



Surrogate duality for quasiconvex vector minimization

Juan Enrique Martínez-Legaz^{1,2} · Christiane Tammer³

Received: 24 December 2025 / Accepted: 20 July 2026
© The Author(s) 2026

Abstract

It is well-known that duality theory is a fundamental tool in various areas of mathematics. There are great advantages to including or using the dual problem and duality statements. Especially, solving the dual problem can be done using other methods of analysis or numerical mathematics. We consider a primal vector optimization problem with an objective function acting between a linear topological space X and a linear topological space Y equipped with a pointed closed convex cone $D \subset Y$ with nonempty interior. The feasible set is supposed to be a closed convex cone. The aim of this paper is to construct a simple and easy-to-handle dual problem by exploiting the special structure of the primal problem and using a suitable nonlinear scalarization. The computation of the dual image set involves the minimization of a nonlinear scalarization of the primal vector-valued objective function subject to only one linear inequality constraint. We introduce a new concept of upper semicontinuity for a vector-valued function and prove (weak and strong) duality statements under the assumption that the vector-valued objective function is D -quasiconvex and D -upper semicontinuous. Furthermore, we study special cases.

Keywords Vector optimization · Minimality · Weak duality · Strong duality

Mathematics Subject Classification Primary 90C29 · Secondary 90C26 · 90C46 · 26B25

1 Introduction

It is well-known that duality theory is a fundamental tool in various areas of mathematics. There are great advantages in including or using dual problems and duality

Dedicated to our dear friend and distinguished mathematician, Professor Marco López

✉ Juan Enrique Martínez-Legaz
juanenrique.martinez.legaz@uab.cat

¹ Department of Economics and Economic History, Autonomous University of Barcelona, Barcelona, Spain

² Bellaterra (Cerdanyola del Vallès), Spain

³ Institute of Mathematics, Martin-Luther-University Halle-Wittenberg, Halle, Germany

statements. Especially, solving the dual problem can be done using other methods of analysis or numerical mathematics. Furthermore, the objective function values of feasible elements of one problem generate bounds for the objective function values of the other problem. Duality statements are used to derive effective primal-dual algorithms to solve the given optimization problem. In addition, important interpretations of dual variables include the Lagrange method, saddle points, equilibrium points in game theory, shadow prices in economics, perturbation methods, and dual variational principles. In order to adequately exploit these advantages, a carefully and appropriately designed dual problem is required. Duality statements are particularly useful when the dual problem has a simpler structure than the primal problem.

As is known from scalar optimization, a (dual) maximization problem can be linked to a given (primal) minimization problem in such a way that, under suitable assumptions, there is no duality gap. Deriving duality statements is also of great interest in vector optimization for theoretical statements and for the development of numerical methods.

A difficult problem in deriving duality statements in vector optimization is the question of how to handle the “inner” optimization problem when using Lagrange techniques or conjugation. Naturally, one obtains a set of solutions of the “inner” optimization problem. The question of how to deal with the infimum and supremum of a subset of a partially ordered space is also very important. There are three main approaches that can be taken to overcome the difficulties that arise when extending duality theory from scalar to vector optimization (see Löhne and Tammer (2007)):

- The first approach is to employ a suitable scalarization in the formulation of the dual problem (see Schönfeld (1970); Breckner (1972); Jahn (1983, 1986, 2011); Boş et al. (2009); Göpfert et al. (2023); Günther et al. (2024a) and references therein). In this approach, scalarization concepts and well-known duality assertions from scalar optimization are applied to derive appropriate dual problems for the primal vector optimization problem and to show duality statements. With this approach, it is possible to derive useful duality assertions, optimality conditions, and primal-dual algorithms based on a scalarization.
- In compliance with the observation that a dual vector optimization problem is naturally set-valued, another group of dual problems is presented in the literature (see Tanino and Sawaragi (1979); Corley (1981); Tanino (1988, 1992); Luc (1989); Nakayama (1999); Tammer (1991); Dolecki and Malivert (1993); Pallaschke and Rolewicz (1997); Song (1998)), where the dual problem to the primal vector optimization problem is considered as set-valued optimization problem. Moreover, the paper by Dolecki and Malivert (1993) employs a set-valued approach in combination with Lagrangian techniques and perturbations of marginal relations to prove duality assertions for general vector optimization problems, where the solution concept is described by a transitive translation-invariant relation. Löhne and Tammer (2007) discussed embedding of the image space of the vector optimization problem into a complete lattice without linear structure, namely a sublattice of the power set of the image space where the primal and dual problems are set-valued.

In this approach, duality assertions are mostly formulated for solutions to set-valued optimization problems, which are more difficult to generate than solutions to vector optimization problems.

- A third approach to duality assertions in vector optimization, where the solution concept is formulated with respect to the supremum and infimum in the sense of a vector lattice, is studied in the literature. Pallaschke and Rolewicz developed a duality theory for objective functions with values in vector lattices using corresponding notions of infimum and supremum in the sense of utopia minimum (maximum) in Pallaschke and Rolewicz (1997). However, these infima and suprema may not be solutions in the sense of vector optimization (Pareto) because they may not belong to the image set of the objective function. In Pallaschke and Rolewicz (1997), notions of vector-convexity, vector-subgradients and vector duality analogously to the scalar case are studied taking into account the order in the vector lattice. Nieuwenhuis (1980) introduced solution concepts based on infimal (supremal) sets closely related to weakly (Pareto) efficient elements. The duality assertions for these solution concepts are presented in Nieuwenhuis (1980) and Tanino (1988, 1992). These concepts were extended by Dolecki and Malivert (1993) to infimal sets, which are also closely related to other kinds of efficiency. A consistent use of the lattice structure in Löhne and Tammer (2007) makes it possible to transfer the formulations, statements, and proof techniques from the scalar optimization theory.

In our paper, we follow the first approach.

We consider a primal vector optimization problem where the vector-valued objective function is D -quasiconvex and D -upper semicontinuous and the feasible set is a closed convex cone. The goal of this paper is to formulate a useful dual problem with a simple structure by exploiting the special structure of the primal problem and using a suitable nonlinear scalarization. The computation of the dual image set involves the minimization of a nonlinear scalarization of the primal objective function subject to only one linear inequality constraint. We prove (weak and strong) duality statements in our paper and study special cases.

To the best of our knowledge, duality for quasiconvex vector minimization over a closed convex cone has not been studied earlier, in spite of its potential applications to portfolio optimization (risk measures are often quasiconvex; see, e.g., the recent paper Bellini et al. (2026) and the references contained therein), among other fields. The scalar case was considered for the first time in Martínez-Legaz and Sosa (2022) as a natural extension of the classical duality for convex minimization over a closed convex cone in Rockafellar (1970, Theorem 31.4). The special structure of the problem we consider has the advantage of yielding a simple dual, involving only one linear inequality constraint, in sharp contrast with the classical vector dual problems. From a computational point of view, the easy handling of a single linear inequality is obvious. To derive weak and strong duality statements for quasiconvex vector optimization problems, we introduce a new concept of upper semicontinuity for a vector-valued function and compare it with the concept of upper semicontinuity for vector-valued functions by Corley (1980).

Our paper is organized as follows. In Sect. 2, we introduce the solution concepts for vector optimization problems. Nonlinear scalarization methods that are important for deriving our duality statements are recalled in Sect. 3. The relationships between vector optimization problems and the corresponding scalarized problems are described in Sect. 4. Our main results are derived in Sect. 5, where we show weak and strong (direct, converse) duality assertions for quasiconvex vector optimization problems. Special scalarization methods and corresponding duality assertions are discussed in Sect. 6. Finally, we give some conclusions in Sect. 7.

2 Preliminaries

In this section, we are going to take a look at some of the well-known solution concepts in vector optimization (see Jahn 2011, Chapter 4 or Khan et al. 2015, Section 2.4) for an overview).

Throughout this paper, let Y be a real topological vector space partially ordered by a pointed closed convex cone D with nonempty interior and $S \subseteq Y$; we recall that D being pointed means that $D \cap (-D) = \{0\}$, which is equivalent to saying that it contains no lines. It is well-known that D induces on Y a partial order relation $\leq_D$ defined, for any two points $y, \bar{y} \in Y$, by

$$y \leq_D \bar{y} \quad :\Longleftrightarrow \quad y \in \bar{y} - D.$$

For convenience in notation, we consider the binary relations $\leq_D^0$ and $<_D$ that are defined, for any two points $y, \bar{y} \in Y$, by

$$\begin{aligned} y \leq_D^0 \bar{y} &:\Longleftrightarrow y \in \bar{y} - (D \setminus \{0\}) \quad (\Longleftrightarrow y \leq_D \bar{y}, y \neq \bar{y}), \\ y <_D \bar{y} &:\Longleftrightarrow y \in \bar{y} - \text{int } D. \end{aligned}$$

Definition 1 (*Minimality*) A point $\bar{y} \in S$ is said to be a minimal element of S w.r.t. D if for any $y \in S$ condition $y \leq_D \bar{y}$ implies $y = \bar{y}$. The set of all minimal elements of S w.r.t. D is denoted by

$$\text{MIN}(S, D) := \{\bar{y} \in S \mid \forall y \in S : y \leq_D \bar{y} \Rightarrow y = \bar{y}\} = \{\bar{y} \in S \mid \nexists y \in S : y \leq_D^0 \bar{y}\}.$$

Definition 2 (*Weak minimality*) A point $\bar{y} \in S$ is said to be a weakly minimal element of S w.r.t. D if there is no $y \in S$ such that $y <_D \bar{y}$. The set of all weakly minimal elements of S w.r.t. D is denoted by

$$\text{WMIN}(S, D) := \{\bar{y} \in S \mid \nexists y \in S : y <_D \bar{y}\} = \{\bar{y} \in S \mid S \cap (\bar{y} - \text{int } D) = \emptyset\}.$$

In order to introduce the solution concept of (weakly) maximal elements of the dual vector optimization problem (see Sect. 5.1), we consider

$$\text{MAX}(R, D) := \text{MIN}(R, -D) \text{ and } \text{WMAX}(R, D) := \text{WMIN}(R, -D),$$

for a nonempty set $R \subset Y$.

We will show duality statements for weakly minimal elements of the primal problem and weakly maximal elements of the dual problem in Sect. 5.

3 Nonlinear scalarization

We are using nonlinear scalarization as the main tool for the construction of the dual problem to the nonconvex primal problem and for the proofs of the duality statements.

Suppose B is a proper closed subset of Y that contains the rays generated by $k \in Y \setminus \{0\}$, i.e.,

$$B + [0, +\infty)k \subseteq B. \quad (1)$$

We consider the nonlinear functional

$$\varphi_{-B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\}. \quad (2)$$

The following properties of the functional $\varphi := \varphi_{-B,k}$ are described in Göpfert et al. (2023, Theorem 2.4.1).

Theorem 3 *Let $B \subset Y$ be a closed proper set and $k \in Y \setminus \{0\}$ be such that (1) holds. Then φ is l.s.c., $\text{dom } \varphi = \mathbb{R}k - B$,*

$$\forall \lambda \in \mathbb{R} : \{y \in Y \mid \varphi(y) \leq \lambda\} = \lambda k - B, \quad (3)$$

and φ is translation invariant along k , i.e.,

$$\forall y \in Y, \forall \lambda \in \mathbb{R} : \varphi(y + \lambda k) = \varphi(y) + \lambda. \quad (4)$$

Moreover,

- (a) *φ is convex if and only if B is convex; $\varphi(\lambda y) = \lambda \varphi(y)$ for all $\lambda > 0$ and $y \in Y$ if and only if B is a cone.*
- (b) *φ is proper if and only if B does not contain lines parallel to k , i.e.,*

$$\forall y \in Y, \exists t \in \mathbb{R} : y + tk \notin B. \quad (5)$$

- (c) *φ is finite-valued if and only if B does not contain lines parallel to k and*

$$\mathbb{R}k - B = Y. \quad (6)$$

- (d) *Let $E \subset Y$; φ is E -monotone (i.e., $y_2 - y_1 \in E \Rightarrow \varphi(y_1) \leq \varphi(y_2)$) if and only if $B + E \subseteq B$.*

Suppose, furthermore, that

$$B + (0, +\infty)k \subseteq \text{int } B. \quad (7)$$

Then

(e) φ is continuous and

$$\forall \lambda \in \mathbb{R} : \{y \in Y \mid \varphi(y) < \lambda\} = \lambda k - \text{int } B, \quad (8)$$

$$\forall \lambda \in \mathbb{R} : \{y \in Y \mid \varphi(y) = \lambda\} = \lambda k - \text{bd } B. \quad (9)$$

(f) If φ is proper and $E \subset Y$, then

$$\varphi \text{ is } E\text{-monotone} \Leftrightarrow B + E \subseteq B \Leftrightarrow \text{bd } B + E \subseteq B.$$

Moreover, if φ is finite-valued, then φ is strictly E -monotone (i.e., $y_2 - y_1 \in E \setminus \{0\} \Rightarrow \varphi(y_1) < \varphi(y_2)$) $\Leftrightarrow B + (E \setminus \{0\}) \subseteq \text{int } B \Leftrightarrow \text{bd } B + (E \setminus \{0\}) \subseteq \text{int } B$.

4 Vector optimization problems, scalarized problems, and their relationships

Let Y be a real topological vector space, B a proper, closed subset of Y with nonempty interior, $S \subseteq Y$ and $y_0 \in S$. Concerning weakly minimal elements of S w.r.t. D introduced in Definition 2, we have the characterization by means of the functional $\varphi_{-B,k}$ in (2) with $-B := a - D$, where $D \subset Y$ is a pointed closed convex cone with nonempty interior, $k \in \text{int } D$ and $a \in Y$, given in Theorem 4 below.

We consider

$$\varphi_{a-D,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk + a - D\}. \quad (10)$$

Using the nonlinear functional in (10), we formulate the scalarized problem

$$\varphi_{a-D,k}(y) \longrightarrow \min_{y \in S}. \quad (P_{a-D,k,S})$$

Furthermore, we consider the

$$\begin{aligned} & t \longrightarrow \min \\ & \text{subject to} \\ & y \in tk + a - D \\ & y \in S, t \in \mathbb{R}. \end{aligned} \quad (11)$$

The equivalence of the scalarized problem $(P_{a-D,k,S})$ and the optimization problem (11) is shown in Tammer and Weidner (2020, Theorem 5.1.3(b)).

The assertions of the following theorem are derived in Tammer and Weidner (2020, Theorem 6.2.37).

Theorem 4 Assume that Y is a linear topological space, $S, D \subset Y$, D is a nontrivial, closed, convex, cone with nonempty interior and there exists $a \in Y$ such that $S \subseteq a + \text{int } D$. Then

$$\text{WMIN}(S, D) = \{y^0 \in S \mid \exists k \in \text{int } D : \varphi_{a-D,k}(y^0) = \min_{y \in S} \varphi_{a-D,k}(y)\}.$$

Furthermore,

$$\text{MIN}(S, D) = \{y^0 \in S \mid \exists k \in \text{int } D : y^0 \text{ is the unique minimizer of } \varphi_{a-D,k} \text{ over } S\}.$$

The functions $\varphi_{a-D,k}$ are finite-valued, convex and continuous.

5 Duality assertions for quasiconvex vector minimization problems

Given two real linear topological spaces X and Y , a nonempty feasible set $K \subseteq X$ supposed to be a closed convex cone, and a vector-valued objective function $f : X \rightarrow Y$, we consider the following primal vector optimization problem:

$$\begin{cases} f(x) \rightarrow \min \text{ w.r.t. } D \\ x \in K, \end{cases} \quad (\text{VP})$$

where the image space Y is partially ordered by a pointed, closed, convex, cone D with nonempty interior and there exists an element $a \in Y$ such that $f(K) \subseteq a + \text{int } D$.

In the study of the vector optimization problem (VP), we are looking for efficient solutions of (VP) belonging to the set

$$\text{Eff}(K, f, D) := f^{-1}(\text{MIN}(f(K), D))$$

and weakly efficient solutions of (VP) belonging to the set

$$\text{WEff}(K, f, D) := f^{-1}(\text{WMIN}(f(K), D)).$$

In the following, we denote the topological dual space of X by X^* with the duality product $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R}$. Furthermore, we denote the dual cone to the cone $K \subseteq X$ by $K^+ := \{x^* \in X^* \mid \forall x \in K : \langle x, x^* \rangle \geq 0\}$. For a function $f : X \rightarrow Y$ from X into another linear topological space Y and $y^* \in Y^*$, we will use the notation $\langle f, y^* \rangle := y^* \circ f$.

5.1 Primal and dual vector optimization problems

We suppose that the objective function $f : X \rightarrow Y$ acting between a linear topological space X and a linear topological space Y equipped with the cone $D \subset Y$ is quasiconvex in the sense of the following definition.

Definition 5 The function $f : X \rightarrow Y$ is called D -quasiconvex, if for any $y \in Y$, the set $f^{-1}(y - D)$ is convex.

Concerning the quasiconvexity of the composition $\varphi_{a-D,k} \circ f$, where $\varphi_{a-D,k}$ is given by (10), the following result by Günther et al. (2024b, Proposition 4.2) is important.

Proposition 6 Suppose that D is a proper, closed, convex cone in Y with nonempty interior and $k \in \text{int } D$. For any vector function $f : X \rightarrow Y$, the following assertions are equivalent:

1. f is D -quasiconvex.
2. For every $a \in Y$, the scalar function $\varphi_{a-D,k} \circ f : X \rightarrow \mathbb{R}$ is quasiconvex.

Furthermore, we assume the upper semicontinuity of the objective function f with respect to D . We will use the following definition.

Definition 7 We will say that $f : X \rightarrow Y$ is u.s.c. with respect to D (for short, D -uppsc) if $f^{-1}(a - \text{int } D)$ is open for every $a \in Y$.

In the case where D is a polyhedral cone, our definition of upper semicontinuity admits a nice characterization, as shown next.

Proposition 8 Suppose that D is a closed, convex cone in Y with nonempty interior, and let $f : X \rightarrow Y$. If f is D -uppsc, then $\langle f, y^* \rangle$ is upper semicontinuous for every $y^* \in D^+$. If, moreover, D is polyhedral, that is, there exist $y_1^*, \dots, y_p^* \in D^+ \setminus \{0\}$ such that

$$D = \{y \in Y : \langle y, y_i^* \rangle \geq 0 \text{ for every } i = 1, \dots, p\}, \quad (12)$$

then the converse also holds true; in fact, in this case, it is sufficient that $\langle f, y_i^* \rangle$ be upper semicontinuous for every $i = 1, \dots, p$. In particular, if $Y = \mathbb{R}^p$ and $D = \mathbb{R}_+^p$, then $f = (f_1, \dots, f_p)$ is D -uppsc if and only if f_i is u.s.c. for every $i = 1, \dots, p$.

Proof Assume that f is D -uppsc, and let $y^* \in D^+$ and $\gamma \in \mathbb{R}$. We will prove that

$$\langle f, y^* \rangle^{-1}((-\infty, \gamma)) = \bigcup_{\substack{a \in Y \\ \langle a, y^* \rangle = \gamma}} f^{-1}(a - \text{int } D). \quad (13)$$

The inclusion $\supseteq$ is immediate. To prove the converse inclusion, let $x \in \langle f, y^* \rangle^{-1}((-\infty, \gamma))$, and take $k \in \text{int } D$. We have $\langle k, y^* \rangle > 0$ and, since $\langle f(x), y^* \rangle < \gamma$, we may assume, without loss of generality, that

$$\langle k, y^* \rangle = \gamma - \langle f(x), y^* \rangle.$$

Then setting $a := f(x) + k$, we have $\langle a, y^* \rangle = \gamma$ and $x \in f^{-1}(a - \text{int } D)$. This proves the inclusion $\subseteq$. Since f is D -uppsc, the equality (13) shows that the set $\langle f, y^* \rangle^{-1}((-\infty, \gamma))$ is open, thus proving the upper semicontinuity of $\langle f, y^* \rangle$.

Suppose now that (12) holds for $y_1^*, \dots, y_p^* \in D^* \setminus \{0\}$ and $\langle f, y_i^* \rangle$ is upper semicontinuous for every $i = 1, \dots, p$, and let $a \in Y$. Using the equality $\text{int } D = \{y \in Y : \langle y, y_i^* \rangle > 0 \text{ for every } i = 1, \dots, p\}$, one can easily prove that $f^{-1}(a - \text{int } D) = \bigcap_{i=1}^p \langle f, y_i^* \rangle^{-1}((-\infty, \langle a, y_i^* \rangle))$, which, by the upper semicontinuity of the functions $\langle f, y_i^* \rangle$, shows that $f^{-1}(a - \text{int } D)$ is open. $\square$

An alternative definition of upper semicontinuity was proposed in Corley (1980), where $f : X \rightarrow Y$ is said to be D -u.s.c. if $f^{-1}(a + D)$ is closed for every $a \in Y$.

These two definitions are clearly equivalent in the scalar case. In the general vector-valued case, our definition is stronger when Y is locally convex, as the following proposition establishes.

Proposition 9 *Under the assumptions of Proposition 8, if Y is locally convex and $f : X \rightarrow Y$ is D -uppsc, then it is also D -u.s.c..*

Proof The local convexity of Y implies the equality

$$D = \{y \in Y : \langle y, y^* \rangle \geq 0 \text{ for every } y^* \in D^+\},$$

using which one can easily prove that, for every $a \in Y$, one has

$$f^{-1}(a + D) = \bigcap_{y^* \in D^+} \langle f, y^* \rangle^{-1}([\langle a, y^* \rangle, +\infty)).$$

Since, by Proposition 8, the functions $\langle f, y^* \rangle$ are upper semicontinuous, this equality shows that $f^{-1}(a + D)$ is closed, thus proving that $f : X \rightarrow Y$ is D -u.s.c.. $\square$

However, in the vector-valued case, these two definitions of upper semicontinuity are not equivalent, as illustrated in the following example.

Example 10 Let $Y = \mathbb{R}^2$, $D = \mathbb{R}_+^2$, and define $f = (f_1, f_2) : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$f_1(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0, \end{cases}$$

$$f_2(x) := \begin{cases} 0 & \text{if } x = 0, \\ \ln |x| & \text{if } x \neq 0. \end{cases}$$

For $a = (a_1, a_2) \in \mathbb{R}^2$, one has

$$f^{-1}(a + \mathbb{R}_+^2) = \begin{cases} \emptyset & \text{if } a_1 > 0, \\ (-\infty, -e^{a_2}] \cup [e^{a_2}, +\infty) \cup \{0\} & \text{if } a_1 \leq 0 \text{ and } a_2 \leq 0, \\ (-\infty, -e^{a_2}] \cup [e^{a_2}, +\infty) & \text{in all other cases.} \end{cases}$$

Thus, f is upper semicontinuous in the sense of Corley. However, since f_1 is not upper semicontinuous, in view of Proposition 8, we see that f is not D -uppsc.

Proposition 11 *Consider $k \in \text{int } D$ and $a \in Y$. If $f : X \rightarrow Y$ is D -uppsc, then $\varphi_{a-D,k} \circ f$ is upper semicontinuous.*

Proof Let $\lambda \in \mathbb{R}$. We have

$$\{x \in X \mid (\varphi_{a-D,k} \circ f)(x) < \lambda\} = f^{-1}(\{y \in Y \mid \varphi_{a-D,k}(y) < \lambda\}), \quad (14)$$

From Theorem 3(e), we get

$$\{y \in Y \mid \varphi_{a-D,k}(y) < \lambda\} = \lambda k + a - \text{int } D. \quad (15)$$

Taking into account (14) and (15), we obtain

$$\{x \in X \mid (\varphi_{a-D,k} \circ f)(x) < \lambda\} = f^{-1}(\lambda k + a - \text{int } D),$$

which, since f is supposed to be D -uppsc, shows that the sets (14) are open, thus proving that $\varphi_{a-D,k} \circ f$ is upper semicontinuous. $\square$

Remark 12 It is possible to relax the assumptions of Proposition 11 by replacing the assumption " $f : X \rightarrow Y$ is D -uppsc" by the weaker condition " $f^{-1}(\lambda k + a - \text{int } D)$ is open for all $\lambda \in \mathbb{R}$ ".

In the following, we consider (VP) with $f : X \rightarrow Y$ being D -quasiconvex and D -uppsc. The **dual vector problem** to (VP) is defined as follows:

$$\begin{cases} y \rightarrow \max \text{ w.r.t. } D \\ y \in \Psi, \end{cases} \quad (\text{VD})$$

where $\Psi \subset Y$ is the dual feasible set. More precisely, we are interested in the following instance of (VD) with $\Psi := \Psi^1$:

Compute an element of $\text{MAX}(\Psi^1, D)$ or $\text{WMAX}(\Psi^1, D)$, where

$$\Psi^1 := \{y \in Y \mid \exists x^* \in K^+, \exists k \in \text{int } D : \inf_{x \in X, \langle x, x^* \rangle \geq 0} \varphi_{a-D,k}(f(x)) \geq \varphi_{a-D,k}(y)\}$$

and $\text{MAX}(\Psi^1, D) = \text{MIN}(\Psi^1, -D)$, $\text{WMAX}(\Psi^1, D) = \text{WMIN}(\Psi^1, -D)$.

Remark 13 The nonlinear functional $\varphi_{a-D,k}$ given by (10) is involved in the construction of the dual image set Ψ^1 . For deriving the duality statements, the properties of $\varphi_{a-D,k}$ (see Theorem 3) are essential.

The choice of the dual set Ψ^1 is motivated by the surrogate duality approach to the scalar case presented in Martínez-Legaz and Sosa (2022), in which the dual objective function incorporates the simple linear inequality constraint $\langle x, x^* \rangle \geq 0$. This constraint is a relaxation of the original constraint $x \in K$, due to the condition $x^* \in K^+$. This is a typical situation in quasiconvex duality; in contrast with convex duality, in which the constraints are incorporated to the objective function using Lagrange multipliers, in the quasiconvex case the multipliers do not yield an unconstrained dual objective function but replace the original constraints with a nonnegative linear combination of them. Thus, in the case of a scalar quasiconvex minimization problem over a closed convex cone K , the optimal multipliers $x^* \in K^+$ represent a homogeneous

halfspace, defined by the linear inequality $\langle x, x^* \rangle \geq 0$, which contains the original constraint set K , and are such that, under the relaxation $\langle x, x^* \rangle \geq 0$, the optimal value remains the same. Note that the condition $x \in K$ can be written equivalently as " $\langle x, x^* \rangle \geq 0$ for every $x^* \in K^+$ ", so that, in this case, the surrogate duality method shows that, under suitable assumptions, infinitely many homogeneous linear inequality constraints may be relaxed by replacing them with a single one. It is worth noticing that no convexity, but just quasiconvexity, of the objective function is required for this approach to work.

5.2 Duality statements for the scalarized primal and dual problem

Furthermore, we study the following scalarizations of (VP) and (VD):

$$\min_{x \in K} \varphi_{a-D,k}(f(x)), \quad (P_{a-D,k,f(K)})$$

where $a \in Y$ and $k \in \text{int } D$. The surrogate dual problem to $(P_{a-D,k,f(K)})$ in the sense of Greenberg and Pierskalla (1970) is

$$\max_{x^* \in K^+} \inf_{x \in X, \langle x, x^* \rangle \geq 0} \varphi_{a-D,k}(f(x)). \quad (D_{a-D,k,f(K)})$$

Using Propositions 6 and 11, from Martínez-Legaz and Sosa (2022, Thms. 9 and 10), one obtains the following results.

Theorem 14 *If $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc, $k \in \text{int } D$ and $a \in Y$, then there is no duality gap between $(P_{a-D,k,f(K)})$ and $(D_{a-D,k,f(K)})$ and $(D_{a-D,k,f(K)})$ has an optimal solution.*

Theorem 15 *Let $k \in \text{int } D$, $a \in Y$, x_0 be a solution to $(P_{a-D,k,f(K)})$, and $x^* \in K^+$. If $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc, then the following equivalence holds true:*

$$\begin{aligned} x^* \text{ is an optimal solution to } (D_{a-D,k,f(K)}) \\ \Leftrightarrow \\ \varphi_{a-D,k}(f(x_0)) = \min_{x \in X, \langle x, x^* \rangle \geq 0} \varphi_{a-D,k}(f(x)). \end{aligned}$$

5.3 Duality assertions for weakly minimal elements

From Theorems 4, 14, and 15, we obtain the following result under the assumption that the objective function is D -quasiconvex and D -uppsc.

Theorem 16 *Let $x_0 \in K$ and suppose that there exists $a \in Y$ such that $f(K) \subseteq a + \text{int } D$. If $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc, then the following conditions are equivalent:*

- (1) x_0 is a solution to $\text{WMIN}(f(K), D)$,
- (2) $\exists x^* \in K^+$ such that x_0 is a weakly minimal solution to

$$\text{WMIN}(f(\tilde{K}), D),$$

where $\tilde{K} := \{x \in X \mid \langle x, x^* \rangle \geq 0\}$.

Theorem 17 (Weak duality) *Consider the pair of dual vector optimization problems (VP) and (VD) with $\Psi := \Psi^1$ and suppose that $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc. Then the weak duality condition holds:*

$$f(K) \cap (\Psi^1 - \text{int } D) = \emptyset. \quad (16)$$

Proof Let y be an arbitrary element of Ψ^1 with the corresponding $x^* \in K^+$ and $k \in \text{int } D$. Then we obtain

$$\varphi_{a-D,k}(y) \leq \inf_{x \in X, \langle x, x^* \rangle \geq 0} \varphi_{a-D,k}(f(x)) \quad (17)$$

$$\leq \varphi_{a-D,k}(f(x)) \text{ for all } x \in X \text{ with } \langle x, x^* \rangle \geq 0, \quad (18)$$

i.e.,

$$\varphi_{a-D,k}(y) \leq \varphi_{a-D,k}(f(x)) \text{ for all } x \in K. \quad (19)$$

Suppose that (16) does not hold. Then there are $\bar{x} \in K$ and $\bar{y} \in \Psi^1$ with the corresponding $\bar{x}^* \in K^+$ and $\bar{k} \in \text{int } D$ such that $f(\bar{x}) \in \bar{y} - \text{int } D$. Taking into account Theorem 3(f), $\varphi_{a-D,\bar{k}}$ is strictly $\text{int } D$ -monotone. This yields

$$\varphi_{a-D,\bar{k}}(f(\bar{x})) < \varphi_{a-D,\bar{k}}(\bar{y}),$$

a contradiction to (19). $\square$

Theorem 18 (Strong (direct) duality) *Consider the problems (VP) and (VD) with $\Psi := \Psi^1$ and assume that $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc.*

If x_0 is a weakly efficient solution of (VP), i.e., $f(x_0) \in \text{WMIN}(f(K), D)$, then $f(x_0)$ is weakly maximal for (VD), i.e., $f(x_0) \in \text{WMAX}(\Psi^1, D)$.

Proof Let x_0 be a weakly efficient solution of (VP), i.e., $f(x_0) \in \text{WMIN}(f(K), D)$. According to our assumptions, there exists an element $a \in Y$ such that $f(K) \subseteq a + \text{int } D$. Then from Theorem 4, we find that there is an element $k \in \text{int } D$ such that x_0 solves the scalar problem $(P_{a-D,k}, f(K))$. Taking into account Theorems 14 and 15, we obtain

$$\begin{aligned} \varphi_{a-D,k}(f(x_0)) &= \max_{x^* \in K^+} \inf_{x \in X, \langle x, x^* \rangle \geq 0} \varphi_{a-D,k}(f(x)) \\ &= \inf_{x \in X, \langle x, x_0^* \rangle \geq 0} \varphi_{a-D,k}(f(x)) \end{aligned}$$

for some $x_0^* \in K^+$.

This yields $f(x_0) \in \Psi^1$ with the corresponding $x_0^* \in K^+$ and $k \in \text{int } D$. From the weak duality assertion in Theorem 17, we can conclude that $f(x_0)$ is a weakly maximal element for Ψ^1 , i.e., $f(x_0) \in \text{WMAX}(\Psi^1, D)$. $\square$

6 Special scalarization

In this section, we study well-known special cases of the functional $\varphi_{a-D,k}$ in (10) involved in the formulation of the dual problem (VD), namely the norm introduced in Jahn (1987) and the Hiriart–Urruty functional introduced in Hiriart–Urruty (1979). Using the general duality statements in Sect. 5, we can derive the corresponding duality statements for these special cases. Our goal is to create a straightforward and practical structure for the dual problem, which could be helpful in solving it more efficiently.

6.1 Norm scalarization

The first goal is to construct the dual problem to the primal problem (VP) using norm scalarization. This is possible because, under certain conditions, the functional $\varphi_{a-D,k}$ in (10) coincides with specific Minkowski functionals and norms.

For $k \in \text{int } D$ and $y \in Y$, we define (see Jahn 1987) the following norm:

$$\|y\|_{D,k} := \inf \left\{ \lambda > 0 : \frac{1}{\lambda} y \in (-k + D) \cap (k - D) \right\}. \quad (20)$$

Remark 19 If $Y = \mathbb{R}^n$, $D = \mathbb{R}_+^n$ and $k = (k_1, \dots, k_n) \in \text{int } \mathbb{R}_+^n$, then $\|y\|_{D,k} = \max_{1 \leq i \leq n} \left| \frac{y_i}{k_i} \right|$ (see Tammer and Weidner 2020, Remark 9.1.9).

The relationship between the scalarized problem $(P_{a-D,k,S})$ and a scalarization by the norm $\|\cdot\|_{D,k}$ in (20) is given in the following proposition as a special case of (Tammer and Weidner, 2020, Proposition 5.5.2).

Proposition 20 Assume that Y is a linear topological space, $S, D \subset Y$, D is a nontrivial, closed, convex, cone with nonempty interior and $k \in \text{int } D$. Let the norm $\|\cdot\|_{D,k}$ be given by (20). Then,

$$\|y - a\|_{D,k} = \varphi_{a-D,k}(y) \text{ if } y \in a + D.$$

Suppose that there exists $a \in Y$ such that $S \subseteq a + \text{int } D$. Then problem

$$\min_{y \in S} \|y - a\|_{D,k}$$

is equivalent to the problem $(P_{a-D,k,S})$.

We obtain the following result as a special case of Theorem 4.

Theorem 21 (Jahn 1987, Thm. 2.1). If there exists $a \in Y$ such that $S \subseteq a + \text{int } D$, for $y_0 \in S$ the following equivalence holds:

$$\begin{aligned} & y_0 \text{ is a weakly minimal element of } S \\ & \Leftrightarrow \\ & \exists k \in \text{int } D \text{ such that } \|y_0 - a\|_{D,k} = \min_{y \in S} \|y - a\|_{D,k}. \end{aligned}$$

Taking into account Proposition 20, we consider the scalarized primal problem

$$\min_{x \in K} \|f(x) - a\|_{D,k}, \quad (P_{\|\cdot\|_{D,k}, f(K)})$$

where $a \in Y$ and $k \in \text{int } D$.

We get the following properties of the objective function in problem $(P_{\|\cdot\|_{D,k}, f(K)})$ from the assertions concerning the general functional $\varphi_{a-D,k}$ derived in Sect. 5.1.

Corollary 22 *Let $k \in \text{int } D$ and $a \in Y$ be such that $S \subseteq a + \text{int } D$. If f is D -quasiconvex, then $\|f - a\|_{D,k}$ is quasiconvex.*

Proof The quasiconvexity of $\|f - a\|_{D,k}$ follows from Proposition 6 and Proposition 20. $\square$

Corollary 23 *Let $k \in \text{int } D$ and $a \in Y$ be such that $S \subseteq a + \text{int } D$. If f is D -uppsc, then $\|f - a\|_{D,k}$ is u.s.c..*

Proof The upper semicontinuity of $\|f - a\|_{D,k}$ follows from Proposition 11 and Proposition 20. $\square$

We now consider the scalarized primal problem $(P_{\|\cdot\|_{D,k}, f(K)})$ under the assumptions that $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc. The corresponding surrogate dual problem to $(P_{\|\cdot\|_{D,k}, f(K)})$ in the sense of Martínez-Legaz and Sosa (2022) is

$$\max_{x^* \in K^+} \inf_{x \in X, \langle x, x^* \rangle \geq 0} \|f(x) - a\|_{D,k}. \quad (D_{\|\cdot\|_{D,k}, f(K)})$$

Using the general results derived in Sect. 5.2, we obtain the following duality statements for the problems $(P_{\|\cdot\|_{D,k}, f(K)})$ and $(D_{\|\cdot\|_{D,k}, f(K)})$.

Corollary 24 *Suppose $f(K) \subseteq a + \text{int } D$ for some $a \in Y$. If $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc, then there is no duality gap between $(P_{\|\cdot\|_{D,k}, f(K)})$ and $(D_{\|\cdot\|_{D,k}, f(K)})$ and $(D_{\|\cdot\|_{D,k}, f(K)})$ has an optimal solution.*

Proof The assertion follows from Theorem 14 and Proposition 20. $\square$

Corollary 25 *Suppose $f(K) \subseteq a + \text{int } D$ for some $a \in Y$. Let $k \in \text{int } D$, x_0 be a solution to $(P_{\|\cdot\|_{D,k}, f(K)})$ and $x^* \in K^+$. If $f : X \rightarrow Y$ is D -quasiconvex and D -uppsc, then the following equivalence holds true:*

$$\begin{aligned} x^* \text{ is an optimal solution to } (D_{\|\cdot\|_{D,k}, f(K)}) \\ \Leftrightarrow \\ \|f(x_0) - a\|_{D,k} = \min_{\langle x, x^* \rangle \geq 0} \|f(x) - a\|_{D,k}. \end{aligned}$$

Proof We obtain the assertion from Theorem 15 and Proposition 20. $\square$

Using these results for the scalarized problems $(P_{\|\cdot\|_{D,k}, f(K)})$ and $(D_{\|\cdot\|_{D,k}, f(K)})$, we can apply the duality assertions in Theorems 17 and 18 derived in Sect. 5.3 to the primal problem (VP) and the dual problem

$$\begin{cases} y \rightarrow \max \text{ w.r.t. } D \\ y \in \Psi_{\|\cdot\|_{D,k}}, \end{cases}$$

with the dual image set given by

$$\Psi_{\|\cdot\|_{D,k}} := \left\{ y \in Y \mid \exists x^* \in K^+, \exists k \in \text{int } D : \inf_{x \in X, \langle x, x^* \rangle \geq 0} \|f(x) - a\|_{D,k} \geq \|y - a\|_{D,k} \right\}.$$

With regard to the aim of this paper, which is to obtain structures for the dual problem that are as simple and manageable as possible, we now return to the finite-dimensional case $Y = \mathbb{R}^n$.

We consider the special case of the primal problem (VP) where $Y = \mathbb{R}^n$, $D = \mathbb{R}_+^n$, $k \in \text{int } \mathbb{R}_+^n$, $f = (f_1, \dots, f_n)$, and $a \in \mathbb{R}^n$ is such that $f(K) \subseteq a + \text{int } \mathbb{R}_+^n$. We denote this special case of the primal problem (VP) by $(VP_{\mathbb{R}_+^n})$.

We consider the scalarized problem of the form $(P_{\|\cdot\|_{D,k}, f(K)})$ to $(VP_{\mathbb{R}_+^n})$

$$\min_{x \in K} \max_{1 \leq i \leq n} \frac{f_i(x) - a_i}{k_i}, \quad (21)$$

where $a \in \mathbb{R}^n$ and $k \in \text{int } \mathbb{R}_+^n$.

Taking into account Remark 19, we obtain the corresponding dual problem to $(VP_{\mathbb{R}_+^n})$ in a very useful structure.

$$\begin{cases} y \rightarrow \max \text{ w.r.t. } \mathbb{R}_+^n \\ y \in \Psi_{\|\cdot\|_{\mathbb{R}_+^n, k}}, \end{cases}$$

with the dual image set given by

$$\Psi_{\|\cdot\|_{\mathbb{R}_+^n, k}} := \left\{ y \in \mathbb{R}^n \mid \exists x^* \in K^+, \exists k \in \text{int } \mathbb{R}_+^n : \inf_{x \in X, \langle x, x^* \rangle \geq 0} \max_{1 \leq i \leq n} \left| \frac{f_i(x) - a_i}{k_i} \right| \geq \max_{1 \leq i \leq n} \left| \frac{y_i - a_i}{k_i} \right| \right\}.$$

The corresponding surrogate dual problem to (21) in the sense of Martínez-Legaz and Sosa (2022) is

$$\max_{x^* \in K^+} \inf_{x \in X, \langle x, x^* \rangle \geq 0} \max_{1 \leq i \leq n} \left| \frac{f_i(x) - a_i}{k_i} \right|. \quad (22)$$

Remark 26 Using our approach to duality, we have thus obtained a dual problem to the vector optimization problem $(VP_{\mathbb{R}^n})$ that is highly amenable to numerical methods.

6.2 Scalarization using the Hiriart–Urruty functional

Now, we consider the Hiriart–Urruty functional (see Hiriart–Urruty (1979)) to derive the dual problem to (VP).

Definition 27 (Hiriart–Urruty 1979) Let Y be a normed space, $d : Y \times Y \rightarrow \mathbb{R}$ the distance function in Y and $S \subseteq Y$. The **Hiriart–Urruty (or oriented distance) function** $\Delta_S(\cdot) : Y \rightarrow \overline{\mathbb{R}}$ is defined by

$$\forall y \in Y : \Delta_S(y) := d(y, S) - d(y, (Y \setminus S)). \quad (23)$$

Remark 28 Under the assumption that Y is equipped with the norm $\|\cdot\|_{D,k}$ (see (20)) given by the Minkowski function of the order interval $[-k, k]_D$ as

$$\|y\|_{D,k} = \inf \left\{ \lambda > 0 : \frac{1}{\lambda} y \in (-k + D) \cap (k - D) \right\}, \quad (24)$$

where $k \in \text{int } D$, the function $\varphi_{-D,k}$ in (10) coincides with the function $\Delta_{-D,k} := \Delta_{-D}$ in (23) corresponding to the distance function d induced by the norm $\|\cdot\|_{D,k}$ in (24), see Tammer and Weidner (2020, Theorem 9.3.5). This means that we have

$$\varphi_{-D,k}(y) = \Delta_{-D,k}(y)$$

for all $y \in Y$.

The corresponding dual problem to (VP) is given by

$$\begin{cases} y \rightarrow \max \text{ w.r.t. } D \\ y \in \Psi_{\Delta_{-D,k}}, \end{cases} \quad (25)$$

with respect to the dual image set

$$\Psi_{\Delta_{-D,k}} := \left\{ y \in Y \mid \exists x^* \in K^+, \exists k \in \text{int } D : \inf_{x \in X, \langle x, x^* \rangle \geq 0} \Delta_{-D,k}(f(x)) \geq \Delta_{-D,k}(y) \right\}.$$

Under the assumption $f(K) \subseteq \text{int } D$, we can employ the duality assertions derived in Sect. 5.3 for the primal problem (VP) and the dual problem (25).

7 Conclusion

In our paper, we derived new duality statements for weakly minimal elements of quasiconvex vector minimization problems using the special structure of the primal problem. For the quasiconvex primal vector optimization problem, it was possible to derive a useful dual problem, where the computation of the dual image set involves the minimization of a nonlinear scalarization of the primal vector-valued objective function subject to only one linear inequality constraint (Theorem 16). We have shown

corresponding weak and strong duality assertions (Theorems 17 and 18, respectively), thus extending the surrogate duality theory for scalar quasiconvex minimization problems over closed convex sets established in Martínez-Legaz and Sosa (2022) to the vector setting.

A limitation of our approach stems from the fact that it relies upon the existence of an ideal point, that is, an element $a \in Y$ such that $f(K) \subseteq a + \text{int } D$. Even though this assumption holds in many cases (for instance, it can be easily proved that it holds when f is continuous, K is compact and $D \supseteq \mathbb{R}_+^n$), it may be worth investigating if it can be relaxed, but this would require a rather different approach.

For further research, it is interesting to prove a converse strong duality statement, showing how every weakly maximal element of the dual problem yields a weakly minimal element of the primal. Furthermore, duality statements for properly minimal elements of the primal quasiconvex vector-valued minimum problem and maximal elements of a suitably constructed dual problem are of interest.

It should be investigated how the duality statements proven in our paper can be used to derive primal-dual algorithms for solving quasiconvex vector optimization problems, especially to get estimates of lower bounds for the primal problem and stopping criteria in primal-dual algorithms. This is a promising avenue given the simple formulation of the dual objective function, involving only a linear constraint.

Another potential field for future research is portfolio optimization, since, as we have mentioned in the Introduction, risk measures are often quasiconvex.

Acknowledgements Juan Enrique Martínez-Legaz has been partially supported by Grant PID2022-136399NB-C22 from the Ministerio de Ciencia e Innovación, Spain, and the European Regional Development Fund, “A way to make Europe”, European Union. We would like to thank the reviewers for carefully reading our manuscript.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability There are no data supporting the results reported in this article.

Code availability No code is available.

Declarations

Conflict of interest Neither financial nor non-financial interests are neither directly nor indirectly related to this work.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

Informed consent This article does not contain any studies with human participants performed by any of the authors.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted

by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Bellini F, Huang M, Wang Q, Wang R (2026) Lambda expected shortfall. [arXiv:2512.23139](https://arxiv.org/abs/2512.23139)
- Boţ RI, Grad SM, Wanka G (2009) Duality in vector optimization. Springer, Berlin, Heidelberg
- Breckner WW (1972) Dualität bei Optimierungsaufgaben in halbgeordneten topologischen Vektorräumen I. *Math Rev Anal Numér Théor Approx* 1:5–35
- Corley HW (1980) An existence result for maximizations with respect to cones. *J Optim Theory Appl* 31:277–281
- Corley HW (1981) Duality theory for maximizations with respect to cones. *J Math Anal Appl* 84:560–568
- Dolecki S, Malivert C (1993) General duality in vector optimization. *Optimization* 27:97–119
- Göpfert A, Riahi H, Tammer C, Zălinescu C (2023) Variational methods in partially ordered spaces, 2nd edn. Springer, New York
- Greenberg HJ, Pierskalla WP (1970) Surrogate mathematical programming. *Oper Res* 18:924–939
- Günther C, Khazayel B, Tammer C (2024a) Duality assertions in vector optimization with respect to relatively solid convex cones in real linear spaces. *Minimax Theory Appl* 9:225–252
- Günther C, Orzan A, Popovici N (2024b) A new concept of semistrict quasiconvexity for vector functions. *Optimization* 74:3573–3601
- Hiriart-Urruty JB (1979) New concepts in nondifferentiable programming. *Bull Soc Math France* 60:57–85
- Jahn J (1983) Duality in vector optimization. *Math Progr* 25:343–353
- Jahn J (1986) Mathematical vector optimization in partially ordered linear spaces. Peter Lang, Frankfurt am Main
- Jahn J (1987) Parametric approximation problems arising in vector optimization. *J Optim Theory Appl* 54:503–516
- Jahn J (2011) Vector optimization: theory, applications, and extensions. Springer, Berlin, Heidelberg
- Khan AA, Tammer C, Zălinescu C (2015) Set-valued optimization: an introduction with applications. Springer, Berlin, Heidelberg
- Löhne A, Tammer C (2007) A new approach to duality in vector optimization. *Optimization* 56:221–239
- Luc DT (1989) Theory of vector optimization. Lecture notes in economics and mathematical systems, vol 319. Springer, Berlin
- Martínez-Legaz JE, Sosa W (2022) Duality for quasiconvex minimization over closed convex cones. *Optimiz Lett* 16:1337–1352
- Nakayama H (1999) Duality in multi-objective optimization. In: Gal T, Stewart TJ, Hanne T (eds) Multicriteria decision making. Kluwer Academic Publishers, Boston/Dordrecht/London, pp 3.1–3.29
- Nieuwenhuis JW (1980) Supremal points and generalized duality. *Math Oper Forsch Stat Ser Optimiz* 11:41–59
- Pallaschke D, Rolewicz S (1997) Foundations of mathematical optimization: convex analysis without linearity. Kluwer Academic Publishers, Dordrecht
- Rockafellar RT (1970) Convex analysis. Princeton University Press, Princeton
- Schönfeld P (1970) Some duality theorems for the nonlinear vector maximum problem. *Unternehmensforschung* 14:51–63
- Song W (1998) A generalization of Fenchel duality in set-valued vector optimization. *Math Methods Oper Res* 48:259–272
- Tammer C (1991) Lagrange-Dualität in der Vektroptimierung. *Wiss Z Tech Hochsch Ilmenau* 37:71–88
- Tammer C, Weidner P (2020) Scalarization and separation by translation invariant functions. Springer, Cham
- Tanino T (1988) On the supremum of a set in a multidimensional space. *J Math Anal Appl* 130:386–397
- Tanino T (1992) Conjugate duality in vector optimization. *J Math Anal Appl* 167(1):84–97
- Tanino T, Sawaragi Y (1979) Duality theory in multiobjective programming. *J Optim Theory Appl* 27:509–529

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.