



Phase Portraits of the Families V to X for the Quadratic Polynomial Differential Systems

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Abstract

If we want to classify a large number of objects, we normally have to divide the set into a few subsets and then classify these subsets one by one. In the study of phase portraits of quadratic systems, different divisions have been tried and some subsets of each division have been classified. So far, no division has been efficient enough to allow a complete classification. In this article we present the most advanced study based on one of these divisions. Specifically, we study a division proposed in [7] which divides quadratic systems into ten different normal forms with six parameters. We will provide the complete classification of phase portraits for six of these normal forms. Altogether with [1], in which a seventh normal form has been studied, we bring together all the current knowledge on this division. Unfortunately the three remaining normal forms are not easy to reduce to simpler forms that allow their complete study.

Keywords Quadratic vector field · Quadratic system · Phase portrait

Mathematics Subject Classification Primary 34C05 · 34A34 · 34C14

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1 Introduction and Statement of the Main Results

A *quadratic polynomial differential system* or simply a *quadratic system* is a differential system of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where P and Q are real polynomials in the variables x and y , such that the biggest degree of the polynomials P and Q is two.

Linear differential systems are easy to study, even beyond planar systems, and covering all dimensions. This problem was already solved by Laplace at the beginning of XIXth century. When one thinks on the increased complexity of quadratic problems with respect to similar linear problems, one tends to think that there may be some difficulties, but not too many, and not too hard. Solving a quadratic polynomial equation is just a bit more complicated than solving a linear equation. Classifying planar conics is a bit harder than studying a simple straight line, and this is known since Greek times. Classifying 3-dimensional (or higher dimension) quadrics takes a bit more of effort, but was already possible two centuries ago. However, the step done from linear differential systems to quadratic differential systems, multiplies by more than 100 the number of phase portraits, and the worst of all, is that we do not know yet which is the exact number of topologically different phase portraits of quadratic systems.

The main difficulty lies in the fact that quadratic systems (and logically systems of higher degree) can have invariant closed orbits called limit cycles and it is not yet known what is the maximum number of limit cycles that a quadratic system can have. So far, examples have been found with up to 4 limit cycles (and this is the commonly conjectured limit), but no proof has yet been given. But even if we separate the problem from the limit cycles, and consider only the set of phase portraits modulo the limit cycles, we still do not know how many topologically different phase portraits can exist in quadratic systems. There is a solid project by the authors and collaborators that comes close to this goal, with more than 1000 phase portraits already classified, but there are still some difficult final cases to study. Anyway they seem feasible according to the algorithm used and we hope to obtain them all in some more years.

As explained in the abstract, a complete classification of phase portraits for quadratic systems must divide the problem into several easier problems, but most ways of doing this lead to several problems that are easy to study, and others that are very difficult or even almost impossible to achieve. In any case, all the efforts are relevant since the phase portraits found by any of these methods are useful in providing the examples needed to study other normal forms.

One of these lines of study was initiated in [7] where quadratic systems were classified, after affine transformations and rescaling of the time in the form:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) = d + ax + by + \ell x^2 + mxy + ny^2,$$

where the polynomial $P(x, y)$ is one of the following ten:

$$\begin{array}{ll} \text{I } \dot{x} = 1 + xy, & \text{VI } \dot{x} = 1 + x^2, \\ \text{II } \dot{x} = xy, & \text{VII } \dot{x} = x^2, \\ \text{III } \dot{x} = y + x^2, & \text{VIII } \dot{x} = x, \\ \text{IV } \dot{x} = y, & \text{IX } \dot{x} = 1, \\ \text{V } \dot{x} = -1 + x^2, & \text{X } \dot{x} = 0. \end{array}$$

This division of the quadratic systems into more simple normal forms, using only six parameters, was useful to study some “easy” cases as chordal quadratic systems [7], or quadratic systems with a single finite singularity [6]. However, it also offered an opportunity to try to obtain all the phase portraits of quadratic systems by means of a complete study of each one of these 10 families. The first serious attempt to do this was [5], in which families VII and VIII were studied, but with the old techniques used in it, it was very difficult to deal with systems having six parameters, and some errors were done and some phase portraits were missed. The second attempt was [1] in which family IV was studied using new techniques which allow to deal with bigger parameter spaces. The authors then decided to correct the errors done for families VII and VIII in [5] and produced [2] using the same technique as in [1].

So the next goal was to complete other families as V, VI, IX and X, already aware that families I–III were the most difficult to deal with. But when we started to work with families V and VI we realized that the bifurcation diagrams that we were obtaining were very similar to others we had already seen before in another paper. In [3] the authors give the complete bifurcation diagram of a 9-parameter family of quadratic systems called Riccati systems which are

$$\begin{aligned} \frac{dx}{dt} &= a + cx + gx^2, \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2. \end{aligned} \quad (2)$$

It is trivial to see that the Riccati systems cover completely families V–X and that the phase portraits of each of the families V–X must be a subset of the 119 phase portraits of Riccati systems, with possibly the same phase portraits appearing in several families and covering in total all the 119 phase portraits, and all the possible branches of the diagram given in [3]. The study of the Riccati systems is one of the biggest classification works done up to now in quadratic systems since it has been able to classify a 9 parameter family. Most works in quadratic systems deal with one to four parameters, and some few as [1] or [5] which deal with six parameters in fact reduce the complexity by producing some normal forms with lower number of parameters (maximum of three). The fact is that a visual bifurcation diagram can only be obtained using two parameters or at most three, using a sliced parameter space, but [3] goes beyond the limitation of visual bifurcation diagrams, and even beyond the complexity of large formulas using 9 parameters, and reduces the problem to compute some global invariants whose sign will determine the phase portrait, even if the system is not written in the Riccati form.

The strength of the Riccati classification is based on a previous work [4] where the authors obtained configurations of invariant straight lines (real or complex) for the family $\mathbf{QSL}^{2p}$ which is the class of *non-degenerate quadratic systems* (systems with no common factor between the equations defining it) that via an affine transformation could be brought to the canonical form (2). The notation $\mathbf{QSL}^{2p}$ brings into focus the principal property of Riccati systems, i.e. that generically they have two parallel invariant straight lines. Thus, based on the strong geometrical property of having two parallel invariant straight lines (even though they may be complex, or double or they may have coalesced with the line at infinity), the authors were able to complete each one of the branches of the bifurcation diagram obtained for invariant straight lines, upon obtaining phase portraits. Some branches lead to a single phase portrait, and others needed to be split with the help of other invariants into several possible phase portraits. Some phase portraits could only be obtained from one of the branches of the diagram, and other phase portraits could come from different branches.

So, once the Riccati study was done, it seemed that it had little sense to study families V–X. However, we decided to do it for several reasons: first, because we wanted to know specifically which phase portraits could be obtained individually from each one of the families V–X; second, because we wanted to test that our classification for families VII–VIII in [2] was right and consistent with this new approach; third, because in this way we could produce an algebraic bifurcation diagram in the 6-parameter space for each one of the families, and finally because we wanted to show the utility of the kind of classifications given in [3] to solve related problems. Moreover, all the effort has become fruitful because we have been able to discover that one of the branches of [3] is empty and should not be in the diagram. This does not affect to the number and the phase portraits because the phase portrait indicated by this branch, is also obtained in another branch. Concretely, the branch given in [3, Diagram 7] for $\tilde{N} \neq 0$, $\mu_0 \neq 0$, $B_2 \neq 0$, $H_{10} > 0$, $\mathbf{D} = 0$, $H_{15} > 0$ and $\mathcal{G}_2 \leq 0$ providing that the phase portrait Ric. 47 is not realizable. Anyway, we may obtain phase portrait Ric. 47 with the conditions of the next branch (with $\mathcal{G}_2 > 0$ and $H_{16} < 0$).

Since the work follows greatly the results and techniques used in [3], the results and diagrams are based on the same invariants used there. For the sake of shortness and avoiding duplicities, we refer the reader to [3, Section 2], for a complete description of the invariants.

Our main results are then (one for each family):

Theorem 1 *A quadratic system of family V has one of the 59 different phase portraits from Figs. 9 and 10 according to the conditions given in the diagrams of Figs. 1, 2 and 3. Concretely, the phase portraits are: Ric. 2, Ric. 3, Ric. 4, Ric. 5, Ric. 6, Ric. 7, Ric. 8, Ric. 9, Ric. 10, Ric. 15, Ric. 16, Ric. 17, Ric. 18, Ric. 19, Ric. 20, Ric. 21, Ric. 22, Ric. 28, Ric. 29, Ric. 30, Ric. 31, Ric. 32, Ric. 33, Ric. 37, Ric. 38, Ric. 40, Ric. 42, Ric. 43, Ric. 44, Ric. 45, Ric. 46, Ric. 47, Ric. 48, Ric. 49, Ric. 54, Ric. 55, Ric. 56, Ric. 57, Ric. 58, Ric. 64, Ric. 65, Ric. 67, Ric. 68, Ric. 69, Ric. 73, Ric. 77, Ric. 78, Ric. 80, Ric. 81, Ric. 89, Ric. 90, Ric. 92, Ric. D₅, Ric. D₇, Ric. D₉, Ric. D₁₅, Ric. D₁₇, Ric. D₁₉ and Ric. D₂₅.*

Theorem 2 *A quadratic system of family VI has one of the 10 different phase portraits from Figs. 9 and 10 according to the conditions given in the diagram of Fig. 4. Con-*

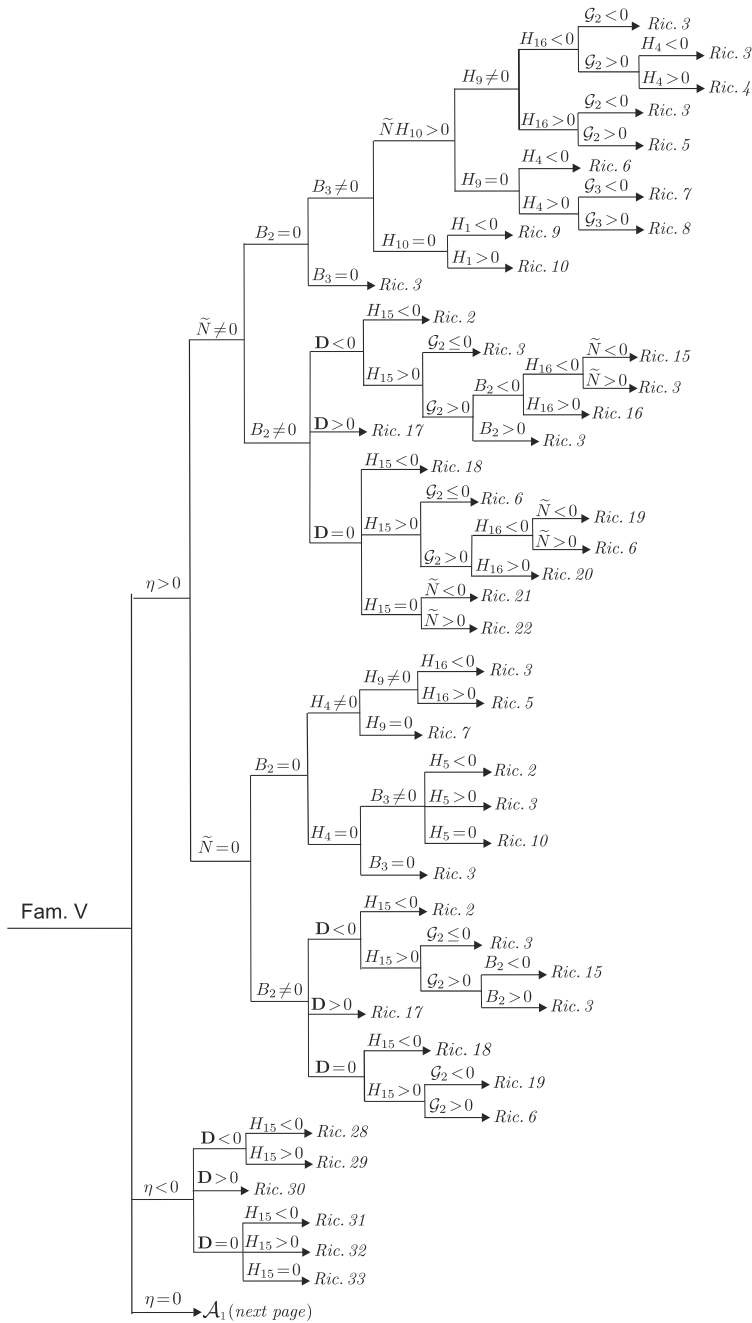


Fig. 1 Diagram Va

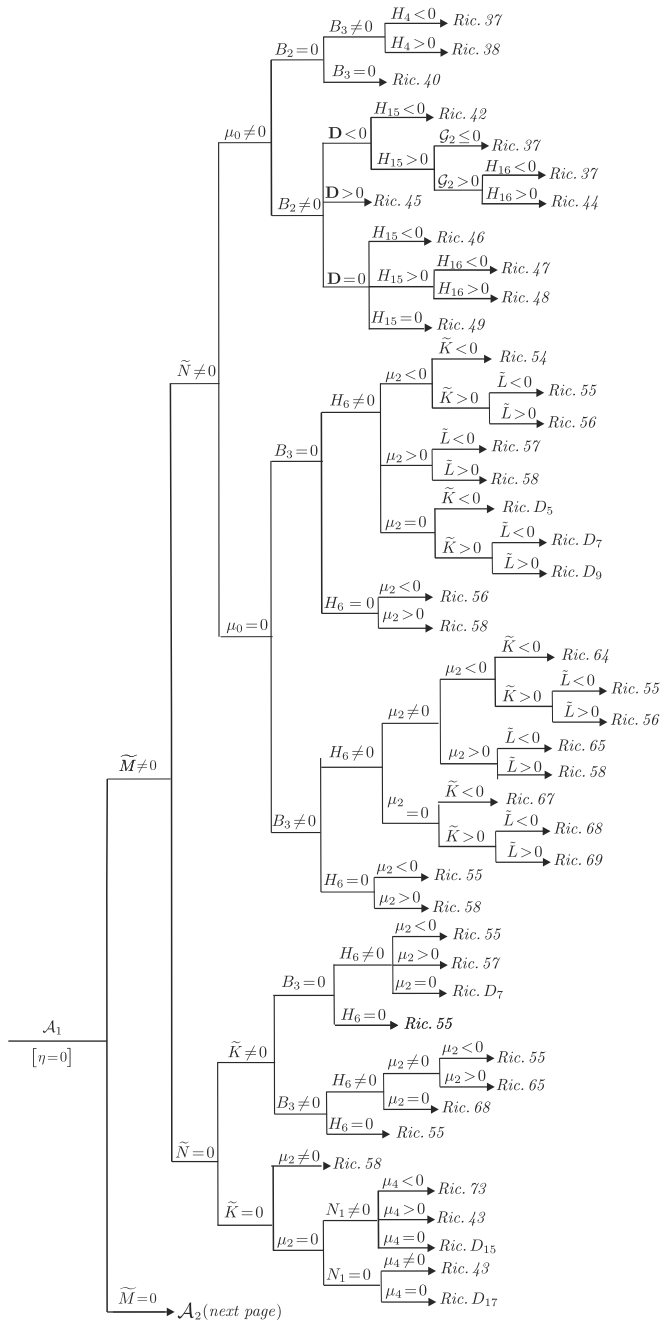
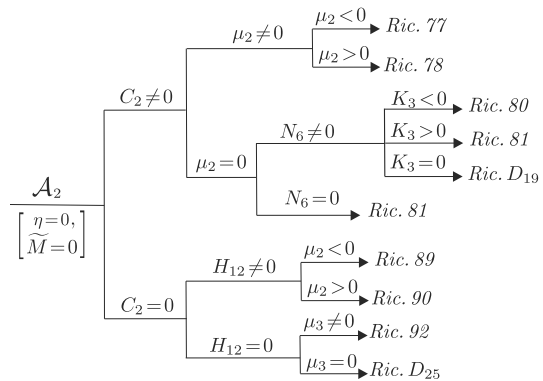


Fig. 2 Diagram Vb

Fig. 3 Diagram Vc

cretely, the phase portraits are: Ric. 1, Ric. 2, Ric. 28, Ric. 36, Ric. 42, Ric. 43, Ric. 52, Ric. 53, Ric. 63 and Ric. 88

Theorem 3 A quadratic system of family VII has one of the 33 different phase portraits from Figs. 9 and 10 according to the conditions given in the diagram of Fig. 5. Concretely, the phase portraits are: Ric. 2, Ric. 9, Ric. 10, Ric. 14, Ric. 18, Ric. 23, Ric. 28, Ric. 31, Ric. 34, Ric. 35, Ric. 36, Ric. 39, Ric. 41, Ric. 42, Ric. 46, Ric. 50, Ric. 51, Ric. 59, Ric. 60, Ric. 63, Ric. 66, Ric. 72, Ric. 79, Ric. 82, Ric. 91, Ric. 93, Ric. D6, Ric. D8, Ric. D10, Ric. D16, Ric. D18, Ric. D20 and Ric. D26.

Theorem 4 A quadratic system of family VIII has one of the 26 different phase portraits from Figs. 9 and 10 according to the conditions given in the diagram of Fig. 6. Concretely, the phase portraits are: Ric. 11, Ric. 12, Ric. 13, Ric. 24, Ric. 25, Ric. 26, Ric. 27, Ric. 28, Ric. 30, Ric. 31, Ric. 53, Ric. 58, Ric. 60, Ric. 61, Ric. 62, Ric. 70, Ric. 71, Ric. 74, Ric. 75, Ric. 76, Ric. 83, Ric. 84, Ric. 85, Ric. 86, Ric. D12 and Ric. D22.

Theorem 5 A quadratic system of family IX has one of the 10 different phase portraits from Figs. 9 and 10 according to the conditions given in the diagram of Fig. 7. Concretely, the phase portraits are: Ric. 1, Ric. 2, Ric. 28, Ric. 36, Ric. 43, Ric. 52, Ric. 53, Ric. 63, Ric. 74 and Ric. 87.

Theorem 6 A quadratic system of family X has one of the 15 different phase portraits from Figs. 9 and 10 according to the conditions given in the diagram of Fig. 8. Concretely, the phase portraits are: Ric. 28, Ric. 35, Ric. 53, Ric. D1, Ric. D2, Ric. D3, Ric. D4, Ric. D11, Ric. D13, Ric. D14, Ric. D17, Ric. D18, Ric. D21, Ric. D23 and Ric. D24.

Corollary 1 Since each branch of Figs. 11, 12, 13, 14 [3, Diagrams 5 to 8] should have to be realizable by at least one of the families V to X, the branch (relative position 110) with $\eta = 0$, $\tilde{M} \neq 0$, $\tilde{N} \neq 0$, $\mu_0 \neq 0$, $B_2 \neq 0$, $H_{10} > 0$, $D = 0$, $H_{15} > 0$ and $G_2 \leq 0$ is not realizable by a Riccati system.

Corollary 2 The phase portraits which belong to more than one family are the following: Ric. 1 in families VI, IX; Ric. 2 in families VI, VII, IX; Ric. 9 in families V, VII;

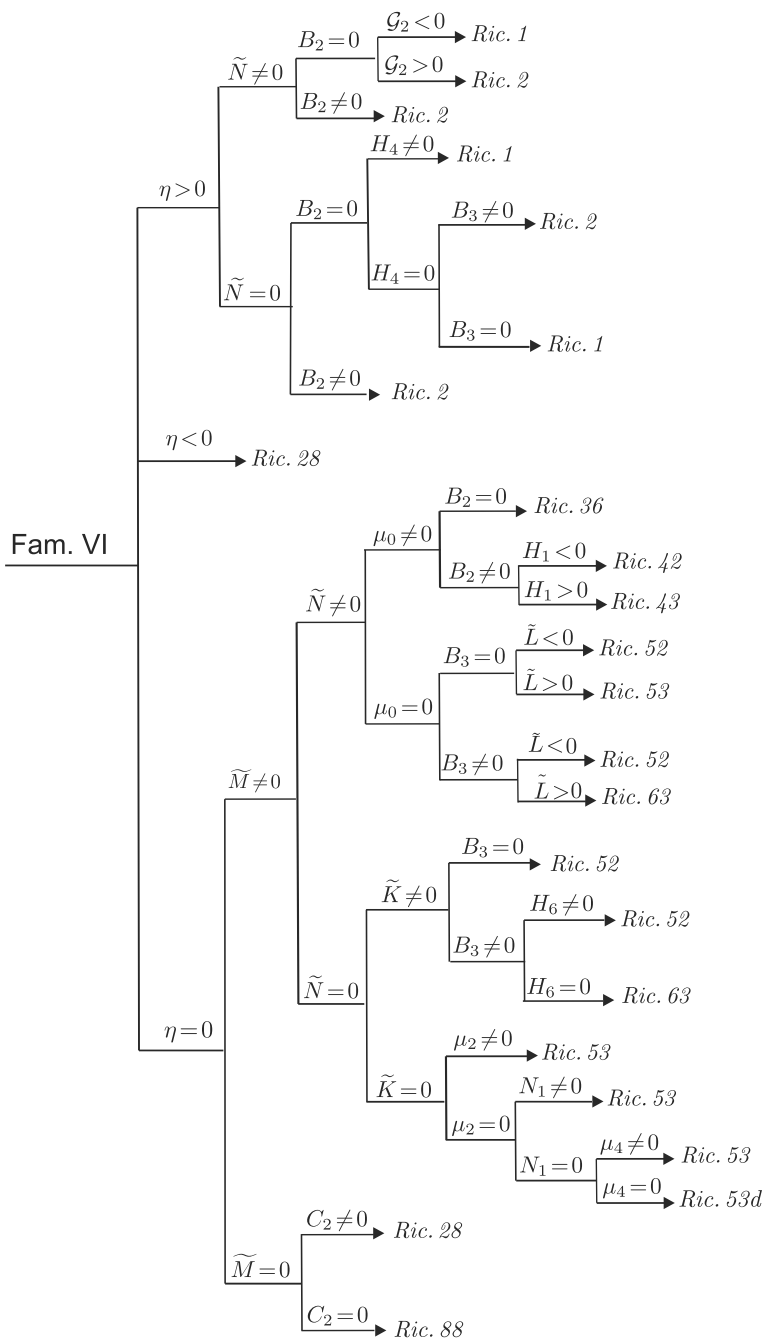


Fig. 4 Diagram VI

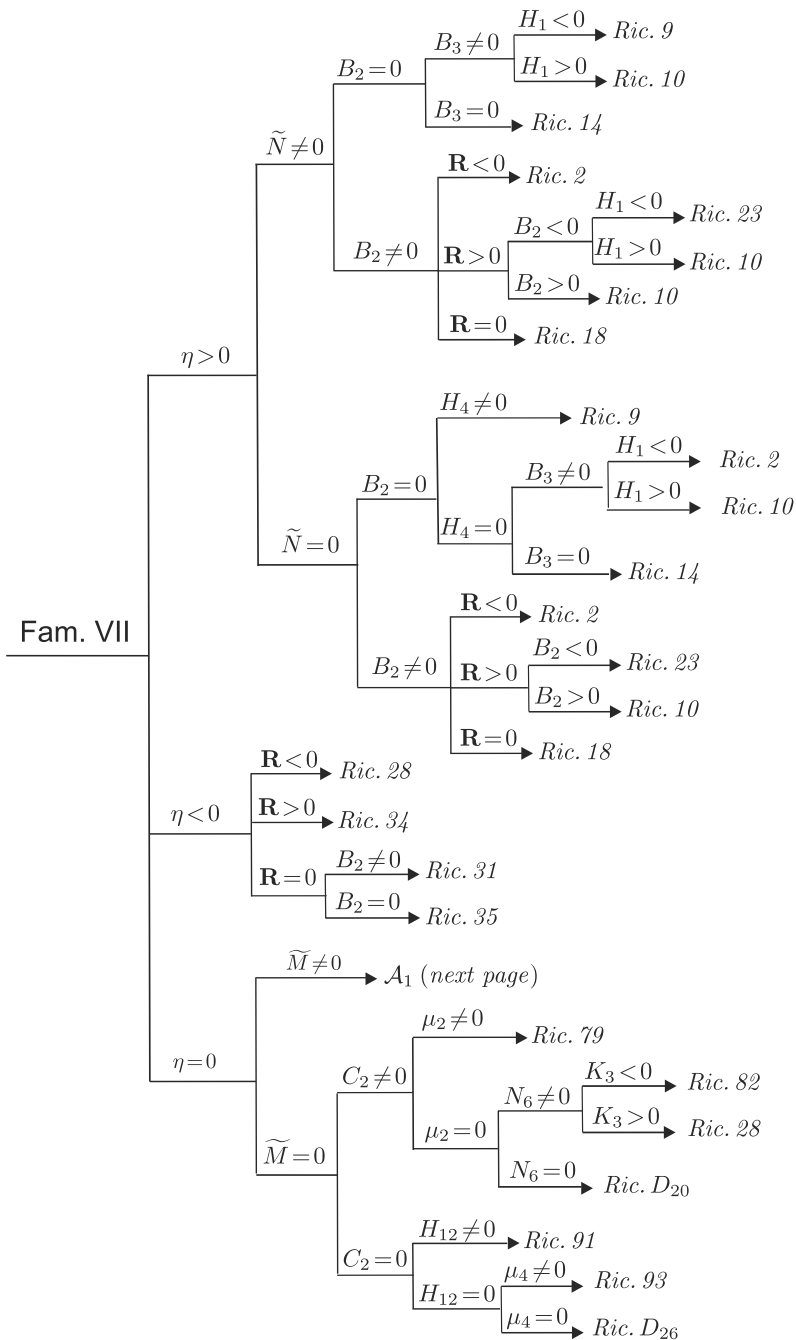


Fig. 5 Diagram VII

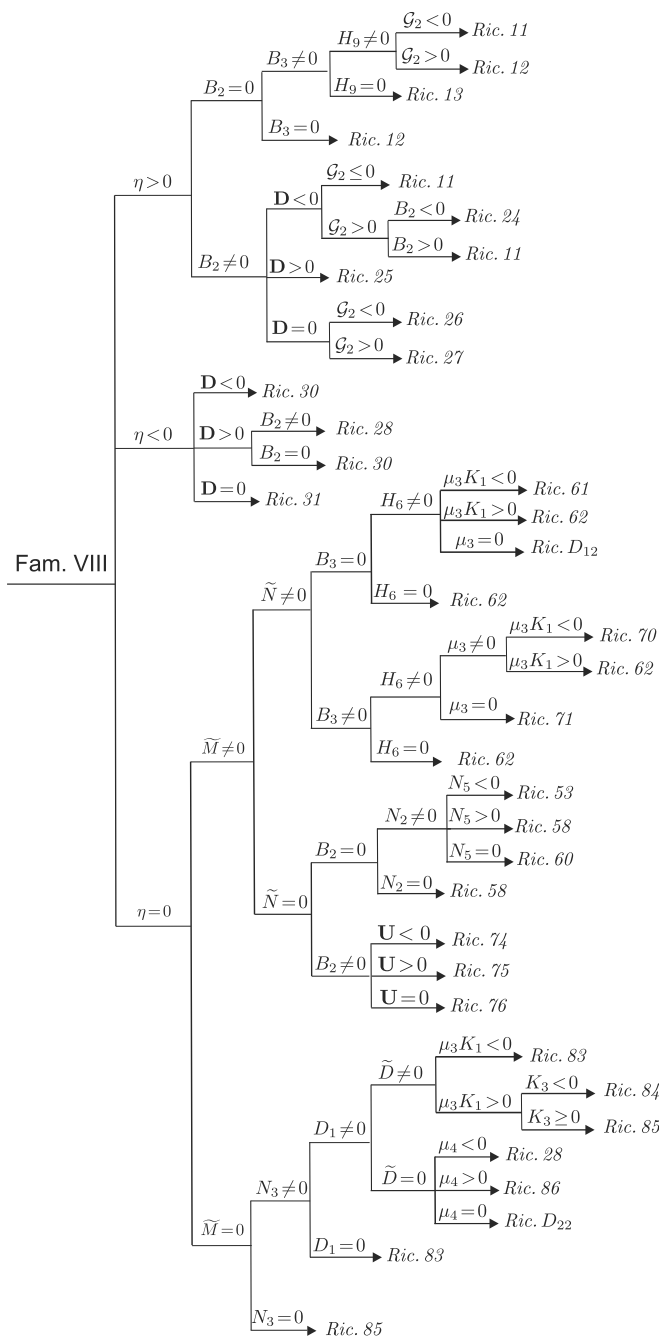


Fig. 6 Diagram VIII

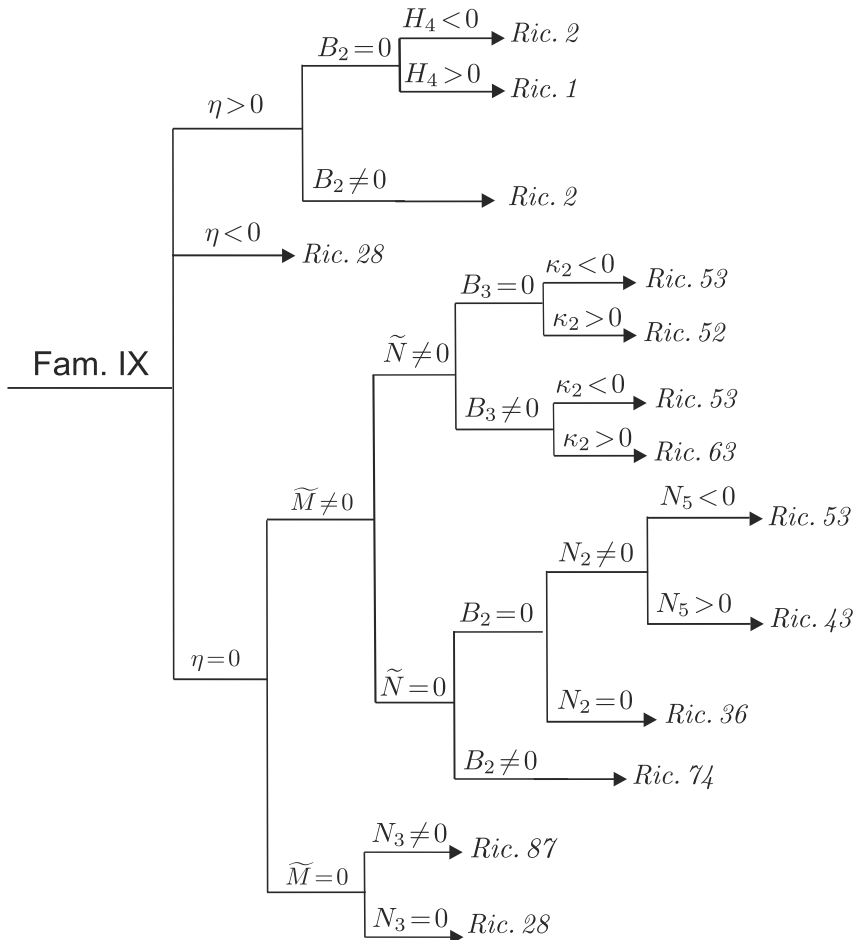


Fig. 7 Diagram IX

Ric. 10 in families V, VII; Ric. 18 in families V, VII; Ric. 28 in families V, VI, VII, VIII, IX; Ric. 30 in families V, VIII; Ric. 31 in families V, VII, VIII; Ric. 36 in families VI, VII, IX; Ric. 42 in families V, VI, VII; Ric. 43 in families V, VI, IX; Ric. 46 in families V, VII; Ric. 52 in families VI, IX; Ric. 53 in families VI, VIII, IX, X; Ric. 58 in families V, VIII; Ric. 60 in families V, VII, VIII; Ric. 63 in families VI, VII, IX; Ric. 74 in families VIII, IX; Ric. D_{17} in families V, X and Ric. D_{18} in families VII, X.

This paper is organized as follows: In Sect. 2 we bring the complete diagrams for the phase portraits of the Riccati systems from [3] as well as the 119 phase portraits. In Sect. 3 we will construct several subfamilies of families V-X which will help to find examples (or prove the impossibility) of several branches when the computation is unable to deal with the main family. In Sect. 4 we will give the technique used to prove each one of the branches of our diagram. Anyway, the complete proof has been produced using Mathematica and relying on its function “FindInstance” which

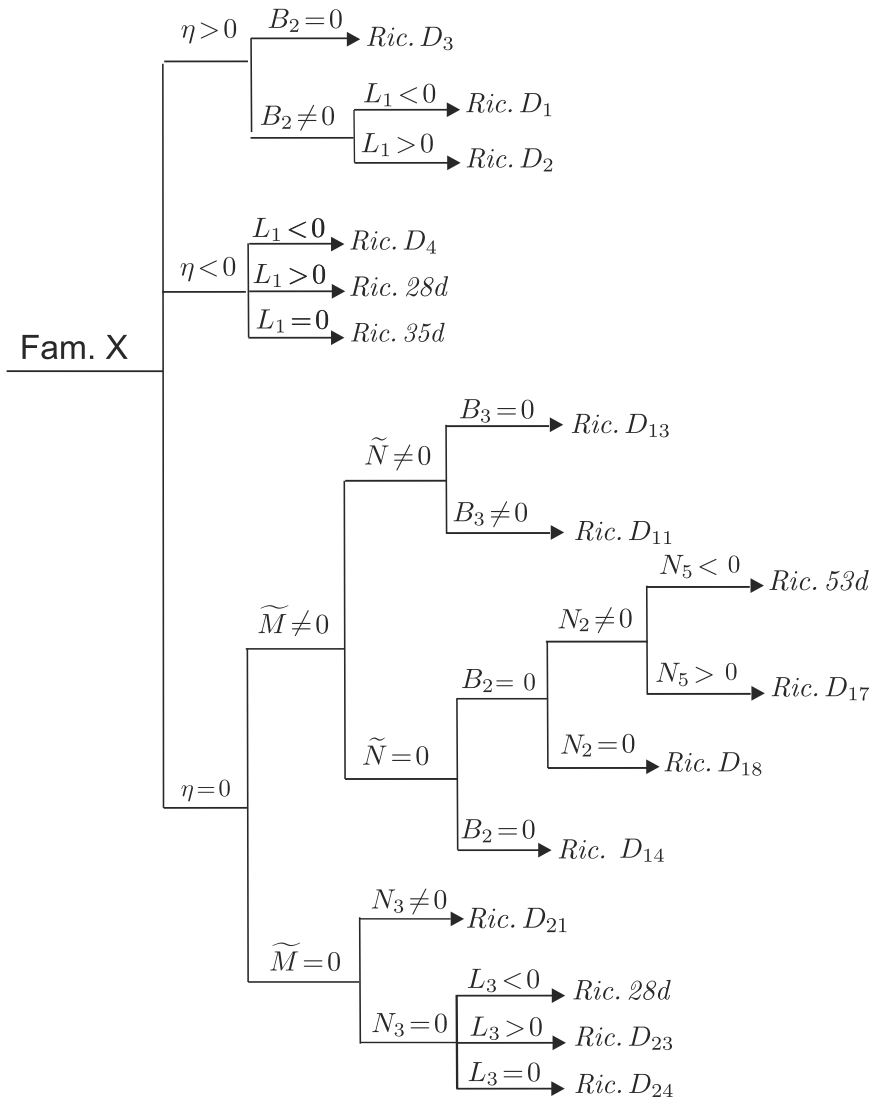


Fig. 8 Diagram X

provides an example of realizable branches and declares empty other branches. We offer the Mathematica file in the link <http://mat.uab.es/~artes/articles/fam-V-X.html> for interested reader and referees to test.

Even though we have relied on the “FindInstance” function of Mathematica to obtain the examples and the first clue that some branches are not achievable by some systems, for each one of these branches, we have added an algebraic proof of its impossibility. We have described in the paper these proofs for some few cases, the rest can be found in the mentioned file.

2 Riccati Classification

In [3] is proved that any system (2) has one of the 119 phase portraits given in Figs. 9 and 10 according to the Diagrams 11 to 14 and using the invariants defined in [3, Section 2].

3 Construction of Subfamilies

In order to detect which branches of the diagrams in Figs. 11, 12, 13 and 14 were achievable by each one of the families V-X, we have relied of the Mathematica function “FindInstance” which is able to provide examples of sets of parameters which hold several algebraic conditions, and also gives empty result when there is some incompatibility. However, in some cases, “FindInstance” is unable to give a result and remains thinking for a longer time we were able to wait, and then we decided to reduce the 6-parameter family into several subfamilies with less parameters which covered all the family. That is, the same technique we used in [1] and [2] to study the bifurcations. Then the “FindInstance” function is able to deal with those simpler normal forms and either provides an example from one of them (and then we do not need to continue with the rest of possibilities), or answers negatively for each of the subfamilies, proving then the impossibility. In order to avoid any distrust on the function “FindInstance” of Mathematica, for each case in which the function has declared empty the set, we have added the proof in the Mathematica file, and from them, we have added in the paper just the few first ones.

We have needed to help Mathematica in three families. Concretely, in the families V, VI and VIII. Thus, we provide here the proof of the construction of the subfamilies needed. In fact, in no case we have needed to go through all the different subfamilies in order to find an example or prove an impossibility. Anyway in the Mathematica file <http://mat.uab.es/~artes/articles/fam-V-X.177html>, we provide the proof to obtain all the subfamilies for each main family.

3.1 Reducing the Number of Parameters of Systems V and VI

These are the families

$$\dot{x} = -1 + x^2, \quad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2, \quad (3)$$

$$\dot{x} = 1 + x^2, \quad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2. \quad (4)$$

Since we start with a family of 6 parameters and this is too many, we will reduce systems (3) and (4) to a set of several families of systems with fewer parameters. Consider first system (3) with $n \neq 0$ and make the change of variables $x = \bar{x}$, $y = \alpha \bar{x} + \beta \bar{y} + \gamma$. So, we have

$$\dot{x} = -1 + x^2, \quad \dot{y} = \bar{d} + \bar{a}x + \bar{b}y + \bar{\ell}x^2 + \bar{m}xy + \bar{n}y^2, \quad (5)$$

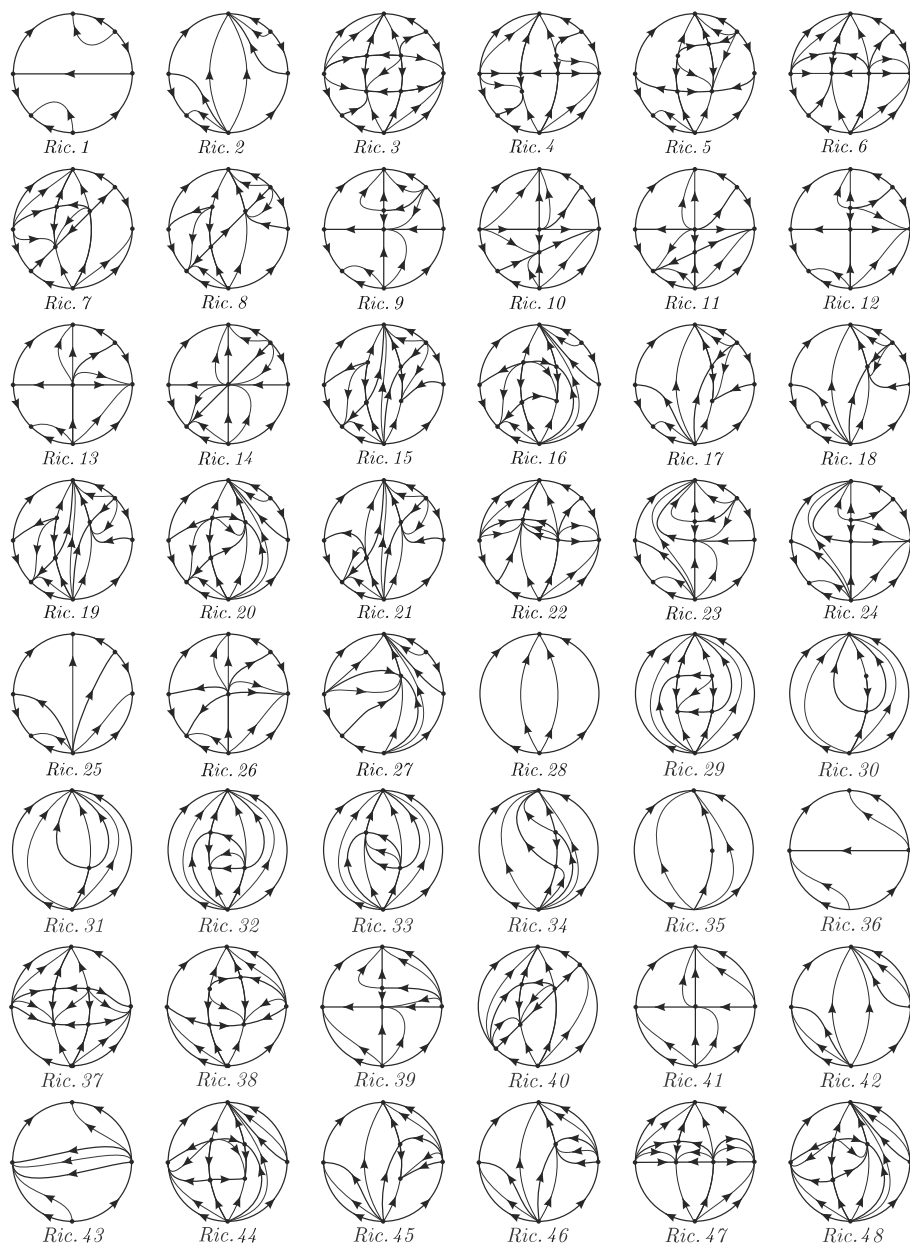


Fig. 9 The phase portraits of Riccati quadratic systems

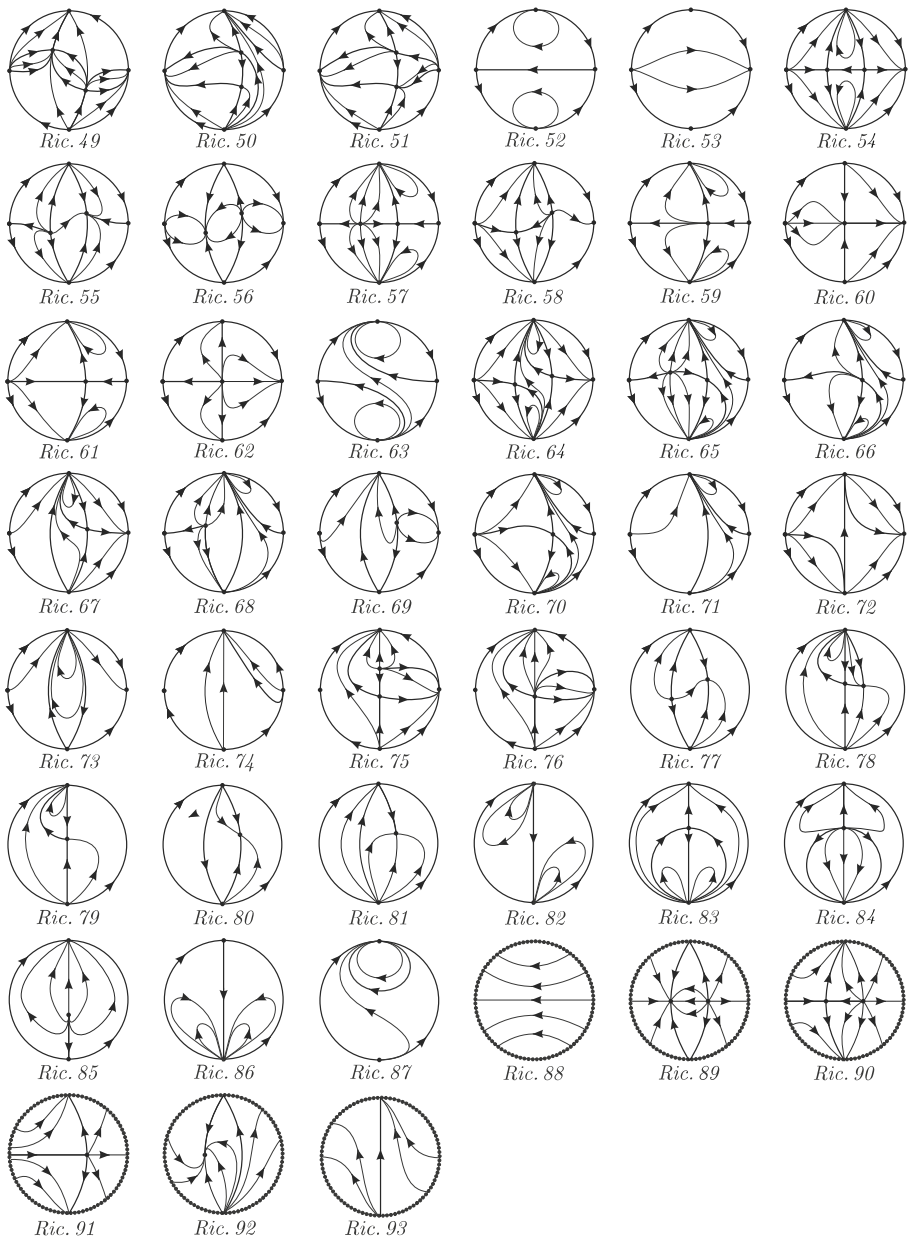


Fig. 9 continued

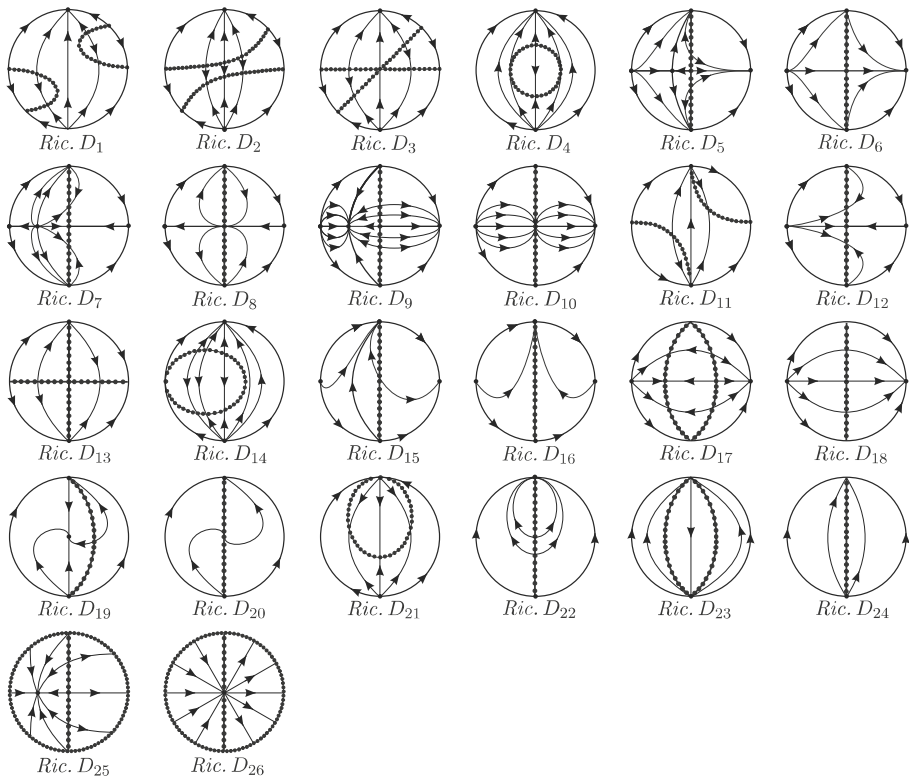


Fig. 10 The phase portraits of Riccati degenerate quadratic systems

with $\bar{b} = b + 2n\gamma$, $\bar{m} = m + 2\alpha n$, $\bar{n} = \beta n$. Now, as α , β , γ are free, we can make $\alpha = -m/(2n)$, $\beta = 1/n$, $\gamma = -b/(2n)$, obtaining, dropping the bar, the system

$$\dot{x} = -1 + x^2, \quad \dot{y} = d + ax + \ell x^2 + y^2, \quad (6)$$

with only the three parameters d , a and ℓ . This constitutes the subfamily $V(A)$. After this, we may also consider the case $n = 0$ and continue the dismantling of system (3) into more subfamilies which cover it completely. However, since we will only need this first subfamily, we will skip the rest of the proof which may be found in the already mentioned link <http://mat.uab.es/~artes/articles/fam-V-X.html> the same applies for the rest of families.

For the system (4) the reduced systems for the second system are the same as for the system (3), so we obtain the subfamily VI(A):

$$\dot{x} = 1 + x^2, \quad \dot{y} = d + ax + \ell x^2 + y^2. \quad (7)$$

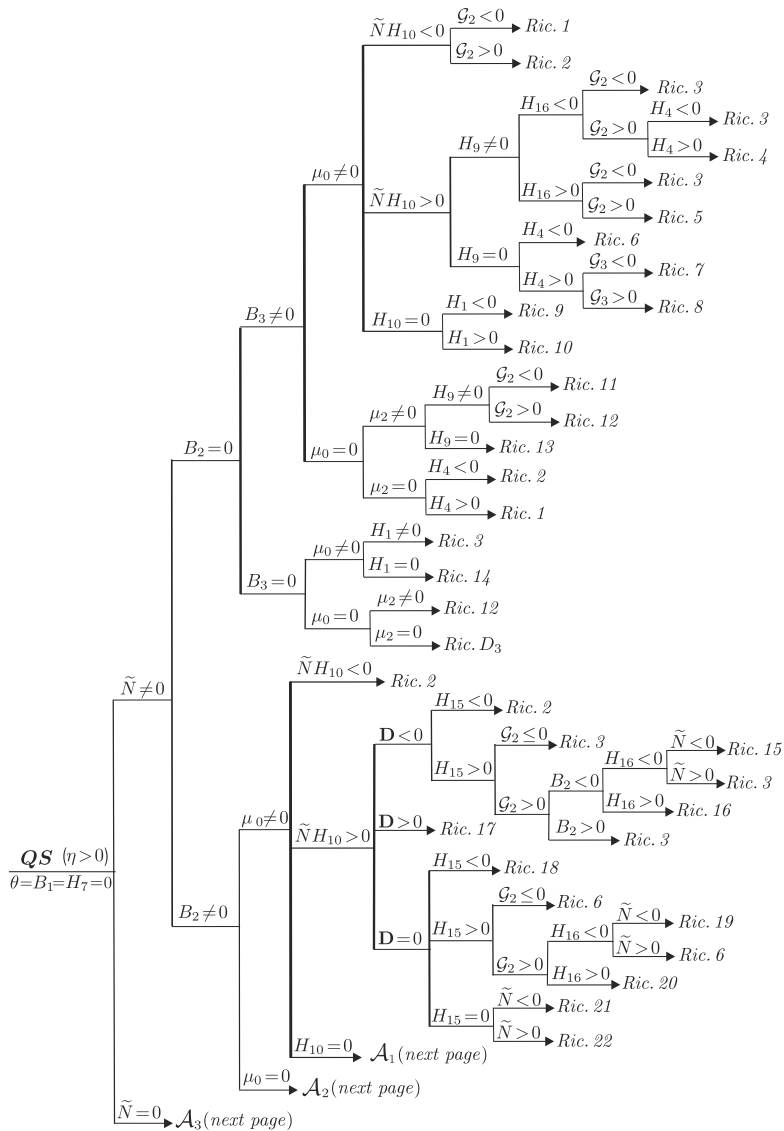


Fig. 11 The invariant criteria for phase portraits of systems in QS_{Ric} (case $\eta > 0$)

3.2 Reducing the Number of Parameters of Systems VIII

This is the family

$$\dot{x} = x, \quad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2, \quad (8)$$

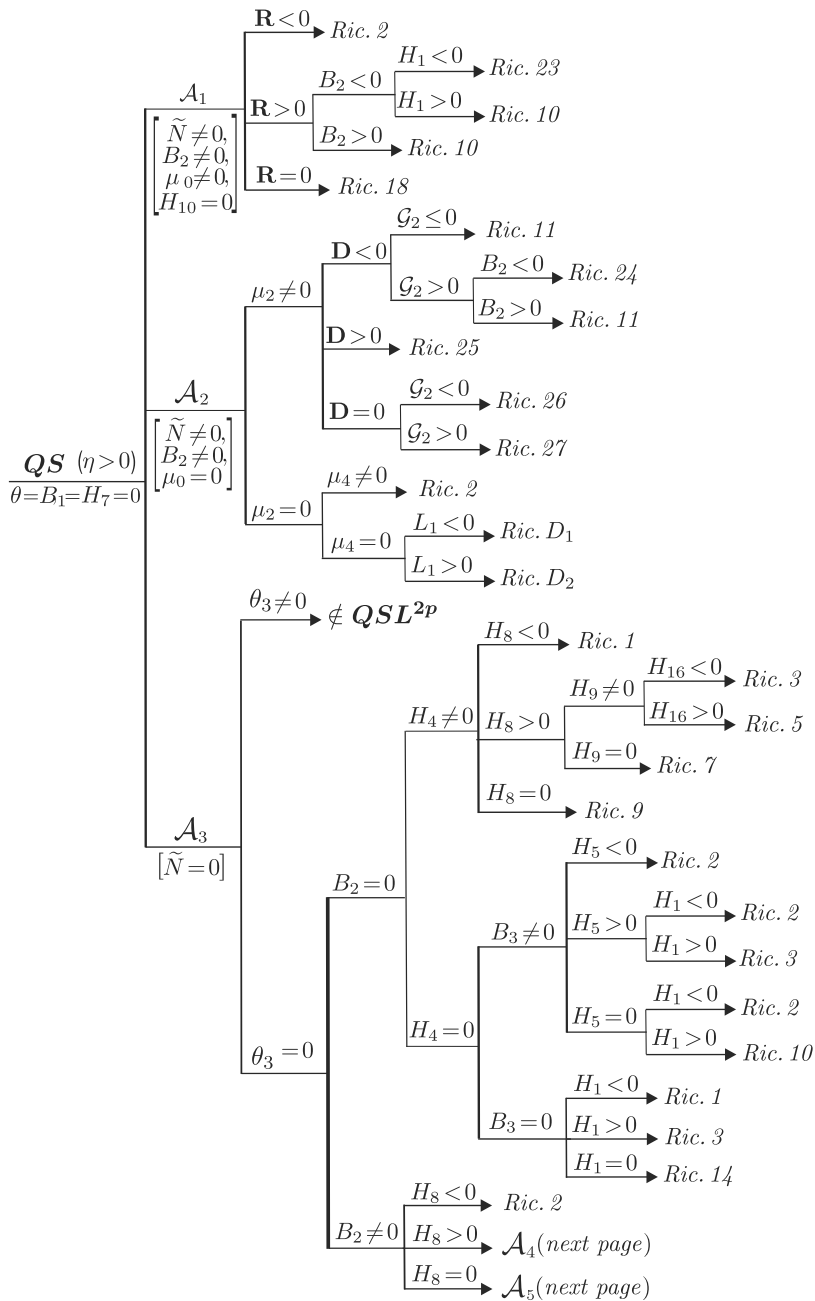


Fig. 11 continued

Fig. 11 continued

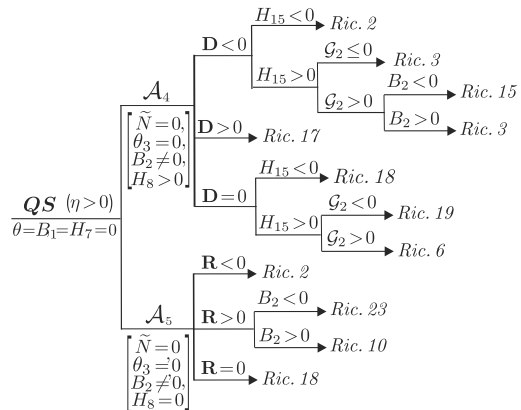
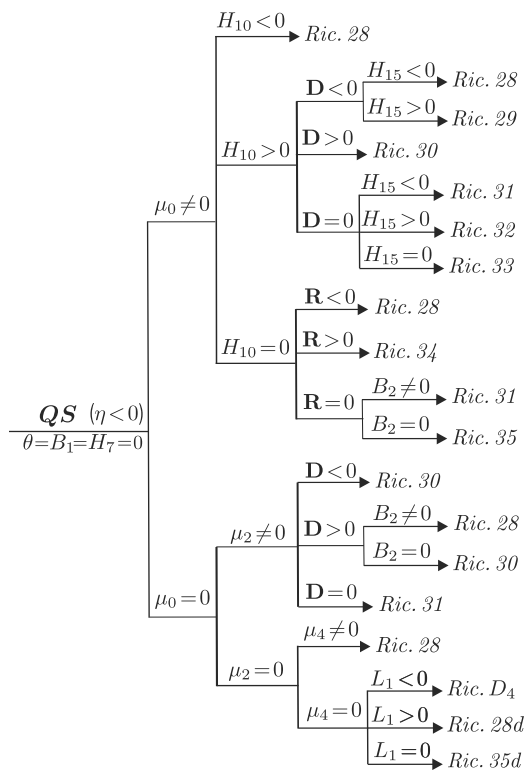


Fig. 12 The invariant criteria for phase portraits of systems in QS_{Ric} (case $\eta < 0$)



Consider system (8) with $mn \neq 0$, then the rescaling $x \rightarrow \bar{x}/m$, $y \rightarrow \bar{y}/n$ leads to a system with $m = n = 1$:

$$\dot{x} = x, \quad \dot{y} = d + ax + by + \ell x^2 + xy + y^2, \quad (9)$$

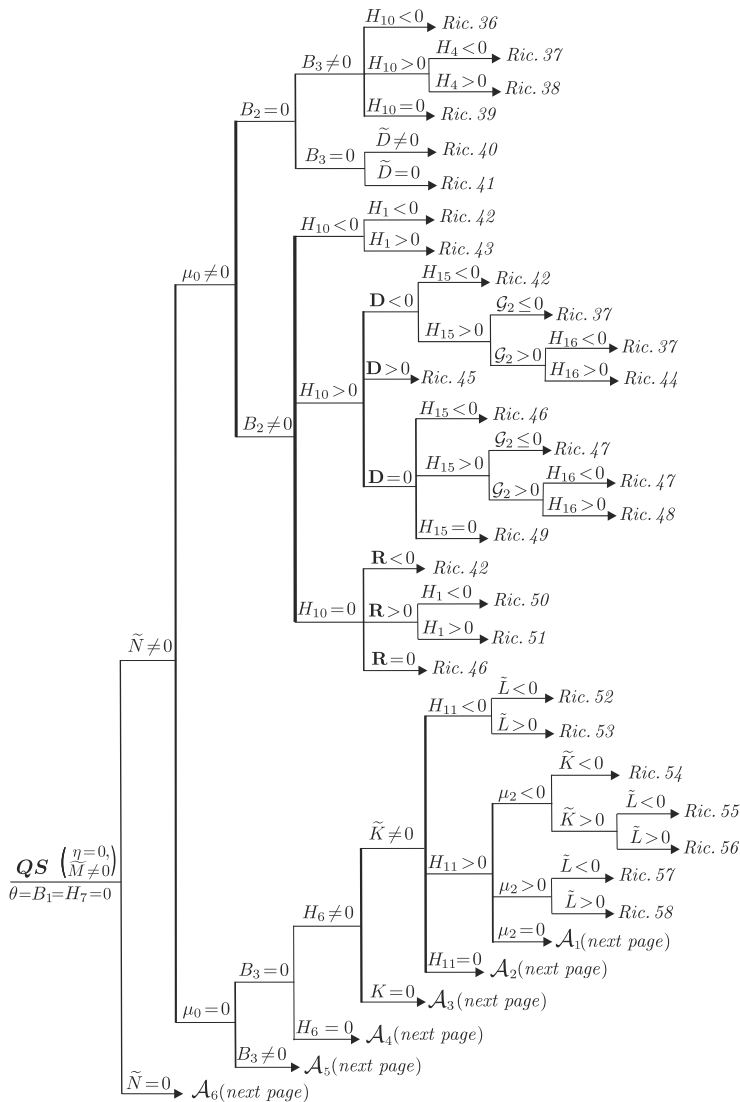


Fig. 13 The invariant criteria for phase portraits of systems in QS_{Ric} (case $\eta = 0 \neq \tilde{M}$)

The translation $y = \bar{y} - a$ eliminates the term in x that is, $a = 0$

$$\dot{x} = x, \quad \dot{y} = d + by + \ell x^2 + xy + y^2, \quad (10)$$

which has just three parameters. Moreover, by means of the change of parameters $(d, b) \rightarrow (d + b/2, b + 1)$ it is easy to detect a symmetry which allows to consider

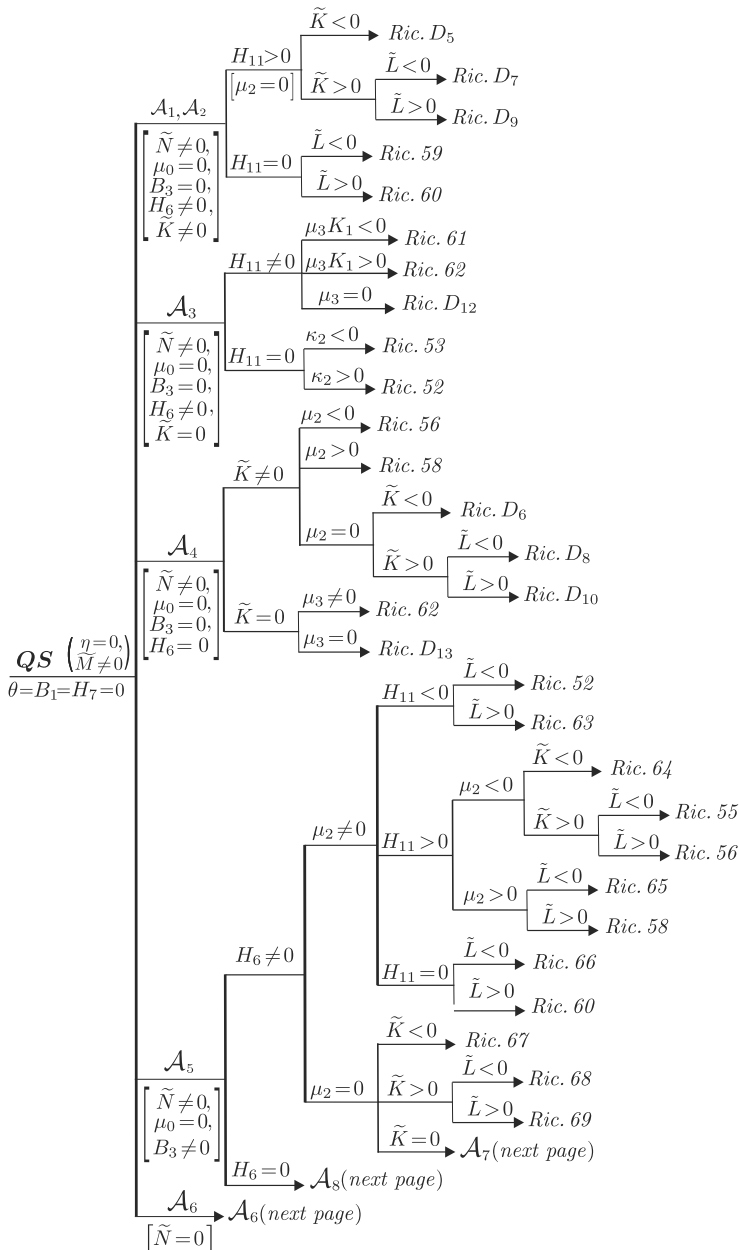


Fig. 13 continued

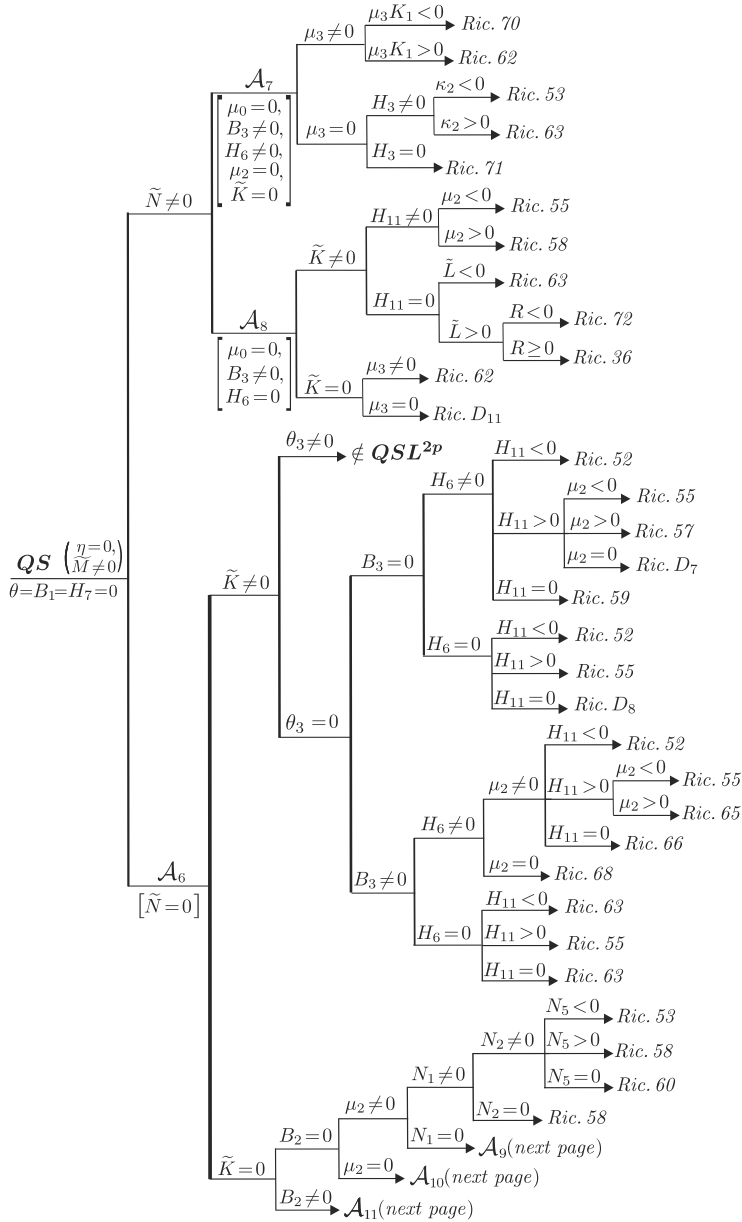


Fig. 13 continued

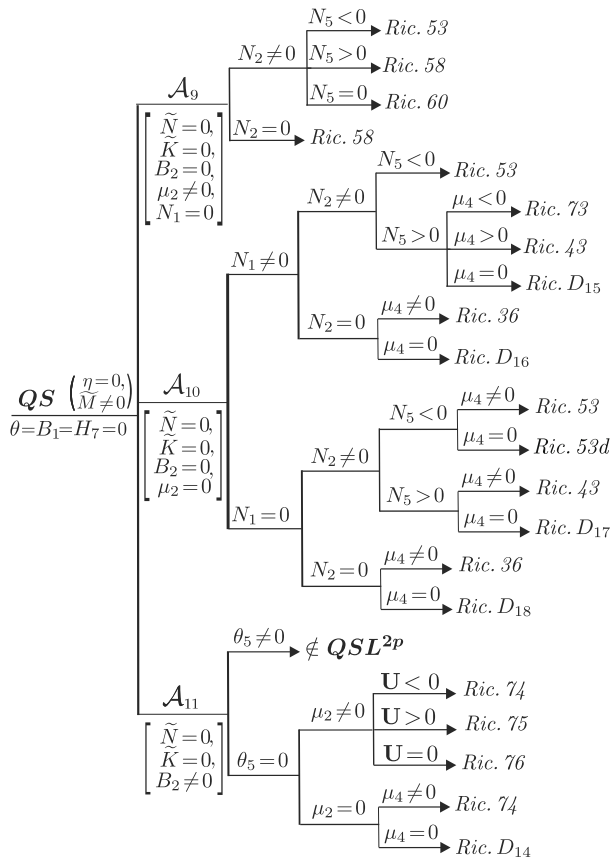


Fig. 13 continued

only the semi-space $b \geq 0$. In summary we have the systems

$$\dot{x} = x, \quad \dot{y} = d + b/2 + (b+1)y + \ell x^2 + xy + y^2, \quad (11)$$

which will be subfamily VIII(A).

4 Proof of the Theorems

The complete proof of the theorems is given in the Mathematica file that the reader may find it in the link <http://mat.uab.es/~artes/articles/fam-V-X.html>. We prefer to provide it in this way that transcribing it to text which would produce a long and tedious work. Moreover the checking of the proof must be done using the same or similar tools.

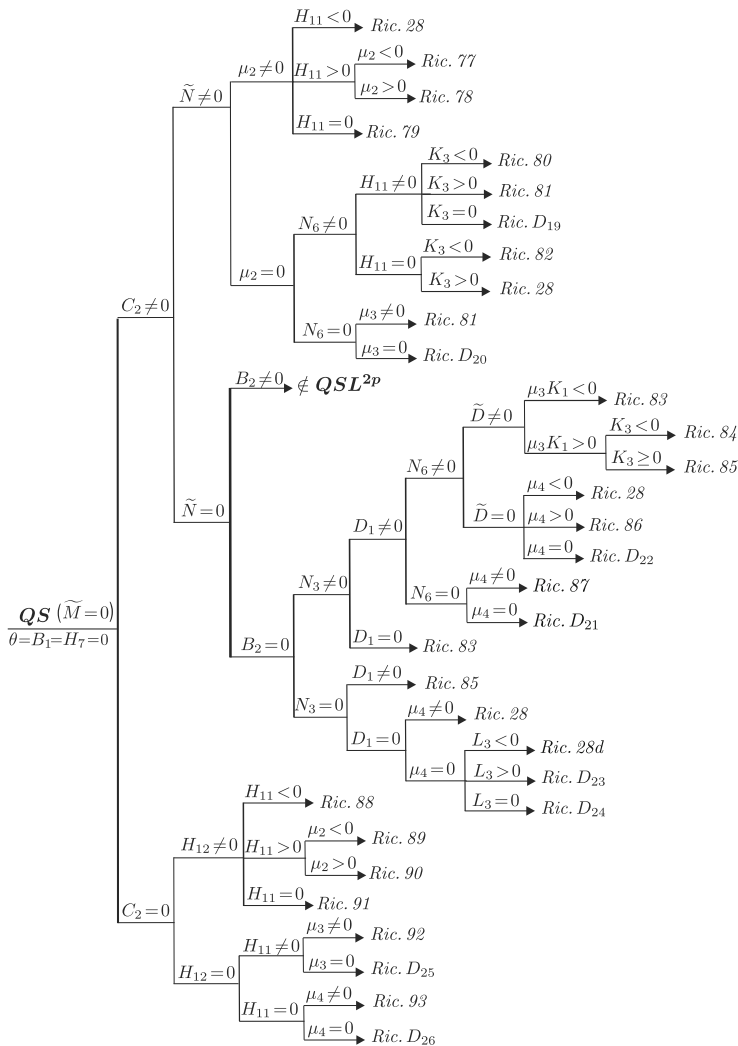


Fig. 14 The invariant criteria for phase portraits of systems in QS_{Ric} (case $\eta = 0 = \tilde{M}$)

Anyway, we will provide here part of the proof of the cases of family V (which is the richer one and most generic) so to help the reader interested in checking the data given in the file.

We first activate a block in the program which contains all the formulas for computing the invariants. We also execute the blocks which produce the needed subfamilies. In fact, we produced subfamilies for all the families, even though finally not all of them were needed.

We start with family V (3) and we compute all the invariants. Then we enter in the subset of branches with $\eta > 0$, that is, we start in Fig. 11. Then we ask Mathematica to find an example of (3) having $\eta > 0 = \theta = B_1 = H_7 = B_2, \tilde{N}B_3\mu_0 \neq 0$ and

$\tilde{N}H_{10} < 0 > \mathcal{G}_2$. If Mathematica is able to propose such an example, we confirm it that it really verifies all the conditions and that the phase portrait is the one predicted by [3]. If Mathematica is unable to produce such an example, we confirm case by case that there is some incompatibility between the set of conditions and the proposed normal form.

For system (3) we have some easy invariants/comitants as

$$\eta = -n^2(-1 + 2m - m^2 + 4\ell n), \quad \theta = B_1 = H_7 = 0, \\ \tilde{N} = -(-2m + m^2 - 4\ell n)x^2, \quad \mu_0 = n^2 \text{ and } \tilde{N}H_{10} = 32n^2(-2m + m^2 - 4\ell n)^2x^2,$$

and we obtain a trivial contradiction directly with $\tilde{N}H_{10} < 0$. In case of more assurance the reader interested in checking the computations, may also use the Mathematica function “Reduce” with the same parameters as “FindInstance” to obtain a definitive answer “False” instead of the null answer of “FindInstance”. That is, the first branch of Fig. 11 is not achievable for family V. This does not prove yet that phase portrait *Ric. I* is not achievable for this family. It might appear in another branch somewhere else.

Obviously, the second branch will neither be achievable because it contains the same condition $\tilde{N}H_{10} < 0$ and the only change is in the sign of $\mathcal{G}_2$.

Now we test the third branch and the problems start. First of all we must explain to the reader the meaning of the comitants and the trick we use to test if a comitant is zero, positive or negative. The comitants are polynomial expressions formed by the parameters (as the invariants) but which include also the variables x and y forming an homogeneous polynomial in (x, y) of some (normally even) degree. Then, when a comitant is required to be zero, this does not mean that we may make use of the variables x and y to make it zero. On the contrary, the comitant must be zero for any x and y , and this means that all the coefficients of the homogeneous polynomial in x and y must be zero. This is a wonderful way that the Sibirskii School has found to accumulate in a single comitant several conditions that the parameters must hold in order to produce some geometrical features. And in the case that a comitant is required to be positive or negative (then the comitant must be always of even degree), this must happen for every couple (x, y) , except maybe a subset of null measure. So there must be some factorization, or the coefficients must be of a very special kind, so that the sign of the comitant is forced.

Since this criteria of determining if a certain comitant is zero, positive or negative may depend on the form and terms that the comitant has, this is a bit complicated to explain to Mathematica, and we use the trick of substituting the variables (x, y) by some transcendent numbers like e and π . If the solution that we obtain from Mathematica includes some of the transcendent numbers we have used, this is a clear clue (even not yet conclusive) that the conditions we have posed are not achievable, and it needs a bit more of test.

Now we apply the conditions from branch 3. Now there is no immediate contradiction as before and the complicated equation $B_2 = 0$ must be taken into account. and Mathematica simply cannot do it, at least the version we have. We have tried it for up to four hours without success. So, we have had to abort the computation and make use

of the subfamilies produced for family V. We try with subfamily V(A) (6) for which $B_2 = -648(4a^2 + a^4 + 4a^2d - 12a^2\ell - 8a^2d\ell + 16d\ell^2 + 16d^2\ell^2 + 16\ell^3)x^4$ is quite simpler than before, but it still implies the resolution of a third degree equation in ℓ . And Mathematica succeeds in giving one answer, but as expected this solution depends on the root of a cubic polynomial. Even though this would be enough for our goal, we want to obtain a simpler solution (in case it is possible) and so we add an extra condition such that the constant term of the cubic polynomial in ℓ to be zero. The success is not guaranteed, but it is worth trying it. And we are lucky and obtain that system (6) with $(a, d, \ell) = (0, -2, -2)$ holds all the conditions of branch 3, and thus it has phase portrait *Ric. 3*. So, there is no need to check other subfamilies of family V.

Now we should have to return to the main family (3) in order to check branch 4. However we will normally find the same difficulties as in the previous branch since the condition that complicates the computation is not the last one, but a middle one. So in order to avoid going up and down with the subfamilies, and having to recompute the invariants for every branch, we decide to stay in the subfamily (6), meanwhile we remain in the same block of the diagrams, and will only move back to the main family when we go out of the condition producing the complicate computations.

Thus for branch 4 we use the same subfamily (6) and the same trick to reduce the cubic polynomial B_2 in ℓ and we see that having $(a, d, \ell) = (0, -1/4, 3/16)$, we are in branch 4 and we obtain (again) phase portrait *Ric. 3*.

The same for branch 5, system (6) with $(a, d, \ell) = (0, 7/64, -497/4096)$ holds all the conditions and thus has phase portrait *Ric. 4*.

The same for branch 6, system (6) with $(a, d, \ell) = (-4, -5, -10 + 6\sqrt{2})$ holds all the conditions and thus has phase portrait *Ric. 3*.

The same for branch 7, system (6) with $a = -31/32$, $d = -5057/4096$ and $\ell = -(4859777 + 97185\sqrt{17345})/33554432$ holds all the conditions and thus has phase portrait *Ric. 5*. We realize that when we increase the number of equalities, the examples become a bit more complicated in their expressions.

We continue with branches 8, 9 and 10 and we obtain examples all of them still in subfamily (6).

We arrive to the branch 11 and we see that “FindInstance” returns a null answer indicating no solution. At this point we have two alternatives: either we continue testing other subfamilies of (3) or we try directly with the 6-parameter family (3). In this case, it is better to return to the main family because we need to obtain the condition $H_{10} = -32n^2(-2m + m^2 - 4\ell n) = 0$, and this implies $\mu_0\tilde{N} = 0$ in clear contradiction with the rest of conditions. The same happens with branch 12.

Up to here we have been in the sub-branch $\mu_0 \neq 0$ but next branch starts the sub-branch $\mu_0 = 0$ which for this normal form implies $n = 0$ and remember that from (4), this implies $\eta = 0$. This kind of reductions normally simplifies the computations because it reduces in a simple way the number of parameters from 6 to 5. This facilitates that we may work with the main family avoiding the subfamilies. Thus we obtain that branches from 13 to 17 are not realizable for family V.

Now from branch 18 we start a new sub-branch $B_3 = 0$ and again the Mathematica fails when trying to find an example. In principle, we should have to solve the system $B_2 = B_3 = 0$ but using [3, Page 8], from the definition of the invariant polynomials B_3 and B_2 , we see that the condition $B_3 = 0$ implies $B_2 = 0$. Thus we only need to

solve $B_3 = 0$. Since $n \neq 0$ by $\eta < 0$, the comitant B_3 has three elements that must be zero, to know, the coefficient of x^4 , the coefficient of x^3y and the coefficient of x^2y^2 .

We have solved apart this system of equations and obtain two rational solutions for $B_3 = 0$ having two parameters in terms of the other four. We must work with both solutions and try branches from 18 to 21. It happens that both solutions are acceptable for branch 18 and we obtain examples of this branch, for example system (3) with $(a, d, \ell, b, m, n) = (-1, 3/2, 1/2, 0, 0, -1)$ holds all the conditions and thus has phase portrait *Ric. 3*. However the other three branches are not realizable for family V with any of the two solutions from the system provided by $B_3 = 0$, and thus become impossible.

In this way we have checked all the branches of the sub-branch $B_2 = 0$ of Fig. 11. Now we prune all the empty branches of this set and produce the equivalent subset for family V that we see in Fig. 1.

Proceeding in the same way, the diagram for family V is completed, and so on, the ones for the rest of families and all theorems are proved.

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References

1. Artés, J.C., Cairó, L., Llibre, J.: New exploration of phase portrait classification of quadratic polynomial differential systems based on invariant theory. *Appl. Math.* **5**(68), 22 (2025)
2. Artés, J.C., Cairó, L., Llibre, J.: Phase portraits of the family IV of the quadratic polynomial differential systems. In: *Qualitative Theory of Dynamical Systems*, pp. 24–66 (2025)
3. Artés, J.C., Llibre, J., Schlomiuk, D., Vulpe, N.: Global analysis of Riccati quadratic differential systems. *Int. J. Bifurc. Chaos* **34**(1), 2450004 (2024)
4. Bujac, C., Schlomiuk, D., Vulpe, N.: On families $QSL_{\geq 2}$ of quadratic systems with invariant lines of total multiplicity at least 2. *Qual. Theory Dyn. Syst.* **21**(4), 133 (2022)
5. Cairó, L., Llibre, J.: Phase portraits of the families VII and VIII of the Quadratic Systems. *Axioms*, 24–66 (2025)

6. Coll, B., Gasull, A., Llibre, J.: Quadratic systems with a unique finite rest point. *Publ. Mat.* **32**, 199–259 (1988)
7. Gasull, A., Sheng, L.-R., Llibre, J.: Chordal quadratic systems. *Rocky Mountain J. Math.* **16**, 751–782 (1986)

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