G'_0(x) - 1) \right).$$

Thus V vanishes at least $k_i + 2$ times at $x = 1/\alpha_i$. We will show that it vanishes at least one more time. Note that from $G_0(x) = (1 - \alpha_i x) \tilde{G}_0(x)$ we have that $G'_0(1/\alpha_i) = -\alpha_i \tilde{G}_0(1/\alpha_i)$. We consider the polynomial

$$\tilde{V}(x) := \tilde{G}_0(x) \left((k_i + 2)(-\alpha_i) \tilde{C}(x) + (1 - \alpha_i x) \tilde{C}'(x) \right) - 2\tilde{C}(x) (G'_0(x) - 1),$$

which verifies $V(x) = (1 - \alpha_i x)^{k_i+2} \tilde{V}(x)$. Observe that

$$\begin{aligned}\tilde{V}(1/\alpha_i) &= (-\alpha_i \tilde{G}_0(1/\alpha_i))(k_i + 2)\tilde{C}(1/\alpha_i) - 2\tilde{C}(1/\alpha_i)(G'_0(1/\alpha_i) - 1) \\ &= \tilde{C}(1/\alpha_i)(G'_0(1/\alpha_i)(k_i + 2) - 2(G'_0(1/\alpha_i) - 1)) = \tilde{C}(1/\alpha_i)(k_i G'_0(1/\alpha_i) + 2).\end{aligned}$$

Since, by assumption, $G'_0(1/\alpha_i) = -2/k_i$, we have that $\tilde{V}(1/\alpha_i) = 0$.

Hence, we have proved that $V(x)$ vanishes at 0 and vanishes at $1/\alpha_i$ with multiplicity at least $k_i + 3$, for $i = 1, 2, \dots, \tilde{n}$. Therefore, the polynomial $V(x)$ vanishes at

$$1 + \sum_{i=1}^{\tilde{n}} (k_i + 3) = 1 + \sum_{i=1}^{\tilde{n}} k_i + 3\tilde{n} = 3\tilde{n} + 3$$

points, counting multiplicity, where we used that $\sum_{i=1}^{\tilde{n}} k_i = 2$. This, due to $\deg V \leq \tilde{n} + 2$, yields $V(x) \equiv 0$, so that (37) holds. $\square$

We are now in position to prove Theorem C.

Proof of Theorem C. The necessity follows by Proposition 5.10. Indeed, since $\alpha_i \in \mathbb{C} \setminus \mathbb{R}$ for all $i = 1, 2, \dots, \tilde{n}$, we obtain (7) from (38) taking into account that G_0 and C are real polynomials and setting $n = \tilde{n}/2$. The sufficiency is a consequence of Proposition 5.12. So it only remains to be proved that if $N(x) \equiv 1/2$ then $M = \frac{G_0^2}{C}$ is a Belyi function (i.e., a rational function with at most three critical values). To this end note that, due to $N = \frac{G_0}{CM}$, the assumption $N \equiv 1/2$ implies $M' = \frac{2G_0}{C}$. Thus $M'(x)$ can only vanish at $x = 0$, $x = 1/\alpha_i$ and $x = 1/\bar{\alpha}_i$ for $i = 1, 2, \dots, n$, and $x = \infty$. Since $M = \frac{G_0^2}{C}$ each one of these points is either a zero or a pole of M or the value $M(\infty)$. Hence M has at most three critical values, which are contained in the set $\{0, \infty, M(\infty)\}$. This concludes the proof of the result. $\square$

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Data availability

No data was used for the research described in the article.

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