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L^p-solvability of boundary value problems for the Laplacian in locally flat unbounded domains

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Abstract

We establish the solvability of the L^p-Dirichlet and $L^{p'}$-Neumann problems for the Laplacian for $p \in (\frac{n}{n-1} - \varepsilon, \frac{2n}{n-1}]$ for some $\varepsilon > 0$ in 2-sided chord-arc domains with unbounded boundary that is sufficiently flat at large scales and outward unit normal vector whose oscillation fails to be small only at finitely many dyadic boundary balls.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an ADR domain (see Section 2.2), and denote its surface measure by

$$\sigma = \sigma_\Omega := \mathcal{H}^n|_{\partial\Omega}.$$

For $1 < p < \infty$, we say that the Dirichlet problem is solvable in L^p (denoted as (D_p) is solvable) in Ω if there is $C_{D_p} \geq 1$ such that for any given $f \in L^p(\sigma)$, there exists $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}u \in L^p(\sigma), \\ u|^{\rm nt}_{\partial\Omega} = f, \ \sigma\text{-a.e.}, \end{cases} \tag{1.1}$$

and

$$\|\mathcal{N}u\|_{L^p(\sigma)} \leq C_{D_p}\|f\|_{L^p(\sigma)}. \tag{1.2}$$

Here $\mathcal{N}$ is the nontangential operator (see (2.2)) and $u|^{\rm nt}_{\partial\Omega}$ is the nontangential limit (see (2.4)). We say that the Neumann problem is solvable in L^p (denoted as (N_p) is solvable) in Ω if there is $C_{N_p} \geq 1$ such that for any given $f \in L^p(\sigma)$, there exists $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\sigma), \\ \partial_{\nu_\Omega} u = f, \ \sigma\text{-a.e.}, \end{cases} \tag{1.3}$$

and

$$\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \leq C_{N_p} \|f\|_{L^p(\sigma)}. \tag{1.4}$$

Here ∂_{ν_Ω} is the interior nontangential derivative (see (2.5)).

We note that the definition of solvability for the Dirichlet and Neumann problems may vary slightly across the literature. We will not distinguish between these variations in the subsequent articles mentioned in the introduction. Furthermore, while several of the results below were originally established for more general divergence-form operators in their respective papers, we restrict our historical overview to the harmonic case.

In ADR domains, it is well known that (D_p) is solvable for some $1 < p < \infty$ if and only if the harmonic measure ω is locally in weak-$A_\infty(\sigma)$. Specifically, there exist constants $C, s > 0$ such that for every ball $B = B(x, r)$ with $x \in \partial\Omega$ and $r < \operatorname{diam}(\partial\Omega)/4$, there holds $\omega^p(E) \leq C(\sigma(E)/\sigma(B))^s \omega^p(2B)$ for any $p \in \Omega \setminus 4B$ and any Borel set $E \subset B$. We refer the reader to [17,19]. For ADR domains satisfying the corkscrew condition (see Definition 2.2), (D_p) is solvable if and only if the harmonic measure satisfies a weak reverse Hölder inequality with exponent $p' := p/(p-1)$ (the Hölder conjugate of p), see, for instance, [40, Theorem 9.2], [39, Proposition 2.14], and the references therein. Consequently, if (D_p) is solvable for some $1 < p < \infty$, then Gehring's lemma guarantees the existence of $\varepsilon > 0$ such that (D_q) is solvable for all $p - \varepsilon \leq q < \infty$.

The solvability of the Dirichlet and Neumann problems is a long-standing and active area of research. In 1963, Lavrent'ev [31] showed that in bounded planar simply connected chord-arc domains (see Definition 2.7), the harmonic measure is locally in weak-$A_\infty(\sigma)$, implying the solvability of (D_p) for some $1 < p < \infty$. However, Jerison [21] later proved in 1983 that for every $1 < p < \infty$, there exists a planar chord-arc domain where (D_p) fails to be solvable. A significant breakthrough came in 1977 with Dahlberg [3], who proved that in any bounded Lipschitz domain Ω, its harmonic measure satisfies a reverse Hölder inequality with exponent 2. Consequently, there exists $\varepsilon_\Omega >$ such that (D_p) is solvable for all $2 - \varepsilon_\Omega \leq p < \infty$. This result is sharp: for every $\varepsilon > 0$ there is a Lipschitz domain Ω_ε where $(D_{2-\varepsilon})$ is not solvable, see [26, pp. 153-54]. In 1978, Fabes, Jodeit and Rivière [12] employed double and single layer potentials (see Sections 3 and 7.1) to establish the solvability of both (D_p) and (N_p) for all $1 < p < \infty$ in bounded C^1 domains. Notably, Dahlberg [4] had already proven the solvability of (D_p) for all $1 < p < \infty$ in bounded C^1 domains without using layer potentials, though this was published later in 1979.[1] Subsequent work by Jerison and Kenig [22,23] simplified the proofs of Dahlberg's results through the application of the so-called Rellich identity [43]. Furthermore, in [24], they resolved (N_2) in bounded Lipschitz domains.

Briefly speaking, the layer potential approach relies on the invertibility of the operators $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ (here K is the double layer potential and K^* is its adjoint) and gives an "explicit" solution to the Dirichlet and Neumann problems, respectively. In $C^{1,\alpha}$ domains, $\alpha > 0$, it is not difficult to see that the double layer potential is compact in $L^p(\sigma)$ and (by Fredholm theory) that those operators are invertible, for all $1 < p < \infty$. For a proof of this, see the lecture notes [7,26]. While this argument does not directly extend to C^1 domains, Fabes, Jodeit and Rivière [12] succeeded in showing that in bounded C^1 domains, the double layer potential K is compact in $L^p(\sigma)$, and both $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ are invertible in $L^p(\sigma)$ for

[1] Fabes, Jodeit and Rivière cite a technical report of [4] in [12].

all $1 < p < \infty$. For Lipschitz domains, however, the compactness of the double layer potential generally fails, as shown by Fabes, Jodeit and Lewis in [11]. Nevertheless, using the Rellich identity mentioned above, Verchota [48] established in 1984 the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ in $L^2(\sigma)$, thereby recovering the solvability of (D_2) and (N_2) for bounded Lipschitz domains, as originally shown in [3,24] respectively.

In 1987, Dahlberg and Kenig [8] established that for every bounded Lipschitz domain Ω, there exists $\varepsilon_\Omega > 0$ such that (N_p) is solvable for all $1 < p \leq 2 + \varepsilon_\Omega$, and in fact, they showed that its solution, as well as the solution of $(D_{p'})$ in Dahlberg's result [3], can be obtained using the method of layer potentials. As in the Dirichlet problem, this range of solvability is sharp: for every $\varepsilon > 0$, there exists a Lipschitz domain Ω_ε for which $(N_{2+\varepsilon})$ fails to be solvable, see [26, pp. 153-54].

Dahlberg's result [3] was extended to chord-arc domains (see Definition 2.7) independently by David and Jerison [6], and Semmes [44] in 1990. They proved that for any bounded chord-arc domain, there exists $1 < p < \infty$ such that the harmonic measure satisfies a reverse Hölder inequality with exponent p. A complete geometric characterization came in 2020 when Azzam, Hofmann, Martell, Mourgoglou, and Tolsa [1] identified the class of ADR domains with interior corkscrews where (D_p) is solvable for some $1 < p < \infty$. These are precisely domains having interior big pieces of chord-arc domains (IBPCAD), as defined in [1, Definition 2.12].

Let us roughly introduce the regularity boundary value problem for an ADR domain Ω. When well-defined, we say that the regularity problem is solvable in L^p (denoted as (R_p) is solvable) in Ω if for any given $f \in C^{0,1}(\partial\Omega)$, there is a harmonic function $u : \Omega \to \mathbb{R}$ such that $u|^{\mathrm{nt}}_{\partial\Omega} = f$ holds σ-a.e. on $\partial\Omega$ and $\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \lesssim \|\nabla f\|_{L^p(\sigma)}$, where the implicit constant does not depend on f. The exact definition may vary slightly across the literature; see [20, Definition 5.35] and [39, Definition 1.1] for technical formulations.

We observe that the regularity problem is closely connected to the Dirichlet problem: for bounded domains with the corkscrew condition and having UR boundary (see Definition 2.1), Mourgoglou, Poggi and Tolsa proved in [39, Theorem 1.13] that $(R_p) \iff (D_{p'})$ for all $p \in (1, \infty)$, see also [40, Theorems 1.2 and 1.6]. One might expect similar results for 2-sided chord-arc domains with unbounded boundaries. However, to the best of our knowledge, such results have not been established in the literature. It is quite likely that the arguments in [40,39] could be extended to domains with unbounded boundaries, but this extension has not yet been documented.

The question of whether (N_p) or (R_p) is solvable for some $1 < p < \infty$ in chord-arc domains was first posed by Kenig [27, Problem 3.2.2] in 1994 and later reintroduced by Toro [46, Question 2.5] at the ICM 2010. While the regularity part of this question was recently resolved in 2023 by Mourgoglou, Poggi, and Tolsa in [39, Corollary 1.16], the Neumann part remains an open problem.

Although the solvability of the Neumann problem remains open for chord-arc domains, significant progress has been made in understanding its extrapolation properties under the assumption that $(D_{p'})$ or/and (R_p) is solvable for some $1 < p < \infty$. As noted earlier, in bounded chord-arc domains (which have UR boundary by Theorem 2.8), the equivalence $(R_p) \iff (D_{p'})$ holds for any $1 < p < \infty$. In 1993, Kenig and Pipher [28, Theorem 6.3] showed the extrapolation $(D_{p'}) + (N_p) \implies (N_q)$ for all $1 < q < p + \varepsilon$, for some $\varepsilon > 0$, for bounded Lipschitz domains. In 2024, Feneuil and Li [13, Corollary 1.22] extended this result to bounded chord-arc domains. In the same year, Mourgoglou and Tolsa showed in [41, Theorem 1.1] that (N_p) is solvable for a fixed $p \in (1, 2)$, whenever Ω is a bounded chord-arc domain such that (R_q) is solvable for some $q > p$, $\partial\Omega$ supports a weak p-Poincaré inequality, and Ω has very big pieces of chord-arc su-

perdomains for which (N_q) is solvable. Most recently in 2025, Hofmann and Sparrius proved in [20, Theorem 5.54] that $(N_p) + (R_p) + (D_{p'}) \implies (N_q)$ for all $1 < q < p$ in 2-sided chord-arc domains with unbounded boundary.

Given the strong connection to the Neumann problem, we briefly address the extrapolation properties of the regularity problem. The equivalence $(R_p) \iff (D_{p'})$ mentioned above, combined with the well-known extrapolation of solvability for the Dirichlet problem, immediately yields extrapolation of solvability of the regularity problem. A more subtle endpoint case was resolved in 2025 by Gallegos, Mourgoglou, and Tolsa, who proved solvability extrapolation for (R_1) (even for $(R_{1-\varepsilon})$) in ADR domains satisfying the interior corkscrew condition, see [16, Theorems 1.3 and 1.6].

Let us now return to our main discussion of the solvability of Dirichlet and Neumann problems. In 2010, Hofmann, M. Mitrea and Taylor studied the solvability of the Dirichlet and Neumann problems (among others) in bounded δ-regular SKT (Semmes-Kenig-Toro) domains. Roughly speaking, a domain is δ-regular SKT if it is Reifenberg flat (with small enough constant) and its geometric measure theoretic outward unit normal vector ν (see Remark 2.3) satisfies $\mathrm{dist}_{\mathrm{BMO}(\sigma)}(\nu, \mathrm{VMO}(\sigma)) \leq \delta$. The precise definition of δ-regular SKT domains can be found in [18, Definition 4.9]. In [18, Section 5], the authors proved that for any $1 < p < \infty$, both (D_p) and $(N_{p'})$ are solvable in δ-regular SKT domains when δ is small enough. Consequently, (D_p) and $(N_{p'})$ are solvable for all $1 < p < \infty$ in regular SKT domains, that is, domains that are δ-regular SKT for all $\delta > 0$, see [18, Definition 4.8].

The approach in [18] employs layer potentials. For any $1 < p < \infty$ and a bounded regular SKT domain, the authors show in [18, Theorem 4.36] that the double layer potential K is compact. More precisely, for bounded δ-regular SKT domains with sufficiently small $\delta = \delta(p) > 0$, [18, Theorem 4.36] establishes that K is close enough in $L^p(\sigma)$ norm to the set of compact operators (see Definition 2.13). This property nevertheless allows the application of Fredholm theory to prove the invertibility of both $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ from the injectivity of $\frac{1}{2}Id + K^*$ in $L^2(\sigma)$, see [18, Proposition 5.11].

Recently in 2022, Marín, Martell, D. Mitrea, I. Mitrea and M. Mitrea [34, Chapter 6] showed the L^p solvability of the Dirichlet and Neumann problems (among others) for 2-sided chord-arc domains with unbounded boundary, under the BMO smallness condition $\|\nu\|_* < \delta$ where $\delta > 0$ is sufficiently small depending on $1 < p < \infty$. They proved that the $L^p(\sigma)$ norm of K tends to zero as $\delta \to 0$. This enabled them to see that both $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ are invertible via Neumann series, thereby solving the Dirichlet and Neumann problems for any $1 < p < \infty$. More specifically, the smallness of $\delta > 0$ depends on the dimension, the chord-arc parameters of the domain and p. We emphasize that the results in [34] hold for weighted L^p spaces and systems of divergence-form operators with constant coefficient matrices.

Throughout this work, we will work in $\mathbb{R}^{n+1}$ with $n \geq 2$, although similar results may hold in the planar case $n = 1$. We focus on domains defined as follows:

Definition 1.1 *(δ-$(s, S; R)$ domain).* A domain $\Omega \subset \mathbb{R}^{n+1}$ is called δ-$(s, S; R)$ domain if it is a 2-sided chord-arc domain (see Definition 2.7) with unbounded boundary and there exist $\delta > 0$, scales $S > s > 0$ and a radius $R > 0$ such that the outward unit normal vector ν (see Remark 2.3) satisfies

$$\fint_{B(x,r)} |\nu(z) - m_{B(x,r)}\nu|\, d\sigma(z) \leq \delta, \text{ provided } \begin{cases} x \in \partial\Omega \setminus B_R(0) \text{ and } r \in (0, \infty), \text{ or} \\ x \in B_R(0) \cap \partial\Omega \text{ and } r \notin (s, S), \end{cases}$$

(here $m_{B(x,r)}\nu = \frac{1}{\sigma(B(x,r))}\int_{B(x,r)}\nu\,d\sigma$) and for all $x \in \partial\Omega$ and $r \geq S$, there holds

$$\beta_{\infty,\partial\Omega}(B(x,r)) := \inf_{n\text{-plane } L\ni x} \sup_{y\in\partial\Omega\cap B(x,r)} \frac{\operatorname{dist}(y,L)}{r} \leq \delta,$$

where the infimum is taken over all n-planes $L \subset \mathbb{R}^{n+1}$ through x.

That is, these are domains whose failure of sufficient flatness is limited to finitely many dyadic boundary balls. We note that the domains studied in [34] satisfy the first condition in Definition 1.1 for every $x \in \partial\Omega$ and every scale $r \in (0,\infty)$, implying $\beta_{\infty,\partial\Omega}(B(x,r)) \lesssim \delta^{\frac{1}{2n}}$ for all $x \in \partial\Omega$ and all $r \in (0,\infty)$, see [34, Theorem 2.2].

In this paper, we study the L^p-solvability of the Dirichlet and Neumann problems in δ-$(s,S;R)$-domains, and we also provide uniqueness results for (1.1) and (1.3) respectively. In the subsequent results, $\mathcal{D}$ denotes the interior double layer potential (see (3.1)), K the (boundary) double layer potential (see (3.2)), K^* the adjoint of K, and $\mathcal{S}_{\text{mod}}$ the modified interior single layer potential (see (7.1)).

Theorem 1.2 *(L^p-Dirichlet problem). Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s,S;R)$ domain. There is $\varepsilon_D = \varepsilon_D(n, CAD) \in (0, \frac{1}{n-1})$ such that for every $p_0 \in (\frac{n}{n-1} - \varepsilon_D, \frac{2n}{n-1}]$ there exists $\delta_0 = \delta_0(n, p_0, CAD) > 0$ such that if $\delta \leq \delta_0$, then $\frac{1}{2}Id + K$ is invertible in $L^{p_0}(\sigma)$ and given $f \in L^{p_0}(\sigma)$, the function*

$$u := \mathcal{D}\left(\left(\frac{1}{2}Id + K\right)^{-1} f\right), \tag{1.5}$$

is the solution of (D_{p_0}). Furthermore, there exists $\varepsilon > 0$ such that (D_p) is solvable for all $p \in (p_0 - \varepsilon, \infty)$, and the solution of (D_p) is the unique solution of the Dirichlet problem (1.1).

Theorem 1.3 *(L^p-Neumann problem). Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s,S;R)$ domain. There is $\varepsilon_N = \varepsilon_N(n, CAD) > 0$ such that for every $p \in [\frac{2n}{n+1}, n + \varepsilon_N)$ there exists $\delta_0 = \delta_0(n, p, CAD) > 0$ such that if $\delta \leq \delta_0$, then $-\frac{1}{2}Id + K^*$ is invertible in $L^p(\sigma)$, (N_p) is solvable and given $f \in L^p(\sigma)$, the function*

$$u := \mathcal{S}_{\text{mod}}\left(\left(-\frac{1}{2}Id + K^*\right)^{-1} f\right), \tag{1.6}$$

is the unique (modulo constants) solution of (N_p). Furthermore, it is the unique (modulo constants) solution of the Neumann problem (1.3).

Theorems 1.2 and 1.3 mainly follow from the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$, respectively. We emphasize that the existence of $\varepsilon_D > 0$ in Theorem 1.2 and $\varepsilon_N > 0$ in Theorem 1.3 guarantees the solvability of both (D_2) and (N_2).

In contrast with [34], the invertibility of $\frac{1}{2}Id + K$ in $L^{p'}(\sigma)$ and $-\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ on which Theorems 1.2 and 1.3 rely holds only for a bounded range of exponents, $p \in [\frac{2n}{n+1}, n + \varepsilon)$.

This is the range for which we can prove the injectivity of $\pm\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ (see Section 7), which is governed by the uniqueness (modulo constants) of harmonic functions with $\mathcal{N}(\nabla u) \in L^p(\sigma)$ (see Section 6). Essentially, the lower endpoint is where the gradient of the solution becomes square-integrable, so that the energy-based regularity estimates apply, and the upper endpoint is where the solution still decays at infinity, forcing it to be constant. We note that, in the bounded case, it would suffice to prove this injectivity at the single endpoint $p = \frac{2n}{n+1}$, with the full range $[\frac{2n}{n+1}, \infty)$ then following by Hölder's inequality.

Let us briefly discuss the extrapolation of solvability. For the Dirichlet problem, the range extends up to $p = \infty$ by Gehring's lemma, whereas for the Neumann problem no comparable extrapolation is available in our setting. While for 2-sided chord-arc domains with unbounded boundary there holds $(N_p) + (R_p) + (D_{p'}) \implies (N_q)$ for all $1 < q < p$, our results for δ-$(s, S; R)$ domains in Theorems 1.2 and 1.3 establish $(D_{p'})$ and (N_p) solvability for a fixed $p \in [2n/(n+1), n+\varepsilon)$ (δ is sufficiently small), but not the solvability of (R_p). Since we have not yet established the solvability of (R_p), we cannot consequently derive that (N_q) is solvable for all $1 < q \leq p$.

Let us outline the proof strategy for Theorems 1.2 and 1.3. Unlike the approach in [34] discussed above, our setting with δ-$(s, S; R)$-domains presents a key difference: the double layer potential K does not a priori have small norm, and thus we cannot directly derive the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$ via Neumann series. As in [18, Theorem 4.36], in the following result (one of the main points in this article), we show that K and its adjoint K^* are close to the set of compact operators (see Definition 2.13). This enables us to employ Fredholm theory to characterize their invertibility.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s, S; R)$ domain, $1 < p < \infty$, and let $p' = p/(p-1)$ be its Hölder conjugate exponent. For all $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon, p, CAD, n) > 0$ such that if $\delta \leq \delta_0$, then there is a compact operator $T : L^p(\sigma) \to L^p(\sigma)$ such that $\|K - T\|_{L^p(\sigma)\to L^p(\sigma)} = \|K^* - T^*\|_{L^{p'}(\sigma)\to L^{p'}(\sigma)} < \varepsilon$.*

Here, $T^* : L^{p'}(\sigma) \to L^{p'}(\sigma)$ is the adjoint of T, which is compact by Schauder's theorem. To prove this theorem we truncate the double layer potential K at small, intermediate ("close" and "far" from $B_R(0)$) and large scales, see (3.6). The operator on the "close" intermediate scales turns out to be compact (Lemma 3.2). The operators at small and "far" intermediate scales have small norm (Lemmas 3.3 and 3.4), by using the already known behavior of the double layer potential at these scales (Theorem 4.5) via Semmes' decomposition Theorem 4.4. One of the main difficulties in this article is establishing the small norm of the operator at large scales (Theorem 3.5), which we address in Section 5.

Let us explain why the large-scale estimate is the delicate point. At small and "far" intermediate scales, Semmes' decomposition shows that, away from an exceptional set of small measure, $\partial\Omega$ coincides with a Lipschitz graph with small constant, which makes the corresponding truncated operators have small norm. At large scales this is no longer available: K has no small norm a priori and the flatness of $\partial\Omega$ may fail at finitely many bad balls. We overcome this by comparing $\partial\Omega$ with a Lipschitz graph that is now only close to it rather than coincident with it, built through a stopping-time argument in the spirit of David-Semmes (Section 5.5). This closeness is ensured by the large-scale smallness of $\beta_{\infty,\partial\Omega}$ and $\|\nu\|_*$.

Combining Theorem 1.4 with the Fredholm alternative Theorem 2.14, in Corollary 3.6, for $\lambda \in \mathbb{R} \setminus \{0\}$ we establish several equivalent conditions in order to see that $\lambda Id + K$ (respectively

$\lambda Id + K^*$) is invertible in $L^p(\sigma)$ (respectively $L^{p'}(\sigma)$). Furthermore, the invertibility of $\lambda Id + K$ in $L^p(\sigma)$ and $\lambda Id + K^*$ in $L^{p'}(\sigma)$ is shown to be equivalent to the injectivity of $\lambda Id + K^*$ in $L^{p'}(\sigma)$.

As previously noted, the solvability of the Dirichlet and Neumann problems mainly relies on the invertibility of $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$. The equivalences discussed in the previous paragraph allow us to reduce this invertibility problem to showing the injectivity of $\pm\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ for some range of p. Using the modified single layer potential (see Section 7.1) and Moser estimates for harmonic functions with vanishing Dirichlet or Neumann boundary conditions, in Section 7 we prove that $\pm\frac{1}{2}Id + K^*$ are indeed injective in $L^p(\sigma)$ when $p \in [2n/(n+1), n+\varepsilon)$, for some $\varepsilon > 0$.

In Section 9, we establish the L^p solvability of the Dirichlet and Neumann problems in δ-$(s, S; R)$ domains in Theorems 1.2 and 1.3. This mainly follows from the invertibility explained above, and also from some already-known properties of the single and double layer potentials. The uniqueness results for (1.1) and (1.3) are proved in Sections 6 and 8.

A natural question is whether Theorems 1.2 to 1.4 generalize to second-order divergence form operators $\operatorname{div}(A\nabla\cdot)$ for uniformly elliptic matrices A with Dini mean oscillation coefficients, in the spirit of recent work [38] by Molero, Mourgoglou, Puliatti, and Tolsa.

2. Preliminaries and definitions

2.1. Notation

- We use $c, C \geq 1$ to denote constants that may depend only on the dimension and the constants appearing in the hypotheses of the results, and whose values may change at each occurrence.
- We write $a \lesssim b$ if there exists a constant $C \geq 1$ such that $a \leq Cb$, and $a \approx b$ if $C^{-1}b \leq a \leq Cb$.
- If we want to stress the dependence of the constant on a parameter η, we write $a \lesssim_\eta b$ or $a \approx_\eta b$ meaning that $C = C(\eta) = C_\eta$.
- The ambient space is $\mathbb{R}^{n+1}$ with $n \geq 2$. We denote by m the Lebesgue measure on $\mathbb{R}^{n+1}$.
- The diameter of a set $E \subset \mathbb{R}^{n+1}$ is denoted by $\operatorname{diam} E$. We allow $\operatorname{diam} E = \infty$ if E is unbounded.
- We denote by $B_r(x)$ or $B(x, r)$ the open ball with center x and radius $r > 0$. We denote $B_r := B_r(0)$.
- Given a domain Ω, we denote the boundary ball centered at $x \in \partial\Omega$ with $r > 0$ by $\Delta(x, r) = \Delta_r(x) := B(x, r) \cap \partial\Omega$.
- Given a ball B, we denote by r_B or $r(B)$ its radius, and by c_B or $c(B)$ its center. Analogously, r_Δ or $r(\Delta)$ and c_Δ or $c(\Delta)$ for a boundary ball Δ.
- Given a ball B and $t > 1$, $tB := B(c_B, tr_B)$. Analogously, $t\Delta = \Delta(c_\Delta, tr_\Delta)$.
- We denote by $Q_r(x)$ or $Q(x, r)$ the open cube with center x and side length $2r$, i.e., $Q_r(x) = Q(x, r) = \{y \in \mathbb{R}^{n+1} : |y_i - x_i| < r \text{ for all } 1 \leq i \leq n+1\}$.
- Given a cube Q, we denote by $\ell(Q)$ its side length, and by c_Q or $c(Q)$ its center. That is, $Q = Q(c_Q, \ell(Q)/2)$.
- Given a cube Q and $t > 1$, $tQ := Q(c_Q, t\ell(Q)/2)$, that is, $c_{tQ} = c_Q$ and $\ell(tQ) = t\ell(Q)$.
- We say that a function f is Hölder continuous with exponent $\alpha \in (0, 1]$ in a set U, or briefly $C^{0,\alpha}(U)$, if there exists a constant $C_\alpha > 0$ (called the Hölder seminorm) such that $|f(x) - f(y)| \leq C_\alpha |x - y|^\alpha$ for all $x, y \in U$. For $\alpha \in (0, 1)$ we write C^α instead of $C^{0,\alpha}$. When $\alpha = 1$ we say that f is "Lipschitz continuous", and we call $C_L := C_1$ its Lipschitz constant. We also say that f is κ-Lipschitz if $C_L \leq \kappa$.

- We denote the characteristic function of a set E by $\mathbf{1}_E$.
- Denote $\mathcal{D}(\mathbb{R}^{n+1})$ the standard dyadic grid. That is, $\mathcal{D}(\mathbb{R}^{n+1}) = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k(\mathbb{R}^{n+1})$ where $\mathcal{D}_k(\mathbb{R}^{n+1})$ is the collection of all cubes of the form

$$\{x \in \mathbb{R}^{n+1} : m_i 2^{-k} \leq x_i < (m_i+1)2^{-k} \text{ for } i = 1, \dots, n+1\},$$

  where $m_i \in \mathbb{Z}$.
- Given $t > 0$ and a set $E \subset \mathbb{R}^{n+1}$, we write $U_t(E) \coloneqq \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, E) < t\}$ for the t-neighborhood of E.
- For $0 \leq s < \infty$, $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure.
- $\mu|_S$ denotes the restriction of the measure μ to a set $S \subset \mathbb{R}^{n+1}$ defined as $\mu|_S(E) \coloneqq \mu(S \cap E)$ for $E \subset \mathbb{R}^{n+1}$.
- Given a measure μ and a set E, if $\mu(E) \neq 0$ then we denote $m_E f \coloneqq \fint_E f \, d\mu \coloneqq \frac{1}{\mu(E)} \int_E f \, d\mu$ for $f \in L^1_{\text{loc}}(\mu)$.
- Let $E \subset \mathbb{R}^{n+1}$ be an ADR closed set (see Section 2.2) and let $\mu \coloneqq \mathcal{H}^n|_E$. Given $f \in L^2_{\text{loc}}(\mu)$, $x \in E$, $R > 0$, we set

$$\|f\|_*(B(x,R)) \coloneqq \sup_{B \subset B(x,R)} \left(\fint_B |f(z) - m_B f|^2 \, d\mu(z) \right)^{1/2},$$

  where the supremum is taken over all balls B centered at E included in $B(x, R)$, and $m_B f \coloneqq \fint_B f \, d\mu$.
- Given an ADR domain Ω satisfying the 2-sided corkscrew condition (see Section 2.2) with outward unit normal ν_Ω (see Remark 2.3), the tangential gradient of a Lipschitz function f in $\partial\Omega$ is

$$\nabla_t f(y) \coloneqq \nabla \widetilde{f}(y) - \langle \nabla \widetilde{f}(y), \nu_\Omega(y) \rangle \nu_\Omega(y) \text{ for } \sigma\text{-a.e. } y \in \partial\Omega,$$

  where $\widetilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}$ is any Lipschitz extension of f to $\mathbb{R}^{n+1}$.

2.2. ADR, UR, uniform, NTA and CAD

We say that a Radon measure μ in $\mathbb{R}^{n+1}$ is (n-dimensional) Ahlfors-David regular if there exists $C \geq 1$ (called the ADR constant) such that

$$C^{-1} r^n \leq \mu(B(x,r)) \leq C r^n \text{ for all } x \in \operatorname{supp}\mu \text{ and } 0 < r < \operatorname{diam}(\operatorname{supp}\mu), \tag{2.1}$$

where $\operatorname{diam}(\operatorname{supp}\mu)$ may be infinite. A closed set $E \subset \mathbb{R}^{n+1}$ is said to be Ahlfors-David regular if $\mathcal{H}^n|_E$ is Ahlfors-David regular. A domain $\Omega \subset \mathbb{R}^{n+1}$ is said to be an Ahlfors-David regular domain if $\partial\Omega$ is Ahlfors-David regular.

Notation. From now on, the term Ahlfors-David regular may be shortened to AD regular or ADR. Moreover, given an ADR domain Ω we will denote its surface measure by

$$\sigma = \sigma_\Omega \coloneqq \mathcal{H}^n|_{\partial\Omega}.$$

Definition 2.1 *(UR set).* A set $E \subset \mathbb{R}^{n+1}$ is called (n-dimensional) uniformly rectifiable, UR for short, if it is ADR and there exist $\varepsilon, M \in (0,\infty)$ (called the UR constants of E) such that for every $x \in E$ and $r \in (0, \operatorname{diam} E)$, there is a Lipschitz map $\varphi = \varphi_{x,r} : \{y \in \mathbb{R}^n : |y| < r\} \to \mathbb{R}^{n+1}$ with Lipschitz constant $\leq M$, such that

$$\mathcal{H}^n(E \cap B(x,r) \cap \varphi(\{y \in \mathbb{R}^n : |y| < r\})) \geq \varepsilon r^n.$$

This is a quantitative version of rectifiability introduced by David and Semmes in [9,10]. It is well known that any UR set is rectifiable, for a detailed proof see [18, p. 2629].

Definition 2.2 *(Corkscrew ball conditions).* We say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies

- the (interior) corkscrew condition if there is a constant $M > 1$ such that for every $x \in \partial\Omega$ and $r \in (0, \operatorname{diam}(\partial\Omega))$ there exists a point $A_r(x) \in \Omega$ such that $B(A_r(x), M^{-1}r) \subset B(x,r) \cap \Omega$. $A_r(x)$ is called the corkscrew point of the point x at radius r.
- the exterior corkscrew condition if $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the corkscrew condition.
- the 2-sided corkscrew condition if it satisfies the interior and exterior corkscrew condition.

Remark 2.3 *(The geometric measure theoretic outward unit normal vector ν).* If $\Omega \subset \mathbb{R}^{n+1}$ is an ADR domain satisfying the 2-sided corkscrew condition, then for σ-a.e. $x \in \partial\Omega$ there exists a unique unit vector $\nu(x)$ (called the geometric measure theoretic outward unit vector) satisfying for $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{R}^{n+1} \setminus \Omega$, both

$$\lim_{r\to 0} \frac{m(\Omega^{\pm} \cap \{y \in B(x,r) : \pm\langle \nu(x), y - x\rangle \geq 0\})}{r^{n+1}} = 0.$$

We remark that the vector ν exists under more general conditions, see for instance [18, Section 2.2] and the references therein.

Definition 2.4 *(Harnack chain condition).* We say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the Harnack chain condition if there is a constant $M > 1$ such that for every $\varepsilon > 0$ and $x_1, x_2 \in \Omega$ with $\operatorname{dist}(x_i, \partial\Omega) \geq \varepsilon$ $(i = 1, 2)$ and $|x_1 - x_2| \leq 2^j\varepsilon$ for some integer $j \geq 1$, there exists a chain of open balls $\{B_k\}_{1\leq k\leq N}$ inside Ω with $N \leq Mj$ satisfying that $x_1 \in B_1$, $x_2 \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ (for $1 \leq k \leq N-1$) and $M^{-1}r(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq Mr(B_k)$ (for $1 \leq k \leq N$).

Definition 2.5 *(Uniform domain).* A domain $\Omega \subset \mathbb{R}^{n+1}$ is called uniform domain if it satisfies the Harnack chain and interior corkscrew conditions.

Definition 2.6 *(NTA domain).* A domain $\Omega \subset \mathbb{R}^{n+1}$ is called nontangentially accessible (NTA for short) domain if it is a uniform domain and it satisfies the exterior corkscrew condition.

Definition 2.7 *(CAD).* A domain $\Omega \subset \mathbb{R}^{n+1}$ is called chord-arc domain (1-sided CAD or CAD for short) if it is an NTA domain and $\partial\Omega$ is ADR. We say that Ω is a 2-sided chord-arc domain (2-sided CAD) if Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ are CAD.

Notation. Given a (2-sided) CAD Ω, we will write $C = C(\text{CAD})$ if the constant C depends on the CAD constants of Ω.

The following is from [6,44], see also [18, Corollary 3.9].

Theorem 2.8. *If $\Omega \subset \mathbb{R}^{n+1}$ is a domain satisfying the 2-sided corkscrew condition and whose boundary is ADR, then $\partial\Omega$ is UR.*

Remark 2.9. Most of the results in [18] are presented for 2-sided local John domains with ADR boundary. However, it was shown in [47] that this apparently weaker condition is, in fact, equivalent to being 2-sided CAD. This allows us to apply the known results in the literature for 2-sided local John domains with ADR boundary when working with 2-sided CAD. For instance, the Semmes decomposition in [18, Theorem 4.16], restated in Theorem 4.4 below.

2.3. Maximal operators and boundary dyadic cubes in ADR domains

Given a domain $\Omega \subset \mathbb{R}^{n+1}$, for each $x \in \partial\Omega$ we define the nontangential approach cone with aperture $\alpha > 0$ as

$$\Gamma(x) = \Gamma_\alpha^\Omega(x) := \{y \in \Omega : |y - x| < (1+\alpha)\mathrm{dist}(y, \partial\Omega)\}.$$

For a function $u : \Omega \to \mathbb{R}$ we define the nontangential maximal function

$$\mathcal{N}u(x) = \mathcal{N}_\alpha u(x) := \sup_{y \in \Gamma_\alpha^\Omega(x)} |u(y)|, \quad x \in \partial\Omega, \tag{2.2}$$

and the δ-nontangential maximal function $\mathcal{N}^\delta u$ of u by

$$\mathcal{N}^\delta u(x) = \mathcal{N}_\alpha^\delta u(x) := \sup_{y \in \Gamma_\alpha^\Omega(x) \cap \overline{B_{2\delta}(x)}} |u(y)|, \quad x \in \partial\Omega. \tag{2.3}$$

For a fixed $\alpha > 0$, we introduce the following definitions whenever well-defined. The nontangential limit is

$$u|_{\partial\Omega}(x) = u|_{\partial\Omega}^{\mathrm{nt}}(x) := \lim_{\Gamma^\Omega(x) \ni z \to x} u(z), \quad x \in \partial\Omega. \tag{2.4}$$

If in addition the outward unit normal ν_Ω exists (see Remark 2.3) and $u \in C^1(\Omega)$, the interior normal nontangential derivative is

$$\partial_{\nu_\Omega}^{\mathrm{int}} u(x) := \lim_{\Gamma^\Omega(x) \ni z \to x} \langle \nu_\Omega(x), \nabla u(z) \rangle, \quad x \in \partial\Omega. \tag{2.5}$$

For shortness, we will also write ∂_{ν_Ω} or ∂_ν instead of $\partial_{\nu_\Omega}^{\mathrm{int}}$. If $u \in C^1(\overline{\Omega}^c)$, the exterior normal nontangential derivative is

$$\partial_{\nu_\Omega}^{\mathrm{ext}} u(x) := \lim_{\Gamma^{\overline{\Omega}^c}(x) \ni z \to x} \langle \nu_\Omega(x), \nabla u(z) \rangle, \quad x \in \partial\Omega. \tag{2.6}$$

Given an $\mathcal{H}^n$-measurable set $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) > 0$ and $1 < q < \infty$, we define the uncentered Hardy-Littlewood maximal operator $\mathcal{M}_{q,E}$, for $f \in L^q_{\mathrm{loc}}(\mathcal{H}^n|_E)$, as

$$\mathcal{M}_{q,E} f(x) := \sup_{\substack{r>0 \\ y\in E \\ B(y,r)\ni x}} \left(\fint_{B(y,r)} |f(z)|^q \, d\mathcal{H}^n|_E(z) \right)^{\frac{1}{q}}, \quad x \in E. \tag{2.7}$$

It is well known that it is bounded from $L^p(\mathcal{H}^n|_E)$ to $L^p(\mathcal{H}^n|_E)$ if $q < p \leq \infty$, with norm $C_{p,q,n} > 0$. If the set E is clear from the context we write $\mathcal{M}_q = \mathcal{M}_{q,E}$. We will use the uncentered Hardy-Littlewood maximal operator with $E = \partial\Omega$, where Ω is an ADR domain.

For ADR domains $\Omega \subset \mathbb{R}^{n+1}$, the $L^p(\sigma)$ norm (with $1 < p < \infty$) of the nontangential maximal function $\mathcal{N}$ in (2.2) does "not" depend on the aperture in the sense that, for every $\alpha, \beta > 0$ and any $u : \Omega \to \mathbb{R}$ there holds

$$\|\mathcal{N}_\alpha u\|_{L^p(\sigma)} \approx_{\alpha,\beta} \|\mathcal{N}_\beta u\|_{L^p(\sigma)}, \tag{2.8}$$

see [18, Proposition 2.2]. For this fact, we will omit the aperture $\alpha > 0$ in $\mathcal{N}_\alpha$ and Γ_α from now on. We may fix $\alpha = 1$ for instance.

The following two lemmas provide the control of interior integrals by the nontangential maximal function. The first is for solid interior integrals (see [18, (2.3.25) in Proposition 2.12]) and the second[2] is for interior sets with n-growth (see [41, Lemma 5.1]).

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain, and fix $\alpha > 0$. Then there exists $C = C(n, \alpha, ADR) > 0$ such that for any measurable function $u : \Omega \to \mathbb{R}$ there holds*

$$\frac{1}{\delta} \int_{U_\delta(\partial\Omega)} |u(z)| \, dm(z) \leq C \|\mathcal{N}_\alpha^\delta u\|_{L^1(\sigma)}, \ 0 < \delta \leq \operatorname{diam}(\Omega).$$

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain, B_0 a ball centered at $\partial\Omega$, and $E \subset B_0 \cap \Omega$ such that*

$$\mathcal{H}^n(B(x,r) \cap E) \leq C_0 r^n \text{ for all } x \in E \text{ and } r > 0.$$

Then, for any Borel function $u : \Omega \to \mathbb{R}$ such that $u \in L^1_{\mathrm{loc}}(\mathcal{H}^n|_E)$,

$$\int_E |u(z)| \, d\mathcal{H}^n(z) \lesssim \int_{2B_0} \mathcal{N}_\beta^{2r(B_0)} u(z) \, d\sigma(z),$$

assuming the aperture $\beta > 0$ to be large enough (depending only on n). The implicit constant above depends only on n, C_0, and the ADR constants of $\partial\Omega$.

For the construction of the Lipschitz graph in Section 5.5 we will follow [9, Section 8]. So, given an ADR domain Ω with surface measure σ of $\partial\Omega$, we consider a dyadic lattice $\mathcal{D}_\sigma$ of "cubes" on $\partial\Omega$ constructed in this generality by Christ [2]. Here we use the formulation of David and Semmes in [10, Chapter 3 of Part I] with codimension 1.

[2] The statement as written here appears in [42, Lemma 5.1].

Lemma 2.12 *(Boundary dyadic cubes). Given an ADR domain Ω with surface measure σ, for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_{\sigma,j}$ of Borel subsets of* $\operatorname{supp}\sigma = \partial\Omega$*, called the dyadic cubes of the j-th generation, with the following properties:*

(1) each $\mathcal{D}_{\sigma,j}$ is a partition of $\partial\Omega$, i.e., $\partial\Omega = \bigcup_{Q\in\mathcal{D}_{\sigma,j}} Q$ with $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{D}_{\sigma,j}$ with $Q \neq Q'$,
(2) if $Q \in \mathcal{D}_{\sigma,i}$ and $Q' \in \mathcal{D}_{\sigma,j}$ for some $i \geq j$, then either $Q \cap Q' = \emptyset$ or $Q \subset Q'$,
(3) for all $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}_{\sigma,j}$, we have that $2^{-j} \leq \operatorname{diam} Q \leq C_{\mathcal{D}} 2^{-j}$ and $C^{-1}2^{-jn} \leq \sigma(Q) \leq C2^{-jn}$, and
(4) for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_{\sigma,j}$ and $0 < \tau < 1$, we have the so-called "thin boundary condition":

$$\sigma(\{x \in Q : \operatorname{dist}(x, \partial\Omega \setminus Q) \leq \tau 2^{-j}\}) + \sigma(\{x \in \partial\Omega \setminus Q : \operatorname{dist}(x, Q) \leq \tau 2^{-j}\}) \leq C_{\mathcal{D}} \tau^{1/C_{\mathcal{D}}} 2^{-jn}.$$

We set $\mathcal{D}_\sigma = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_{\sigma,j}$. The constants $C_{\mathcal{D}}, C \geq 1$ in (3) and (4) above do not depend on j, Q or τ.

We follow the convention that the size of the cubes decreases as the generation index grows, that is, $\operatorname{diam} Q \approx 2^{-j}$ for $Q \in \mathcal{D}_{\sigma,j}$, in accordance with the standard dyadic grid of $\mathbb{R}^{n+1}$ in Section 2.1.

2.4. Compact operators

Let us briefly recall the definition of compact operators and the Fredholm alternative.

Definition 2.13. Given Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, a bounded linear operator $T : X \to Y$ is called compact if for every bounded sequence $\{x_k\}_{k\geq 1} \subset X$, the sequence $\{Tx_k\}_{k\geq 1} \subset Y$ has a convergent subsequence.

Theorem 2.14 *(Fredholm alternative). Let $(X, \|\cdot\|_X)$ be a Banach space, $T : X \to X$ be a compact operator, and $\lambda \in \mathbb{C} \setminus \{0\}$. Then exactly one of the following holds:*

- *the equation $Tv - \lambda v = 0$ has a non-zero solution $v \in X$, or*
- *for every $u \in X$, the equation $Tv - \lambda v = u$ has a unique solution $v \in X$. In this case, the solution v depends continuously on u.*

Remark 2.15. Since the composition $T \circ B$ of a compact operator T with a bounded operator B is again compact, the same holds when replacing λv by $\mathcal{I}v$ for an invertible bounded operator $\mathcal{I} : X \to X$.

2.5. Calderón-Zygmund operators and the Riesz transform

We say that $k : \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : x \neq y\} \to \mathbb{C}$ is a Calderón-Zygmund kernel if there exist constants $C \geq 1$ and $0 < \tau \leq 1$ such that for all $x, x', y \in \mathbb{R}^{n+1}$ with $x \neq y$, $x' \neq y$, there holds

$$|k(x,y)| \leq C\frac{1}{|x-y|^n}, \text{ and}$$
$$|k(x,y)-k(x',y)|+|k(y,x)-k(y,x')| \leq C\frac{|x-x'|^\tau}{|x-y|^{n+\tau}}, \text{ if } |x-x'| \leq |x-y|/2.$$

Given a Radon measure μ and a Calderón-Zygmund kernel k, we define

$$T^k\mu(x) := \int k(x,y)\,d\mu(y), \quad x \in \mathbb{R}^{n+1} \setminus \operatorname{supp}\mu,$$

and as this may not converge for $x \in \operatorname{supp}\mu$, for $\varepsilon > 0$ we define the truncated operator

$$T^k_\varepsilon\mu(x) := \int\limits_{|y-x|>\varepsilon} k(x,y)\,d\mu(y), \quad x \in \mathbb{R}^{n+1}.$$

Given a Radon measure μ and $f \in L^1_{\mathrm{loc}}(\mu)$, we define

$$T^k_\mu f(x) := T^k(f\mu)(x), \text{ for } x \in \mathbb{R}^{n+1} \setminus \operatorname{supp}\mu,$$
$$T^k_{\mu,\varepsilon} f(x) := T^k_\varepsilon(f\mu)(x), \text{ for } \varepsilon > 0 \text{ and } x \in \mathbb{R}^{n+1},$$

and the maximal operator

$$T^k_{\mu,*} f(x) := \sup_{\varepsilon>0} |T^k_{\mu,\varepsilon} f(x)|, \quad x \in \operatorname{supp}\mu. \tag{2.9}$$

We say that T^k_μ is bounded in $L^p(\mu)$, $1 < p < \infty$, if the truncated operators $T^k_{\mu,\varepsilon}$ are bounded in $L^p(\mu)$ uniformly in $\varepsilon > 0$. In this case, we write $T^k_\mu : L^p(\mu) \to L^p(\mu)$ is bounded. We remark that, if μ has growth of degree n (i.e., μ satisfies the upper bound in (2.1)), then the boundedness of T^k_μ in $L^p(\mu)$ is equivalent to the boundedness of the maximal operator $T^k_{\mu,*}$ in $L^p(\mu)$, by [45, Theorem 2.16][3] and Cotlar's inequality (take $s = 1$ in [45, (2.26)] for instance).

Definition 2.16 *(Riesz transform).* The (n-dimensional) Riesz kernel is the Calderón-Zygmund vector-valued kernel (with $\tau = 1$)

$$\widetilde{k}_{\mathcal{R}}(x) := \frac{x}{|x|^{n+1}} \text{ for } x \in \mathbb{R}^{n+1} \setminus \{0\}.$$

The (n-dimensional) Riesz transform $\mathcal{R}$ is defined as

$$\mathcal{R} := T^k, \text{ with } k(x,y) = \widetilde{k}_{\mathcal{R}}(x-y) \text{ for } x \neq y.$$

By [5, Proposition 4 bis], if $\Omega \subset \mathbb{R}^{n+1}$ is an ADR domain with the 2-sided corkscrew condition (in particular $\partial\Omega$ is UR by Theorem 2.8), then

[3] A quick inspection of its proof reveals that the same holds if $L^2(\mu)$ is replaced by $L^p(\mu)$ for any $1 < p < \infty$.

$$\|\mathcal{R}_{\sigma,*} f\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR}} \|f\|_{L^p(\sigma)} \text{ for all } f \in L^p(\sigma),$$

see also [18, Proposition 3.18] for instance.

3. The double layer potential

Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition (in particular $\partial\Omega$ is UR by Theorem 2.8), let $\nu := \nu_\Omega$ be the geometric measure theoretic outward unit vector of Ω (see Remark 2.3), and let $f \in L^1\left(\frac{d\sigma(x)}{1+|x|^n}\right)$. The interior double layer potential operator associated with Ω is

$$\mathcal{D} f(x) = \mathcal{D}_\Omega f(x) := \frac{1}{w_n} \int_{\partial\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \mathbb{R}^{n+1} \setminus \partial\Omega. \tag{3.1}$$

Here w_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$. The double layer potential satisfies $\Delta(\mathcal{D} f) = 0$ in $\mathbb{R}^{n+1} \setminus \partial\Omega$.

Remark 3.1. Note that $L^p(\sigma) \subset L^1\left(\frac{d\sigma(x)}{1+|x|^n}\right)$ for all $p > 1$, as $\partial\Omega$ is ADR.

The boundary double layer potential, that is, the principal value version of the interior double layer potential, is defined as

$$K f(x) = K_\Omega f(x) := \lim_{\varepsilon \to 0^+} K_\varepsilon f(x), \quad x \in \partial\Omega, \tag{3.2}$$

where

$$K_\varepsilon f(x) := \frac{1}{w_n} \int_{\{y \in \partial\Omega : |y-x| > \varepsilon\}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{3.3}$$

The principal value in (3.2) exists for σ-a.e. $x \in \partial\Omega$, see [35, Proposition 5.6.7].

Also, the maximal operator of the boundary double layer potential is defined as

$$K_* f(x) := \sup_{\varepsilon > 0} |K_\varepsilon f(x)|, \quad x \in \partial\Omega.$$

We also define

$$K^*_\varepsilon f(x) := \frac{1}{w_n} \int_{\{y \in \partial\Omega : |y-x| > \varepsilon\}} \frac{\langle \nu(x), x-y\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

and the maximal operator

$$K^*_* f(x) := \sup_{\varepsilon > 0} |K^*_\varepsilon f(x)|, \quad x \in \partial\Omega.$$

A quick computation shows that the operator K^* defined as

$$K^* f(x) \coloneqq \lim_{\varepsilon \to 0^+} K^*_\varepsilon f(x), \quad x \in \partial\Omega,$$

is the adjoint operator of K.

The interior double layer potential satisfies the jump relation

$$(\mathcal{D} f)|^{\mathrm{nt}}_{\partial\Omega}(x) = \left(\frac{1}{2} Id + K\right) f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{3.4}$$

see [34, (3.31)]. If in addition $f \in L^p(\sigma)$ with $1 < p < \infty$, then the boundary and interior double layer potentials satisfy

$$\|K_* f\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR}} \|f\|_{L^p(\sigma)}, \tag{3.5a}$$

$$\|\mathcal{N}(\mathcal{D} f)\|_{L^p(\sigma)} \lesssim_{p,\mathrm{UR}} \|f\|_{L^p(\sigma)}, \tag{3.5b}$$

see [18, (3.3.5) and (3.3.6)] respectively. Here UR denotes that the constant depends on the UR constants of $\partial\Omega$. The second estimate also depends on the aperture of $\mathcal{N}$.

3.1. Proof of Theorem 1.4 and consequences

In order to study the boundary double layer potential K in Theorem 1.4, for $0 < t < T < \infty$ we define its truncations by

$$K_s f(x) \coloneqq \lim_{\varepsilon \to 0^+} \frac{1}{w_n} \int\limits_{\{y \in \partial\Omega : \varepsilon < |y-x| \leq t\}} \frac{\langle \nu(y), y - x\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

$$K_i f(x) \coloneqq \frac{1}{w_n} \int\limits_{\{y \in \partial\Omega : t < |y-x| \leq T\}} \frac{\langle \nu(y), y - x\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

$$K_l f(x) \coloneqq \frac{1}{w_n} \int\limits_{\{y \in \partial\Omega : |y-x| > T\}} \frac{\langle \nu(y), y - x\rangle}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

where s, i and l stand for small, intermediate and large scales respectively. If we want to stress the scales we will write $K_{s(t)}$, $K_{i(t,T)}$ and $K_{l(T)}$ respectively. We define $K^*_{s(t)}$, $K^*_{i(t,T)}$ and $K^*_{l(T)}$ analogously from the definition of K^*.

For $\widetilde{R} > 0$ we also break the intermediate scales operator as $K_i = \mathbf{1}_{B_{\widetilde{R}}(0)} K_i + \mathbf{1}_{B_{\widetilde{R}}(0)^c} K_i$. So, for $f \in L^p(\sigma)$ and $x \in \partial\Omega$ we will decompose the double layer potential as

$$Kf(x) = K_s f(x) + K_l f(x) + \mathbf{1}_{B_{\widetilde{R}}(0)}(x) K_i f(x) + \mathbf{1}_{B_{\widetilde{R}}(0)^c}(x) K_i f(x). \tag{3.6}$$

Next, we present a series of results to summarize the relevant properties of the operators in the decomposition (3.6).

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition. For any $0 < t < T < \infty$ and $\widetilde{R} > 0$, the operator $\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)} : L^p(\sigma) \to L^p(\sigma)$ is compact for all $p \in (1, \infty)$.*

We write the proof in Section 3.2. The compact operator in Theorem 1.4 will be in fact $T = \mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$, for some choice of parameters.

The following two results control the $L^p(\sigma)$ norm of the boundary double layer potential at small scales and, respectively, at scales far from the "bad" balls in the definition of δ-$(s, S; R)$ domains (see Definition 1.1).

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD. Assume also that there is $\delta > 0$ and $s > 0$ such that*

$$\sup_{\substack{x\in\partial\Omega \\ 0<r\leq s}} \fint_{B(x,r)} |\nu - m_{B(x,r)}\nu|\, d\sigma \leq \delta.$$

Given $p \in (1, \infty)$, there exists $t_0 = t_0(s, \delta, p, CAD, n) > 0$ such that for every $0 < t \leq t_0$ there holds

$$\|K_{s(t)}\|_{L^p(\sigma)\to L^p(\sigma)} \lesssim \delta^{1/4},$$

where the involved constant depends on n, the CAD constants of Ω and p.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-$(s, S; R)$ domain. Given $p \in (1, \infty)$ and $t > 0$, there exists $\widetilde{R} = \widetilde{R}(R, \delta, t, p, CAD, n)$ such that there holds*

$$\|\mathbf{1}_{B_{\widetilde{R}}(0)^c} K_{s(t)}\|_{L^p(\sigma)\to L^p(\sigma)} \lesssim \delta^{1/4},$$

where the involved constant depends on n, the CAD constants of Ω and p.

We prove the preceding two lemmas in Section 4. Note that, to obtain the claimed bound, the first lemma gives a sufficiently small parameter $t > 0$, while the second one provides the truncation parameter $\widetilde{R}$ given a scale parameter $t > 0$, which is not assumed to be small in this case.

The next result is the main work in this article, and provides the small norm of the large scales double layer potential maximal operator, defined for $f \in L^p(\sigma)$ as

$$K_{l,*} f(x) := \sup_{\varepsilon \geq T} |K_\varepsilon f(x)|, \quad x \in \partial\Omega,$$

assuming flatness conditions at large scales. The proof is deferred to Section 5.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD with unbounded boundary. Assume also that there are $\delta_\beta, \delta_* > 0$ and $S > 0$ such that the following two conditions hold:*

- *Small Jones' $\beta_{\infty,\partial\Omega}$ coefficient at large scales: for $x \in \partial\Omega$ and $r \geq S$,*

$$\beta_{\infty,\partial\Omega}(B(x, r)) \leq \delta_\beta. \tag{3.7}$$

- *Small* BMO *norm of* ν *at large scales: for* $x \in \partial\Omega$ *and* $r \geq S$,

$$\fint_{B(x,r)} |\nu(z) - m_{B(x,r)}\nu| \, d\sigma(z) \leq \delta_*. \tag{3.8}$$

Denote $\delta = \max\{\delta_\beta, \delta_*\}$. *Given* $1 < p < \infty$, *there exists* $\theta = \theta(n, p) > 0$ *(see (5.4)) and* $T = T(p, \delta, S, CAD, n) \gg 100S$ *such that*

$$\|K_{l(T),*}\|_{L^p(\sigma)\to L^p(\sigma)} \lesssim \delta^\theta, \tag{3.9}$$

where the involved constant depends on n, the CAD constants of Ω *and p.*

Although the truncated operators appearing in (3.6) and their properties in Lemmas 3.2 to 3.4 and Theorem 3.5 have not yet been studied and proved, we now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Given $\varepsilon > 0$, by Lemmas 3.3 and 3.4 and Theorem 3.5 there exists $\delta_0 = \delta_0(\varepsilon, p, \text{CAD}, n)$ and $0 < t \ll 1 \ll T \ll \widetilde{R}$ such that if $\delta \leq \delta_0$, then for all $f \in L^p(\sigma)$ there holds

$$\|Kf - \mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)} f\|_{L^p(\sigma)} \overset{(3.6)}{=} \|K_{s(t)} f + K_{l(T)} f + \mathbf{1}_{B_{\widetilde{R}}(0)^c} K_{i(t,T)} f\|_{L^p(\sigma)} < \varepsilon,$$

where we used that $|K_{i(t,T)} f| \leq |K_{s(t)} f| + |K_{s(T)} f|$ σ-a.e. on $\partial\Omega$. By Lemma 3.2, the operator $T = \mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$ is compact (with abuse of notation using T for both the compact operator and the scale). Finally, since T is compact, its adjoint T^* is also compact by Schauder's theorem and moreover

$$\|K^* - T^*\|_{L^{p'}(\sigma)\to L^{p'}(\sigma)} = \|K - T\|_{L^p(\sigma)\to L^p(\sigma)} < \varepsilon,$$

as claimed. □

As a consequence of Theorem 1.4, in the following result we obtain that injectivity implies invertibility for $\frac{1}{2}Id + K$ and $-\frac{1}{2}Id + K^*$, under the assumption of enough flatness.

Corollary 3.6. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a* δ*-*$(s, S; R)$ *domain (see Definition 1.1),* $\lambda_0 > 0$, $1 < p < \infty$ *and* $p' = p/(p-1)$ *its Hölder conjugate exponent. There exists* $\delta_0 = \delta_0(\lambda_0, p, CAD, n)$ *such that if* $\delta \leq \delta_0$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| \geq \lambda_0$, *then the following are equivalent:*

(1) $\lambda Id + K$ *is invertible in* $L^p(\sigma)$,
(2) $\lambda Id + K$ *is injective in* $L^p(\sigma)$,
(3) $\lambda Id + K$ *is surjective in* $L^p(\sigma)$,
(4) $\lambda Id + K^*$ *is invertible in* $L^{p'}(\sigma)$,
(5) $\lambda Id + K^*$ *is injective in* $L^{p'}(\sigma)$,
(6) $\lambda Id + K^*$ *is surjective in* $L^{p'}(\sigma)$.

Proof. By Theorem 1.4 there is $\delta_0 = \delta_0(\lambda_0, p, \text{CAD}, n)$ such that if $\delta \leq \delta_0$ then there is a compact operator T such that $\|K - T\|_{L^p(\sigma)\to L^p(\sigma)} = \|K^* - T^*\|_{L^{p'}(\sigma)\to L^{p'}(\sigma)} < \lambda_0 \leq |\lambda|$.

By Neumann series, the operator $\mathcal{I} \coloneqq \lambda Id + K - T$ is invertible. By the Fredholm alternative Theorem 2.14 and Remark 2.15, if either $\lambda Id + K$ is injective or surjective in $L^p(\sigma)$ we get that it is bijective, and by the bounded inverse theorem we conclude that $(\lambda Id + K)^{-1} : L^p(\sigma) \to L^p(\sigma)$ is a linear bounded operator. This concludes the equivalence between items (1), (2) and (3).

The same argument holds mutatis mutandis with $\lambda Id + K^*$ in $L^{p'}(\sigma)$, whence we get the equivalences between (4), (5) and (6).

Now, since an (arbitrary) operator U is injective if its adjoint U^* is surjective, in particular (6) implies (2), and (3) implies (5). □

3.2. *The compact operator: proof of Lemma 3.2*

We conclude this section by seeing that $\mathbf{1}_{B_{\widetilde{R}}(0)} K_i$ is compact.

Proof of Lemma 3.2. First note that the kernels of $\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)})^*$ are

$$\mathbf{1}_{B_{\widetilde{R}}(0)}(x)\mathbf{1}_{\{y\in\partial\Omega: t<|y-x|\leq T\}}(y)\frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}}, \text{ and}$$
$$\mathbf{1}_{\{y\in B_{\widetilde{R}}(0)\cap\partial\Omega: t<|y-x|\leq T\}}(y)\frac{\langle \nu(x), x-y\rangle}{|x-y|^{n+1}},$$

respectively. In particular, both satisfy the so-called Hilbert-Schmidt condition, see the line before [14, Theorem 0.45] for instance. Hence, by [14, Theorem 0.45] we have that both $\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)})^*$ are bounded and compact from $L^2(\sigma)$ to $L^2(\sigma)$. This concludes the proof for the case $p = 2$.

For $p \in (1, 2)$, fix any $p_0 \in (1, p)$. Since the boundary is UR, we have that both $\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)})^*$ are bounded from $L^{p_0}(\sigma)$ to $L^{p_0}(\sigma)$. By the interpolation theorem in [29] (see also [30, Theorem 3.10]) between compact operators and bounded operators, we conclude that both $\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)}$ and $(\mathbf{1}_{B_{\widetilde{R}}(0)} K_{i(t,T)})^*$ are compact from $L^p(\sigma)$ to $L^p(\sigma)$. This gives the case $p \in (1, 2)$.

The case $p \in (2, \infty)$ follows from the result obtained in the previous paragraph and Schauder's theorem, which states that a bounded linear operator between Banach spaces is compact if and only if its adjoint is compact. This concludes the proof of the lemma. □

4. The double layer potential at small scales. Proofs of Lemmas 3.3 and 3.4

In this section we study the domain and the double layer potential for a fixed scale. We conclude the section by proving Lemmas 3.3 and 3.4.

Theorem 4.1 *([47, Theorem 1.3]). Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD. Then the following weak 1-Poincaré inequality for Lipschitz functions on $\partial\Omega$ holds: there exist constants[4] $C_P \geq 1$ and $\Lambda \geq 1$ such that for every Lipschitz function f on $\partial\Omega$, every $x \in \partial\Omega$ and every $r > 0$, for $\Delta \coloneqq \Delta(x, r)$ we have*

[4] The notation C_P is to specify the constant appearing in the Poincaré inequality.

$$\fint_{\Delta} |f(z) - m_{\Delta} f| \, d\sigma(z) \leq C_P r \fint_{\Lambda\Delta} |\nabla_t f(z)| \, d\sigma(z),$$

where ∇_t is the tangential gradient of f.

This is a no-tail version of [18, Proposition 4.13] for Lipschitz functions, which is enough for our applications.

By the same proof in [18, Theorem 4.14], but using the refined Poincaré inequality above, we obtain the following estimate for the theoretical unit normal vector ν. We provide the proof for completeness.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD. Then there exist $C'_P = C(C_P) > 0$ such that for every $\alpha \in (0,1)$, every $x \in \partial\Omega$ and every $r > 0$, for $\Delta := \Delta(x,r)$ there holds*

$$\sup_{y \in 2\Delta} r^{-1} |\langle x - y, m_{\Delta}\nu \rangle| \leq C'_P \left(\fint_{\Lambda\Delta} |\nu(z) - m_{\Delta}\nu|^{\frac{n}{1-\alpha}} \, d\sigma(z) \right)^{\frac{1-\alpha}{n}},$$

with $\Lambda \geq 1$ as in the weak 1-Poincaré inequality in Theorem 4.1.

Proof. Define the Lipschitz function $g_x(z) := \langle x - z, m_{\Delta}\nu \rangle$ for $z \in \mathbb{R}^{n+1}$. As in [18, (4.2.25)], the claim follows from the particular case $y' = x$ of

$$|g_x(y) - g_x(y')| \leq C'_P r^{1-\alpha} |y - y'|^{\alpha} \left(\fint_{\Lambda\Delta} |\nu(z) - m_{\Delta}\nu|^{\frac{n}{1-\alpha}} \, d\sigma(z) \right)^{\frac{1-\alpha}{n}}, \tag{4.1}$$

for all $y, y' \in 2\Delta$ and each $\alpha \in (0,1)$.

Let us see this. Fixed $\alpha \in (0,1)$, let $p = n/(1-\alpha) > 1$, equivalently $\alpha = 1 - n/p$. For σ-a.e. $z \in \partial\Omega$ we have

$$\begin{aligned} |\nabla_t g_x(z)| &= |m_{\Delta}\nu - \langle m_{\Delta}\nu, \nu(z)\rangle \nu(z)| \\ &= |m_{\Delta}\nu - \nu(z) - \langle m_{\Delta}\nu - \nu(z), \nu(z)\rangle \nu(z)| \leq 2|\nu(z) - m_{\Delta}\nu|. \end{aligned}$$

Now, for any arbitrary boundary ball $\Delta_s \subset \Delta$ (centered at $\partial\Omega$) of radius s, by the 1-Poincaré inequality in Theorem 4.1 we have

$$\begin{aligned} \frac{1}{s} \fint_{\Delta_s} |g_x(z) - m_{\Delta_s} g_x| \, d\sigma(z) &\leq C_P \fint_{\Lambda\Delta_s} |\nabla_t g_x(z)| \, d\sigma(z) \leq 2C_P \fint_{\Lambda\Delta_s} |\nu(z) - m_{\Delta}\nu| \, d\sigma(z) \\ &\leq 2C_P \left(\fint_{\Lambda\Delta_s} |\nu(z) - m_{\Delta}\nu|^p \, d\sigma(z) \right)^{1/p} \lesssim C_P \frac{r^{\frac{n}{p}}}{s^{\frac{n}{p}}} \left(\fint_{\Lambda\Delta} |\nu(z) - m_{\Delta}\nu|^p \, d\sigma(z) \right)^{1/p}. \end{aligned}$$

By the choice of p in terms of α, for all $\Delta_s \subset \Delta$ we get

$$\frac{1}{s^\alpha}\fint_{\Delta_s}|g_x(z)-m_{\Delta_s}g_x|\,d\sigma(z)\le C_P' r^{1-\alpha}\left(\fint_{\Lambda\Delta}|\nu(z)-m_\Delta\nu|^{\frac{n}{1-\alpha}}\,d\sigma(z)\right)^{\frac{1-\alpha}{n}}.$$

This implies the Hölder regularity in (4.1) by Meyer's criterion in [33]. □

From Theorem 4.2 we deduce the following estimate at large scales.

Corollary 4.3. *Under the assumptions of Theorem 4.2 and (3.8), for every $x\in\partial\Omega$ and every $r\ge S/\Lambda$,*

$$\sup_{y\in\Delta(x,2r)} r^{-1}|\langle x-y, m_{\Delta(x,r)}\nu\rangle|\lesssim_{\Lambda,C_P}\delta_*^{\frac{1-\alpha}{n}}. \tag{4.2}$$

Proof. By Theorem 4.2, it suffices to bound its right-hand side, which is controlled by

$$\left(\fint_{\Lambda\Delta}|\nu(z)-m_\Delta\nu|^{\frac{n}{1-\alpha}}\,d\sigma(z)\right)^{\frac{1-\alpha}{n}}\lesssim_\Lambda\left(\fint_{\Lambda\Delta}|\nu(z)-m_{\Lambda\Delta}\nu|^{\frac{n}{1-\alpha}}\,d\sigma(z)\right)^{\frac{1-\alpha}{n}}+\fint_{\Lambda\Delta}|\nu(z)-m_{\Lambda\Delta}\nu|\,d\sigma(z).$$

Since $|\nu(z)-m_{\Lambda\Delta}\nu|\le 2$, we uniformly have

$$\left(\fint_{\Lambda\Delta}|\nu(z)-m_{\Lambda\Delta}\nu|^{\frac{n}{1-\alpha}}\,d\sigma(z)\right)^{\frac{1-\alpha}{n}}\lesssim\left(\fint_{\Lambda\Delta}|\nu(z)-m_{\Lambda\Delta}\nu|\,d\sigma(z)\right)^{\frac{1-\alpha}{n}}. \tag{4.3}$$

Since $\Lambda r\ge S$, by (3.8) both terms in the display above are $\lesssim\delta_*^{\frac{1-\alpha}{n}}$, which proves the claim. □

For completeness, we state the Semmes decomposition, as in [18, Theorem 4.16] (see Remark 2.9 for the applicability to 2-sided CAD).

Theorem 4.4. *Let $\Omega\subset\mathbb{R}^{n+1}$ be a 2-sided CAD. Then there exist $C_*\ge 1$ and $C_1,C_2,C_3,C_4>0$ (depending on the CAD constants of Ω and n) with the property that if for every compact set $\mathcal{K}\subset\mathbb{R}^{n+1}$, there exists $R_{\mathcal{K}}>0$ for which*

$$\sup_{x\in\mathcal{K}\cap\partial\Omega}\|\nu\|_*(\Delta(x,R_{\mathcal{K}}))\le\delta\le 1/C_*,$$

then for every compact set $\mathcal{K}\subset\mathbb{R}^{n+1}$ and $T\ge 1$, setting

$$\widetilde{\mathcal{K}}_T:=\{y\in\mathbb{R}^{n+1}:\operatorname{dist}(y,\mathcal{K})\le T\},$$

and

$$R_{*,T,\mathcal{K}} = \min\{\delta R_{\widetilde{\mathcal{K}}_T}/C_*, \operatorname{diam}(\partial\Omega)/C_*, T\}, \tag{4.4}$$

for $x \in \mathcal{K} \cap \partial\Omega$ and $0 < r \leq R_{*,T,\mathcal{K}}$ the following holds:

(1) There exists a unit vector $\vec{n}_{x,r}$ and a Lipschitz function

$$h : H(x,r) \coloneqq \langle \vec{n}_{x,r} \rangle^{\perp} \to \mathbb{R} \text{ with } \|\nabla h\|_{L^\infty} \leq C_3\sqrt{\delta},$$

and whose graph

$$\mathcal{G} \coloneqq \left\{ y = x + \zeta + t\vec{n}_{x,r} : \zeta \in H(x,r), t = h(\zeta) \right\}$$

(in the coordinate system $y = (\zeta, t) \Longleftrightarrow y = x + \zeta + t\vec{n}_{x,r}$, $\zeta \in H(x,r)$, $t \in \mathbb{R}$) is a good approximation of $\partial\Omega$ in the cylinder

$$\mathcal{C}(x,r) \coloneqq \left\{ x + \zeta + t\vec{n}_{x,r} : \zeta \in H(x,r), |\zeta| \leq r, |t| \leq r \right\}$$

in the sense

$$\sigma\left(\mathcal{C}(x,r) \cap ((\partial\Omega \setminus \mathcal{G}) \cup (\mathcal{G} \setminus \partial\Omega))\right) \leq C_1 w_n r^n \exp\left(-C_2/\sqrt{\delta}\right).$$

(2) There exist two disjoint sets $G(x,r)$ ("good") and $E(x,r)$ ("evil") such that

$$\mathcal{C}(x,r) \cap \partial\Omega = G(x,r) \cup E(x,r) \text{ with } G(x,r) \subset \mathcal{G},$$
$$\sigma(E(x,r)) \leq C_1 w_n r^n \exp\left(-C_2/\sqrt{\delta}\right),$$

and moreover, if $\Pi : \mathbb{R}^{n+1} \to H(x,r)$ is defined by $\Pi(y) = \zeta$ if $y = x + \zeta + t\vec{n}_{x,r} \in \mathbb{R}^{n+1}$ with $\zeta \in H(x,r)$ and $t \in \mathbb{R}$, then

$$\left| y - \left(x + \Pi(y) + h(\Pi(y))\vec{n}_{x,r} \right) \right| \leq C_3\sqrt{\delta} \operatorname{dist}(\Pi(y), \Pi(G(x,r))) \text{ for all } y \in E(x,r),$$

and

$$\mathcal{C}(x,r) \cap \partial\Omega \subset \left\{ x + \zeta + t\vec{n}_{x,r} : |t| \leq C_3\sqrt{\delta} r, \zeta \in H(x,r) \right\}$$
$$\Pi(\mathcal{C}(x,r) \cap \partial\Omega) = \{\zeta \in H(x,r) : |\zeta| < r\}.$$

(3) $\left(1 - C_4\sqrt{\delta}\right) w_n r^n \leq \sigma(\Delta(x,r)) \leq \left(1 + C_4\sqrt{\delta}\right) w_n r^n.$

A few comments are in order. The case $T \geq 1$ follows from the case $T = 1$ by scaling. For the case $T = 1$, a careful inspection of [18, Proof of Theorem 4.16] reveals the following:

(1) The constants $C_* \geq 1$ and $C_1, C_2, C_3, C_4 > 0$ in [18, Theorem 4.16] do not depend on the compact set $\mathcal{K}$.

(2) In [18, p. 2703, l. 6] the authors define

$$R_* = \min\{\delta R_{\widetilde{\mathcal{K}}_1}/(8C), R_{\widetilde{\mathcal{K}}_1}/8, R_0/100, 1\},$$

where R_0 is the constant used in the statement of [18, Theorem 4.14] and $C > 0$ is the geometrical constant appearing in [18, (4.2.20)], i.e., $C = C(\mathrm{CAD})$. It turns out that $R_0 \approx_{\mathrm{CAD}} \mathrm{diam}(\partial\Omega)$ since Ω is a 2-sided CAD, by Definition 2.7 and the fact that being a 2-sided local John domain (see [18, Definition 3.12] with $R = \mathrm{diam}(\partial\Omega)$) with ADR boundary is equivalent to being 2-sided CAD, see [47, Theorem 1.2 and Corollary 1.5].

All in all, reusing the same notation for the constants, there exists a constant $C_* \geq 1$ such that the conclusions of the theorem hold with the choice of $R_{*,T,\mathcal{K}}$ in (4.4).

Given a 2-sided CAD $\Omega \subset \mathbb{R}^{n+1}$, for a boundary ball $\Delta_0 := \Delta(\xi, r_0)$, i.e., $\xi \in \partial\Omega$ and $r_0 > 0$, let $k_0 \in \mathbb{Z}$ the maximal index satisfying $r_0 \leq 2^{-k_0}$ and we define

$$I_0(\Delta_0) := \bigcup_{Q \in \mathcal{Q}_0(\Delta_0)} Q, \text{ where } \mathcal{Q}_0(\Delta_0) := \{Q \in \mathcal{D}_{\sigma,k_0} : Q \cap 2\Delta_0 \neq \emptyset\}, \tag{4.5}$$

recall $\mathcal{D}_{\sigma,k_0}$ is the family of dyadic cubes of $\partial\Omega$ in Lemma 2.12.

Here we state the localized $L^p(\sigma)$ norm of the double layer potential. This estimate is not stated as a separate result in [18], but it is obtained by specializing the argument in the proof of [18, Theorem 4.36] to our localized setting. More precisely, it is the localized bound [18, (4.5.59)] obtained in the course of that proof, which follows from the good lambda inequality [18, (4.5.17)]; the latter relies in turn on Semmes' decomposition [18, Theorem 4.16] (see Theorem 4.4) and on the L^p bounds for the double layer potential on Lipschitz graphs [18, Theorem 4.34].

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 2-sided CAD, $p \in (1, \infty)$, $\xi_0 \in \partial\Omega$ and $r_0 > 0$. There exists $C_0 = C_0(ADR, p, n) \geq 1$ such that if the conclusions in the Semmes decomposition Theorem 4.4 are valid with $\delta > 0$ for all $x \in I_0(\Delta(\xi_0, r_0))$ and all $0 < r \leq C_0 r_0$, and*

$$\sup_{x \in I_0(\Delta(\xi_0,r_0))} \|\nu\|_*(\Delta(x, C_0 r_0)) \leq \delta,$$

then

$$\int_{I_0(\Delta(\xi_0,r_0))} |K_* f|^p \, d\sigma \lesssim \delta^{p/4} \int_{\Delta(\xi_0,r_0)} |f|^p \, d\sigma \text{ for any } f \in L^p(\Delta(\xi_0, r_0)),$$

where the involved constant depends on the CAD constants of Ω, p and n.

Remark 4.6. When invoking Theorem 4.5, we may suppose that $C_0 = C_0(\mathrm{ADR}, p, n)$ is sufficiently large so that $I_0(\Delta(x, r)) \subset \Delta(x, C_0 r)$ for all $x \in \partial\Omega$ and all $r > 0$.

Using Theorem 4.5, we are now ready to prove Lemmas 3.3 and 3.4.

Proof of Lemma 3.3. By the John-Nirenberg inequality, let $C_1 \geq 1$ be a constant satisfying

$$\sup_{x\in\partial\Omega} \|\nu\|_*(\Delta(x, s/2)) \leq C_1\delta.$$

Let C_* and C_0 be the constants in Theorems 4.4 and 4.5 respectively. We can assume $\delta \leq 1/(C_1C_*)$, since otherwise, for any $t > 0$, we have

$$\|K_{s(t)}\|_{L^p(\sigma)\to L^p(\sigma)} \leq 2\|K_*\|_{L^p(\sigma)\to L^p(\sigma)} \overset{(3.5a)}{\lesssim} 1 \lesssim \delta^{1/4}.$$

We fix

$$t_0 = \frac{1}{2C_0}\min\{s/2, \delta s/(2C_*), \operatorname{diam}(\partial\Omega)/C_*, 1\},$$

(recall $R_{*,1,\cdot}$ from (4.4)) so that for all $0 < t \leq t_0$ we have

$$\sup_{x\in\partial\Omega} \|\nu\|_*(B(x, 2C_0t)) \leq C_1\delta,$$

and the conclusions in the Semmes decomposition Theorem 4.4 hold (with $C_1\delta$) for all $x \in \partial\Omega$ and all $0 < r \leq 2C_0t$. For a fixed $0 < t \leq t_0$ and any $x \in \partial\Omega$, applying Theorem 4.5 we obtain

$$\begin{aligned}\|\mathbf{1}_{I_0(\Delta(x,2t))}K_{s(t)}(f\mathbf{1}_{\Delta(x,2t)})\|_{L^p(\sigma)} &\leq 2\|\mathbf{1}_{I_0(\Delta(x,2t))}K_*(f\mathbf{1}_{\Delta(x,2t)})\|_{L^p(\sigma)}\\ &\lesssim \delta^{1/4}\|f\mathbf{1}_{\Delta(x,2t)}\|_{L^p(\sigma)},\end{aligned} \tag{4.6}$$

with $I_0(\cdot)$ as in (4.5).

By the $5R$-covering theorem, let $\{\Delta_i\}_{i\in\mathbb{N}}$ be a subfamily of $\{\Delta(x,t)\}_{x\in\partial\Omega}$ such that $\partial\Omega \subset \bigcup_{i\in\mathbb{N}} \Delta_i$ and $\{\Delta_i/5\}_{i\in\mathbb{N}}$ is pairwise disjoint. Since $\{\Delta_i/5\}_{i\in\mathbb{N}}$ is pairwise disjoint and all balls have the same radius, we have that the family $\{2\Delta_i\}_{i\in\mathbb{N}}$ has finite overlap, with constant depending only on the dimension. For any $f \in L^p(\sigma)$ we have

$$\|K_{s(t)}f\|^p_{L^p(\sigma)} \leq \sum_{i\in\mathbb{N}} \|\mathbf{1}_{\Delta_i}K_{s(t)}f\|^p_{L^p(\sigma)} = \sum_{i\in\mathbb{N}} \|\mathbf{1}_{\Delta_i}K_{s(t)}(f\mathbf{1}_{2\Delta_i})\|^p_{L^p(\sigma)}.$$

For each $i \in \mathbb{N}$, let $I_i \coloneqq I_0(2\Delta_i) \supset 4\Delta_i$. Applying (4.6), we obtain

$$\|\mathbf{1}_{\Delta_i}K_{s(t)}(f\mathbf{1}_{2\Delta_i})\|^p_{L^p(\sigma)} \leq \|\mathbf{1}_{I_i}K_{s(t)}(f\mathbf{1}_{2\Delta_i})\|^p_{L^p(\sigma)} \lesssim \delta^{p/4}\|f\mathbf{1}_{2\Delta_i}\|^p_{L^p(\sigma)}.$$

By the finite overlap of the family $\{2\Delta_i\}_{i\in\mathbb{N}}$, we conclude

$$\|K_{s(t)}f\|_{L^p(\sigma)} \lesssim \delta^{1/4}\|f\|_{L^p(\sigma)}$$

as claimed. □

The proof of Lemma 3.4 follows a similar approach to the proof of Lemma 3.3. However, in the previous proof, we chose the small truncation parameter to ensure that Theorem 4.5 is satisfied. In contrast, here, given a truncation parameter for the "small" scales, we must choose a sufficiently large radius $\widetilde{R}$ so that Theorem 4.5 holds in the complement of the ball $B_{\widetilde{R}}(0)$.

Proof of Lemma 3.4. By the John-Nirenberg inequality, let $C_1 \geq 1$ be a constant such that for any $x \in \partial\Omega$ and any $s > 0$ there holds

$$\|\nu\|_*(B(x, s/2)) \leq C_1 \sup_{\substack{B \subset B(x,s) \\ c_B \in \partial\Omega}} \fint_B |\nu(z) - m_B\nu| \, d\sigma(z),$$

where the supremum is taken over all balls B centered on $\partial\Omega$ and contained in $B(x, s)$.

Let C_* and C_0 be the constants in Theorems 4.4 and 4.5 respectively. As in the proof of Lemma 3.3, we may assume $\delta \leq 1/(C_1C_*)$ and the lemma will follow by finding $\widetilde{R} = \widetilde{R}(t, R)$ such that for all $\xi_0 \in \partial\Omega \setminus B_{\widetilde{R}}(0)$ there holds

$$\|\mathbf{1}_{I_0(\Delta(\xi_0,2t))} K_*(f\mathbf{1}_{\Delta(\xi_0,2t)})\|_{L^p(\sigma)} \lesssim \delta^{1/4} \|f\mathbf{1}_{\Delta(\xi_0,2t)}\|_{L^p(\sigma)},$$

with $I_0(\cdot)$ as in (4.5). By Theorem 4.5, to see this it suffices to find $\widetilde{R}$ such that for all $\xi_0 \in \partial\Omega \setminus B_{\widetilde{R}}(0)$, we have

$$\sup_{x \in I_0(\Delta(\xi_0,2t))} \|\nu\|_*(\Delta(x, 2C_0t)) \leq C_1\delta,$$

and the conclusions of the Semmes decomposition Theorem 4.4 hold (with $C_1\delta$) for all $x \in I_0(\Delta(\xi_0, 2t))$ and all $0 < r \leq 2C_0t$.

We now determine $\widetilde{R}$ to ensure these two conditions hold. By the John-Nirenberg inequality, the δ-$(s, S; R)$ domain Ω satisfies the assumption of the Semmes decomposition Theorem 4.4. Recall that for every $x \in \partial\Omega \setminus B_R(0)$ we have

$$\fint_{B(x,r)} |\nu(z) - m_{B(x,r)}\nu| \, d\sigma(z) \leq \delta, \text{ for all } r \in (0, \infty).$$

In particular, by the John-Nirenberg inequality, we have

$$\sup_{x \in \partial\Omega \setminus B_{R+T}(0)} \|\nu\|_*(\Delta(x, T)) \leq C_1\delta. \tag{4.7}$$

This implies that for any compact set $\mathcal{K} \subset \partial\Omega \setminus B_{R+T}(0)$, we can take $R_{\widetilde{\mathcal{K}}_T} = T$ with $C_1\delta \leq 1/C_*$ in Theorem 4.4. Consequently, the value in (4.4) is $R_{*,T,\mathcal{K}} = \delta T/C_*$, since $\operatorname{diam}(\partial\Omega) = \infty$.

We fix $T = \max\{2C_*C_0t/\delta, 2C_0t\}$ and we take any

$$\widetilde{R} > R + T + 2C_0t.$$

With this choice, for any $\xi_0 \in \partial\Omega \setminus B_{\widetilde{R}}(0)$, we have

$$I_0(\Delta(\xi_0, 2t)) \subset \Delta(\xi_0, 2C_0 t) \subset \partial\Omega \setminus B_{R+T}(0),$$

and therefore,

$$\sup_{x \in I_0(\Delta(\xi_0,2t))} \|v\|_*(\Delta(x, 2C_0 t)) \leq \sup_{x \in \partial\Omega \setminus B_{R+T}(0)} \|v\|_*(\Delta(x, T)) \overset{(4.7)}{\leq} C_1\delta,$$

and the conclusions in the Semmes decomposition Theorem 4.4 hold (with $C_1\delta$) for all $x \in \partial\Omega \setminus B_{R+T}(0) \supset I_0(\Delta(\xi_0, 2t))$ and all $0 < r \leq 2C_0 t$. This concludes the proof. □

5. The double layer potential at large scales. Proof of Theorem 3.5

This section is dedicated to the proof of Theorem 3.5. During this section we assume $\Omega \subset \mathbb{R}^{n+1}$ as in the statement of Theorem 3.5. Briefly, Ω is a 2-sided CAD with unbounded boundary satisfying the small Jones' $\beta_{\infty,\partial\Omega}$ coefficient and small BMO norm of v conditions (3.7) and (3.8) at large scales.

We allow the constants to depend on n, $1 < p < \infty$ and the CAD constants of Ω, and this will not be specified in the computations from now on in this section.

5.1. Relation between v and the unit normal vector from $\beta_{\infty,\partial\Omega}$

Given by (3.7), for each $x \in \partial\Omega$ and $r \geq S$, we fix an n-plane $L_{B(x,r)} \ni x$ such that

$$\sup_{y \in \partial\Omega \cap B(x,r)} \frac{\operatorname{dist}(y, L_{B(x,r)})}{r} \leq 2\delta_\beta, \tag{5.1}$$

and we denote its orthogonal unit vectors as $\pm N_{B(x,r)}$. The orientation of $N_{B(x,r)}$ is fixed below in (5.3).

Lemma 5.1. *Let $x \in \partial\Omega$ and $r \geq S$. For all $y \in B(x,r) \cap \partial\Omega$,*

(1) $|\langle x - y, N_{B(x,r)}\rangle| \leq 2\delta_\beta r$, *and*

(2) $|\langle x - y, m_{B(x,r)} v\rangle| \lesssim \delta_*^{\frac{1}{2n}} r$.

Proof. The second item is proved in (4.2). For the first item, write $B = B(x,r)$ and let N_B and $L_B \ni x$ be as in (5.1). Hence $\operatorname{dist}(y, L_B) \leq 2\delta_\beta r$ for all $y \in B(x,r) \cap \partial\Omega$. For each $y \in B(x,r)$ let $\alpha_y = \angle(N_B, y - x)$ denote the angle between N_B and $y - x$. Therefore, $|\cos\alpha_y| = |\sin(\pi/2 - \alpha_y)| = \operatorname{dist}(y, L_B)/|x - y| \leq 2\delta_\beta r/|x - y|$, and so $|\langle x - y, N_B\rangle| = |x - y||\cos\alpha_y| \leq 2\delta_\beta r$. □

Under the conditions of the lemma above, as in [9, Lemma 5.8], if A is big enough, depending only on the dimension and the ADR constant, then there exists $\{y_j\}_{j=0}^n \subset B(x,r) \cap \partial\Omega$ with $y_0 = x$ and $\operatorname{dist}(y_j, L_{j-1}) \geq A^{-1} r$, where L_k is the k-plane passing through the points $y_0, \ldots, y_k$. In particular $|x - y_j| \approx_A r$ and by Lemma 5.1 we have

$$\left|\left\langle \frac{x-y_j}{|x-y_j|}, N_{B(x,r)} \right\rangle\right| \leq 2A\delta_\beta, \text{ and}$$
$$\left|\left\langle \frac{x-y_j}{|x-y_j|}, m_{B(x,r)}\nu \right\rangle\right| \lesssim A\delta_*^{\frac{1}{2n}}.$$

Therefore, (recall $\delta = \max\{\delta_\beta, \delta_*\}$)

$$|\langle m_{B(x,r)}\nu, N_{B(x,r)}\rangle| \geq 1 - C_A \max\{\delta_\beta, \delta_*^{\frac{1}{2n}}\} \geq 1 - C_A\delta^{\frac{1}{2n}}, \tag{5.2}$$

meaning that $N_{B(x,r)}$ and $\nu_{B(x,r)}$ are almost parallel provided both δ's are small enough. By (5.2), choosing appropriately the orientation of $N_{B(x,r)}$, we assume that

$$|m_{B(x,r)}\nu - N_{B(x,r)}|^2 \lesssim \delta^{\frac{1}{2n}}. \tag{5.3}$$

5.2. Notation and parameters for the proof

Here we write the notation we use in the following sections.

- $S > 0$: the first (large) scale where we have the smallness condition on the Jones' $\beta_{\infty,\partial\Omega}$ coefficient (bound given by δ_β) in (3.7) and the BMO norm of the unit normal vector ν (bound given by δ_*) in (3.8).
- $T \gg 100S$: scale where we truncate the kernel on the large scales.
- L, L_B, L_j, etc: planes, depending on the situation.
- N_B or similar: unit normal vector in the Jones' $\beta_{\infty,\partial\Omega}$ coefficient in (5.1).
- M: big parameter in the proof of (5.8a).
- α: the stopping condition in the construction of the Lipschitz graph, to obtain the Lipschitz graph with norm $\lesssim \alpha$.
- We fix

$$\gamma := \frac{1}{2}\min\left\{1, p-1, \frac{1}{2n-1}\right\} > 0.$$

  In particular, $\gamma/(1+\gamma) < 1/2n$. We will use $\tau^{\frac{1}{2n}} < \tau^{\frac{\gamma}{1+\gamma}}$ when $\tau \in (0,1)$ repeatedly.
- Let $\mathcal{M}_{1+\gamma} = \mathcal{M}_{1+\gamma,\partial\Omega}$ denote the uncentered Hardy-Littlewood maximal function defined in (2.7). Recall it is bounded from $L^p(\sigma)$ to $L^p(\sigma)$ with norm C_p, since $1+\gamma < p \leq \infty$ and $\gamma > 0$ is already fixed depending on p.

Given $S > 0$ and $\delta = \max\{\delta_\beta, \delta_*\}$ we take

$$A = \delta^{-\theta} \text{ with } \theta = \frac{\gamma}{2(n+3)(1+\gamma)}, \tag{5.4}$$

$M = A^{1+\frac{1}{n+1}}$, $\alpha = \delta^{\frac{\gamma}{3(1+\gamma)}}$ and $T = SA^2 \gg 100S$. With this choice, we claim that, as $\delta \to 0$, there holds:

(C1) $A \to \infty$,

(C2) $A/M \to 0$,
(C3) $\delta^{\frac{\gamma}{1+\gamma}} A \le \delta^{\frac{1}{4n}(1-\frac{1}{\sqrt{1+\gamma}})} A^2 \le \delta^{\frac{\gamma}{1+\gamma}} M^{n+1} A \to 0$,
(C4) $\delta^{\frac{\gamma}{1+\gamma}} \le \alpha^3 \ll \alpha \to 0$,
(C5) $\alpha A^2 \to 0$, and
(C6) $AS/T \to 0$.

Indeed, (C1), (C2), (C4) and (C6) are immediate from the choice of parameters. For (C3) we have

$$\delta^{\frac{\gamma}{1+\gamma}} M^{n+1} A = \delta^{\frac{\gamma}{1+\gamma}} A^{(1+\frac{1}{n+1})(n+1)+1} = \delta^{\frac{\gamma}{1+\gamma}-\theta(n+3)} = \delta^{\frac{1}{2}\frac{\gamma}{1+\gamma}} \to 0.$$

For (C5),

$$\alpha A^2 = \delta^{\frac{\gamma}{3(1+\gamma)}-2\theta} = \delta^{(\frac{1}{3}-\frac{1}{n+3})\frac{\gamma}{1+\gamma}} \to 0.$$

5.3. Reduction to a good lambda inequality

In order to estimate the $L^p(\sigma)$ norm of $K_{l,*} = K_{l(T),*}$ in (3.9), it suffices to prove the following good lambda inequality

$$\sigma(\{x \in \partial\Omega : K_{l,*} f(x) > 101\lambda, \mathcal{M}_{1+\gamma} f(x) \le A\lambda\}) \le c_\delta \sigma(\{x \in \partial\Omega : K_{l,*} f(x) > \lambda\}), \quad (5.5)$$

for the parameters fixed above in (5.4) with δ small enough, and $c_\delta \to 0$ as $\delta \to 0$. A standard and routine computation using Fubini's theorem (see [32, Theorem 1.15] for instance), the L^p bound of the maximal operator $\mathcal{M}_{1+\gamma}$ (see (2.7)) and the good lambda inequality (5.5) provides that there exists $\delta_p > 0$ such that if $\delta \le \delta_p$ then

$$\|K_{l,*} f\|_{L^p(\sigma)} \lesssim_{p,n} \frac{1}{A} \|f\|_{L^p(\sigma)}. \quad (5.6)$$

By the choice of $A = \delta^{-\theta}$ in (5.4), we obtain (3.9) with constants depending only on the dimension and p. In the other case, i.e., if $\delta > \delta_p$, then (3.9) follows directly from (3.5a), with constants depending also on the CAD constants of Ω.

5.4. Localization of the good lambda inequality (5.5)

First, let us see the classical Whitney decomposition.

Lemma 5.2 *(Whitney decomposition). If $U \subset \mathbb{R}^{n+1}$ is open, $U \neq \mathbb{R}^{n+1}$, then U can be covered as*

$$U = \bigcup_{i \in I} Q_i,$$

where Q_i, $i \in I$, are (classical) dyadic cubes in $\mathcal{D}(\mathbb{R}^{n+1})$ with disjoint interiors such that the following holds:

(1) $5Q_i \subset U$ *for each* $i \in I$.
(2) $11Q_i \cap U^c \neq \emptyset$ *for each* $i \in I$.
(3) If $3Q_i \cap 3Q_j \neq \emptyset$, $i, j \in I$, *then* $1/(7\sqrt{n+1}) \leq \ell(Q_i)/\ell(Q_j) \leq 7\sqrt{n+1}$.
(4) The family $\{3Q_i\}_{i\in I}$ *has finite overlap with constant depending only on the dimension.*

Proof. The family $\mathcal{F} = \{Q_i\}_{i\in I}$ of maximal dyadic cubes $Q \in \mathcal{D}(\mathbb{R}^{n+1})$ such that $5Q \subset U$ satisfies the above properties by standard arguments. □

For $\lambda > 0$, let W_λ be the family of dyadic cubes given by Lemma 5.2 of the open set $\Omega_\lambda := \mathbb{R}^{n+1} \setminus \{x \in \partial\Omega : K_{l,*}f(x) \leq \lambda\}$. We will refer to W_λ as the Whitney decomposition of Ω_λ. So, we can write

$$V_\lambda := \partial\Omega \cap \Omega_\lambda = \{x \in \partial\Omega : K_{l,*}f(x) > \lambda\} = \bigcup_{Q\in W_\lambda} Q \cap \partial\Omega.$$

Note that $11Q \cap (\partial\Omega \setminus V_\lambda) \neq \emptyset$ for all $Q \in W_\lambda$ and $\{3Q\}_{Q\in W_\lambda}$ has finite overlap.

Notation. We will sometimes use Q instead of $Q \cap \partial\Omega$ when it is clear from the context.

Lemma 5.3. *For* $Q \in W_\lambda$, *there holds*

$$\{x \in Q \cap \partial\Omega : K_{l,*}f(x) > 101\lambda, \mathcal{M}_{1+\gamma}f(x) \leq A\lambda\} \subset \{x \in Q \cap \partial\Omega : K_{l,*}(f\mathbf{1}_{3Q})(x) > 50\lambda\}. \tag{5.7}$$

We note that if A were small, the proof of (5.7) would be standard. However, its dependence on δ makes the proof more involved, relying on the flatness assumptions (3.7) and (3.8).

Proof of Lemma 5.3. Let $x \in Q \cap \partial\Omega$ satisfy $K_{l,*}f(x) > 101\lambda$ and $\mathcal{M}_{1+\gamma}f(x) \leq A\lambda$, and let $z \in 11Q \cap (\partial\Omega \setminus V_\lambda) \neq \emptyset$ so that $K_{l,*}f(z) \leq \lambda$. We have $x, z \in 11Q \subset B(z, 11(n+1)^{1/2}\ell(Q)) =: B$, and so $r(B) \approx \ell(Q)$.

The lemma will follow from

$$K_{l,*}(f\mathbf{1}_{B^c})(x) \leq 41\lambda, \text{ and} \tag{5.8a}$$
$$K_{l,*}(f\mathbf{1}_{B\setminus 3Q})(x) \leq 10\lambda, \tag{5.8b}$$

because

$$K_{l,*}(f\mathbf{1}_{3Q})(x) \geq K_{l,*}f(x) - K_{l,*}(f\mathbf{1}_{B^c})(x) - K_{l,*}(f\mathbf{1}_{B\setminus 3Q})(x) \geq 50\lambda.$$

We first prove (5.8a). For $M > 0$ big enough (auxiliary parameter only used in this proof, already defined depending on A, see (C2) and (C3)) write

$$\begin{aligned} K_{l,*}(f\mathbf{1}_{B^c})(x) &= K_{l,*}(f\mathbf{1}_{B^c})(z) + K_{l,*}(f\mathbf{1}_{B^c})(x) - K_{l,*}(f\mathbf{1}_{B^c})(z) \\ &\leq K_{l,*}(f\mathbf{1}_{B^c})(z) + \left|K_{l,*}(f\mathbf{1}_{MQ\setminus B})(x) - K_{l,*}(f\mathbf{1}_{MQ\setminus B})(z)\right| \\ &\quad + \left|K_{l,*}(f\mathbf{1}_{(MQ)^c})(x) - K_{l,*}(f\mathbf{1}_{(MQ)^c})(z)\right| \\ &=: \boxed{i} + \boxed{ii} + \boxed{iii}. \end{aligned}$$

Since B is centered at z and by the choice of z we have

$$\boxed{i} \leq K_{l,*} f(z) \leq \lambda.$$

Let us bound the term $\boxed{ii}$. We simply have

$$\boxed{ii} \lesssim \frac{1}{r_B^{n+1}} \left(\int_{MQ\setminus B} |\langle \nu(y), y-x\rangle| |f(y)|\, d\sigma(y) + \int_{MQ\setminus B} |\langle \nu(y), y-z\rangle| |f(y)|\, d\sigma(y) \right).$$

For $\xi = x$ or $\xi = z$, we have

$$\int_{MQ\setminus B} |\langle \nu(y), y-\xi\rangle| |f(y)|\, d\sigma(y) \leq \int_{B(\xi, \sqrt{n+1}M\ell(Q))} |\langle \nu(y), y-\xi\rangle| |f(y)|\, d\sigma(y) =: \boxed{ii_\xi}.$$

Set $B_\xi := B(\xi, \sqrt{n+1}M\ell(Q))$. By the triangle inequality (after adding $\pm m_{B_\xi}\nu$), Hölder's inequality, (4.2) and $\mathcal{M}_{1+\gamma} f(x) \leq A\lambda$ we have

$$\begin{aligned}
\boxed{ii_\xi} &\leq \int_{B_\xi} |\langle \nu(y) - m_{B_\xi}\nu, y-\xi\rangle| |f(y)|\, d\sigma(y) + \int_{B_\xi} |\langle m_{B_\xi}\nu, y-\xi\rangle| |f(y)|\, d\sigma(y) \\
&\lesssim M^{n+1}\ell(Q)^{n+1} \left(\fint_{B_\xi} |\nu(y) - m_{B_\xi}\nu|^{\frac{1+\gamma}{\gamma}}\, d\sigma(y) \right)^{\frac{\gamma}{1+\gamma}} \left(\fint_{B_\xi} |f(y)|^{1+\gamma}\, d\sigma(y) \right)^{\frac{1}{1+\gamma}} \\
&\quad + \int_{B_\xi} |\langle m_{B_\xi}\nu, y-\xi\rangle| |f(y)|\, d\sigma(y) \\
&\overset{(4.2)}{\lesssim} M^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}} \mathcal{M}_{1+\gamma} f(x) + M^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{1}{2n}} \mathcal{M}_1 f(x) \\
&\lesssim M^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}} A\lambda.
\end{aligned}$$

Note that even though we are studying the term $\boxed{ii_\xi}$, in any case we controlled $\fint_{B_\xi} |f|\, d\sigma$ and $(\fint_{B_\xi} |f|^{1+\gamma}\, d\sigma)^{1/(1+\gamma)}$ by $\mathcal{M}_1 f(x)$ and $\mathcal{M}_{1+\gamma} f(x)$ respectively, and so we finally used that $\mathcal{M}_{1+\gamma} f(x) \leq A\lambda$. Hence,

$$\boxed{ii} \lesssim \frac{1}{r_B^{n+1}} M^{n+1}\ell(Q)^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}} A\lambda \approx M^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}} A\lambda.$$

Let us bound $\boxed{iii}$. For $\varepsilon \geq T$, we have

$$|K_\varepsilon(f\mathbf{1}_{(MQ)^c})(x) - K_\varepsilon(f\mathbf{1}_{(MQ)^c})(z)| \leq \boxed{A} + \boxed{B} + \boxed{C},$$

where

$$\boxed{A} := \left| \int\limits_{\substack{|y-x|>\varepsilon \\ |y-z|>\varepsilon}} \left\langle \nu(y) f(y) \mathbf{1}_{(MQ)^c}(y), \frac{y-x}{|x-y|^{n+1}} - \frac{y-z}{|z-y|^{n+1}} \right\rangle d\sigma(y) \right|,$$

$$\boxed{B} := \left| \int\limits_{\substack{|y-x|>\varepsilon \\ |y-z|\leq\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}} f(y) \mathbf{1}_{(MQ)^c}(y)\, d\sigma(y) \right|, \text{ and}$$

$$\boxed{C} := \left| \int\limits_{\substack{|y-z|>\varepsilon \\ |y-x|\leq\varepsilon}} \frac{\langle \nu(y), y-z\rangle}{|z-y|^{n+1}} f(y) \mathbf{1}_{(MQ)^c}(y)\, d\sigma(y) \right|.$$

We first study the term $\boxed{A}$. Using the cancellation property of the Calderón-Zygmund kernel first (this is just the mean value theorem), and $B(x, M\ell(Q)/2) \subset MQ$ and $|x-z| \lesssim \ell(Q)$ second, we have

$$\boxed{A} \lesssim \int\limits_{y\notin MQ} |f(y)| \frac{|x-z|}{|x-y|^{n+1}}\, d\sigma(y) \lesssim \ell(Q) \int\limits_{y\notin B(x, \frac{M}{2}\ell(Q))} \frac{|f(y)|}{|x-y|^{n+1}}\, d\sigma(y).$$

Note that the latter integral does not depend on z. Breaking the latter integral into annuli and by the AD regularity of σ, using standard estimates we get

$$\boxed{A} \lesssim \frac{1}{M} \mathcal{M}_{1+\gamma} f(x) \leq \frac{A}{M} \lambda.$$

We now turn to the terms $\boxed{B}$ and $\boxed{C}$. First we note that we can assume that $\varepsilon \geq \frac{M-10}{2}\ell(Q) \geq \frac{M}{4}\ell(Q)$, otherwise $B(z,\varepsilon) \subset MQ$ and so $\boxed{B} = 0$. For $\varepsilon > \frac{M}{4}\ell(Q)$ we have $B(z,\varepsilon) \subset B(x, 2\varepsilon)$, and we get

$$\begin{aligned}
\boxed{B} &\leq \int\limits_{B(x,2\varepsilon)\setminus\overline{B(x,\varepsilon)}} \frac{|\langle \nu(y), y-x\rangle|}{|x-y|^{n+1}} |f(y)|\, d\sigma(y) \lesssim \frac{1}{\varepsilon} \fint\limits_{B(x,2\varepsilon)} |\langle \nu(y), y-x\rangle| |f(y)|\, d\sigma(y) \\
&\leq \frac{1}{\varepsilon} \fint\limits_{B(x,2\varepsilon)} |\langle \nu(y) - m_{B(x,2\varepsilon)}\nu, y-x\rangle| |f(y)|\, d\sigma(y) \\
&\quad + \frac{1}{\varepsilon} \fint\limits_{B(x,2\varepsilon)} |\langle m_{B(x,2\varepsilon)}\nu, y-x\rangle| |f(y)|\, d\sigma(y) \\
&\lesssim \left(\fint\limits_{B(x,2\varepsilon)} |\nu(y) - m_{B(x,2\varepsilon)}\nu|^{\frac{1+\gamma}{\gamma}}\, d\sigma(y) \right)^{\frac{\gamma}{1+\gamma}} \left(\fint\limits_{B(x,2\varepsilon)} |f(y)|^{1+\gamma}\, d\sigma(y) \right)^{\frac{1}{1+\gamma}}
\end{aligned}$$

$$+\left(\sup_{y\in\partial\Omega\cap B(x,2\varepsilon)}\frac{|\langle x-y,m_{B(x,2\varepsilon)}\nu\rangle|}{\varepsilon}\right)\left(\fint_{B(x,2\varepsilon)}|f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}}$$

$$\overset{(4.2)}{\lesssim}(\delta_*^{\frac{\gamma}{1+\gamma}}+\delta_*^{\frac{1}{2n}})\mathcal{M}_{1+\gamma}f(x)\lesssim\delta_*^{\frac{\gamma}{1+\gamma}}\mathcal{M}_{1+\gamma}f(x)\le\delta_*^{\frac{\gamma}{1+\gamma}}A\lambda.$$

By symmetry, but using that $B(z,\varepsilon)\subset B(x,2\varepsilon)$ (recall we can assume $\varepsilon>\frac{M}{4}\ell(Q)$) before applying the bound of the maximal operator $\mathcal{M}_{1+\gamma}$, we have the same bound for $\boxed{C}$, that is, $\boxed{C}\lesssim\delta_*^{\frac{\gamma}{1+\gamma}}\mathcal{M}_{1+\gamma}f(x)\le\delta_*^{\frac{\gamma}{1+\gamma}}A\lambda$. Therefore, the term $\boxed{iii}$ is controlled by

$$\boxed{iii}\le\boxed{A}+\boxed{B}+\boxed{C}\lesssim\left(\frac{A}{M}+\delta_*^{\frac{\gamma}{1+\gamma}}A\right)\lambda$$

From the bound of $\boxed{i}$, $\boxed{ii}$ and $\boxed{iii}$, and conditions (C2) and (C3) we conclude

$$K_{l,*}(f\mathbf{1}_{B^c})(x)\le\lambda+C\left(M^{n+1}\delta_*^{\frac{\gamma}{1+\gamma}}A+\frac{A}{M}+\delta_*^{\frac{\gamma}{1+\gamma}}A\right)\lambda\le 41\lambda,$$

as claimed in (5.8a).

It remains to prove (5.8b). For $\varepsilon\ge T$, we may assume $\{y:|y-x|>\varepsilon\}\cap B\neq\emptyset$, otherwise $K_\varepsilon(f\mathbf{1}_{B\setminus 3Q})(x)=0$. Thus, $r(B)\gtrsim\varepsilon\ge T$, allowing us to apply (4.2). Using that $\ell(Q)\approx r(B)$ and that $|y-x|\gtrsim\ell(Q)$ if $y\notin 3Q$, and arguing in the last inequality as in the bound for $\boxed{ii_\xi}$, we get

$$\begin{aligned}K_\varepsilon(f\mathbf{1}_{B\setminus 3Q})(x)&=\left|\int_{|y-x|>\varepsilon}\frac{\langle\nu(y),y-x\rangle}{|x-y|^{n+1}}f(y)\mathbf{1}_{B\setminus 3Q}(y)\,d\sigma(y)\right|\\&\lesssim\frac{1}{\ell(Q)}\left(\fint_B|\langle\nu(y)-m_B\nu,y-x\rangle||f(y)|\,d\sigma(y)+\fint_B|\langle m_B\nu,y-x\rangle||f(y)|\,d\sigma(y)\right)\\&\lesssim\delta_*^{\frac{\gamma}{1+\gamma}}A\lambda,\end{aligned}$$

uniformly in $\varepsilon\ge T$. Then (5.8b) follows by condition (C3). □

By (5.7), in order to prove (5.5) we claim that it suffices to see

$$\sigma(\{x\in Q:K_{l,*}(f\mathbf{1}_{3Q})(x)>50\lambda,\mathcal{M}_{1+\gamma}f(x)\le A\lambda\})\le c_\delta\sigma(3Q),\tag{5.9}$$

for all $Q\in W_\lambda$. Indeed,

$$\begin{aligned}&\sigma(\{x\in\partial\Omega:K_{l,*}f(x)>101\lambda,\mathcal{M}_{1+\gamma}f(x)\le A\lambda\})\\&\quad\le\sum_{Q\in W_\lambda}\sigma(\{x\in Q:K_{l,*}f(x)>101\lambda,\mathcal{M}_{1+\gamma}f(x)\le A\lambda\})\end{aligned}$$

$$\overset{(5.7)}{\leq} \sum_{Q\in W_\lambda} \sigma(\{x\in Q : K_{l,*}(f\mathbf{1}_{3Q})(x) > 50\lambda, \mathcal{M}_{1+\gamma} f(x) \leq A\lambda\})$$

$$\overset{(5.9)}{\leq} c_\delta \sum_{Q\in W_\lambda} \sigma(3Q) \lesssim c_\delta \sigma(\Omega_\lambda) = c_\delta \sigma(V_\lambda),$$

where in the last step we used the finite overlap of the family $\{3Q\}_{Q\in W_\lambda}$.

Remark 5.4. From now on we can assume without loss of generality that $\ell(Q) \geq \frac{T}{10(n+1)^{1/2}}$, otherwise, if $x \in Q$ then $\{y \in 3Q : |y-x| > T\} = \emptyset$ and hence

$$K_{l,*}(f\mathbf{1}_{3Q})(x) = \sup_{\varepsilon\geq T} \left| \int_{y\in 3Q\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^{n+1}} f(y)\mathbf{1}_{3Q}(y)\, d\sigma(y) \right| = 0.$$

5.5. Construction of a Lipschitz graph

During this section, let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with unbounded boundary, and let $\mathcal{D}_\sigma$ denote its dyadic lattice in Lemma 2.12. For $t > 1$ and $Q \in \mathcal{D}_\sigma$ we set

$$tQ := \{x \in \partial\Omega : \operatorname{dist}(x, Q) \leq (t-1)\operatorname{diam} Q\}. \tag{5.10}$$

Notation *(Dependence of parameters).* Fixed $m > 1$, we choose $k_0 = k_0(m) \gg m$ and $k = k(k_0) \gg k_0$. We consider $\varepsilon, \alpha \ll 1$ such that $\varepsilon \ll \alpha$ and $\varepsilon \ll 1/k_0$. Throughout this section, we allow the constant C to depend on the ADR constant, but not on the parameters mentioned above.

Following [9, Section 8], in this section we construct a Lipschitz approximating graph at large scales (see Proposition 5.9), which only uses the following flatness assumption at large scales: We assume that for each cube $Q \in \mathcal{D}_\sigma$ with $\operatorname{diam} Q \geq S$, there exists an n-plane L_Q such that

$$\operatorname{dist}(x, L_Q) \leq \varepsilon \operatorname{diam} Q \text{ for all } x \in kQ. \tag{5.11}$$

We will see in Remark 5.16 in Section 5.6 below that this is guaranteed under the assumption (3.7).

We recall that the n-plane in (5.11) is almost unique in the following sense.

Lemma 5.5 *(See [9, Lemma 5.13]). Let $Q \in \mathcal{D}_\sigma$ and assume that L_1 and L_2 are two n-planes such that* $\operatorname{dist}(x, L_i) \leq \varepsilon' \operatorname{diam} Q$ *for all $x \in Q$, $i = 1, 2$. Then $\angle(L_1, L_2) \leq C\varepsilon'$.*

Let us fix $R \in \mathcal{D}_{\sigma,j_0}$ with $j_0 \in \mathbb{Z}$ so that $2^{-j_0} \geq S$; in particular $\operatorname{diam} Q \geq S$ for all $Q \in \mathcal{D}_{\sigma,j_0}$. Fixed $m > 1$, we define the "m-neighborhood cubes" of R as the collection of cubes in $\mathcal{D}_{\sigma,j_0}$ touching mR. That is,

$$U_m(R) := \{Q \in \mathcal{D}_{\sigma,j_0} : Q \cap mR \neq \emptyset\}.$$

So, for every $Q \in U_m(R)$ there holds $\operatorname{dist}(Q, R) \leq (m-1)\operatorname{diam} R$. For any two cubes $Q_1, Q_2 \in U_m(R)$, if $x \in Q_1$ then

$$\operatorname{dist}(x, Q_2) \leq \operatorname{diam} Q_1 + \operatorname{dist}(Q_1, R) + \operatorname{diam} R + \operatorname{dist}(R, Q_2) \leq C_{\mathcal{D}} 2m \operatorname{diam} Q_2,$$

and therefore

$$Q_1 \subset (C_{\mathcal{D}} 2m + 1) Q_2.$$

Fix a constant $k \gg m$ (at least $k \geq C_{\mathcal{D}} 2m + 1$), so that the inclusion above implies

$$Q_1 \subset k Q_2 \text{ for any } Q_1, Q_2 \in U_m(R).$$

As a consequence of this, (5.11) and Lemma 5.5, we have

$$\measuredangle(L_{Q_1}, L_{Q_2}) \lesssim \varepsilon \text{ for any } Q_1, Q_2 \in U_m(R). \tag{5.12}$$

Definition 5.6. We say that $Q \in \bigcup_{j \geq j_0} \mathcal{D}_{\sigma,j}$ is a stopping cube with respect to R, and we write $Q \in \mathsf{Stop}(R)$, if $Q \subset \bigcup_{Q' \in U_m(R)} Q'$ and Q is maximal such that either

(1) $\operatorname{diam} Q < S$, or
(2) $\operatorname{diam} Q \geq S$ and $\measuredangle(L_R, L_Q) > \alpha$, recall L_R and L_Q as in (5.11).

Definition 5.7. We define the set $\mathsf{Tree}(R)$ as the collection of cubes $P \in \bigcup_{j \geq j_0} \mathcal{D}_{\sigma,j}$ such that

(1) $P \subset \bigcup_{Q' \in U_m(R)} Q'$, and
(2) $P \not\subset Q$ for any $Q \in \mathsf{Stop}(R)$.

In particular, every $Q \in \mathsf{Tree}(R)$ satisfies $\operatorname{diam} Q \geq S$ and $\measuredangle(L_R, L_Q) \leq \alpha$.

Remark 5.8. In contrast to the construction in [9, Section 8], where only dyadic subcubes of R are considered, here we take dyadic subcubes of $U_m(R)$ in the construction of $\mathsf{Tree}(R)$. In our situation, since $\varepsilon \ll \alpha$ and $\operatorname{diam} Q \geq S$ for all $Q \in U_m(R)$, we note that (5.12) ensures that $Q \notin \mathsf{Stop}(R)$ for all $Q \in U_m(R)$. Consequently, given a cube in $\mathsf{Stop}(R)$, its parent is in $\mathsf{Tree}(R)$.

Consider

$$d(x) = d_R(x) := \inf_{Q \in \mathsf{Tree}(R)} \{\operatorname{dist}(x, Q) + \operatorname{diam} Q\}, \quad x \in \mathbb{R}^{n+1}.$$

Since $\#\mathsf{Tree}(R) < \infty$, the infimum is in fact a minimum. Note that $d(\cdot) \geq S$ and is 1-Lipschitz, as it is the infimum of 1-Lipschitz functions. Clearly, if $x \in Q \in \mathsf{Tree}(R)$ then $d(x) \leq \operatorname{diam} Q$.

In this section, we prove the following result, which is the analog of [9, Proposition 8.2]. In this lemma and throughout this section, $L_R^\perp$ denotes the orthogonal line to the n-plane L_R (see (5.11)), $\Pi = \Pi_R$ denotes the orthogonal projection onto L_R, and $\Pi^\perp = \Pi_R^\perp$ denotes the orthogonal projection onto $L_R^\perp$.

Proposition 5.9. *For $R \in \mathcal{D}_{\sigma,j_0}$ with $j_0 \in \mathbb{Z}$ so that $2^{-j_0} \geq S$ (in particular $\operatorname{diam} R \geq S$), there exists $A : L_R \to L_R^{\perp}$ Lipschitz graph with norm $\leq C\alpha$ such that*

$$\operatorname{dist}(x, (\Pi_R(x), A(\Pi_R(x)))) \leq C\varepsilon d_R(x) \text{ for all } x \in k_0 R. \tag{5.13}$$

We define in L_R the 1-Lipschitz function

$$D(p) = D_R(p) \coloneqq \inf_{x \in \Pi^{-1}(p)} d_R(x), \quad p \in L_R, \tag{5.14}$$

which can be rewritten as

$$D(p) = \inf_{x \in \Pi^{-1}(p)} \inf_{Q \in \mathsf{Tree}(R)} \{\operatorname{dist}(x, Q) + \operatorname{diam} Q\} = \inf_{Q \in \mathsf{Tree}(R)} \{\operatorname{dist}(p, \Pi_R(Q)) + \operatorname{diam} Q\}.$$

As $d \geq S$ in $\mathbb{R}^{n+1}$, in particular $D \geq S$ in L_R. As before, since $\#\mathsf{Tree}(R) < \infty$, the latter infimum is in fact a minimum.

Arguing as in the proof of [9, Lemma 8.4], we obtain the following, as the proof relies only on the fact that every $Q \in \mathsf{Tree}(R)$ satisfies $\angle(L_R, L_Q) \leq \alpha$.

Lemma 5.10 *([9, Lemma 8.4]). If $x, y \in 10k_0 R$ satisfy $|x - y| \geq 10^{-3} \min\{d(x), d(y)\}$, then*

$$|\Pi^{\perp}(x) - \Pi^{\perp}(y)| \leq 2\alpha |\Pi(x) - \Pi(y)|.$$

Now we decompose L_R in classical dyadic cubes using the function D_R. First, we shall identify L_R with $\mathbb{R}^n$, and in particular we equip L_R with classical dyadic cubes $\mathcal{D}_{L_R} \coloneqq \mathcal{D}_{\mathbb{R}^n}$. For each $x \in L_R$ (recall $D_R \geq S > 0$ in L_R), let $R_x \in \mathcal{D}_{L_R}$ be the largest dyadic cube containing x and satisfying

$$\operatorname{diam} R_x \leq 20^{-1} \inf_{u \in R_x} D(u). \tag{5.15}$$

We relabel them without repetition as $\{R_i\}_{i \in I}$. Thus, the family $\{R_i\}_{i \in I}$ is pairwise disjoint and covers L_R. (As in [9, Section 8], we use the convention that dyadic cubes are closed but are called disjoint if their interiors are disjoint.)

Recall that the definition of tQ is slightly different depending on whether $Q \in \mathcal{D}_\sigma$ or $Q \in \mathcal{D}_{L_R} = \mathcal{D}_{\mathbb{R}^n}$, see (5.10) and Section 2.1 respectively.

From the definition of the family $\{R_i\}_{i \in I}$ using (5.15) and that D is 1-Lipschitz in L_R, by the same proof of [9, Lemma 8.7] we have:

Lemma 5.11 *([9, Lemma 8.7]). For all $y \in 10R_i$, $i \in I$,*

$$10 \operatorname{diam} R_i \leq D(y) \leq 60 \operatorname{diam} R_i, \tag{5.16}$$

and in particular, if $10R_i \cap 10R_j \neq \emptyset$, $i, j \in I$, then

$$6^{-1} \operatorname{diam} R_i \leq \operatorname{diam} R_j \leq 6 \operatorname{diam} R_i. \tag{5.17}$$

Let $U_0 \coloneqq L_R \cap B(\Pi(x_R), 2k_0 \operatorname{diam} R)$, where x_R is any fixed point in R, and $I_0 \coloneqq \{i \in I : R_i \cap U_0 \neq \emptyset\}$. We fix $k_0 \gg m$ so that $\bigcup_{Q' \in U_m(R)} \Pi(Q') \subset U_0$. For each $i \in I_0$, let $Q(i) \in \mathsf{Tree}(R)$ be such that[5]

$$C^{-1} \frac{1}{k_0} \operatorname{diam} R_i \leq \operatorname{diam} Q(i) \leq C \operatorname{diam} R_i, \text{ and} \tag{5.18a}$$

$$\operatorname{dist}(\Pi(Q(i)), R_i) \leq C \operatorname{diam} R_i. \tag{5.18b}$$

Such cubes $Q(i)$ exist, because if $p \in R_i$, then there exists $Q \in \mathsf{Tree}(R)$ such that

$$\operatorname{dist}(p, \Pi(Q)) + \operatorname{diam} Q \leq 2D(p) \overset{(5.16)}{\leq} 120 \operatorname{diam} R_i,$$

and we can take $Q(i)$ to be the maximal cube in $\mathsf{Tree}(R)$ with $Q \subset Q(i)$ and $\operatorname{diam} Q(i) \leq 120 \operatorname{diam} R_i$.

Definition 5.12. For $i \in I_0$, let $A_i : L_R \to L_R^{\perp}$ denote the affine function whose graph is the n-plane $L_{Q(i)}$. By the stopping condition, the Lipschitz norm is $\leq 2\alpha$.

We consider a partition of unity for $V = \bigcup_{i \in I_0} 2R_i$. That is, a family $\{\phi_i\}_{i \in I_0}$ satisfying for all $i \in I_0$ that $\phi_i \geq 0$, $\phi_i \in C_c^1(3R_i)$ (hence $\{\operatorname{supp} \phi_i\}_{i \in I_0}$ has finite overlap, by (5.17)) and

$$|\nabla \phi_i| \lesssim \frac{1}{\operatorname{diam} R_i}.$$

Finally we define A on V by

$$A(p) \coloneqq \sum_{i \in I_0} \phi_i(p) A_i(p) \text{ for } p \in V.$$

By the same proof in [9, Lemma 8.17] we have:

Lemma 5.13 *([9, Lemma 8.17]). If $10R_i \cap 10R_j \neq \emptyset$, then $\operatorname{dist}(Q(i), Q(j)) \leq C \operatorname{diam} R_j$ and*

$$|A_i(q) - A_j(q)| \leq C\varepsilon \operatorname{diam} R_j \text{ for all } q \in 100R_j. \tag{5.19}$$

Using the previous lemma, the same proof in [9, Section 8, p. 46] applies to see that the restriction of A to $2R_j$, $j \in I_0$, is 3α-Lipschitz.

Lemma 5.14 *([9, (8.19)]). If $p, q \in 2R_j$, $j \in I_0$, then*

$$|A(p) - A(q)| \leq 3\alpha |p - q|.$$

[5] The correspondence $I_0 \ni i \mapsto Q(i)$ may not be injective.

We now aim to show that A is $C\alpha$-Lipschitz in U_0. We remark that this is the analog of [9, (8.20)], with the difference that in our case, we have $D(\cdot) \geq S > 0$ in L_R. For the sake of completeness, we provide the full details of the proof.

Lemma 5.15. *If $p, q \in U_0$, then $|A(p) - A(q)| \leq C\alpha|p - q|$.*

Proof. Since $D \geq S > 0$, all points in U_0 belong to some (unique) R_k, where $k \in I_0$. Let $i, j \in I_0$ be such that $p \in R_i$ and $q \in R_j$. If $p, q \in 2R_i$ or $p, q \in 2R_j$, then $|A(p) - A(q)| \leq 3\alpha|p - q|$ by Lemma 5.14, and we are done. Hence, from now on, we may assume that $q \notin 2R_i$ and $p \notin 2R_j$. In particular, this implies

$$|p - q| \geq \max\{\operatorname{diam} R_i, \operatorname{diam} R_j\}. \tag{5.20}$$

Let $y \in Q(i)$ and $z \in Q(j)$ be such that $|y - z| = \sup_{a\in Q(i), b\in Q(j)} |a - b|$, which in particular implies

$$|y - z| \geq \frac{1}{2}\min\{\operatorname{diam} Q(i), \operatorname{diam} Q(j)\}. \tag{5.21}$$

We have

$$\begin{aligned} |A(p) - A(q)| \leq\ & |A(p) - A_i(p)| + |A_i(p) - A_i(\Pi(y))| + |A_i(\Pi(y)) - \Pi^{\perp}(y)| \\ & + |\Pi^{\perp}(y) - \Pi^{\perp}(z)| \\ & + |A(q) - A_j(q)| + |A_j(q) - A_j(\Pi(z))| + |A_j(\Pi(z)) - \Pi^{\perp}(z)|. \end{aligned}$$

Let us see that all the terms are bounded by $\lesssim \alpha$. We bound the first four terms, and the last three follow by symmetry.

Term $|A(p) - A_i(p)|$: Using the partition of unity in the second equality, and by (5.19) in Lemma 5.13 (since $p \in R_i$, the fact that $\phi_j(p) \neq 0$ implies that $\operatorname{supp}\phi_j \cap R_i \neq \emptyset$, and in particular $3R_j \cap R_i \neq \emptyset$) and since the last sum runs over the index $j \in I_0$ around R_i (in particular a finite number of candidates), we have

$$\begin{aligned} |A(p) - A_i(p)| &= \left|\left(\sum_{j\in I_0} \phi_j(p)A_j(p)\right) - A_i(p)\right| = \left|\sum_{j\in I_0} \phi_j(p)(A_j(p) - A_i(p))\right| \\ &\overset{(5.19)}{\leq} C\varepsilon\operatorname{diam} R_i \overset{(5.20)}{\leq} C\varepsilon|p - q| \leq \alpha|p - q|. \end{aligned} \tag{5.22}$$

Term $|A_i(p) - A_i(\Pi(y))|$: First, $|A_i(p) - A_i(\Pi(y))| \leq 2\alpha|p - \Pi(y)|$ since A_i is 2α-Lipschitz, and second $|p - \Pi(y)| \leq \operatorname{diam} R_i + \operatorname{dist}(R_i, \Pi(Q(i))) + \operatorname{diam} Q(i)$, which by the choice of $Q(i)$ in (5.18a) and (5.18b) the last two terms are controlled by $\lesssim \operatorname{diam} R_i$. We conclude this term by using (5.20).

Term $|A_i(\Pi(y)) - \Pi^{\perp}(y)|$: Since $\angle(L_{Q(i)}, L_R) \leq \alpha$ is small and by (5.11), we have $|A_i(\Pi(y)) - \Pi^{\perp}(y)| \leq 2\operatorname{dist}(y, L_{Q(i)}) \leq 2\varepsilon\operatorname{diam} Q(i)$. By the choice of $Q(i)$ in (5.18a) this is controlled by $\leq C\varepsilon\operatorname{diam} R_i$, and by (5.20) the last term is controlled by $\leq C\varepsilon|p - q| \leq \alpha|p - q|$.

Term $|\Pi^{\perp}(y)-\Pi^{\perp}(z)|$: First, $y\in Q(i)$ implies that $d(y)\leq \operatorname{diam} Q(i)$, and $z\in Q(j)$ implies $d(z)\leq \operatorname{diam} Q(j)$. Thus, from (5.21) and this we have $|y-z|\geq \frac{1}{2}\min\{\operatorname{diam} Q(i), \operatorname{diam} Q(j)\}\geq \frac{1}{2}\min\{d(y),d(z)\}$. Using Lemma 5.10 in the first inequality we get

$$|\Pi^{\perp}(y)-\Pi^{\perp}(z)|\leq 2\alpha|\Pi(y)-\Pi(z)|\leq 2\alpha\left(|\Pi(y)-p|+|p-q|+|q-\Pi(z)|\right).$$

As in the bound of the second term (that is, $|A_i(p)-A_i(\Pi(y))|$), the first term inside the brackets is bounded by $\leq C\operatorname{diam} R_i$. By symmetry, the third term is controlled by $\leq C\operatorname{diam} R_j$. This term follows by (5.20) as well. □

Now we can use the Whitney extension theorem to extend A from U_0 to a $C\alpha$-Lipschitz function on all L_R. This gives the existence of the $C\alpha$-Lipschitz graph in Proposition 5.9. It remains to see now that this $C\alpha$-Lipschitz graph approximates k_0R in the (5.13) sense.

Proof of (5.13). Let $x\in k_0R$ and set $p=\Pi(x)$. Recall $D(p)\geq S>0$, and let R_i, $i\in I_0$, so that $p\in R_i$. We first break

$$\operatorname{dist}(x,(\Pi(x),A(\Pi(x))))\leq |\Pi^{\perp}(x)-A_i(\Pi(x))|+|A(\Pi(x))-A_i(\Pi(x))|.$$

Applying [9, Lemma 8.21] with $r=D(p)$ and $Q=Q(i)$, we get that $x\in C_{k_0}Q(i)$ for some C_{k_0} depending on k_0, so taking $k\gg C_{k_0}$ we have $x\in kQ(i)$. Using that $\angle(L_R,L_{Q(i)})\leq\alpha$ is small, the flatness condition in (5.11) and the choice of $Q(i)$ in (5.18a) respectively, we have

$$|\Pi^{\perp}(x)-A_i(\Pi(x))|\leq 2\operatorname{dist}(x,L_{Q(i)})\leq 2\varepsilon\operatorname{diam} Q(i)\lesssim \varepsilon\operatorname{diam} R_i.$$

The term $|A(\Pi(x))-A_i(\Pi(x))|$ is simply the first term in the proof of Lemma 5.15, whence we obtain directly from (5.22) that

$$|A(\Pi(x))-A_i(\Pi(x))|\leq C\varepsilon\operatorname{diam} R_i.$$

All in all, $\operatorname{dist}(x,(\Pi(x),A(\Pi(x))))\leq C\varepsilon\operatorname{diam} R_i$. From the definition of the family $\{R_i\}_{i\in I}$ in (5.15), we have $\operatorname{diam} R_i\leq 20^{-1}D(p)$. Just from the definition D in (5.14), $D(p)\leq d(x)$. This concludes the proof of (5.13). □

5.6. *Proof of the localized good lambda inequality (5.9)*

Let Ω be as in the statement of Theorem 3.5. To apply the notation of the section of the Lipschitz graph construction, for $\lambda>0$ we write $R_W\in W_\lambda$ for the cube from the Whitney covering of Ω_λ (the letters Q, R are reserved for boundary dyadic cubes $\mathcal{D}_\sigma$), and we rewrite the good lambda inequality (5.9) as

$$\sigma(\{x\in R_W: K_{l,*}(f\mathbf{1}_{3R_W})(x)>50\lambda, \mathcal{M}_{1+\gamma}f(x)\leq A\lambda\})\leq c_\delta\sigma(3R_W). \tag{5.23}$$

We will use without mention that we assume that there exists $x_0\in R_W$ with $\mathcal{M}_{1+\gamma}f(x_0)\leq A\lambda$, otherwise the left-hand side in (5.23) is zero and we are done. Recall also from Remark 5.4 that we have $\ell(R_W)\gtrsim T\gg S$.

The thin boundary condition (4) in Lemma 2.12 implies, for each $Q \in \mathcal{D}_\sigma$, the existence of a point $c_Q \in Q$, called the center of Q, such that $\operatorname{dist}(c_Q, \partial\Omega \setminus Q) \gtrsim_{\mathrm{ADR},n} \operatorname{diam} Q$, see [10, Lemma 3.5 of Part I]. We denote

$$B(Q) := B(c_Q, c_1 \operatorname{diam} Q), \quad B_Q := B(c_Q, \operatorname{diam} Q) \supset Q, \tag{5.24}$$

where $c_1 \in (0, 1]$ is chosen so that $B(Q') \cap \partial\Omega \subset Q'$ for all $Q' \in \mathcal{D}_\sigma$, satisfying also that $B(Q_1) \cap B(Q_2) = \emptyset$ for all disjoint $Q_1, Q_2 \in \mathcal{D}_\sigma$. Indeed, assuming without loss of generality that $\operatorname{diam} Q_2 \leq \operatorname{diam} Q_1$, if $B(Q_1) \cap B(Q_2) \neq \emptyset$, then

$$\operatorname{diam} Q_1 \lesssim_{\mathrm{ADR},n} \operatorname{dist}(c_{Q_1}, \partial\Omega \setminus Q_1) \leq |c_{Q_1} - c_{Q_2}| \leq c_1(\operatorname{diam} Q_1 + \operatorname{diam} Q_2) \leq 2c_1 \operatorname{diam} Q_1,$$

which is not possible if c_1 is chosen to be sufficiently small, depending only on the ADR character of $\partial\Omega$.

We begin the proof of (5.23). Given $R_W \in W_\lambda$, let $j \in \mathbb{Z}$ be so that $\ell(R_W) = 2^{-j}$ (recall $R_W \in \mathcal{D}(\mathbb{R}^{n+1})$), fix any $x \in R_W \cap \partial\Omega$ (if such point does not exist then the left-hand side of (5.23) is zero), and let $R \in \mathcal{D}_{\sigma,j}$ be the cube with $x \in R$. We fix $m = 3$ so that $\partial\Omega \cap 3R_W \subset mR$. Indeed, since $\operatorname{diam} R \geq 2^{-j}$, for all $y \in R_W$ we have $|y - x| \leq 2\operatorname{diam} R_W = 2 \cdot 2^{-j} \leq 2\operatorname{diam} R$, thus $\operatorname{dist}(y, R) \leq |y - x| \leq 2\operatorname{diam} R$, see (5.10).

For every $Q \in U_m(R)$ and every $x \in Q$,

$$|x - c_R| \leq \operatorname{diam} Q + (m-1)\operatorname{diam} R + \operatorname{diam} R \leq C_{\mathcal{D}}(m+1)\operatorname{diam} R,$$

where $C_{\mathcal{D}}$ is the constant in (3) in the boundary dyadic lattice Lemma 2.12. Also, if $x \in B_Q \supset B(Q)$ (not necessarily in Q), then

$$|x - c_R| \leq |x - c_Q| + |c_Q - c_R| \leq \operatorname{diam} Q + C_{\mathcal{D}}(m+1)\operatorname{diam} R \leq C_{\mathcal{D}}(m+2)\operatorname{diam} R.$$

This implies that $\bigcup_{Q \in U_m(R)} B_Q \supset \bigcup_{Q \in U_m(R)} B(Q)$ and $\bigcup_{Q \in U_m(R)} Q$ are subsets of

$$\mathfrak{B}_R := C_{\mathcal{D}}(m+2) B_R, \tag{5.25}$$

with B_R as in (5.24).

Since σ is ADR, we will use that

$$\sigma(3R_W) \approx \sigma(R) \approx_m \sigma(\mathfrak{B}_R) \approx_m \sigma\left(\bigcup_{Q \in U_m(R)} Q\right).$$

Remark 5.16. Under the choice in (5.1), the estimate (5.11) holds with $\varepsilon = 2k\delta_\beta$ and $L_Q = L_{kB_Q}$. Indeed, as $kQ \subset kB_Q$ we have

$$\sup_{x \in kQ} \operatorname{dist}(x, L_Q) \leq \sup_{x \in \partial\Omega \cap kB_Q} \operatorname{dist}(y, L_Q) \overset{(5.1)}{\leq} 2\delta_\beta k \operatorname{diam} Q.$$

Recall in Section 5.5 we needed $\varepsilon = 2k\delta_\beta \ll \alpha$, which is granted by (C4).

We define the following subfamilies of $\mathsf{Stop}(R)$ (see Definition 5.6) of big and small angle respectively as

$$\mathsf{BA}(R) \coloneqq \{Q \in \mathsf{Stop}(R) : \angle(L_R, L_Q) > \alpha\}, \text{ and}$$
$$\mathsf{SA}(R) \coloneqq \{Q \in \mathsf{Stop}(R) : \angle(L_R, L_Q) \leq \alpha\},$$

so that $\mathsf{Stop}(R) = \mathsf{BA}(R) \cup \mathsf{SA}(R)$ and the union is disjoint. By the stopping conditions in Definition 5.6 in the construction of the Lipschitz graph, we have

$$\mathsf{BA}(R) \supset \{Q \in \mathsf{Stop}(R) : \operatorname{diam} Q \geq S\}, \text{ and}$$
$$\mathsf{SA}(R) \subset \{Q \in \mathsf{Stop}(R) : \operatorname{diam} Q < S\}.$$

Remark 5.17. If $Q \in \mathsf{SA}(R)$, then $\operatorname{diam} Q < S$ but its parent $\widehat{Q} \in \mathsf{Tree}(R)$ satisfies $\operatorname{diam} \widehat{Q} \geq S$ as $\widehat{Q} \notin \mathsf{Stop}(R)$, and therefore $\operatorname{diam} Q \approx S$.

Let us define

$$G_R \coloneqq \bigcup_{Q \in \mathsf{SA}(R)} Q.$$

The notation G_R stands for "good" in the following sense:

Lemma 5.18. *We have* $\sigma\left(\bigcup_{Q \in U_m(R)} Q \setminus G_R\right) = \sigma\left(\bigcup_{Q \in \mathsf{BA}(R)} Q\right) \lesssim \delta^{1/3} \sigma(3R_W)$.

Proof. For each $Q \in \mathsf{BA}(R)$, let L_Q be as in (5.11) (with $\varepsilon = 2k\delta_\beta$) and L_{B_Q} as in (5.1). By the same computations in Remark 5.16, both L_Q and L_{B_Q} satisfy the assumptions of Lemma 5.5 with $\varepsilon = 2k\delta_\beta$, and hence we conclude that $\angle(L_Q, L_{B_Q}) \lesssim k\delta_\beta$. By the same argument, $\angle(L_R, L_{\mathfrak{B}_R}) \lesssim k\delta_\beta$, where $\mathfrak{B}_R$ is the ball in (5.25) so that $\bigcup_{Q \in U_m(R)} Q \subset \mathfrak{B}_R$.

From $\angle(L_Q, L_{B_Q}) \lesssim k\delta_\beta$, $\angle(L_R, L_{\mathfrak{B}_R}) \lesssim k\delta_\beta$, and the stopping condition $\angle(L_R, L_Q) > \alpha$, we get $\angle(L_{\mathfrak{B}_R}, L_{B_Q}) > \alpha/2$ provided $\delta_\beta \ll \alpha/k$, see (C4). From this and (5.2) we have $\angle(m_{\mathfrak{B}_R}\nu, m_{B_Q}\nu) > \alpha/3$ provided $\delta^{\frac{1}{2n}} \ll \alpha$, which is granted by (C4). In particular $|m_{\mathfrak{B}_R}\nu - m_{B_Q}\nu| \gtrsim 1 - \cos\alpha$ by the law of cosines.

Therefore, by Chebysheff's inequality we have

$$\begin{aligned}
\sigma\left(\bigcup_{Q \in \mathsf{BA}(R)} Q\right) &= \sum_{Q \in \mathsf{BA}(R)} \sigma(Q) \lesssim \frac{1}{1-\cos\alpha} \sum_{Q \in \mathsf{BA}(R)} \int_Q |m_{B_Q}\nu - m_{\mathfrak{B}_R}\nu| \, d\sigma \\
&\leq \frac{1}{1-\cos\alpha} \left(\sum_{Q \in \mathsf{BA}(R)} \int_Q |\nu - m_{B_Q}\nu| \, d\sigma + \sum_{Q \in \mathsf{BA}(R)} \int_Q |\nu - m_{\mathfrak{B}_R}\nu| \, d\sigma \right) \\
&\leq \frac{1}{1-\cos\alpha} \left(\sum_{Q \in \mathsf{BA}(R)} \int_{B_Q} |\nu - m_{B_Q}\nu| \, d\sigma + \int_{\mathfrak{B}_R} |\nu - m_{\mathfrak{B}_R}\nu| \, d\sigma \right),
\end{aligned}$$

where we simply used in the last step that $Q \subset B_Q$ and $\bigcup_{Q\in \mathsf{BA}(R)} Q \subset \bigcup_{Q\in U_m(R)} Q \subset \mathfrak{B}_R$. The first term is controlled by

$$\sum_{Q\in \mathsf{BA}(R)} \int_{B_Q} |\nu - m_{B_Q}\nu|\, d\sigma \overset{(3.8)}{\leq} \delta_* \sum_{Q\in \mathsf{BA}(R)} \sigma(B_Q) \approx \delta_* \sum_{Q\in \mathsf{BA}(R)} \sigma(Q)$$
$$= \delta_* \sigma\left(\bigcup_{Q\in \mathsf{BA}(R)} Q\right) \leq \delta_* \sigma\left(\bigcup_{Q\in U_m(R)} Q\right) \lesssim \delta_* \sigma(\mathfrak{B}_R).$$

The other term is controlled by

$$\int_{\mathfrak{B}_R} |\nu - m_{\mathfrak{B}_R}\nu|\, d\sigma \overset{(3.8)}{\leq} \delta_* \sigma(\mathfrak{B}_R).$$

All in all, from the last three inline equations, and using $\delta_* \leq \delta$, $\delta \leq \alpha^3$ (see (C4)) and $\frac{\delta}{1-\cos(\delta^{1/3})} \leq 3\delta^{1/3}$ for small enough δ, we have

$$\sigma\left(\bigcup_{Q\in \mathsf{BA}(R)} Q\right) \lesssim \frac{\delta_*}{1-\cos\alpha} \sigma(\mathfrak{B}_R) \lesssim \delta^{1/3} \sigma(\mathfrak{B}_R).$$

The lemma follows as $\sigma(3R_W) \approx_m \sigma(\mathfrak{B}_R)$. □

Using the set G_R defined above, we decompose

$$\sigma(\{x \in R_W : K_{l,*}(f\mathbf{1}_{3R_W})(x) > 40\lambda, \mathcal{M}_{1+\gamma} f(x) \leq A\lambda\}) \leq \boxed{1} + \boxed{2} + \boxed{3},$$

where

$$\boxed{1} := \sigma(R_W \setminus G_R),$$
$$\boxed{2} := \sigma(\{x \in G_R : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 20\lambda, \mathcal{M}_{1+\gamma} f(x) \leq A\lambda\}), \text{ and}$$
$$\boxed{3} := \sigma(\{x \in \partial\Omega : K_{l,*}(f\mathbf{1}_{3R_W\setminus G_R})(x) > 20\lambda\}).$$

Using the control of the measure of $\bigcup_{Q\in U_m(R)} Q \setminus G_R$ in Lemma 5.18, we estimate $\boxed{1}$ and $\boxed{3}$:

Lemma 5.19. *We have* $\boxed{1} \lesssim \delta^{1/3}\sigma(3R_W)$.

Proof. This is just by the inclusion $\partial\Omega \cap 3R_W \subset mR \subset \bigcup_{Q\in U_m(R)} Q$ and Lemma 5.18. □

Lemma 5.20. *We have* $\boxed{3} \lesssim A^{\sqrt{1+\gamma}} \delta^{\frac{1}{3}(1-\frac{1}{\sqrt{1+\gamma}})} \sigma(3R_W)$. *In particular* $\boxed{3} \leq c_\delta \sigma(3R_W)$ *with* $c_\delta \to 0$ *as* $\delta \to 0$, *by (C3).*

Proof. By Chebysheff's inequality and the fact that $K_{l,*}g \le K_*g$ for any $g \in L^p(\sigma)$, we have

$$\boxed{3} \lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_{\partial\Omega} |K_{l,*}(f\mathbf{1}_{3R_W\setminus G_R})|^{\sqrt{1+\gamma}}\, d\sigma \le \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_{\partial\Omega} |K_*(f\mathbf{1}_{3R_W\setminus G_R})|^{\sqrt{1+\gamma}}\, d\sigma.$$

Since $L^p(\sigma) \subset L^{\sqrt{1+\gamma}}_{\mathrm{loc}}(\sigma)$ (as we are assuming that $0 < \gamma < p-1$), by (3.5a) (since $\partial\Omega$ is uniformly rectifiable) and Hölder's inequality respectively, the right-hand side term is bounded by

$$\begin{aligned}
&\frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_{\partial\Omega} |K_*(f\mathbf{1}_{3R_W\setminus G_R})|^{\sqrt{1+\gamma}}\, d\sigma \\
&\qquad \lesssim \frac{1}{\lambda^{\sqrt{1+\gamma}}} \int_{\partial\Omega} |f\mathbf{1}_{3R_W\setminus G_R}|^{\sqrt{1+\gamma}}\, d\sigma \\
&\qquad \lesssim \frac{\sigma(3R_W\setminus G_R)^{1-\frac{1}{\sqrt{1+\gamma}}}}{\lambda^{\sqrt{1+\gamma}}} \left(\int_{3R_W} |f|^{1+\gamma}\, d\sigma \right)^{\frac{1}{\sqrt{1+\gamma}}} \\
&\qquad = \frac{\sigma(3R_W\setminus G_R)^{1-\frac{1}{\sqrt{1+\gamma}}}\, \sigma(3R_W)^{\frac{1}{\sqrt{1+\gamma}}}}{\lambda^{\sqrt{1+\gamma}}} \left(\fint_{3R_W} |f|^{1+\gamma}\, d\sigma \right)^{\frac{1}{\sqrt{1+\gamma}}}.
\end{aligned}$$

Using that $\left(\fint_{3R_W} |f|^{1+\gamma}\, d\sigma\right)^{\frac{1}{\sqrt{1+\gamma}}} \lesssim \mathcal{M}_{1+\gamma} f(x_0)^{\sqrt{1+\gamma}}$, $\mathcal{M}_{1+\gamma} f(x_0) \le A\lambda$ and $3R_W \subset U_m(R)$ in the latter term, we conclude

$$\boxed{3} \lesssim A^{\sqrt{1+\gamma}} \left(\frac{\sigma\left(\bigcup_{Q\in U_m(R)} Q \setminus G_R\right)}{\sigma(3R_W)} \right)^{1-\frac{1}{\sqrt{1+\gamma}}} \sigma(3R_W),$$

and the lemma follows by applying Lemma 5.18 in the last term. □

It remains to study the term $\boxed{2}$. As in the terms $\boxed{1}$ and $\boxed{3}$, we want to see:

Lemma 5.21. *We have* $\boxed{2} \lesssim (\alpha A^2 + \delta^{\frac{1-1/\sqrt{1+\gamma}}{4n}} A^2)\sigma(3R_W)$. *In particular* $\boxed{2} \le c_\delta \sigma(3R_W)$ *with* $c_\delta \to 0$ *as* $\delta \to 0$, *by (C3) and (C5).*

The rest of the section is devoted to the proof of Lemma 5.21. By Proposition 5.9 there exists a Lipschitz graph Γ_R given by $A : L_R \to L_R^\perp$ with norm $\le C\alpha$ such that

$$\operatorname{dist}(x, (\Pi_R(x), A(\Pi_R(x)))) \le Ck\delta_\beta d_R(x) \text{ for all } x \in k_0 R \supset mR \supset 3R_W.$$

In particular, for $x \in Q \in \mathsf{SA}(R)$, if $\widehat{Q} \supset Q$ is the parent of Q then $\widehat{Q} \in \mathsf{Tree}(R)$, and hence

$$d_R(x) = \inf_{Q \in \mathsf{Tree}(R)} \{\operatorname{dist}(x, Q) + \operatorname{diam} Q\} \leq \operatorname{diam} \widehat{Q} \approx \operatorname{diam} Q \approx S$$

and

$$\operatorname{dist}(x, (\Pi_R(x), A(\Pi_R(x)))) \leq Ck\delta_\beta d_R(x) \lesssim k\delta_\beta S.$$

So, if $\delta_\beta \ll 1/k$ is small enough we have

$$\Gamma_R \cap \frac{1}{2} B(Q) \neq \emptyset;$$

see (5.24) for the definition of $B(Q)$. This in particular implies

$$\mathcal{H}^n|_{\Gamma_R}(B(Q)) \approx \sigma(Q).$$

Given $f\mathbf{1}_{3R_W \cap G_R} \in L^p(\sigma)$, let us define an auxiliary function f_{Γ_R} in Γ_R. First, for each $Q \in \mathsf{SA}(R)$, let

$$\begin{aligned} f_{\Gamma_R}(Q) &:= \frac{1}{\mathcal{H}^n(\Gamma_R \cap B(Q))} \int_Q f(y)\mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y) \\ &\approx \fint_Q f(y)\mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y), \end{aligned} \tag{5.26}$$

and we define the function f_{Γ_R} in Γ_R as

$$f_{\Gamma_R}(x) := \sum_{Q \in \mathsf{SA}(R)} f_{\Gamma_R}(Q)\mathbf{1}_{B(Q)}(x), \quad x \in \Gamma_R. \tag{5.27}$$

Note that f_{Γ_R} is a piecewise constant function, since $\{B(Q)\}_{Q \in \mathsf{SA}(R)}$ is a pairwise disjoint family. This follows from the fact that $\{Q\}_{Q \in \mathsf{SA}(R)}$ is a pairwise disjoint family and the choice of c_1 in (5.24). By definition of f_{Γ_R} and (5.25), we have

$$\operatorname{supp} f_{\Gamma_R} \subset \bigcup_{Q \in \mathsf{SA}(R)} \overline{B(Q)} \subset \mathfrak{B}_R. \tag{5.28}$$

From the definition of f_{Γ_R} from $f\mathbf{1}_{3R_W \cap G_R}$ in (5.27) we have that both $f\mathbf{1}_{3R_W \cap Q}\, d\sigma$ and $f_{\Gamma_R}\mathbf{1}_{B(Q)}\, d\mathcal{H}^n|_{\Gamma_R}$ have the same mass for each $Q \in \mathsf{SA}(R)$. That is,

$$\int_{\Gamma_R} f_{\Gamma_R}\mathbf{1}_{B(Q)}\, d\mathcal{H}^n = \mathcal{H}^n(\Gamma_R \cap B(Q)) f_{\Gamma_R}(Q) = \int_Q f\mathbf{1}_{3R_W \cap Q}\, d\sigma. \tag{5.29}$$

Before seeing some properties of f_{Γ_R}, let us define the subfamily I_R of $\mathsf{SA}(R)$ as

$$I_R := \{Q \in \mathsf{SA}(R) : \exists x \in Q \text{ with } K_{l,*}(f\mathbf{1}_{3R_W \cap G_R})(x) > 6\lambda \text{ and } \mathcal{M}_{1+\gamma} f(x) \leq A\lambda\}.$$

For each $Q \in I_R$, we fix $z_Q \in Q$ (do not confuse with c_Q, the "center" of Q) such that

$$\mathcal{M}_{1+\gamma} f(z_Q) \leq A\lambda. \tag{5.30}$$

The following lemma shows that f_{Γ_R} inherits a localized maximal bound from f, restricted to the cubes $Q \in I_R$.

Lemma 5.22. *For each $Q \in I_R$ we have*

$$\left(\fint_{\Gamma_R \cap B(Q)} |f_{\Gamma_R}|^{1+\gamma} \, d\mathcal{H}^n \right)^{\frac{1}{1+\gamma}} \lesssim A\lambda \quad \text{and} \quad \left(\fint_{\Gamma_R \cap \mathfrak{B}_R} |f_{\Gamma_R}|^{1+\gamma} \, d\mathcal{H}^n \right)^{\frac{1}{1+\gamma}} \lesssim A\lambda.$$

Proof. Let z_Q denote the point in Q such that $\mathcal{M}_{1+\gamma} f(z_Q) \leq A\lambda$, see (5.30). The first inequality follows from the definition of f_{Γ_R}, (5.26) and $Q \in I_R$. That is,

$$\left(\fint_{\Gamma_R \cap B(Q)} |f_{\Gamma_R}(y)|^{1+\gamma} \, d\mathcal{H}^n(y) \right)^{\frac{1}{1+\gamma}} = |f_{\Gamma_R}(Q)| \lesssim \fint_Q |f(y)| \, d\sigma(y) \lesssim \mathcal{M}_{1+\gamma} f(z_Q) \leq A\lambda.$$

Similarly, the second inequality is obtained as

$$\begin{aligned}
&\fint_{\Gamma_R \cap \mathfrak{B}_R} |f_{\Gamma_R}(y)|^{1+\gamma} \, d\mathcal{H}^n(y) \\
&\quad \lesssim \frac{1}{\mathcal{H}^n(\Gamma_R \cap \mathfrak{B}_R)} \int_{\Gamma_R \cap \mathfrak{B}_R} \left(\sum_{Q \in \mathsf{SA}(R)} |f_{\Gamma_R}(Q)| \mathbf{1}_{B(Q)}(y) \right)^{1+\gamma} d\mathcal{H}^n(y) \\
&\quad \approx \frac{1}{\ell(R)^n} \sum_{Q \in \mathsf{SA}(R)} \int_{\Gamma_R \cap B(Q)} |f_{\Gamma_R}(Q)|^{1+\gamma} \, d\mathcal{H}^n(y) \\
&\quad \overset{(5.26)}{\lesssim} \frac{S^n}{\ell(R)^n} \sum_{Q \in \mathsf{SA}(R)} \left(\fint_Q |f(y)| \, d\sigma(y) \right)^{1+\gamma} \\
&\quad \overset{\text{(Jensen)}}{\lesssim} \frac{1}{\ell(R)^n} \sum_{Q \in \mathsf{SA}(R)} \int_Q |f(y)|^{1+\gamma} \, d\sigma(y) \overset{(5.25)}{\lesssim} \fint_{\mathfrak{B}_R} |f(y)|^{1+\gamma} \, d\sigma(y) \\
&\quad \lesssim \mathcal{M}_{1+\gamma} f(z_Q)^{1+\gamma} \leq (A\lambda)^{1+\gamma},
\end{aligned}$$

where we took any z_Q with $Q \in I_R$ in the last inequality. □

To shorten the notation, for $\varepsilon > 0$ we denote

$$\Phi_\varepsilon(z) := \frac{z}{|z|^{n+1}} \mathbf{1}_{\{|z|>\varepsilon\}} \text{ for } z \in \mathbb{R}^{n+1} \setminus \{0\}.$$

We present two lemmas that we will need in the proof of Lemma 5.21.

Lemma 5.23. *For all $x \in Q \in I_R$, all $x' \in B(Q) \cap \Gamma_R$ and every $\varepsilon \geq T$,*

$$\left| \int_{\partial\Omega} \Phi_\varepsilon(x-y) f(y) \mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y) - \int_{\Gamma_R} \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n(y) \right| \leq \frac{1}{2}\lambda.$$

Proof. We write

$$\begin{aligned}
\boxed{M} &= \left| \int \left(\Phi_\varepsilon(x-y) f(y) \mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y) - \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y) \right) \right| \\
&\leq \left| \int_{\partial\Omega} (\Phi_\varepsilon(x-y) - \Phi_\varepsilon(x'-y)) f(y) \mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y) \right| \\
&\quad + \left| \int \Phi_\varepsilon(x'-y) f(y) \mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y) - \int \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y) \right| \\
&=: \boxed{M1} + \boxed{M2}.
\end{aligned}$$

Let us bound the term $\boxed{M1}$. Since $x \in Q$ and $x' \in B(Q)$ satisfy $|x - x'| \leq 2\operatorname{diam} Q < 2S$, and Φ_ε is the truncated kernel of big scales $\varepsilon \geq T \gg 100S$, in particular $|x-y| \geq T \gg 100S$, we have $|x-x'|/|x-y| \leq S/T \leq 1/2$. Thus, we can use the cancellation property of the Calderón-Zygmund kernel (or just the mean value theorem) to obtain

$$|\Phi_\varepsilon(x-y) - \Phi_\varepsilon(x'-y)| \lesssim \frac{|x-x'|}{|x-y|^{n+1}} \mathbf{1}_{B(x,\varepsilon/2)^c}(y) \lesssim \frac{S}{|x-y|^{n+1}} \mathbf{1}_{B(x,T/2)^c}(y),$$

where we used in the last step that $\varepsilon \geq T$ and that $x \in Q$ and $x' \in B(Q)$ satisfy $|x-x'| \leq 2S$. From this we get

$$\begin{aligned}
\boxed{M1} &\leq \int_{\partial\Omega} |\Phi_\varepsilon(x-y) - \Phi_\varepsilon(x'-y)||f(y)| \mathbf{1}_{G_R}(y)\, d\sigma(y) \lesssim S \int_{B(x,T/2)^c} \frac{|f(y)|}{|x-y|^{n+1}}\, d\sigma(y) \\
&= S \sum_{k=0}^{\infty} \int_{B(x,2^k T) \setminus B(x,2^{k-1}T)} \frac{|f(y)|}{|x-y|^{n+1}}\, d\sigma(y) \lesssim \frac{S}{T} \sum_{k=0}^{\infty} \frac{1}{2^k} \fint_{B(x,2^k T)} |f|\, d\sigma(y) \overset{(5.30)}{\lesssim} \frac{S}{T} A\lambda,
\end{aligned}$$

where we used in the last step that $Q \in I_R$.

It remains to bound the term $\boxed{M2}$. We have

$$\begin{aligned}\boxed{M2} &= \left| \int \Phi_\varepsilon(x'-y) f(y) \mathbf{1}_{3R_W \cap G_R}(y)\, d\sigma(y) - \int \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y) \right| \\ &= \left| \sum_{\widetilde{Q}\in \mathsf{SA}(R)} \int_{\widetilde{Q}} \Phi_\varepsilon(x'-y) f(y) \mathbf{1}_{3R_W}(y)\, d\sigma(y) - \sum_{\widetilde{Q}\in \mathsf{SA}(R)} \int_{B(\widetilde{Q})} \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y) \right| \\ &\leq \sum_{\widetilde{Q}\in \mathsf{SA}(R)} \left| \int \Phi_\varepsilon(x'-y) f(y) \mathbf{1}_{3R_W\cap \widetilde{Q}}(y)\, d\sigma(y) - \int \Phi_\varepsilon(x'-y) f_{\Gamma_R}(y) \mathbf{1}_{B(\widetilde{Q})}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y) \right|.\end{aligned}$$

To shorten the notation, we write $\boxed{M2} \leq \sum_{\widetilde{Q}\in \mathsf{SA}(R)} \boxed{M2(\widetilde{Q})}$ where

$$\boxed{M2(\widetilde{Q})} := \left| \int \Phi_\varepsilon(x'-y) \left(f(y) \mathbf{1}_{3R_W\cap \widetilde{Q}}(y)\, d\sigma(y) - f_{\Gamma_R}(y) \mathbf{1}_{B(\widetilde{Q})}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y) \right) \right|.$$

Let us study $\boxed{M2(\widetilde{Q})}$ for each $\widetilde{Q} \in \mathsf{SA}(R)$. Since we have that $f(y)\mathbf{1}_{3R_W\cap \widetilde{Q}}(y)\, d\sigma(y) - f_{\Gamma_R}(y)\mathbf{1}_{B(\widetilde{Q})}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y)$ has zero mass for each $\widetilde{Q} \in \mathsf{SA}(R)$, see (5.29), we can add $\Phi_\varepsilon(x'-c_{\widetilde{Q}})$ inside the integral sign in the last term, where $c_{\widetilde{Q}}$ is the "center" of $\widetilde{Q}$,[6] and therefore

$$\boxed{M2(\widetilde{Q})} = \left| \int (\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}}))(f(y)\mathbf{1}_{3R_W\cap \widetilde{Q}}(y)\, d\sigma(y) - f_{\Gamma_R}(y)\mathbf{1}_{B(\widetilde{Q})}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y)) \right|.$$

With this we have

$$\begin{aligned}\boxed{M2(\widetilde{Q})} \leq & \int |\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}})||f(y)|\mathbf{1}_{3R_W\cap \widetilde{Q}}(y)\, d\sigma(y) \\ & + \int |\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}})||f_{\Gamma_R}(y)|\mathbf{1}_{B(\widetilde{Q})}(y)\, d\mathcal{H}^n|_{\Gamma_R}(y).\end{aligned}$$

For $y \in \widetilde{Q}$ or $y \in B(\widetilde{Q})$, note first that $|y - c_{\widetilde{Q}}| \leq \operatorname{diam} \widetilde{Q} < S$, and second, $|x'-y| > \varepsilon \geq T \gg 100S$ by the truncation of the kernel Φ_ε. Hence, $|y-c_{\widetilde{Q}}|/|y-x'| \leq 1/2$. Therefore, by the cancellation of the Calderón-Zygmund kernel Φ_ε we have

$$|\Phi_\varepsilon(x'-y) - \Phi_\varepsilon(x'-c_{\widetilde{Q}})| \lesssim \frac{|y-c_{\widetilde{Q}}|}{|x'-y|^{n+1}} \leq \frac{S}{|x'-y|^{n+1}}.$$

Thus, defining

$$\begin{aligned}\boxed{M2(\widetilde{Q})_\sigma} &:= \int_{\widetilde{Q}\cap\{|y-x'|\geq T/2\}} \frac{S}{|x'-y|^{n+1}} |f(y)|\, d\sigma(y), \text{ and} \\ \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}} &:= \int_{B(\widetilde{Q})\cap\Gamma_R\cap\{|y-x'|\geq T/2\}} \frac{S}{|x'-y|^{n+1}} |f_{\Gamma_R}(y)|\, d\mathcal{H}^n(y),\end{aligned}$$

[6] Any point in $\widetilde{Q}$ would do the job.

we have

$$\boxed{M2(\widetilde{Q})} \lesssim \boxed{M2(\widetilde{Q})_\sigma} + \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}}.$$

Summing $\boxed{M2(\widetilde{Q})_\sigma}$ over $\widetilde{Q} \in \mathsf{SA}(R)$, as we did in the bound of $\boxed{M1}$ we have

$$\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_\sigma} \le S \int_{B(x',T/2)^c} \frac{|f(y)|}{|x'-y|^{n+1}}\, d\sigma(y) \lesssim \frac{S}{T}\sum_{k=0}^{\infty}\frac{1}{2^k} \fint_{B(x',2^kT)} |f|\, d\sigma \overset{(5.30)}{\lesssim} \frac{S}{T} A\lambda,$$

where we used in the last step that $Q \in I_R$.

We obtain the same bound for $\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}}$. Indeed, by the same computations above we have

$$\begin{aligned}\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}} &\lesssim S\sum_{k=0}^{\infty} \int_{B(x',T/2)^c} \frac{|f_{\Gamma_R}(y)|}{|x'-y|^{n+1}}\, d\mathcal{H}^n|_{\Gamma_R}(y)\\ &\lesssim \frac{S}{T}\sum_{k=0}^{\infty}\frac{1}{2^k} \fint_{B(x',2^kT)} |f_{\Gamma_R}|\, d\mathcal{H}^n|_{\Gamma_R}. \end{aligned} \tag{5.31}$$

For each $k \ge 0$, we define

$$\mathsf{SA}_k(R) := \{\widetilde{Q} \in \mathsf{SA}(R) : \widetilde{Q} \cap B(x', 2^kT) \neq \emptyset\}.$$

Since $|f_{\Gamma_R}(\widetilde{Q})|\mathcal{H}^n|_{\Gamma_R}(B(\widetilde{Q})) \le \int_{\widetilde{Q}} |f|\, d\sigma$ for all $\widetilde{Q} \in \mathsf{SA}(R)$ (see (5.26)) and $\widetilde{Q} \subset B(x', 2^{k+1}T)$ if $\widetilde{Q} \in \mathsf{SA}_k(R)$ (because $\operatorname{diam}\widetilde{Q} < S \ll T$ for all $\widetilde{Q} \in \mathsf{SA}(R)$), we have

$$\begin{aligned}\int_{B(x',2^kT)} |f_{\Gamma_R}|\, d\mathcal{H}^n|_{\Gamma_R} &\le \sum_{\widetilde{Q}\in\mathsf{SA}_k(R)} \int_{B(\widetilde{Q})} |f_{\Gamma_R}|\, d\mathcal{H}^n|_{\Gamma_R} = \sum_{\widetilde{Q}\in\mathsf{SA}_k(R)} |f_{\Gamma_R}(\widetilde{Q})|\mathcal{H}^n|_{\Gamma_R}(B(\widetilde{Q}))\\ &\lesssim \sum_{\widetilde{Q}\in\mathsf{SA}_k(R)} \int_{\widetilde{Q}} |f|\, d\sigma \le \int_{B(x',2^{k+1}T)} |f|\, d\sigma. \end{aligned} \tag{5.32}$$

By (5.31) and (5.32), using now that $Q \in I_R$ (i.e., (5.30)), we conclude

$$\sum_{\widetilde{Q}\in\mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}} \lesssim \frac{S}{T}\sum_{k=0}^{\infty}\frac{1}{2^k} \fint_{B(x',2^{k+1}T)} |f|\, d\sigma \lesssim \frac{S}{T} A\lambda, \tag{5.33}$$

as claimed.

All in all, we conclude

$$\boxed{M} \le \boxed{M1} + \boxed{M2} \le C\frac{S}{T}A\lambda \le \frac{1}{2}\lambda,$$

where we use (C6) in the last step. □

From now on, we write the double layer potential in Γ_R as K^{Γ_R}, replacing $\partial\Omega$ and its surface measure σ by Γ_R and $\mathcal{H}^n|_{\Gamma_R}$ in (3.3).

The second lemma for the proof of Lemma 5.21 is about a continuity-type result for the large scale double layer potential in Γ_R.

Lemma 5.24. *For any $Q \in I_R$ we have $|K^{\Gamma_R}_{l,*}(f_{\Gamma_R})(x) - K^{\Gamma_R}_{l,*}(f_{\Gamma_R})(z)| \leq \lambda$ for all $x, z \in B(Q) \cap \Gamma_R$.*

Proof. For $\varepsilon \geq T$, since $x, z \in B(Q) \cap \Gamma_R$, $\operatorname{diam} Q < S$ and $\varepsilon \geq T \gg S$, we have $|x-y| \approx |z-y|$ if $|y-x| > \varepsilon$. So, using the cancellation of the Calderón-Zygmund kernel, we have

$$\begin{aligned} |K^{\Gamma_R}_{\varepsilon}(f_{\Gamma_R})(x) - K^{\Gamma_R}_{\varepsilon}(f_{\Gamma_R})(z)| &= \left| \int \langle \Phi_\varepsilon(x-y) - \Phi_\varepsilon(z-y), \nu_{\Gamma_R}(y)\rangle f_{\Gamma_R}\, d\mathcal{H}^n|_{\Gamma_R}(y) \right| \\ &\lesssim \int\limits_{|y-x|\geq T/2} \frac{|x-z|}{|x-y|^{n+1}} |f_{\Gamma_R}(y)|\, d\mathcal{H}^n|_{\Gamma_R}(y) \\ &\leq S \int\limits_{|y-x|\geq T/2} \frac{1}{|x-y|^{n+1}} |f_{\Gamma_R}(y)|\, d\mathcal{H}^n|_{\Gamma_R}(y). \end{aligned}$$

Note that the last element is in fact $\sum_{\widetilde{Q}\in \mathsf{SA}(R)} \boxed{M2(\widetilde{Q})_{\mathcal{H}^n}}$ in the proof of Lemma 5.23, replacing x' by x. Thus, by the same computations in (5.31), (5.32) and (5.33), we have $|K^{\Gamma_R}_{\varepsilon}(f_{\Gamma_R})(x) - K^{\Gamma_R}_{\varepsilon}(f_{\Gamma_R})(z)| \lesssim \frac{S}{T} A\lambda$. By the condition (C6), we conclude that this is bounded by $\leq \lambda$. □

For the proof of Lemma 5.21, we will compare the double layer potential in $\partial\Omega$ with the double layer potential in the Lipschitz graph Γ_R, so that we can use Lemmas 5.23 and 5.24, and known properties of K^{Γ_R}. The following is a simplified version of [18, Theorem 4.34], see also [18, (4.4.9)].

Theorem 5.25. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, $\Gamma = \{(x, \varphi(x)) : x \in \mathbb{R}^n\}$ its graph, and $1 < p < \infty$. Then*

$$\|K^{\Gamma}_*\|_{L^p(\Gamma)\to L^p(\Gamma)} \leq C_{n,p} \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} (1 + \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)})^N, \tag{5.34}$$

for some $N = N(n)$.

Let us see the proof of Lemma 5.21. During the proof, $\mathcal{R} = \mathcal{R}^{\partial\Omega}$ and $\mathcal{R}^{\Gamma_R}$ denote the Riesz transform in $\partial\Omega$ and Γ_R respectively, see Definition 2.16, and $\mathcal{R}_*$ and $\mathcal{R}^{\Gamma_R}_*$ their respective maximal operators as in (2.9).

Proof of Lemma 5.21. For each $Q \in I_R$, let us study first

$$\sigma(\{x \in Q : K_{l,*}(f \mathbf{1}_{3R_W \cap G_R})(x) > 20\lambda, \mathcal{M}_{1+\gamma} f(x) \leq A\lambda\}).$$

For any $x \in Q \in I_R$ and any $z \in B(Q) \cap \Gamma_R$, we have

$$\begin{aligned} K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) \le K^{\Gamma_R}_{l,*}(f_{\Gamma_R})(z) &+ \mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(x) \\ &+ \mathcal{R}^{\Gamma_R}_*(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) \\ &+ \sup_{\varepsilon\ge T}\left| \int_{\partial\Omega} \Phi_\varepsilon(x-\cdot) f\mathbf{1}_{3R_W\cap G_R}\, d\sigma - \int_{\Gamma_R} \Phi_\varepsilon(z-\cdot) f_{\Gamma_R}\, d\mathcal{H}^n \right|, \end{aligned}$$

where $N_{\mathfrak{B}_R}$ is the orthogonal unit vector to $L_{\mathfrak{B}_R}$ in (5.1) for the ball $\mathfrak{B}_R$ defined in (5.25). Therefore, if we fix (once and for all, for the cube Q) a point $z(Q) \in B(Q) \cap \Gamma_R$ satisfying

$$\mathcal{R}^{\Gamma_R}_*(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z(Q)) \le \inf_{z\in B(Q)\cap\Gamma_R} \mathcal{R}^{\Gamma_R}_*(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) + \frac{\lambda}{2},$$

then we have that[7]

$$\begin{aligned} \sigma(\{x\in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda\}) \le\ & \sigma(\{x\in Q : K^{\Gamma_R}_{l,*}(f_{\Gamma_R})(z(Q)) > 2\lambda\}) \\ &+ \sigma(\{x\in Q : \mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(x) > \lambda\}) \\ &+ \sigma(\{x\in Q : \inf_{z\in B(Q)\cap\Gamma_R} \mathcal{R}^{\Gamma_R}_*(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) > \lambda\}) \\ &+ \sigma(\{x\in Q : \sup_{\varepsilon\ge T}\left| \int_{\partial\Omega} \Phi_\varepsilon(x-y) f(y)\mathbf{1}_{3R_W\cap G_R}(y)\, d\sigma(y) \right. \\ &\qquad \left. - \int_{\Gamma_R} \Phi_\varepsilon(z(Q)-y) f_{\Gamma_R}(y)\, d\mathcal{H}^n(y) \right| > \lambda\}). \end{aligned}$$

By Lemma 5.23, the last term drops out as the corresponding set is empty, and, since $z(Q) \in B(Q) \cap \Gamma_R$, Lemma 5.24 implies that the first is controlled by $\sigma(\{x \in Q : \inf_{z\in B(Q)\cap\Gamma_R} K^{\Gamma_R}_{l,*}(f_{\Gamma_R})(z) > \lambda\})$. That is,

$$\begin{aligned} \sigma(\{x\in Q : K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x) > 6\lambda\}) \le\ & \sigma(\{x\in Q : \inf_{z\in B(Q)\cap\Gamma_R} K^{\Gamma_R}_{l,*}(f_{\Gamma_R})(z) > \lambda\}) \\ &+ \sigma(\{x\in Q : \mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot) - N_{\mathfrak{B}_R}))(x) > \lambda\}) \\ &+ \sigma(\{x\in Q : \inf_{z\in B(Q)\cap\Gamma_R} \mathcal{R}^{\Gamma_R}_*(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot) - N_{\mathfrak{B}_R}))(z) > \lambda\}) \\ =:\ & \boxed{2_{Q,i}} + \boxed{2_{Q,ii}} + \boxed{2_{Q,iii}}. \end{aligned}$$

Using the infimum property, $\boxed{2_{Q,i}}$ is treated as

[7] The reader may notice that the first set on the right-hand side below is either empty or all of Q, since $z(Q)$ does not depend on x. In the latter case we will see that it is nonetheless small, as the continuity-type estimate Lemma 5.24 reduces it to the infimum of $K^{\Gamma_R}_{l,*}(f_{\Gamma_R})$ over $B(Q) \cap \Gamma_R$, which is controlled below.

$$\begin{aligned}\boxed{2_{Q,i}} &\leq \frac{\sigma(Q)}{\lambda^{1+\gamma}}\left(\inf_{z\in B(Q)\cap\Gamma_R} K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z)\right)^{1+\gamma}\\ &\approx \frac{\mathcal{H}^n|_{\Gamma_R}(B(Q))}{\lambda^{1+\gamma}}\left(\inf_{z\in B(Q)\cap\Gamma_R} K_{l,*}^{\Gamma_R}(f_{\Gamma_R})(z)\right)^{1+\gamma}\\ &\leq \frac{1}{\lambda^{1+\gamma}}\int\limits_{B(Q)\cap\Gamma_R} |K_*^{\Gamma_R}(f_{\Gamma_R})(y)|^{1+\gamma}\,d\mathcal{H}^n(y),\end{aligned}$$

in the last step we removed the restriction at large scales. By the same computations above,

$$\boxed{2_{Q,iii}} \lesssim \frac{1}{\lambda^{1+\gamma}}\int\limits_{B(Q)\cap\Gamma_R} |\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot)-N_{\mathfrak{B}_R}))(y)|^{1+\gamma}\,d\mathcal{H}^n(y).$$

Using Chebysheff's inequality, the term $\boxed{2_{Q,ii}}$ is controlled directly by

$$\boxed{2_{Q,ii}} \leq \frac{1}{\lambda^{\sqrt{1+\gamma}}}\int\limits_{Q} |\mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot)-N_{\mathfrak{B}_R}))(y)|^{\sqrt{1+\gamma}}\,d\sigma(y).$$

All in all,

$$\begin{aligned}\sigma(\{x\in Q: K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x)>6\lambda\}) &\lesssim \frac{1}{\lambda^{1+\gamma}}\int\limits_{B(Q)\cap\Gamma_R} |K_*^{\Gamma_R}(f_{\Gamma_R})(y)|^{1+\gamma}\,d\mathcal{H}^n(y)\\ &+\frac{1}{\lambda^{\sqrt{1+\gamma}}}\int\limits_{Q} |\mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(\cdot)-N_{\mathfrak{B}_R}))(y)|^{\sqrt{1+\gamma}}\,d\sigma(y)\\ &+\frac{1}{\lambda^{1+\gamma}}\int\limits_{B(Q)\cap\Gamma_R} |\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(\cdot)-N_{\mathfrak{B}_R}))(y)|^{1+\gamma}\,d\mathcal{H}^n(y).\end{aligned}$$

Summing over all $Q\in I_R\subset \mathsf{SA}(R)$ and using the above inequality in the last step, we have

$$\begin{aligned}\boxed{2} &= \sigma(\{x\in G_R: K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x)>20\lambda, \mathcal{M}_{1+\gamma}f(x)\leq A\lambda\})\\ &= \sum_{Q\in I_R}\sigma(\{x\in Q: K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x)>20\lambda, \mathcal{M}_{1+\gamma}f(x)\leq A\lambda\})\\ &\leq \sum_{Q\in I_R}\sigma(\{x\in Q: K_{l,*}(f\mathbf{1}_{3R_W\cap G_R})(x)>6\lambda\})\\ &\lesssim \frac{1}{\lambda^{1+\gamma}}\int\limits_{\Gamma_R} |K_*^{\Gamma_R}(f_{\Gamma_R})(y)|^{1+\gamma}\,d\mathcal{H}^n(y)\\ &\quad+\frac{1}{\lambda^{\sqrt{1+\gamma}}}\int\limits_{\partial\Omega} |\mathcal{R}_*(f\mathbf{1}_{3R_W\cap G_R}(\nu_{\partial\Omega}(y)-N_{\mathfrak{B}_R}))(y)|^{\sqrt{1+\gamma}}\,d\sigma(y)\end{aligned}$$

$$+\frac{1}{\lambda^{1+\gamma}}\int_{\Gamma_R}|\mathcal{R}_*^{\Gamma_R}(f_{\Gamma_R}(\nu_{\Gamma_R}(y)-N_{\mathfrak{B}_R}))(y)|^{1+\gamma}\,d\mathcal{H}^n(y)$$

$$=:\boxed{2_i}+\boxed{2_{ii}}+\boxed{2_{iii}}.$$

The smallness of the term $\boxed{2_i}$ will come from the $L^p(\Gamma_R)$ norm of $K_*^{\Gamma_R}$ in the Lipschitz graph with small constant $\lesssim\alpha$, while we will bound the terms $\boxed{2_{ii}}$ and $\boxed{2_{iii}}$ using the smallness assumption on the unit normal vectors, after using that the Riesz transform is bounded in L^p (with constant depending on p). We do this in the rest of the proof.

Using that the Lipschitz norm of Γ_R is $\lesssim\alpha$ small, by Theorem 5.25 we have that the $L^q(\Gamma_R)$ norm (in every $1<q<\infty$) of $K_*^{\Gamma_R}$ is $\lesssim_q\alpha$, and hence we get

$$\boxed{2_i}\overset{(5.34)}{\lesssim}\frac{\alpha^{1+\gamma}}{\lambda^{1+\gamma}}\int_{\Gamma_R}|f_{\Gamma_R}(y)|^{1+\gamma}\,d\mathcal{H}^n(y)\approx\frac{\alpha^{1+\gamma}}{\lambda^{1+\gamma}}\sigma(\mathfrak{B}_R)\fint_{\Gamma_R\cap\mathfrak{B}_R}|f_{\Gamma_R}(y)|^{1+\gamma}\,d\mathcal{H}^n(y),$$

where we used in the last step that $\operatorname{supp} f_{\Gamma_R}\subset\mathfrak{B}_R$, see (5.28), and we allow the implicit constant to depend on γ. By Lemma 5.22, the last averaged integral is controlled by $\lesssim(A\lambda)^{1+\gamma}$ and hence

$$\boxed{2_i}\lesssim\alpha^{1+\gamma}A^{1+\gamma}\sigma(\mathfrak{B}_R)\leq\alpha A^2\sigma(\mathfrak{B}_R)\approx\alpha A^2\sigma(3R_W),$$

where the second-to-last inequality is simply because $0<\gamma<1$.

Using that the Riesz transform is bounded on UR sets, Hölder's inequality, and that there exists some point $z\in 3R_W\subset\mathfrak{B}_R$ such that $\mathcal{M}_{1+\gamma}f(z)\leq A\lambda$, we have

$$\begin{aligned}\boxed{2_{ii}}&\lesssim\frac{1}{\lambda^{\sqrt{1+\gamma}}}\int_{3R_W}(|f(y)||\nu_{\partial\Omega}(y)-N_{\mathfrak{B}_R}|)^{\sqrt{1+\gamma}}\,d\sigma(y)\\&\lesssim\frac{1}{\lambda^{\sqrt{1+\gamma}}}\sigma(3R_W)\left(\fint_{\mathfrak{B}_R}|f(y)|^{1+\gamma}\,d\sigma(y)\right)^{1/\sqrt{1+\gamma}}\left(\fint_{\mathfrak{B}_R}|\nu_{\partial\Omega}(y)-N_{\mathfrak{B}_R}|\,d\sigma(y)\right)^{1-1/\sqrt{1+\gamma}}\\&\lesssim\frac{1}{\lambda^{\sqrt{1+\gamma}}}\sigma(3R_W)(A\lambda)^{\sqrt{1+\gamma}}\left(\fint_{\mathfrak{B}_R}|\nu_{\partial\Omega}(y)-N_{\mathfrak{B}_R}|\,d\sigma(y)\right)^{1-1/\sqrt{1+\gamma}}.\end{aligned}$$

In the second step we also used that $|\nu_{\partial\Omega}(\cdot)-N_{\mathfrak{B}_R}|\leq 2$ σ-a.e. to reduce the exponent of $|\nu_{\partial\Omega}(y)-N_{\mathfrak{B}_R}|$ to 1 (as in (4.3)). Using the smallness assumption of the oscillation of the unit normal vector in (3.8) and that $|m_{\sigma,\mathfrak{B}_R}\nu_{\partial\Omega}-N_{\mathfrak{B}_R}|\lesssim\delta^{\frac{1}{4n}}$ by (5.3), we have

$$\fint_{\mathfrak{B}_R}|\nu_{\partial\Omega}(y)-N_{\mathfrak{B}_R}|\,d\sigma(y)\leq\fint_{\mathfrak{B}_R}|\nu_{\partial\Omega}(y)-m_{\sigma,\mathfrak{B}_R}\nu_{\partial\Omega}|\,d\sigma(y)+|m_{\sigma,\mathfrak{B}_R}\nu_{\partial\Omega}-N_{\mathfrak{B}_R}|\lesssim\delta^{\frac{1}{4n}}.$$

We conclude then that

$$\boxed{2_{ii}} \leq C\delta_*^{\frac{1-1/\sqrt{1+\gamma}}{4n}} A^2\sigma(3R_W).$$

The term $\boxed{2_{iii}}$ is treated similarly, using the L^∞ bound of the oscillation of the unit vector as Γ_R is a Lipschitz graph of norm $\lesssim \alpha$. That is, since Γ_R is the Lipschitz graph of $A : L_R \to L_R^\perp$ in Proposition 5.9 with norm $\lesssim \alpha$, then $\angle(\nu_{\Gamma_R}(y), N_R) \lesssim \alpha$ for all $y \in \Gamma_R$, where $N_R \perp L_R$, and on the other hand, since $\operatorname{dist}(x, L_R) \leq 2k\delta_\beta \operatorname{diam} R$ for all $x \in kR \supset R$ by (5.11) and Remark 5.16, and $\operatorname{dist}(x, L_{\mathfrak{B}_R}) \leq 2\delta_\beta r_{\mathfrak{B}_R}$ for all $x \in \partial\Omega \cap \mathfrak{B}_R \supset R$ by (5.1), by Lemma 5.5 we have $\angle(N_{\mathfrak{B}_R}, N_R) = \angle(L_{\mathfrak{B}_R}, L_R) \lesssim k\delta_\beta \leq \alpha$, provided $\delta_\beta \leq \alpha/k$. So, $\angle(\nu_{\Gamma_R}(y), N_{\mathfrak{B}_R}) \lesssim \alpha$ for all $y \in \Gamma_R$, and by the law of cosines we conclude

$$|\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}|^2 = 2 - 2|\cos(\angle(\nu_{\Gamma_R}(y), N_{\mathfrak{B}_R}))| \leq 2 - 2\cos(C\alpha) \approx \alpha^2.$$

Using that the Riesz transform is bounded on Lipschitz graphs, $\operatorname{supp} f_{\Gamma_R} \subset \mathfrak{B}_R$ and $|\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}| \lesssim \alpha$, and Lemma 5.22, we have

$$\begin{aligned}
\boxed{2_{iii}} &\lesssim \frac{1}{\lambda^{1+\gamma}} \int_{\Gamma_R} (|f_{\Gamma_R}(y)||\nu_{\Gamma_R}(y) - N_{\mathfrak{B}_R}|)^{1+\gamma}\, d\mathcal{H}^n(y) \\
&\lesssim \alpha^{1+\gamma} \mathcal{H}^n|_{\Gamma_R}(\mathfrak{B}_R) \frac{1}{\lambda^{1+\gamma}} \fint_{\Gamma_R \cap \mathfrak{B}_R} |f_{\Gamma_R}(y)|^{1+\gamma}\, d\mathcal{H}^n(y) \\
&\lesssim \alpha^{1+\gamma} A^{1+\gamma} \sigma(3R_W) \leq \alpha A^2 \sigma(3R_W).
\end{aligned}$$

All in all,

$$\boxed{2} \leq C\left(\boxed{2_i} + \boxed{2_{ii}} + \boxed{2_{iii}}\right) \leq C(\alpha A^2 + \delta_*^{\frac{1-1/\sqrt{1+\gamma}}{4n}} A^2)\sigma(3R_W),$$

and the lemma follows since $\delta_* \leq \max\{\delta_\beta, \delta_*\} = \delta$. □

With this we have that Lemmas 5.19 to 5.21 are proved. Recall that this implies the good lambda inequality (5.23), a rewrite of the good lambda inequality (5.9). As we already saw below (5.9), it implies the good lambda inequality (5.5), and in particular (5.6) and (3.9). This concludes the proof of Theorem 3.5. □

6. Uniqueness of the solution of the Neumann problem

In this section we prove the uniqueness (modulo constant) for the Neumann problem in terms of weak derivatives in unbounded 1-sided CAD with unbounded boundary for the range of $2n/(n+1) \leq p < n + \varepsilon_1$.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded 1-sided CAD with unbounded boundary. Then there exists $\varepsilon_1 = \varepsilon_1(n, CAD) > 0$ such that if $u : \Omega \to \mathbb{R}$ satisfies $\mathcal{N}(\nabla u) \in L^p(\sigma)$ with $p \in [2n/(n+1), n+\varepsilon_1)$ and*

$$\int_\Omega \nabla u(z) \nabla\varphi(z)\, dm(z) = 0, \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}),$$

then u is constant.

In what follows, the nontangential maximal function is taken with its aperture fixed to $\alpha = 1$, as agreed in (2.8).

Before the proof of Proposition 6.1, let us state some fundamental properties that will be useful in this section. By [18, Proposition 3.24], if Ω is an ADR domain which is either bounded or has an unbounded boundary and $p \in (0, \infty)$, then

$$\|u\|_{L^{p(n+1)/n}(\Omega)} \lesssim \|\mathcal{N}u\|_{L^p(\sigma)}, \tag{6.1}$$

where the involved constant depends on the ADR parameter, independent of the function $u : \Omega \to \mathbb{R}$. If Ω is a 1-sided CAD, by [34, (2.609)], there exists a sufficiently large $C > 1$ such that if $\mathcal{N}^\varepsilon(\nabla u) \in L^p_{\mathrm{loc}}(\sigma)$ for some $p \in (0, \infty]$ and some $\varepsilon > 0$, then

$$\mathcal{N}^{\varepsilon/C} u \in L^p_{\mathrm{loc}}(\sigma). \tag{6.2}$$

In particular, for $p \in (0, \infty)$ and a 1-sided CAD Ω that is either bounded or has unbounded boundary, we claim

$$\mathcal{N}(\nabla u) \in L^p(\sigma) \implies u \in W^{1,p(n+1)/n}(\Omega \cap B) \text{ for any ball } B \text{ centered at } \partial\Omega. \tag{6.3}$$

Indeed, directly from (6.1) we obtain $\nabla u \in L^{p(n+1)/n}(\Omega)$. For a ball B centered at $\partial\Omega$, again by (6.1) we have $\|u\|_{L^{p(n+1)/n}(\Omega\cap B)} \lesssim \|\mathcal{N}^{r_B}(u\mathbf{1}_B)\|_{L^p(\sigma)}$. We note that $\mathrm{supp}(\mathcal{N}^{r_B}(u\mathbf{1}_B)) \subset 3B$, whence we get $\|\mathcal{N}^{r_B}(u\mathbf{1}_B)\|_{L^p(\sigma)} \leq \|\mathcal{N}^{r_B} u\|_{L^p(\sigma|_{3B})}$, which is bounded using that we can take any $\varepsilon > 0$ in (6.2).

For the type of domains in Proposition 6.1, the following lemma relates the nontangential and the weak derivatives.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition, and let $u : \Omega \to \mathbb{R}$ be harmonic. If $\mathcal{N}(\nabla u) \in L^p(\sigma)$ for some $p \in (1, \infty)$ and the pointwise nontangential limit $(\nabla u)|_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$, then*

$$\int_\Omega \nabla u(z) \nabla\varphi(z)\, dm(z) = \int_{\partial\Omega} \varphi(z) \partial_\nu u(z)\, d\sigma(z), \text{ for all } \varphi \in C^\infty_c(\mathbb{R}^{n+1}).$$

Proof. If Ω is bounded or has unbounded boundary, we define $D = \Omega$, and if Ω is unbounded and has bounded boundary, then we take a ball B so that $\partial\Omega \subset B/2$ and we define $D = \Omega \cap B$. Note that ∂D is ADR, and we denote $\sigma_D = \mathcal{H}^n|_{\partial D}$. During the proof $\mathcal{N}_\Omega$ and $\mathcal{N}_D$ denote the nontangential maximal operators in Ω and D respectively. We claim

$$\mathcal{N}_D(\nabla u) \in L^p(\sigma_D). \tag{6.4}$$

This is by assumption when $D = \Omega$. If Ω is unbounded with bounded boundary, we have $\mathcal{N}_D(\nabla u) \leq \mathcal{N}_\Omega(\nabla u)$ on $\partial\Omega$, and since u is harmonic in Ω (in particular $u \in C^\infty(\Omega)$) and for $x \in \partial B$ we have that $\Gamma_D(x)$ is uniformly far from $\partial\Omega$ (depending on r_B), we therefore get $\mathcal{N}_D(\nabla u) < +\infty$ on ∂B. This gives (6.4).

Fixing $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and assuming also $B \supset \operatorname{supp}\varphi$, this is an almost direct consequence of the divergence theorem in [18, Theorem 2.8], applied to the domain D and the vector field $\varphi\nabla u$. Let us recall it in the form we use. Let $U \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition, write $\sigma_U = \mathcal{H}^n|_{\partial U}$, and let ν be the outward unit normal of U. Let $v \in C^0(U)$ be a vector field with $\operatorname{div} v \in L^1(U)$, $\mathcal{N}v \in L^p(\sigma_U)$ for some $p \in (1,\infty)$, and whose pointwise nontangential trace $v|_{\partial U}$ exists σ_U-a.e. on ∂U. If either U is bounded or v has bounded support in U (the latter case being noted in [18, lines 1-6 in proof of Theorem 3.25]), then Green's formula

$$\int_U \operatorname{div} v \, dm = \int_{\partial U} \langle \nu, v|_{\partial U} \rangle \, d\sigma_U \tag{6.5}$$

holds.

We apply this to D and $v = \varphi\nabla u$, whose support is bounded since $B \supset \operatorname{supp}\varphi$. Let us check these hypotheses. First, ∂D is ADR and D satisfies the 2-sided corkscrew condition, inherited from Ω. Next, $\varphi\nabla u \in C^0(D)$ holds because u is harmonic (and hence smooth) in Ω. For the nontangential maximal function, $\mathcal{N}_D(\varphi\nabla u) \in L^p(\sigma)$ follows from the bound $\mathcal{N}_D(\varphi\nabla u) \le C_\varphi \mathcal{N}_D(\nabla u)$ and the fact that $\mathcal{N}_D(\nabla u) \in L^p(\sigma_D)$ by (6.4). The pointwise nontangential limit $(\varphi\nabla u)|_{\partial D}$ exists σ_D-a.e., since $(\nabla u)|_{\partial\Omega}$ exists σ-a.e. by assumption. Finally, $\operatorname{div}(\varphi\nabla u) \in L^1(D)$ because

$$\int_D |\operatorname{div}(\varphi\nabla u)| \, dm = \int_D |\nabla u \nabla\varphi + \varphi\Delta u| \, dm \le C_\varphi \int_{D\cap B} |\nabla u| \, dm < \infty,$$

where we used $\Delta u = 0$ in D and that $\nabla u \in L^{p(n+1)/n}(D)$ by (6.4) and (6.1).

Having checked these conditions, as $\Delta u = 0$ in Ω and by Green's formula (6.5) for the domain D and the vector field $\varphi\nabla u$ (with bounded support in D), we have

$$\int_D \nabla u \nabla\varphi \, dm = \int_D \operatorname{div}(\varphi\nabla u) \, dm = \int_{\partial\Omega} \varphi \partial_\nu u \, d\sigma,$$

as claimed. □

As a consequence of Proposition 6.1 and Lemma 6.2, we get the uniqueness of the Neumann problem with zero nontangential derivative.

Corollary 6.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ and $\varepsilon_1 > 0$ be as in Proposition 6.1. If $p \in [2n/(n+1), n+\varepsilon_1)$, then constant functions in Ω are the unique solutions of the Neumann boundary value problem*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\sigma), \\ \partial_\nu u = 0, \ \sigma\text{-a.e. on } \partial\Omega. \end{cases}$$

The rest of the section is devoted to the proof of Proposition 6.1. The proof will use the extra regularity in (6.3) for functions satisfying $\mathcal{N}(\nabla u) \in L^p(\sigma)$, the Hölder regularity (up to the

boundary) of $W^{1,2}$-solutions of the Neumann problem in terms of weak derivatives (see Theorem 6.4 below), and the well-known Poincaré inequality for uniform domains (see Lemma 6.5 below).

Theorem 6.4 *([20, Corollary 4.49]). Let $\Omega \subset \mathbb{R}^{n+1}$ be as in Proposition 6.1 and let $B := B(\xi, R)$ be a ball of radius $R > 0$ centered at $\xi \in \partial\Omega$. There exist $K = K(CAD, n) \geq 1$, $C = C(CAD, n) < \infty$ and $\alpha = \alpha(CAD, n) > 0$ such that if $u \in W^{1,2}(\Omega \cap B_{KR}(\xi))$ satisfies*

$$\int_{\Omega} \nabla u(z) \nabla \varphi(z)\, dm(z) = 0, \text{ for all } \varphi \in W^{1,2}_c(B_{KR}(\xi)),$$

then for all $0 < r < R/2$ there holds

$$|u(x) - u(y)| \leq C \left(\frac{|x-y|}{r}\right)^{\alpha} \left(\fint_{\Omega \cap B_{2r}(\xi)} u(z)^2\, dm(z) \right)^{1/2} \text{ for all } x, y \in \Omega \cap B_r(\xi).$$

Lemma 6.5 *(Poincaré inequality). Let $\Omega \subset \mathbb{R}^{n+1}$ be a uniform domain and let B be a ball of radius $R > 0$ centered at $\partial\Omega$. There exists $K > 2$, depending only on the uniformity constants of Ω, such that if $u \in W^{1,p}(KB \cap \Omega)$, for $1 < p < n+1$, then with $p + \varepsilon_p = p(n+1)/(n+1-p)$ there holds*

$$\left(\fint_{\Omega \cap B} |u(z) - m_{\Omega \cap B} u|^{p+\varepsilon_p}\, dm(z) \right)^{1/(p+\varepsilon_p)} \lesssim R \left(\fint_{\Omega \cap KB} |\nabla u(z)|^p\, dm(z) \right)^{1/p}. \tag{6.6}$$

Proof. See [41, Theorem 4.1] for instance. A quick inspection of [41, p. 25, l. 5] reveals that $p + \varepsilon_p = p(n+1)/(n+1-p)$. □

We now turn to the proof of Proposition 6.1.

Proof of Proposition 6.1. Let us fix a constant $K > 2$ for which both Theorem 6.4 and the Poincaré inequality in Lemma 6.5 hold, and we allow all subsequent involved constants to depend on K. Fixed $\xi \in \partial\Omega$, we write $B_R := B_R(\xi)$ for $R > 0$. By (6.3) we have $u \in W^{1,p(n+1)/n}(\Omega \cap KB_{10R}) \subset W^{1,p}(\Omega \cap KB_{10R})$, in particular $u \in W^{1,2}(\Omega \cap KB_{10R})$ since we are assuming $p \geq 2n/(n+1)$. By assumption (and a density argument since $u \in W^{1,2}(\Omega \cap KB_{10R})$), it follows that we can apply Theorem 6.4 and therefore, for any $x, y \in B(\xi, R/2)$ we have

$$|u(x) - u(y)| \lesssim \left(\frac{|x-y|}{R}\right)^{\alpha} \left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^2\, dm \right)^{1/2}, \tag{6.7}$$

with $\alpha = \alpha(n, \text{CAD}) > 0$.

If $p \geq 2$, then by the Poincaré inequality (6.6) and Hölder's inequality we have

$$\left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^2 \, dm \right)^{1/2} \lesssim R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^2 \, dm \right)^{1/2} \leq R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^p \, dm \right)^{1/p}.$$

If $2n/(n+1) \leq p < 2$, then by Hölder's inequality and the Poincaré inequality (6.6) (with $p + \varepsilon_p := (n+1)p/(n+1-p)$ which is ≥ 2 since $p \geq 2n/(n+1) \geq 2(n+1)/(n+3)$),

$$\begin{aligned} \left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R(\xi)} u|^2 \, dm \right)^{1/2} &\leq \left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^{p+\varepsilon_p} \, dm \right)^{1/(p+\varepsilon_p)} \\ &\lesssim R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^p \, dm \right)^{1/p}. \end{aligned}$$

That is, in any case we obtain the same bound. From this, Hölder's inequality and (6.1) we have

$$\begin{aligned} \left(\fint_{\Omega \cap B_R} |u - m_{\Omega \cap B_R} u|^2 \, dm \right)^{1/2} &\lesssim R \left(\fint_{\Omega \cap B_{10RK}} |\nabla u|^p \, dm \right)^{1/p} \\ &\lesssim \frac{R}{R^{(n+1)\frac{n}{p(n+1)}}} \|\nabla u\|_{L^{p(n+1)/n}(\Omega)} \lesssim R^{1-n/p} \|\mathcal{N}(\nabla u)\|_{L^p(\sigma)}. \end{aligned} \tag{6.8}$$

From (6.7) and (6.8) we get

$$|u(x) - u(y)| \lesssim |x-y|^\alpha R^{1-n/p-\alpha} \|\mathcal{N}(\nabla u)\|_{L^p(\sigma)}.$$

Having fixed $x, y \in \Omega$ and $\xi \in \partial\Omega$, this bound holds as long as $R > 0$ is big enough so that $x, y \in B(\xi, R/2)$. Taking $R \to \infty$, we obtain that u is constant in Ω provided $p < n/(1-\alpha)$, i.e., $p < n + \varepsilon_1$ with $\varepsilon_1 = \varepsilon_1(\alpha, n) = \varepsilon_1(\text{CAD}, n) > 0$. □

7. Injectivity of $\pm\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ for $p \in [2n/(n+1), n+\varepsilon)$

Recall that for Theorems 1.2 and 1.3, we are interested in the invertibility of the operators $\frac{1}{2}Id + K$ in $L^{p'}(\sigma)$ and $-\frac{1}{2}Id + K^*$ in $L^p(\sigma)$. By Corollary 3.6, it suffices to show that $\frac{1}{2}Id + K^*$ and $-\frac{1}{2}Id + K^*$ are injective in $L^p(\sigma)$.

More precisely, in this section we see the injectivity of $\pm\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ with $2n/(n+1) \leq p < n + \varepsilon_3$, for 2-sided CAD's with unbounded boundary. This is stated in Corollary 7.2, and follows from the injectivity of $-\frac{1}{2}Id + K^*$ for unbounded 1-sided CAD with unbounded boundary.

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ and $\varepsilon_1 > 0$ be as in Proposition 6.1. Then there exists $0 < \varepsilon_2 = \varepsilon_2(n, CAD) \leq \varepsilon_1$ such that $-\frac{1}{2}Id + K^*$ is injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_2)$.*

Proposition 7.1 is proved in Section 7.2. Applying this result to $\overline{\Omega}^c$ we deduce the following corollary.

Corollary 7.2. *Assume that $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ is an unbounded 1-sided CAD with unbounded boundary, and let $\varepsilon_2 > 0$ be as in Proposition 7.1 for the domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Then $\frac{1}{2}Id + K^*$ is injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_2)$.*

Proof. A quick computation reveals that $K^*_\Omega = -K^*_{\overline{\Omega}^c}$. Hence, $\frac{1}{2}Id + K^*_\Omega = \frac{1}{2}Id - K^*_{\overline{\Omega}^c}$, which is injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_2)$ by Proposition 7.1. □

Corollary 7.3. *If $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded 2-sided CAD with unbounded boundary, then there exists $\varepsilon_3 = \varepsilon_3(n, CAD) > 0$ such that both $\pm\frac{1}{2}Id + K^*$ are injective in $L^p(\sigma)$ for all $p \in [2n/(n+1), n+\varepsilon_3)$.*

Proof. Take $\varepsilon_3 := \min\{\varepsilon_2, \widetilde{\varepsilon}_2\}$, with ε_2 and $\widetilde{\varepsilon}_2$ as in Proposition 7.1 and Corollary 7.2 for the domains Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ respectively. □

By Corollary 3.6, this immediately implies that for $p \in [2n/(n+1), n+\varepsilon_3)$, the operators $\pm\frac{1}{2}Id + K$ and $\pm\frac{1}{2}Id + K^*$ are invertible in $L^{p'}(\sigma)$ and $L^p(\sigma)$ respectively, under small enough flatness assumption of the δ-$(s, S; R)$ domain Ω, see Definition 1.1.

Remark 7.4. For bounded 2-sided corkscrew ADR domains, the injectivity of $\frac{1}{2}Id + K^*$ in $L^p(\sigma)$ for $p \in [2n/(n+1), \infty)$ is proved in [37, Theorem 1.7.2 (4)]. Since the domain is bounded, the injectivity for the full range $[2n/(n+1), \infty)$ follows from the particular case $p = 2n/(n+1)$ and Hölder's inequality. The case $p = 2$ was already proved in [18, Proposition 5.11]. Both proofs are based on the divergence theorem. In fact, for the so-called regular SKT domains (see [18, Definition 4.8]), the injectivity in $L^2(\sigma)$ of $\frac{1}{2}Id + K^*$ is enough to show that all four operators $(\pm)\frac{1}{2}Id + K^{(*)}$ are invertible in $L^p(\sigma)$, for all $1 < p < \infty$, see [18, Proposition 5.12].

7.1. The single layer potential

Let $\Omega \subset \mathbb{R}^{n+1}$ be an ADR domain with the 2-sided corkscrew condition (in particular $\partial\Omega$ is UR by Theorem 2.8) and $f \in L^1\left(\frac{d\sigma(x)}{1+|x|^n}\right)$. The modified single layer potential operator associated with Ω is

$$\mathcal{S}_{\text{mod}} f(x) = \mathcal{S}_{\text{mod},\Omega} f(x) := \frac{1}{w_n(1-n)} \int_{\partial\Omega} \left(\frac{1}{|x-y|^{n-1}} - \frac{\mathbf{1}_{B_1(0)^c}(y)}{|y|^{n-1}} \right) f(y)\, d\sigma(y),$$

$$x \in \mathbb{R}^{n+1} \setminus \partial\Omega, \tag{7.1}$$

and its boundary version is defined as

$$S_{\mathrm{mod}} f(x) = S_{\mathrm{mod},\Omega} f(x) \coloneqq \frac{1}{w_n(1-n)} \int_{\partial\Omega} \left(\frac{1}{|x-y|^{n-1}} - \frac{\mathbf{1}_{B_1(0)^c}(y)}{|y|^{n-1}} \right) f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

Remark 7.5. The classical single layer potential $\mathscr{S}$ as well as its boundary version S are defined as $\mathscr{S}_{\mathrm{mod}}$ and S_{mod} but replacing their kernel by $\frac{1}{|x-y|^{n-1}}$. However, this only makes sense for domains with bounded boundary, and in this case, the difference between the classical and the modified single layer potential is constant.

The single layer potential satisfies

$$\nabla \mathscr{S}_{\mathrm{mod}} f(x) = \frac{1}{w_n} \int_{\partial\Omega} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\sigma(y), \quad x \in \mathbb{R}^{n+1} \setminus \partial\Omega,$$

see [34, (3.39)]. Note that this is (modulo a multiplicative constant) the Riesz transform $\mathcal{R}_\sigma f$. Moreover, it satisfies $\Delta(\mathscr{S}_{\mathrm{mod}} f) = 0$ in $\mathbb{R}^{n+1} \setminus \partial\Omega$, see [34, (3.40)], and $\mathscr{S}_{\mathrm{mod}} f$ is continuous through the boundary $\partial\Omega$ in the nontangential sense

$$(\mathscr{S}_{\mathrm{mod}} f)|^{\mathrm{nt}}_{\partial\Omega}(z) = S_{\mathrm{mod}} f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{7.2}$$

see [34, (3.47)].

Recall the interior and exterior normal nontangential derivatives in (2.5) and (2.6). There is a jump formula for the interior derivative, see [34, (3.67)], which is

$$\partial^{\mathrm{int}}_{\nu_\Omega} \mathscr{S}_{\mathrm{mod}} f(x) = \left(-\frac{1}{2} Id + K^* \right) f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{7.3}$$

A quick computation reveals that $K^*_{\overline{\Omega}^c} = -K^*_\Omega$, and using this in the last equality, for σ-a.e. $x \in \partial\Omega$ we have the jump formula for the exterior derivative:

$$\begin{aligned} \partial^{\mathrm{ext}}_{\nu_\Omega} \mathscr{S}_{\mathrm{mod}} f(x) &= \lim_{\substack{z \to x \\ z \in \Gamma^{\overline{\Omega}^c}(x)}} \langle \nu(x), \nabla \mathscr{S}_{\mathrm{mod}} f(z) \rangle = - \lim_{\substack{z \to x \\ z \in \Gamma^{\overline{\Omega}^c}(x)}} \langle \nu_{\overline{\Omega}^c}(x), \nabla \mathscr{S}_{\mathrm{mod},\overline{\Omega}^c} f(z) \rangle \\ &\overset{(7.3)}{=} - \left(-\frac{1}{2} Id + K^*_{\overline{\Omega}^c} \right) f(x) = \left(\frac{1}{2} Id + K^* \right) f(x). \end{aligned} \tag{7.4}$$

Given any $\varepsilon > 0$, by [34, (3.41)] we have

$$\mathcal{N}^\varepsilon(\mathscr{S}_{\mathrm{mod}} f) \in \bigcap_{0 < p < \frac{n}{n-1}} L^p_{\mathrm{loc}}(\sigma). \tag{7.5}$$

Arguing as in the proof of (6.3) (using (7.5) with $p = 1$ instead of (6.2)) we obtain $\mathscr{S}_{\mathrm{mod}} f \in L^{(n+1)/n}(\Omega \cap B)$ for any ball B centered at $\partial\Omega$. Note that the same holds for $\overline{\Omega}^c$ (as it satisfies the same assumptions as Ω), and therefore we have

$$\mathscr{S}_{\mathrm{mod}} f \in L^{(n+1)/n}_{\mathrm{loc}}(\mathbb{R}^{n+1}). \tag{7.6}$$

For $f \in L^p(\sigma)$ with $1 < p < \infty$, the single layer potential satisfies

$$\|\mathcal{N}(\nabla \mathcal{S}_{\text{mod}} f)\|_{L^p(\sigma)} \lesssim_{p,\text{UR},\alpha} \|f\|_{L^p(\sigma)}, \tag{7.7}$$

see [34, (3.127)]. By (7.7) (and arguing as in (6.4) if Ω is unbounded with bounded boundary) and (6.1), the single layer potential satisfies

$$\nabla(\mathcal{S}_{\text{mod}} f) \in L^{p(n+1)/n}(\Omega \cap B) \text{ for any ball } B \text{ centered at } \partial\Omega. \tag{7.8}$$

In fact, if Ω is bounded or has unbounded boundary, a direct application of (7.7) and (6.1) gives $\nabla(\mathcal{S}_{\text{mod}} f) \in L^{p(n+1)/n}(\Omega)$. As before, the same holds for $\overline{\Omega}^c$. Arguing as in the proof of Lemma 6.2,[8] weak derivatives of the single layer potential in $\mathbb{R}^{n+1}$ exist and its gradient is $\mathbf{1}_{\mathbb{R}^{n+1}\setminus\partial\Omega}\nabla\mathcal{S}_{\text{mod}} f$, whence we get from (7.8) (with both domains Ω and $\overline{\Omega}^c$) that in the weak sense there holds

$$\nabla(\mathcal{S}_{\text{mod}} f) \in L^{p(n+1)/n}_{\text{loc}}(\mathbb{R}^{n+1}), \tag{7.9}$$

and in fact $\nabla(\mathcal{S}_{\text{mod}} f) \in L^{p(n+1)/n}(\mathbb{R}^{n+1})$ if $\partial\Omega$ is unbounded, i.e., both Ω and $\overline{\Omega}^c$ are unbounded.

7.2. *Proof of Proposition 7.1*

The proof of Proposition 7.1 is presented in several steps. In Step 0 we define the notation used throughout the proof, in Step 1 we show that the single layer potential is constant in Ω and σ-a.e. on $\partial\Omega$, in Step 2 we find a submean value property for the modulus of the single layer potential minus the constant found in Step 1, and from this we deduce in Step 3 that the single layer potential is also constant in the complement of Ω, with the same value. From this, in Step 4 we conclude the proof of the injectivity.

Step 0: Let $\varepsilon_2 \in (0, \varepsilon_1]$, with $\varepsilon_1 > 0$ as in Corollary 6.3. During the proof we write $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{R}^{n+1} \setminus \overline{\Omega}$. Let $f \in L^p(\sigma)$ satisfy $(-\frac{1}{2}Id + K^*)f = 0$ in $L^p(\sigma)$, equivalently,

$$\left(-\frac{1}{2}Id + K^*\right) f = 0 \ \sigma\text{-a.e. on } \partial\Omega. \tag{7.10}$$

We want to show that $f = 0$ σ-a.e. on $\partial\Omega$. To this end, we consider the modified single layer potential

$$u := \mathcal{S}_{\text{mod}} f,$$

which is harmonic in $\mathbb{R}^{n+1} \setminus \partial\Omega$.

[8] By the divergence theorem in [18, Theorem 2.8] using now (7.2), (7.5), (7.6) and (7.8).

Step 1: In this step we prove

$$u \equiv c_0 \text{ constant in } \Omega \text{ and } \sigma\text{-a.e. on } \partial\Omega. \tag{7.11}$$

By (7.3) and (7.10) we have $\partial_{\nu_\Omega} u = (-\frac{1}{2} Id + K^*) f = 0$ σ-a.e. on $\partial\Omega$. Recall that $u : \Omega \to \mathbb{R}$ is harmonic and $\mathcal{N}(\nabla u) \in L^p(\sigma)$ by (7.7). All in all, the assumptions of Corollary 6.3 are satisfied, and therefore we conclude that u is constant in Ω. By (7.2), we conclude the proof of (7.11).
Step 2: In this step we prove the following submean value property

$$|u(z) - c_0| \leq \fint_{B_r(z)} |u(x) - c_0|\, dm(x) \text{ for all } z \in \mathbb{R}^{n+1} \setminus \partial\Omega \text{ and } r > 0, \tag{7.12}$$

where c_0 is the constant in (7.11). Note that if u were a continuous function in $\mathbb{R}^{n+1}$, this would immediately follow from the fact that h is harmonic in $\overline{\Omega}^c$ and $h = 0$ in $\overline{\Omega}$. However, due to the lack of continuity, a more careful argument is required.

We define $h(\cdot) := u(\cdot) - c_0$ to shorten the notation. To prove (7.12), we first see that it satisfies

$$\int_{\mathbb{R}^{n+1}} \nabla|h| \nabla\varphi \, dm \leq 0 \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^{n+1}) \text{ with } \varphi \geq 0, \tag{7.13}$$

called weak subharmonic condition.

Note that, by (7.9) and since $|\nabla|h|| = |\nabla u|$ m-a.e. in $\mathbb{R}^{n+1}$, we have

$$\nabla|h| \in L^{p(n+1)/n}_{\text{loc}}(\mathbb{R}^{n+1}). \tag{7.14}$$

Let us fix $B \supset \operatorname{supp}\varphi$ (not necessarily centered at $\partial\Omega$), and write $\Omega_B^- := \Omega^- \cap B$ from now on. By $\operatorname{supp}\varphi \subset B$ and $h = u - c_0 \equiv 0$ in Ω and σ-a.e. in $\partial\Omega$, we have

$$\int_{\mathbb{R}^{n+1}} \nabla|h| \nabla\varphi \, dm = \int_{\Omega_B^-} \nabla|h| \nabla\varphi \, dm. \tag{7.15}$$

As in [18, (2.3.37)] for the domain Ω_B^-, for $\delta > 0$ we define

$$\chi_\delta(x) := \begin{cases} 1 & \text{if } x \in \Omega_B^- \setminus U_\delta(\partial\Omega_B^-), \\ 2\delta^{-1}\operatorname{dist}(x, \partial U_{\delta/2}(\partial\Omega_B^-)) & \text{if } x \in \Omega_B^- \cap (U_\delta(\partial\Omega_B^-) \setminus U_{\delta/2}(\partial\Omega_B^-)), \\ 0 & \text{if } x \in U_{\delta/2}(\partial\Omega_B^-) \cup (\mathbb{R}^{n+1} \setminus \Omega_B^-). \end{cases}$$

Moreover, we fix $\psi \in C_c^\infty(B_1(0))$ satisfying $\psi \geq 0$ and $\int_{B_1(0)} \psi \, dm = 1$, and define

$$\psi_\varepsilon(\cdot) := \frac{1}{\varepsilon^{n+1}} \psi\left(\frac{\cdot}{\varepsilon}\right), \text{ for } \varepsilon > 0. \tag{7.16}$$

Since by Hölder's inequality and (7.14) we have

$$\int_{\Omega_B^-} \left|(1-\chi_\delta)\nabla|h|\nabla\varphi\right| dm \lesssim \|\nabla|h|\|_{L^p(m|_B)} m(U_\delta(\partial\Omega_B^-))^{1-\frac{1}{p}} \lesssim \|\nabla|h|\|_{L^p(m|_B)} \delta^{1-\frac{1}{p}} \xrightarrow{\delta\to 0} 0,$$

we get

$$\int_{\Omega_B^-} \nabla|h|\nabla\varphi\, dm = \lim_{\delta\to 0} \int_{\Omega_B^-} \chi_\delta \nabla|h|\nabla\varphi\, dm. \tag{7.17}$$

Also, since $\nabla|h| \in L^1(m|_B)$ by (7.14), for every $\delta > 0$ we have

$$\int_{\Omega_B^-} \chi_\delta \nabla|h|\nabla\varphi\, dm = \lim_{\varepsilon\to 0} \int_{\Omega_B^-} \chi_\delta \nabla(|h| * \psi_\varepsilon)\nabla\varphi\, dm \tag{7.18}$$

For every $\delta > 0$ and every $0 < \varepsilon < \delta/2$, by $\nabla(\chi_\delta\varphi) = \varphi\nabla\chi_\delta + \chi_\delta\nabla\varphi$ and $\operatorname{div}(\chi_\delta\varphi\nabla(|h| * \psi_\varepsilon)) = \nabla(\chi_\delta\varphi)\nabla(|h| * \psi_\varepsilon) + \chi_\delta\varphi\Delta(|h| * \psi_\varepsilon)$, we have

$$\begin{aligned}\int_{\Omega_B^-} \chi_\delta \nabla(|h| * \psi_\varepsilon)\nabla\varphi\, dm &= \int_{\Omega_B^-} \operatorname{div}(\chi_\delta\varphi\nabla(|h| * \psi_\varepsilon))\, dm \\ &\quad - \int_{\Omega_B^-} \chi_\delta\varphi\Delta(|h| * \psi_\varepsilon)\, dm - \int_{\Omega_B^-} \varphi\nabla(|h| * \psi_\varepsilon)\nabla\chi_\delta\, dm\end{aligned}$$

The first term on the right-hand side is zero by the classical divergence theorem. Also, we have that $-\Delta(|h| * \psi_\varepsilon) \le 0$ in $\Omega^- \setminus U_\delta(\partial\Omega)$ for $\varepsilon < \delta/2$, because $|h| * \psi_\varepsilon$ is smooth and $(|h| * \psi_\varepsilon)(z) \le m_{B(z,r)}(|h| * \psi_\varepsilon)$ for all $z \in \Omega^- \setminus U_\delta(\partial\Omega)$ and $0 < r < \delta/2$, which follows from the fact that $|h|(z) \le m_{B(z,r)}|h|$ for all $z \in \Omega^-$ and $0 < r < \operatorname{dist}(z, \partial\Omega)$ (since $\Delta h = 0$ in Ω^-). Hence, we get

$$\int_{\Omega_B^-} \chi_\delta \nabla(|h| * \psi_\varepsilon)\nabla\varphi\, dm \le -\int_{\Omega_B^-} \varphi\nabla(|h| * \psi_\varepsilon)\nabla\chi_\delta\, dm. \tag{7.19}$$

From (7.17), (7.18), (7.19), and that the latter integral in (7.19) converges to $-\int_{\Omega_B^-} \varphi\nabla|h|\nabla\chi_\delta\, dm$ as $\varepsilon \to 0$ because $\nabla|h| \in L^1(m|_B)$, we get

$$\int_{\Omega_B^-} \nabla|h|\nabla\varphi\, dm \le \lim_{\delta\to 0} -\int_{\Omega_B^-} \nabla\chi_\delta(\varphi\nabla|h|)\, dm. \tag{7.20}$$

Rewriting the latter integral using the notation $\nu_\delta(x) := -\nabla(\operatorname{dist}(x, \partial U_{\delta/2}(\partial\Omega_B^-)))$ and $\widetilde{U}_\delta(\partial\Omega_B^-) := \Omega_B^- \cap (U_\delta(\partial\Omega_B^-) \setminus U_{\delta/2}(\partial\Omega_B^-))$, we now claim (and prove below) that

$$\lim_{\delta\to 0}\frac{2}{\delta}\int\limits_{\widetilde{U}_\delta(\partial\Omega_B^-)}\langle v_\delta,\varphi\nabla|h|\rangle\,dm=\int\limits_{\partial\Omega_B^-}\langle \nu_{\Omega_B^-},(\varphi\nabla|h|)|_{\partial\Omega_B^-}\rangle\,d\mathcal{H}^n. \tag{7.21}$$

Before its proof, let us conclude the proof of (7.13). Using that $\varphi\equiv 0$ on ∂B and the fact that $\partial_{\nu_{\Omega^-}}|h|\le 0$ because $|h|\ge 0$ converges to zero nontangentially σ-a.e. at $\partial\Omega$ (by (7.11) in Step 1), we have

$$\int\limits_{\partial\Omega_B^-}\langle \nu_{\Omega_B^-},(\varphi\nabla|h|)|_{\partial\Omega_B^-}\rangle\,d\mathcal{H}^n=\int\limits_{B\cap\partial\Omega}\varphi\partial_{\nu_{\Omega^-}}|h|\,d\sigma\le 0.$$

This, together with (7.15), (7.20) and (7.21), concludes the proof of (7.13), in the absence of the justification of (7.21).

Let us see the claim in (7.21). We first need to check that

$$\mathcal{N}(\nabla|h|)\in L^p(\sigma)\text{ and }(\nabla|h|)|_{\partial\Omega}\text{ exists }\sigma\text{-a.e. on }\partial\Omega. \tag{7.22}$$

The first condition is satisfied by $|\nabla|h||=|\nabla u|$ m-a.e. in Ω^- and (7.7). Regarding the second condition, we want to see that for σ-a.e. $x\in\partial\Omega$ there exists a vector v_x such that, for all $\varepsilon>0$ there is $\delta>0$ satisfying that

$$z\in\Gamma^{\Omega^-}(x)\cap B_\delta(x)\implies|\nabla|h|-v_x|<\varepsilon.$$

By the jump formula (7.4) and the assumption $(-\frac{1}{2}Id+K^*)f=0$, see (7.10), we have $\partial_{\nu_{\Omega^-}}h=f$ for σ-a.e., in particular the pointwise nontangential limit $(\nabla h)|_{\partial\Omega}$ exists σ-a.e., which we denote by w_x for such $x\in\partial\Omega$. If $w_x=0$, then $v_x=w_x$ does the job as $|\nabla|h||=|\nabla h|$. Since $h\equiv 0$ σ-a.e., if $w_x\neq 0$, then for $\delta>0$ small enough we have that either $h>0$ or $h<0$ in $\Gamma^{\Omega^-}(x)\cap B_\delta(x)$. If $\pm h>0$, then we take $v_x=\pm w_x$, and hence we have

$$|\nabla|h|-\pm w_x|=|\pm\nabla u\mp w_x|=|\nabla u-w_x|<\varepsilon,$$

where we used that $\nabla|h|=\frac{h}{|h|}\nabla u=\pm\nabla u$ in $\Gamma^{\Omega^-}(x)\cap B_\delta(x)$ (since we are in the case $\pm h>0$ there) and the fact that the pointwise nontangential limit exists σ-a.e. and is $(\nabla u)|_{\partial\Omega}=w_x$.

As argued in [18, p. 2596, lines 6-8], the equality (7.21) directly holds if one replaces $\varphi\nabla|h|$ by $v\in C^{0,1}(\overline{\Omega_B^-})$. We adapt the end of Step I in the proof of [35, Theorem 1.3.1] to our situation. By the density of $C_c^\infty(\mathbb{R}^{n+1})$ functions in $L^1(\sigma|_{B\cap\partial\Omega})$, for any $\eta>0$ there exists $w\in C_c^\infty(\mathbb{R}^{n+1})$ such that

$$\|(\nabla|h|)|_{\partial\Omega}-w\|_{L^1(\sigma|_{B\cap\partial\Omega})}<\eta.$$

First,

$$\int\limits_{\partial\Omega_B^-}|\langle \nu_{\Omega_B^-},\varphi w-(\varphi\nabla|h|)|_{\partial\Omega}\rangle|\,d\mathcal{H}^n\lesssim\|(\nabla|h|)|_{\partial\Omega}-w\|_{L^1(\sigma|_{B\cap\partial\Omega})}<\eta,$$

second, as we already noted, since φw is in particular Lipschitz in $\overline{\Omega_B^-}$, the equality in (7.21) holds in this case and so we have

$$\lim_{\delta\to 0}\frac{2}{\delta}\int\limits_{\widetilde{U}_\delta(\partial\Omega_B^-)}\langle \nu_\delta,\varphi w\rangle\,dm=\int\limits_{\partial\Omega_B^-}\langle \nu_{\Omega_B^-},\varphi w\rangle\,d\mathcal{H}^n,$$

and third, by Lemma 2.10, for small enough $\delta>0$ so that $\varphi=0$ in $B\setminus(1-\delta)B$ we have

$$\frac{2}{\delta}\int\limits_{\widetilde{U}_\delta(\partial\Omega_B^-)}|\langle \nu_\delta,\varphi\nabla|h|-\varphi w\rangle|\,dm\lesssim\|\mathcal{N}^\delta(\nabla|h|-w)\|_{L^1(\sigma|_{B\cap\partial\Omega})},$$

recall the definition of $\mathcal{N}^\delta$ in (2.3). Note that the second condition in (7.22) implies that $\mathcal{N}^\delta(\nabla|h|-w)(x)\to(\nabla|h|)|_{\partial\Omega}(x)-w(x)$ as $\delta\to 0$ for σ-a.e. $x\in\partial\Omega$. From this and $\mathcal{N}(\nabla|h|)\in L^1(\sigma|_{B\cap\partial\Omega})$, see (7.22), by the dominated convergence theorem we have

$$\lim_{\delta\to 0}\|\mathcal{N}^\delta(\nabla|h|-w)\|_{L^1(\sigma|_{B\cap\partial\Omega})}=\|(\nabla|h|)|_{\partial\Omega}-w\|_{L^1(\sigma|_{B\cap\partial\Omega})}<\eta.$$

All in all, we have

$$\left|\lim_{\delta\to 0}\frac{2}{\delta}\int\limits_{\widetilde{U}_\delta(\partial\Omega_B^-)}\langle \nu_\delta,\varphi\nabla|h|\rangle\,dm-\int\limits_{\partial\Omega_B^-}\langle \nu_{\Omega_B^-},(\varphi\nabla|h|)|_{\partial\Omega}\rangle\,d\mathcal{H}^n\right|\lesssim\eta,$$

with uniform constant, and as $\eta>0$ is arbitrary, we conclude the claim in (7.21), and therefore the proof of (7.13) is now complete.

Let us see now how (7.13) implies the submean value property (7.12). With ψ_ε as in (7.16), the function $|h|*\psi_\varepsilon$ is smooth and for $z\in\mathbb{R}^{n+1}$ there holds

$$\begin{aligned}-\Delta(|h|*\psi_\varepsilon)(z)&=\sum_{i=1}^{n+1}-(\partial_{x_i}|h|*\partial_{x_i}\psi_\varepsilon)(z)=-\int\nabla|h|(y)\nabla\psi_\varepsilon(z-y)\,dm(y)\\&=\int\nabla|h|(y)\nabla(\psi_\varepsilon(z-y))\,dm(y)\overset{(7.13)}{\leq}0,\end{aligned}\tag{7.23}$$

whence we get that $|h|*\psi_\varepsilon$ is subharmonic in $\mathbb{R}^{n+1}$ and in particular we get the submean value property

$$(|h|*\psi_\varepsilon)(z)\leq\fint\limits_{B_r(z)}(|h|*\psi_\varepsilon)(x)\,dm(x)\text{ for all }z\in\mathbb{R}^{n+1}\text{ and }r>0.$$

For $z\in\mathbb{R}^{n+1}\setminus\partial\Omega$, the left-hand side converges to $|h(z)|$ because $h\in C^0(\mathbb{R}^{n+1}\setminus\partial\Omega)$. On the other hand, since $\||h|*\psi_\varepsilon-|h|\|_{L^1(m|_{B_r(z)})}\to 0$ as $\varepsilon\to 0$ by (7.6), the right-hand side term converges to $\fint_{B_r(z)}|h(x)|\,dm(x)$ as $\varepsilon\to 0$. This concludes the proof of the submean value property (7.12). □

Step 3: In this step we prove

$$u \equiv c_0 \text{ constant in } \Omega^-. \tag{7.24}$$

For a fixed $z \in \Omega^-$, taking $r > 2\text{dist}(z, \partial\Omega)$ and $\xi \in \partial\Omega$ so that $|z - \xi| = \text{dist}(z, \partial\Omega)$, we have $B(z, r) \subset B(\xi, 2r)$ and therefore

$$|u(z) - c_0| \overset{(7.12)}{\leq} \fint_{B(z,r)} |u(x) - c_0|\, dm(x) \lesssim \fint_{B(\xi,2r)} |u(x) - c_0|\, dm(x).$$

Note that c_0 is the mean of u over any ball inside Ω. So, taking $\widetilde{B}$ the interior corkscrew ball of $B(\xi, 2r)$, we have that $\widetilde{B} \subset B(\xi, 2r) \cap \Omega$ with $r_{\widetilde{B}} \approx r$, and therefore we get that the last term above is

$$\fint_{B(\xi,2r)} |u(x) - c_0|\, dm(x) = \fint_{B(\xi,2r)} |u(x) - m_{\widetilde{B}} u|\, dm(x).$$

Adding $\pm m_{B(\xi,2r)} u$, by the classical Poincaré inequality since $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^{n+1})$ (see (7.6) and (7.14)), Hölder's inequality and the estimates (6.1) and (7.7), the latter term is controlled by

$$\begin{aligned}\fint_{B(\xi,2r)} |u(x) - m_{\widetilde{B}} u|\, dm(x) &\lesssim \fint_{B(\xi,2r)} |u(x) - m_{B(\xi,2r)} u|\, dm(x) \lesssim r \fint_{B(\xi,2r)} |\nabla u(x)|\, dm(x). \\ &\lesssim r^{1-n/p} \|\nabla u\|_{L^{p(n+1)/n}(\Omega)} \overset{(6.1),(7.7)}{\lesssim} r^{1-n/p} \|f\|_{L^p(\sigma)}\end{aligned}$$

All in all, for any $z \in \Omega^-$ and any $r > 2\text{dist}(z, \partial\Omega)$ we get

$$|u(z) - c_0| \lesssim r^{1-n/p} \|f\|_{L^p(\sigma)}. \tag{7.25}$$

In the case $2n/(n+1) \leq p < n$, letting $r \to \infty$ we deduce that $u(z) = c_0$, as claimed. For $n \leq p < n + \varepsilon_2$, we will use the following lemma to see that (7.25) implies a slightly better bound, as in (6.7).

Lemma 7.6. *Let $D \subset \mathbb{R}^{n+1}$ be a domain satisfying the exterior corkscrew condition with constant $M > 1$. There are $\alpha = \alpha(M, n) > 0$ and $C = C(M, n) \geq 1$ such that for every non-negative smooth subharmonic function u in $\mathbb{R}^{n+1}$, all $\xi \in \partial\Omega$ and all $r > 0$, if $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus U_{r/(4M)^{k_0}}(D)$ for some integer $k_0 \geq 1$, then*

$$u(x) \leq C \left(\frac{|x - \xi|}{r}\right)^{\alpha} \sup_{B_r(\xi)} u, \text{ for all } x \in B(\xi, r) \setminus \overline{B(\xi, r/(4M)^{k_0+1})}. \tag{7.26}$$

This is well known for harmonic functions vanishing continuously on $\partial\Omega$, even for more general domains. However, to keep track of the parameters, the proof is written in full detail at the end of this section. Assume the lemma to be true for the moment.

Let us fix $z \in \Omega^-$, and take $r > 2\text{dist}(z, \partial\Omega)$ and $\xi \in \partial\Omega$ so that $\text{dist}(z, \partial\Omega) = |z - \xi|$. With ψ_ε as in (7.16) for $\varepsilon > 0$, as we already saw in (7.23), $|h| * \psi_\varepsilon$ is nonnegative, smooth and subharmonic in $\mathbb{R}^{n+1}$, and identically zero in $\mathbb{R}^{n+1} \setminus U_\varepsilon(\Omega^-)$. So, by Lemma 7.6 there is $\alpha_2 > 0$ such that if $0 < \varepsilon \ll r$, then

$$(|h| * \psi_\varepsilon)(z) \overset{(7.26)}{\lesssim} \left(\frac{|z-\xi|}{r}\right)^{\alpha_2} \sup_{B(\xi,r)} (|h| * \psi_\varepsilon).$$

Also, for $x \in B(\xi, r)$ we have

$$(|h| * \psi_\varepsilon)(x) = \int_{B(x,\varepsilon)} |h|(y)\psi_\varepsilon(x-y)\, dm(y) \overset{(7.25)}{\lesssim} r^{1-n/p}\|f\|_{L^p(\sigma)},$$

as long as $\varepsilon < r$. All in all, and as $(|h| * \psi_\varepsilon)(z) \to |h(z)| = |u(z) - c_0|$ as $\varepsilon \to 0$ because $h \in C^0(\Omega^-)$, we conclude

$$|u(z) - c_0| \lesssim |z-\xi|^{\alpha_2} r^{1-\frac{n}{p}-\alpha_2}\|f\|_{L^p(\sigma)}.$$

This goes to zero as $r \to \infty$ if $1 < p < n/(1-\alpha_2)$. The parameter $\varepsilon_2 > 0$ is taken so that $n + \varepsilon_2 = \min\{n + \varepsilon_1, n/(1-\alpha_2)\}$. We conclude that $u \equiv c_0$ in Ω^-. □

Step 4: Let us finish the proof by seeing that $f \equiv 0$ in $L^p(\sigma)$. By (7.4) and since $u \equiv c_0$ in Ω^- (see (7.24) in Step 3), we have $(\frac{1}{2}Id + K^*)f = \partial_{\nu_{\overline{\Omega^c}}} u = 0$. On the other hand, recall that $(-\frac{1}{2}Id + K^*)f = 0$ by assumption, see (7.10). We conclude that

$$f = \left(\frac{1}{2}Id + K^*\right)f - \left(-\frac{1}{2}Id + K^*\right)f = 0 \quad \sigma\text{-a.e. on } \partial\Omega,$$

as claimed. □

We conclude this section with the proof of Lemma 7.6. We adapt the standard argument to show the Hölder continuity of harmonic functions that vanish continuously on the boundary of a domain satisfying the exterior corkscrew condition.

Proof of Lemma 7.6. We first claim that there is $c_1 = c_1(M, n) \in (0, 1)$ such that for a fixed $\xi \in \partial\Omega$ and $s > 0$, there holds

$$\sup_{B(\xi, s/(4M))} u \leq (1 - c_1) \sup_{B(\xi,s)} u, \text{ as long as } u \equiv 0 \text{ in } \mathbb{R}^{n+1} \setminus U_{s/(2M)}(D). \tag{7.27}$$

Indeed, by the exterior corkscrew condition there is $B(A_s(\xi), s/M) \subset B(\xi, s) \setminus \overline{D}$, and therefore there exists $\rho \in (s/(2M), s)$ such that

$$\mathcal{H}^n(\partial B(\xi, \rho) \setminus D) \geq \mathcal{H}^n(\partial B(\xi, \rho) \cap B(A_s(\xi), s/(2M))) \gtrsim_{M,n} \rho^n. \tag{7.28}$$

We are using $2M > 2$ to ensure that $u \equiv 0$ in $B(A_s(\xi), s/(2M)) \subset \mathbb{R}^{n+1} \setminus U_{s/(2M)}(D)$. Let H_u be the harmonic extension in $B(\xi, \rho)$ of $u|_{\partial B(\xi,\rho)}$. For $x \in B(\xi, \rho)$, using the Poisson kernel and that $u \equiv 0$ in $B(A_s(\xi), s/(2M)) \subset \mathbb{R}^{n+1} \setminus U_{s/(2M)}(D)$, we write H_u as

$$H_u(x) = \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)} \frac{u(z)}{|x-z|^{n+1}}\, d\mathcal{H}^n(z)$$
$$= \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)\setminus B(A_s(\xi),s/(2M))} \frac{u(z)}{|x-z|^{n+1}}\, d\mathcal{H}^n(z),$$

where $w_n = \mathcal{H}^n(\partial B(0,1))$. We bound $H_u(x)$ for $x \in B(\xi, \rho/2)$. Bounding u by its supremum in $B(\xi, s) \supset B(\xi, \rho)$, we have

$$H_u(x) \le \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)\setminus B(A_s(\xi),s/(2M))} \frac{\sup_{B(\xi,s)} u}{|x-z|^{n+1}}\, d\mathcal{H}^n(z)$$
$$= \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)} \frac{\sup_{B(\xi,s)} u}{|x-z|^{n+1}}\, d\mathcal{H}^n(z)$$
$$- \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)\cap B(A_s(\xi),s/(2M))} \frac{\sup_{B(\xi,s)} u}{|x-z|^{n+1}}\, d\mathcal{H}^n(z)$$
$$= \left(1 - \frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)\cap B(A_s(\xi),s/(2M))} \frac{1}{|x-z|^{n+1}}\, d\mathcal{H}^n(z)\right) \sup_{B(\xi,s)} u,$$

where we used in the last equality that the harmonic extension (using the Poisson kernel) of a constant function is the constant itself. Now, using that $|x-\xi| \le \rho/2$ and $|x-z| \le 3\rho/2$ if $x \in B(\xi, \rho/2)$ and $z \in \partial B(\xi, \rho)$, we get

$$\frac{\rho^2 - |x-\xi|^2}{w_n \rho} \int\limits_{\partial B(\xi,\rho)\cap B(A_s(\xi),s/(2M))} \frac{1}{|x-z|^{n+1}}\, d\mathcal{H}^n(z)$$
$$\ge \frac{2^{n-1}}{3^n w_n \rho^n} \mathcal{H}^n(\partial B(\xi,\rho) \cap B(A_s(\xi), s/(2M))) \overset{(7.28)}{\ge} c_1 \in (0,1),$$

for some $c_1 = c_1(M,n) \in (0,1)$. All in all, by the inclusion, the maximum principle, and the estimates above, we get

$$\sup_{B(\xi,s/(4M))} u \le \sup_{B(\xi,\rho/2)} u \le \sup_{B(\xi,\rho/2)} H_u \le (1-c_1) \sup_{B(\xi,s)} u,$$

provided that $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus U_{s/(2M)}(D)$, as claimed in (7.27).

As we are assuming $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus U_{r/(4M)^{k_0}}(D)$, iterating (7.27) we get

$$\sup_{B(\xi,r/(4M)^k)} u \le (1-c_1)^k \sup_{B(\xi,r)} u \text{ for all } 0 \le k \le k_0.$$

This readily proves the lemma. □

8. Uniqueness of the solution of the Dirichlet problem

Given a Wiener regular[9] domain Ω (not necessarily bounded) and $x \in \Omega$, we denote by $g_x = g_x^\Omega$ the harmonic Green function of Ω with pole at x, and by $\omega^x = \omega_\Omega^x$ the harmonic measure of Ω with pole at x. If the domain Ω is ADR and ω^x is absolutely continuous with respect to σ ($\omega^x \ll \sigma$ for short) for some (and hence for all) $x \in \Omega$, then the Radon-Nikodym derivative $d\omega^x/d\sigma$ (also known as the Poisson kernel) exists, with

$$\omega^x(E) = \int_E \frac{d\omega^x}{d\sigma}\, d\sigma, \text{ for any Borel set } E \subset \mathbb{R}^{n+1},$$

see [32, Theorem 2.17] for instance.

This section is dedicated to the uniqueness of the solution of the L^p Dirichlet problem in (1.1) under the assumption that the harmonic measure satisfies the p'-reverse Hölder inequality.

Proposition 8.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD, $x \in \Omega$, $p \in (1,\infty)$, and let p' so that $1/p + 1/p' = 1$. Assume $\omega_\Omega^x \ll \sigma$ and that there exists $\gamma > 0$ such that*

$$\left(\fint_{B(\xi,r)} \left| \frac{d\omega_\Omega^x}{d\sigma} \right|^{p'} d\sigma \right)^{1/p'} \lesssim \frac{\omega^x(B(\xi,r))}{\sigma(B(\xi,r))}, \tag{8.1}$$

for every $\xi \in \partial\Omega$ and $0 < r \le \gamma$. Then the zero function is the unique solution of the homogeneous Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ \mathcal{N} u \in L^p(\sigma) \\ u|_{\partial\Omega}^{\mathrm{nt}} = 0 \ \sigma\text{-a.e. on } \partial\Omega. \end{cases} \tag{8.2}$$

This is well known when the domain is assumed to be bounded, see for instance [36, Theorem 5.7.7]. For completeness, we provide a detailed proof in the general case, where Ω is not necessarily bounded. Its proof is a direct combination of Lemmas 8.3 and 8.4 below. A quick inspection of the proof of Lemma 8.3 below reveals that the NTA condition is only used for balls with sufficiently small radii.

We state the extrapolation of the solvability of (D_p) with uniqueness of (1.1).

Corollary 8.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD. If (D_{p_0}) is solvable for some $1 < p_0 < \infty$, then there exists $\varepsilon > 0$ such that (D_p) is solvable for all $p \in (p_0 - \varepsilon, \infty)$ and the solution of (D_p) is the unique solution of (1.1).*

[9] This ensures that the Green function is well-defined. For instance, domains with the exterior corkscrew condition or whose boundary is ADR are Wiener regular. In fact, satisfying one of these conditions at small scales is sufficient.

Proof. Using that the harmonic measure is doubling in NTA domains (see [25, Lemma 4.9][10]), this follows by the well-known equivalence between the solvability of the Dirichlet problem and the reverse Hölder for the harmonic measure (see [39, Proposition 2.14][11] for instance), Gehring's lemma (see [15, Theorem 6.38] for instance), and Proposition 8.1. □

We now turn to the first step in the proof of Proposition 8.1.

Lemma 8.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD domain, $x \in \Omega$, $p \in (1,\infty)$, and let p' so that $1/p + 1/p' = 1$. Assume that there exists $\gamma > 0$ such that (8.1) holds for every $\xi \in \partial\Omega$ and $0 < r \leq \gamma$. Then for any $\delta < \min\{\operatorname{diam}(\partial\Omega), \operatorname{dist}(x,\partial\Omega)\}/100$ there holds*

$$\|\mathcal{N}^\delta g_x^\Omega\|_{L^{p'}(\sigma)} \lesssim \gamma^{-n/p}\delta,$$
$$\|\mathcal{N}^\delta(\nabla g_x^\Omega)\|_{L^{p'}(\sigma)} \lesssim \gamma^{-n/p},$$

where the involved constant depends on p, the CAD character of Ω, the constant in (8.1) (and the aperture on the definition of $\mathcal{N}^\delta$ in (2.3)).

Proof. We define the Hardy-Littlewood maximal function

$$M_\sigma \omega^x(\xi) := \sup_{r>0} \frac{\omega^x(B_r(\xi))}{\sigma(B_r(\xi))} = \sup_{r>0} \fint_{B_r(\xi)} \frac{d\omega^x}{d\sigma}\, d\sigma, \quad \xi \in \partial\Omega.$$

Let us fix $\xi \in \partial\Omega$. For any $y \in \Gamma(\xi) \cap \overline{B_{2\delta}(\xi)}$, in particular, $|y-\xi| < (1+\alpha)\operatorname{dist}(y,\partial\Omega)$, we have that y is essentially a corkscrew point at scale $2|y-\xi|$ in the sense that both y and $A_{2|y-\xi|}(\xi)$ (the corkscrew point of ξ at scale $2|y-\xi|$) belong to $B_{2|y-\xi|}(\xi)$ and are uniformly far from the boundary with distance $\gtrsim 2|y-\xi|$. Thus, by Harnack inequality and the relation in [25, Lemma 4.8][10] between the Green function and the harmonic measure in NTA domains, we have

$$g_x(y) \approx g_x(A_{2|y-\xi|}(\xi)) \lesssim \frac{\omega^x(B(\xi, 2|y-\xi|))}{|y-\xi|^{n-1}}.$$

Therefore, from this and the definition of $M_\sigma\omega^x$, we have

$$\mathcal{N}^\delta g_x(\xi) = \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} g_x(y) \lesssim \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} |y-\xi| \frac{\omega^x(B(\xi,2|y-\xi|))}{|y-\xi|^n} \lesssim \delta M_\sigma\omega^x(\xi).$$

So, from this and the $L^{p'}$-bound of the Hardy-Littlewood maximal function $M_\sigma\omega^x$ we have

[10] This is originally for bounded domains, but the same continues to hold for unbounded domains.

[11] The solvability of the Dirichlet problem in [39, Definition 1.1] is stated for $C_c(\partial\Omega)$ functions, instead of $L^p(\sigma)$. However, it is well known that this is equivalent to the L^p solvability definition of (D_p) in (1.1) and (1.2), by a density argument.

$$\int_{\partial\Omega} |\mathcal{N}^\delta g_x|^{p'}\, d\sigma \lesssim \delta^{p'} \int_{\partial\Omega} |M_\sigma \omega^x|^{p'}\, d\sigma \lesssim_{p'} \delta^{p'} \int_{\partial\Omega} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma.$$

Given by the $5R$-covering theorem, let $\{B_k\}_{k\geq 1}$ be a subfamily of $\{B(\xi,\gamma)\}_{\xi\in\partial\Omega}$ such that $\partial\Omega \subset \bigcup_{k\geq 1} B_k$ and $\{B_k/5\}_{k\geq 1}$ is pairwise disjoint. So, (8.1) applies for each B_k since $r(B_k)=\gamma$. Also, since $\{B_k/5\}_{k\geq 1}$ is a pairwise disjoint family and each ball has the same radius, we also have that the family $\{B_k\}_{k\geq 1}$ has finite overlap. Thus, we have

$$\begin{aligned}\int_{\partial\Omega} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma &\leq \sum_{k\geq 1} \int_{B_k} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma \approx \gamma^n \sum_{k\geq 1} \fint_{B_k} \left|\frac{d\omega^x}{d\sigma}\right|^{p'} d\sigma \lesssim \gamma^n \sum_{k\geq 1} \left(\frac{\omega^x(B_k)}{\sigma(B_k)}\right)^{p'} \\ &\approx \gamma^{n(1-p')} \sum_{k\geq 1} \omega^x(B_k)^{p'} \leq \gamma^{n(1-p')} \sum_{k\geq 1} \omega^x(B_k) \lesssim \gamma^{n(1-p')}.\end{aligned}$$

All in all, $\|\mathcal{N}^\delta g_x\|_{L^{p'}(\sigma)} \lesssim \gamma^{-n/p}\delta$.

The $L^{p'}(\sigma)$-norm of the nontangential maximal operator of ∇g_x follows from the same computations above, using that $|\nabla g_x(\cdot)| \lesssim g_x(\cdot)/\mathrm{dist}(\cdot,\partial\Omega)$ far from the pole. Indeed, we now have

$$\mathcal{N}^\delta(\nabla g_x)(\xi) = \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} |\nabla g_x(y)| \lesssim \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} \frac{g_x(y)}{\mathrm{dist}(y,\partial\Omega)} \lesssim \sup_{y\in\Gamma(\xi)\cap\overline{B_{2\delta}(\xi)}} \frac{g_x(y)}{|y-\xi|},$$

and the other computations that are needed have already been done. □

We now proceed to the second step in the proof of Proposition 8.1.

Lemma 8.4. *Let $\Omega\subset\mathbb{R}^{n+1}$ be an ADR domain and let u be a solution of (8.2) with $1<p<\infty$. Take p' so that $1/p+1/p'=1$, and fix $z\in\Omega$ and $g_z=g_z^\Omega$. If $\|\mathcal{N}^\delta g_z\|_{L^{p'}(\sigma)} \lesssim \delta$ for small enough $\delta>0$, then $u\equiv 0$ in Ω.*

Proof. First note that $\|\mathcal{N}^\delta g_x\|_{L^{p'}(\sigma)} \lesssim \delta$ holds for every $x\in\Omega$ with constant depending on $x\in\Omega$ and provided δ is small enough also depending on $x\in\Omega$, because by the symmetry of the harmonic Green function we have

$$g_x(y) = g_y(x) \approx_{x,z} g_y(z) = g_z(y).$$

Let us fix $x\in\Omega$ and $R\geq 2\mathrm{dist}(x,\partial\Omega)$, and define $\Omega_R := \Omega\cap B_R(x)$. We remark that the auxiliary domain Ω_R and its Green function are only necessary if the domain is unbounded, either with bounded or unbounded boundary. If the domain Ω is bounded, one can directly remove the dependence on R by taking $R=2\mathrm{diam}(\Omega)$ on the computations below.

We will make use of three Green's functions: g_x^Ω, $g_x^{\Omega_R}$ and $g_x^{B_R(x)}$ are the Green functions with pole at x of Ω, Ω_R and $B_R(x)$ respectively. Moreover, it holds that $g_x^{\Omega_R}\leq g_x^\Omega$ and $g_x^{\Omega_R}\leq g_x^{B_R(x)}$ in $\Omega_R\setminus\{x\}$.

Let us fix a dimensional constant $C_1\geq 1$. For $\delta<\mathrm{dist}(x,\partial\Omega_R)/(4C_1)$, as in [18, (7.1.10) and (7.1.11)], let $\psi_\delta^\Omega\in C^\infty(\Omega)$ such that $0\leq\psi_\delta^\Omega\leq 1$, $\psi_\delta^\Omega=1$ in $\Omega\setminus U_{C_1\delta}(\partial\Omega)$, $\psi_\delta^\Omega=0$ in $U_{C_1\delta/2}(\partial\Omega)\cap\Omega$, and $|\partial^\alpha\psi_\delta^\Omega| \lesssim_\alpha (C_1\delta)^{-|\alpha|}$ for all multi-indices α. Similarly for $B_R(x)$

(with δ instead of $C_1\delta$), define $\psi_\delta^{B_R(x)} \in C_c^\infty(B_R(x))$ such that $0 \le \psi_\delta^{B_R(x)} \le 1$, $\psi_\delta^{B_R(x)} = 1$ in $B_R(x) \setminus U_\delta(\partial B_R(x))$, $\psi_\delta^{B_R(x)} = 0$ in $U_{\delta/2}(\partial B_R(x)) \cap B_R(x)$, and $|\partial^\alpha \psi_\delta^{B_R(x)}| \lesssim_\alpha \delta^{-|\alpha|}$ for all multi-indices α. Finally, we define

$$\psi_\delta := \psi_\delta^\Omega \psi_\delta^{B_R(x)} \in C_c^\infty(\Omega_R),$$

which in particular satisfies $0 \le \psi_\delta \le 1$, $|\partial^\alpha \psi_\delta| \lesssim_{\alpha, C_1} \delta^{-|\alpha|}$ for all multi-indices α, and moreover

$$\operatorname{dist}(y, \partial\Omega_R) \ge \delta/2 \text{ for all } y \in \operatorname{supp} \nabla\psi_\delta.$$

As in [18, (7.1.12)], we have

$$u(x) = u(x)\psi_\delta(x) = \int_{\Omega_R} \langle \nabla g_x^{\Omega_R}, \nabla(u\psi_\delta)\rangle \, dm = 2\int_{\Omega_R} \langle \nabla g_x^{\Omega_R}, \nabla\psi_\delta\rangle u \, dm + \int_{\Omega_R} g_x^{\Omega_R} u \Delta\psi_\delta \, dm.$$

Using that $|\nabla g_x^{\Omega_R}(\cdot)| \lesssim g_x^{\Omega_R}(\cdot)/\operatorname{dist}(\cdot, \partial\Omega_R) \lesssim g_x^{\Omega_R}(\cdot)/\delta$ in $\operatorname{supp}\nabla\psi_\delta$, and

$$\operatorname{supp}\nabla\psi_\delta \subset (U_{C_1\delta}(\partial\Omega) \cap B_R(x)) \cup ((B_R(x) \setminus \overline{B_{R-\delta}(x)}) \setminus U_{C_1\delta}(\partial\Omega)),$$

we have

$$\begin{aligned} |u(x)| &\lesssim \frac{1}{\delta^2} \int_{\operatorname{supp}\nabla\psi_\delta} g_x^{\Omega_R} |u| \, dm|_\Omega \\ &\le \frac{1}{\delta^2} \int_{U_{C_1\delta}(\partial\Omega)\cap B_R(x)} g_x^{\Omega_R} |u| \, dm|_\Omega + \frac{1}{\delta^2} \int_{(B_R(x)\setminus\overline{B_{R-\delta}(x)})\setminus U_{C_1\delta}(\partial\Omega)} g_x^{\Omega_R} |u| \, dm|_\Omega \qquad (8.3) \\ &=: \mathrm{I}_\delta + \mathrm{II}_\delta. \end{aligned}$$

Let us study the term I_δ. Using that $g^{\Omega_R} \le g^\Omega$ in $\Omega_R \setminus \{x\}$, Lemma 2.10 and Hölder's inequality respectively, we have

$$\begin{aligned} \mathrm{I}_\delta &\le \frac{1}{\delta^2} \int_{U_{C_1\delta}(\partial\Omega)\cap B_R(x)} g^\Omega |u| \, dm|_\Omega \lesssim \frac{1}{\delta} \int_{\partial\Omega} \mathcal{N}^{C_1\delta}(g^\Omega |u|) \, d\sigma \\ &\le \frac{1}{\delta} \|\mathcal{N}^{C_1\delta} g^\Omega\|_{L^{p'}(\sigma)} \|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)}. \qquad (8.4) \end{aligned}$$

We now turn to the term II_δ, which is directly zero if the domain is bounded and $R \ge 2\operatorname{diam}(\Omega)$. Before, recall that

$$g_x^{B_R(x)}(y) = \frac{1}{\kappa_n(n-1)} \left(\frac{1}{|y-x|^{n-1}} - \frac{1}{R^{n-1}} \right),$$

where κ_n is the surface area of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. So, for $y \in B_R(x) \setminus \overline{B_{R-\delta}(x)}$ we get

$$g_x^{B_R(x)}(y) \lesssim \frac{1}{|R-\delta|^{n-1}} - \frac{1}{R^{n-1}} \lesssim \frac{\delta}{R^n}.$$

Hence, using that $g^{\Omega_R} \leq g^{B_R}$ in $\Omega_R \setminus \{x\}$ and this, we have

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n}\frac{1}{\delta} \int\limits_{(B_R(x)\setminus\overline{B_{R-\delta}(x)})\setminus U_{C_1\delta}(\partial\Omega)} |u|\,dm|_\Omega.$$

For shortness we write $F_\delta := (B_R(x) \setminus \overline{B_{R-\delta}(x)}) \setminus U_{C_1\delta}(\partial\Omega)$. Let W_Ω denote the family of Whitney cubes of Ω, as in Lemma 5.2, and

$$W_\Omega^\delta := \{Q \in W_\Omega : Q \cap F_\delta \neq \emptyset\}.$$

Then we have

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n}\frac{1}{\delta} \int\limits_{F_\delta} |u|\,dm|_\Omega = \frac{1}{R^n}\frac{1}{\delta} \sum_{Q\in W_\Omega^\delta} \int\limits_{Q\cap F_\delta} |u|\,dm.$$

Since $m(Q\cap F_\delta) \lesssim \frac{\delta}{\ell(Q)} m(Q) = \delta\ell(Q)^n$ for each $Q \in W_\Omega^\delta$, we have

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n}\frac{1}{\delta} \sum_{Q\in W_\Omega^\delta} \int\limits_{Q\cap F_\delta} |u|\,dm \lesssim \frac{1}{R^n} \sum_{Q\in W_\Omega^\delta} \ell(Q)^n \sup_Q |u|.$$

Note that having fixed $C_1 \geq 1$ big enough depending on the dimension, then every $Q \in W_\Omega^\delta$ satisfies $\ell(Q) \geq 100\delta$, and therefore $3Q \cap \partial B_R(x) \neq \emptyset$ with $\mathcal{H}^n(3Q\cap\partial B_R(x)) \approx \ell(Q)^n$. So, we get

$$\mathrm{II}_\delta \lesssim \frac{1}{R^n} \sum_{Q\in W_\Omega^\delta} \ell(Q)^n \sup_Q |u| \lesssim \frac{1}{R^n} \sum_{Q\in W_\Omega} \mathcal{H}^n(3Q\cap\partial B_R(x)) \sup_Q |u|, \tag{8.5}$$

uniformly in $\delta > 0$, where we also used in the last step that $W_\Omega^\delta \subset W_\Omega$.

Let us bound the sum on the right-hand side. Defining

$$f(y) := \sum_{Q\in W_\Omega} \mathbf{1}_{3Q\cap\partial B_R(x)}(y) \sup_Q |u|, \quad y \in \Omega,$$

we have

$$\begin{aligned} \sum_{Q\in W_\Omega} \mathcal{H}^n(3Q\cap\partial B_R(x)) \sup_Q |u| &= \sum_{Q\in W_\Omega} \int\limits_{3Q\cap\partial B_R(x)} \sup_Q |u|\,d\mathcal{H}^n \\ &\leq \sum_{Q\in W_\Omega} \int\limits_{3Q\cap\partial B_R(x)} f\,d\mathcal{H}^n \lesssim \int\limits_{\partial B_R(x)} f\,d\mathcal{H}^n, \end{aligned} \tag{8.6}$$

where we used the finite overlap of $\{3Q\}_{Q\in W_\Omega}$ in the last step. Applying Lemma 2.11 to f (with $\xi_x \in \partial\Omega$ such that $\operatorname{dist}(x,\partial\Omega) = |x-\xi_x|$, $C_2 \geq 1$ is a fixed big enough dimensional constant so that $\operatorname{supp} f \subset B(\xi, C_2R)$, and with aperture β_1), the latter term is bounded by

$$\int_{\partial B_R(x)} f\, d\mathcal{H}^n \lesssim \int_{2B(\xi_x, C_2R)\cap\partial\Omega} \mathcal{N}_{\beta_1} f\, d\sigma. \tag{8.7}$$

Given $\xi \in \partial\Omega$, let

$$W_\Omega(\xi) := \{Q \in W_\Omega : 3Q \cap \Gamma_{\beta_1}(\xi) \neq \emptyset\}.$$

Using this definition, for any $y \in \Gamma_{\beta_1}(\xi)$, the sum in the definition of $f(y)$ runs over the cubes in $W_\Omega(\xi)$ (instead of W_Ω), which in particular implies

$$\mathcal{N}_{\beta_1} f(\xi) = \sup_{y\in\Gamma_{\beta_1}(\xi)} f(y) = \sup_{y\in\Gamma_{\beta_1}(\xi)} \sum_{Q\in W_\Omega(\xi)} \mathbf{1}_{3Q\cap\partial B_R(x)}(y) \sup_Q |u|. \tag{8.8}$$

By the relation $\ell(Q) \approx \operatorname{dist}(Q,\partial\Omega)$ for any $Q \in W_\Omega$, it is clear that there exists an aperture $\beta_2 \geq \beta_1$ (depending on β_1 and the dimension) such that

$$\bigcup_{Q\in W_\Omega(\xi)} 3Q \subset \Gamma_{\beta_2}(\xi).$$

From (8.8), this and the finite overlap of $\{3Q\}_{Q\in W_\Omega}$, we get

$$\mathcal{N}_{\beta_1} f(\xi) \lesssim \mathcal{N}_{\beta_2} u.$$

Therefore, from (8.6), (8.7) and this, we conclude

$$\sum_{Q\in W_\Omega} \mathcal{H}^n(3Q\cap\partial B_R(x)) \sup_Q |u| \lesssim \int_{2B(\xi_x, C_2R)\cap\partial\Omega} \mathcal{N}_{\beta_2} u\, d\sigma. \tag{8.9}$$

All in all, we have

$$\mathrm{II}_\delta \overset{(8.5)}{\lesssim} \frac{1}{R^n} \sum_{Q\in W_\Omega} \mathcal{H}^n(3Q\cap\partial B_R(x)) \sup_Q |u| \overset{(8.9)}{\lesssim} \frac{1}{R^n} \int_{2B(\xi_x, C_2R)\cap\partial\Omega} \mathcal{N}_{\beta_2} u\, d\sigma,$$

and by Hölder's inequality (and (2.8)) we get

$$\mathrm{II}_\delta \lesssim R^{-n/p} \|\mathcal{N}u\|_{L^p(\sigma)}, \tag{8.10}$$

and we conclude the control of the term II_δ.

Collecting estimates (8.4) and (8.10) in (8.3), we obtain, for all $R \geq 2\operatorname{dist}(x,\partial\Omega)$ and all $\delta < \operatorname{dist}(x,\partial\Omega_R)/(4C_1)$,

$$|u(x)| \lesssim \frac{1}{\delta}\|\mathcal{N}^{C_1\delta} g^{\Omega}\|_{L^{p'}(\sigma)}\|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)} + R^{-n/p}\|\mathcal{N}u\|_{L^p(\sigma)}.$$

First, for $\delta > 0$ small enough (depending on x), by Lemma 8.3 we have

$$\frac{1}{\delta}\|\mathcal{N}^{C_1\delta} g^{\Omega}\|_{L^{p'}(\sigma)}\|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)} \lesssim \|\mathcal{N}^{C_1\delta} u\|_{L^p(\sigma)}.$$

Since $u|^{\rm nt}_{\partial\Omega} = 0$ σ-a.e. on $\partial\Omega$ implies $\mathcal{N}^{C_1\delta}u(x) \to 0$ as $\delta \to 0$ for σ-a.e. $x \in \partial\Omega$, and $\mathcal{N}u \in L^p(\sigma)$, by the dominated convergence theorem we have $\|\mathcal{N}^{C_1\delta}u\|_{L^p(\sigma)} \to 0$ as $\delta \to 0$. Second, as we are assuming $\|\mathcal{N}u\|_{L^p(\sigma)} < \infty$ we have $R^{-n/p}\|\mathcal{N}u\|_{L^p(\sigma)} \to 0$ as $R \to \infty$. That is, $u(x) = 0$. Since the point $x \in \Omega$ is arbitrary, we conclude that u is identically zero in Ω. □

9. The Dirichlet and Neumann problems

In this section we solve the Dirichlet and Neumann problems, stated in Theorems 1.2 and 1.3 respectively.

Proof of Theorem 1.2. Let $\varepsilon_D > 0$ be so that $n/(n-1) - \varepsilon_D$ is the Hölder conjugate exponent of $n + \varepsilon_2$, with $\varepsilon_2 > 0$ as in Corollary 7.2. Fix $n/(n-1) - \varepsilon_D < p_0 \leq 2n/(n+1)$, and let $\delta_0 = \delta_0(p_0, n, \mathrm{CAD}) > 0$ be given by Corollary 3.6.

By Corollaries 3.6 and 7.2 we have that $\left(\frac{1}{2}Id + K\right)^{-1} : L^{p_0}(\sigma) \to L^{p_0}(\sigma)$ is a bounded linear operator, as claimed in the theorem. Hence, $\left(\frac{1}{2}Id + K\right)^{-1} f \in L^{p_0}(\sigma)$ with $\|\left(\frac{1}{2}Id + K\right)^{-1} f\|_{L^{p_0}(\sigma)} \lesssim \|f\|_{L^{p_0}(\sigma)}$, where the involved constant does not depend on $f \in L^{p_0}(\sigma)$. It is clear then that the function $u = \mathcal{D}\left(\left(\frac{1}{2}Id + K\right)^{-1} f\right)$ in (1.5) is harmonic in Ω. Moreover, by the jump formula (3.4), we have

$$u|^{\rm nt}_{\partial\Omega}(x) = f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega,$$

whence we conclude that it solves the Dirichlet problem in (1.1) with p_0 instead of p. Finally, by (3.5b) and as $\left(\frac{1}{2}Id + K\right)^{-1}$ is bounded in $L^{p_0}(\sigma)$, we conclude

$$\|\mathcal{N}u\|_{L^{p_0}(\sigma)} \lesssim \left\|\left(\frac{1}{2}Id + K\right)^{-1} f\right\|_{L^{p_0}(\sigma)} \lesssim \|f\|_{L^{p_0}(\sigma)},$$

as claimed in (1.2). That is, (D_{p_0}) is solvable with solution u.

By Corollary 8.2, there is $\varepsilon > 0$ such that (D_p) is solvable for all $p \in (p_0 - \varepsilon, \infty)$, and the solution of (D_p) is the unique solution of (1.1). □

Proof of Theorem 1.3. Let $\varepsilon_N = \varepsilon_2$ with $\varepsilon_2 > 0$ as in Proposition 7.1. We fix $2n/(n+1) \leq p < n + \varepsilon_N$, and let $\delta_0 = \delta_0(p, n, \mathrm{CAD})$ be given by Corollary 3.6.

By Corollary 3.6 and Proposition 7.1 we have that $\left(-\frac{1}{2}Id + K^*\right)^{-1} : L^p(\sigma) \to L^p(\sigma)$ is a bounded linear operator. Hence, $\left(-\frac{1}{2}Id + K^*\right)^{-1} f \in L^p(\sigma)$ with $\|\left(-\frac{1}{2}Id + K^*\right)^{-1} f\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)}$, where the involved constant does not depend on $f \in L^p(\sigma)$. It is clear then that the

function $u = \mathscr{S}_{\text{mod}}\left(\left(-\frac{1}{2}Id + K^*\right)^{-1} f\right)$ in (1.6) is harmonic in Ω. Moreover, by the jump formula (7.3) we have

$$\partial_{\nu_\Omega} u(x) = f(x), \text{ for } \sigma\text{-a.e. } x \in \partial\Omega,$$

whence we conclude that it solves the Neumann problem in (1.3). Finally, by (7.7) and as $\left(-\frac{1}{2}Id + K^*\right)^{-1}$ is bounded in $L^p(\sigma)$, we conclude

$$\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \lesssim \left\| \left(-\frac{1}{2}Id + K^*\right)^{-1} f \right\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)},$$

as claimed in (1.4). That is, (N_p) is solvable with solution u, and by Corollary 6.3, it is the unique (modulo constant) solution of (1.3). □

Remark 9.1. It is straightforward to verify that any solution u of the Dirichlet problem (1.1) (respectively, Neumann problem (1.3)) satisfies $\|\mathcal{N}u\|_{L^p(\sigma)} \geq \|f\|_{L^p(\sigma)}$ (respectively, $\|\mathcal{N}(\nabla u)\|_{L^p(\sigma)} \geq \|f\|_{L^p(\sigma)}$).

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Data availability

No data was used for the research described in the article.

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