On embedding properties of some extrapolation spaces *

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Abstract

Given a sublinear operator T satisfying that $\|Tf\|_{L^p(\nu)} \leq \frac{C}{p-1}\|f\|_{L^p(\mu)}$, for every $1 < p \leq p_0$, with C independent of f and p, it has been recently proved that $T : L\log L \to M(\varphi)$, where $M(\varphi)$ is the maximal Lorentz space with $\varphi(t) = t(1+\log^+ t)^{-1}$. Also, if T satisfies that $\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)}$, for every $p \geq p_0$, then $T : \Lambda^1\big(\min(t^{-1}, 1)\big) \cap L^\infty \to M(\phi)$, where $\phi(t) = \big(1+\log^+(1/t)\big)^{-1}$.

The purpose of this note, is to study embedding properties of the extrapolation spaces $L\log L$ and $M(\varphi)$ with respect to L^1, and also embedding properties of $\Lambda^1\big(\min(t^{-1}, 1)\big) \cap L^\infty$ and $M(\phi)$ with respect to L^∞. We shall also extend these type of results to more general extrapolation theorems.

1 Introduction

In 1951, Yano (see [6]) proved that for every sublinear operator satisfying that

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

is bounded, for every $1 < p \leq p_0$, with constant less than or equal to $\frac{C}{p-1}$, where μ and ν are two finite measure, it holds that $T : L\log L(\mu) \longrightarrow L^1(\nu)$ is bounded. If the measures involved are not finite, then an easy modification in the proof of this result shows that $T : L\log L(\mu) \longrightarrow L^1(\nu) + L^\infty(\nu)$ is bounded.

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This theorem has recently been improved in [3] and [4], showing that, if μ and ν are σ-finite measures and T satisfies that

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu),$$

is bounded with constant less than or equal to $\frac{C}{p-1}$, where $L^{p,\infty}(\nu)$ is endowed with the norm $\|f\|_{L^{p,\infty}} = \sup_t \left(t^{1/p} f_\nu^{**}(t)\right)$, then

$$T : L \log L(\mu) \longrightarrow M(\varphi; \nu)$$

where $\varphi(t) = t(1 + \log^+ t)^{-1}$ and, the maximal Lorentz space $M(\varphi) = M(\varphi; \nu)$ is defined (see [1], p. 69]) as the set of measurable functions such that

$$\|f\|_{M(\varphi)} = \sup_{t>0} \Big(\varphi(t) f_\nu^{**}(t)\Big) < \infty,$$

where $f_\nu^{**}(t) = \frac{1}{t}\int_0^t f_\nu^*(s)\, ds$ and f_ν^* is the decreasing rearrangement of f with respect to the measure ν, (in what follows, we shall omit the subindices ν or μ whenever it is clear the measure we are working with). In particular, if $\varphi(t) = t^{1/p}$, $M(\varphi) = L^{p,\infty}$.

Also, in the setting of Lorentz spaces, it holds that $L \log L$ is the minimal Lorentz space $\Lambda(\varphi)$, where $\varphi(t) = t(1 + \log^+(1/t))$ and

$$\|f\|_{\Lambda(\varphi)} = \int_0^\infty f^*(t)\, d\varphi(t)\cdot$$

If $\varphi(t) = t^{1/p}$, $\Lambda(\varphi)$ is the Lorentz space $L^{p,1}$, where

$$\|f\|_{L^{p,1}} = \frac{1}{p}\int_0^\infty f^*(t) t^{1/p-1}\, dt\cdot \tag{1}$$

Therefore, in this context of minimal-maximal Lorentz spaces, the new version of Yano's theorem can be stated as follows:

Theorem 1.1 *(Yano) Let $\varphi_\theta(t) = t^{1-\theta}$ and let T be a sublinear operator such that*

$$T : \Lambda(\varphi_\theta; \mu) \to M(\varphi_\theta; \nu)$$

is bounded with $\|T\| \le C/\theta$, $(0 < \theta < \theta_0 \le 1)$. Then

$$T : \Lambda(\varphi_{D_+}; \mu) \to M(\varphi_{R_+}; \nu),$$

where $\varphi_{D_+}(t) = t(1 + \log^+ \frac{1}{t})$ and $\varphi_{R_+}(t) = t(1 + \log^+ t)^{-1}$.

We also have a dual version. That is, if

$$\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)},$$

for every $p \geq p_0$, then, it was proved in [4] that

$$T : \Lambda^1(\min(t^{-1}, 1); \mu) \cap L^\infty(\mu) \to M(\phi; \nu),$$

where $\phi(t) = 1/(1+\log^+(1/t))$, improving a previous result due to Zygmund (see [7], p. 119). The formulation of this result in the above terminology is the following:

Theorem 1.2 *(Zygmund) Let $\varphi_\theta(t) = t^{1-\theta}$ and let T be a sublinear operator such that*

$$T : \Lambda(\varphi_\theta; \mu) \to M(\varphi_\theta; \nu)$$

is bounded with $\|T\| \leq C/(1-\theta)$, $(\theta_0 < \theta < 1)$. Then

$$T : \Lambda(\varphi_{D_-}; \mu) \to M(\varphi_{R_-}; \nu),$$

where $\varphi_{D_-}(t) = (1 + \log^+ t)$ and $\varphi_{R_-}(t) = (1 + \log^+(1/t))^{-1}$.

Now, let us consider **compatible pairs** of Banach spaces $\bar{A} = (A_0, A_1)$. That is, we assume that there is a large topological vector space $\mathcal{V}$ such that $A_i \subset \mathcal{V}$, $i = 0, 1$, continuously. Usually we drop the terms "compatible" and "Banach" and refer to a compatible Banach pair simply as a "pair".

Let us recall that given a pair $\bar{A} = (A_0, A_1)$, the Peetre $K-$**functional** is defined, for $a \in A_0 + A_1$ and $t > 0$, by

$$K(a,t) = K(a,t;\bar{A}) = \inf\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a = a_0 + a_1,\ a_i \in A_i\}\cdot$$

It is easy to see that $K(t, a)$ is a nonnegative and concave function of $t > 0$, (and thus also continuous). Therefore

$$K(a,t;\bar{A}) = K(a, 0^+; \bar{A}) + \int_0^t k(a,s;\bar{A})\, ds,$$

where the $k-$functional, $k(a,s;\bar{A}) = k(a,s)$, is a uniquely defined, nonnegative, decreasing and right-continuous function of $s > 0$.

In particular, if $\bar{A} = (L^1(\nu), L^\infty(\nu))$, we have that $k(a,s;\bar{A}) = f^*(s)$ and

$$K(f,t) = \int_0^t f^*(s)\, ds\cdot$$

The new point of view of Yano's and Zygmund's theorems presented above gives us the idea of defining, for a pair $\bar{A}$, the corresponding minimal and maximal spaces as follows:

Definition 1.1 *The minimal Lorentz space, $\Lambda(\varphi;\bar{A})$, is the set of elements $a\in A_0+A_1$ such that $K(a,0^+;\bar{A})=0$ and*

$$\|a\|_{\Lambda(\varphi;\bar{A})}=\int_0^\infty k(a,s;\bar{A})\,d\varphi(s)<\infty,$$

and the maximal Lorentz space, $M(\varphi;\bar{A})$, is the set of elements $a\in A_0+A_1$ such that

$$\|a\|_{M(\varphi;\bar{A})}=\sup_{t>0}\left(\frac{K(a,t;\bar{A})}{t}\varphi(t)\right)<\infty\cdot$$

Then, the two following extrapolation results have been obtained in [5]

Theorem 1.3 *Let $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ be two pairs and let T be a linear operator such that*

$$T:\Lambda(\varphi_\theta;\bar{A})\to M(\varphi_\theta;\bar{B})$$

is bounded with $\|T\|\leq\frac{C}{\theta}$, $(0<\theta<\theta_0)$. Then

$$T:\Lambda(\varphi_{D_+};\bar{A})\to M(\varphi_{R_+};\bar{B})\cdot$$

Theorem 1.4 *Let $\bar{A}=(A_0,A_1)$ and $\bar{B}=(B_0,B_1)$ be two pairs and let T be a linear operator such that*

$$T:\Lambda(\varphi_\theta;\bar{A})\to M(\varphi_\theta;\bar{B})$$

is bounded with $\|T\|\leq\frac{C}{1-\theta}$, $(\theta_0<\theta<1)$. Then

$$T:\Lambda(\varphi_{D_-};\bar{A})\to M(\varphi_{R_-};\bar{B})\cdot$$

The purpose of this note is to study embedding properties of the extrapolation spaces $\Lambda(\varphi_{D_+};\bar{A})$, $M(\varphi_{R_+};\bar{B})$, $\Lambda(\varphi_{D_-};\bar{A})$ and $M(\varphi_{R_-};\bar{B})$ with respect to the corresponding end-point spaces A_0, A_1, B_0 and B_1 respectively. For example: it is clear that the domain space $\Lambda(\varphi_{D_+};\bar{A})\subset A_0$, while the opposite embedding does not clearly hold. However, if we consider the Lions-Peetre real interpolation spaces $\bar{A}_{\theta,p}$ defined by (see [1])

$$\|a\|_{\bar{A}_{\theta,p}}=\left(\theta(1-\theta)\int_0^\infty\left(\frac{K(t,a;\bar{A})}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p},$$

then, for every $0<\theta<1$ and $p\geq 1$,

$$\bar{A}_{\theta,1}\subset\cdots\bar{A}_{\theta,p}\subset\cdots\bar{A}_{\theta,\infty},$$

and, we obtain (see Theorem 2.3 below) that if we intersect A_0 with the biggest space of the above chain then

$$A_0 \cap \bar{A}_{\theta,\infty} \subset \Lambda(\varphi_{D_+}; \bar{A}).$$

Similar results will be proved for the other three extrapolation spaces.

Constants such as C will denote universal constants (independent of the parameters involved) and may change from one occurrence to the next. As usual, the symbol $f \approx g$ will indicate the existence of a universal positive constant C so that $f/C \leq g \leq Cf$, while the symbol $f \preceq g$ means that $f \leq Cg$.

2 Relationship betwe en the extrapolation and the end-point spaces.

Let us start by analyzing the case $\bar{L} = (L^1, L^\infty)$.

Proposition 2.1 *For every $p > 1$,*

$$L^1 \cap L^{p,\infty} \subset L \log L \subset L^1,$$

and,

$$L^1 \subset M(\varphi_{R_+}) \subset L^1 + L^{p,1},$$

where the constant of the first and last embeddings are less than or equal to $Cp/(p-1)$.

Proof: To show the first embedding, we observe that

$$\|f\|_{L \log L} \approx \|f\|_1 + \int_0^1 f^{**}(t)\, dt,$$

and therefore

$$\begin{aligned} \|f\|_{L \log L} &\preceq \|f\|_{L^1} + \int_0^1 \frac{t^{1/p} f^{**}(t)}{t^{1/p}}\, dt \leq \|f\|_{L^1} + \|f\|_{L^{p,\infty}} \int_0^1 t^{-1/p}\, dt \\ &= \|f\|_{L^1} + \frac{p}{p-1} \|f\|_{L^{p,\infty}}. \end{aligned}$$

The second and third embeddings are trivial. To prove the last embedding, let $f \in M(\varphi_{R_+})$. Then, for every $t > 0$,

$$\int_0^t f^* \leq \|f\|_{M(\varphi_{R_+})}(1 + \log^+ t).$$

Then, if we define $\overline{f} = f\chi_{\{|f|>f^*(1)\}}$, we have that

$$\|\overline{f}\|_{L^1} = \int_0^1 f^*(t)\, dt \leq \|f\|_{M(\varphi_{R_+})}.$$

Now, set $\underline{f} = f - \overline{f}$ and recall that the norm in $L^{p,1}$ is given by (1). Then, if $p > 1$, an integration by parts shows that

$$\begin{aligned}
\|\underline{f}\|_{L^{p,1}} &= \frac{1}{p}\int_0^\infty \underline{f}^*(t)t^{1/p-1}\, dt \leq f^*(1) + \frac{1}{p}\int_1^\infty f^*(t)t^{1/p-1}\, dt \\
&\preceq \|f\|_{M(\varphi_{R_+})} + \frac{p-1}{p}\|f\|_{M(\varphi_{R_+})} + \frac{p-1}{p}\int_1^\infty \Big(\int_0^t f^*\Big)t^{1/p-2}\, dt \\
&\leq \frac{p-1}{p}\|f\|_{M(\varphi_{R_+})} + \|f\|_{M(\varphi_{R_+})}\frac{p-1}{p}\int_1^\infty (1 + \log^+ t)t^{1/p-2}\, dt \\
&\approx \frac{p}{p-1}\|f\|_{M(\varphi_{R_+})}
\end{aligned}$$

from which the result follows. □

Proposition 2.2 *For every $p > 1$, it holds that*

$$L^\infty \cap L^{p,\infty} \subset \Lambda(\varphi_{D_-}) \subset L^\infty,$$

and,

$$L^\infty \subset M(\varphi_{R_-}) \subset L^\infty + L^{p,1},$$

where the constants of the first and last embedding are less than or equal to Cp.

Proof: The proof follows the same pattern than Proposition 2.1. Also, it can be deduced using duality in Proposition 2.1, since the associated space of $L^{p,1}$ is equal to $L^{p',\infty}$ and it was proved in [4] that $\Lambda(\varphi_{D_-})$ is the associated space of $M(\varphi_{R_+})$ and $M(\varphi_{R_-})$ is the associated space of $L \log L$. □

Let us consider now, the general case $\bar{A} = (A_0, A_1)$.

Theorem 2.1 *Let $\bar{A} = (A_0, A_1)$ be a pair. Then, for every $0 < \theta < 1$,*

$$A_0 \cap \bar{A}_{\theta,\infty} \subset \Lambda(\varphi_{D_+}; \bar{A}) \subset A_0,$$

and,

$$A_0 \subset M(\varphi_{R_+}; \bar{A}) \subset A_0 + \bar{A}_{\theta,1},$$

where the constants of the first and last embedding are less than or equal to C/θ.

Proof: The second and the third embeddings are trivial. The first embedding follows from the fact that if $a \in A_0 \cap \bar{A}_{\theta,\infty}$, then $k(.;a) \in L^1 \cap L^{1/(1-\theta),\infty}$, and, by Proposition 2.1, we have that $k(.;a) \in L \log L$, which is equivalent to

$$\|a\|_{\Lambda(\varphi_{D_+}; \bar{A})} \approx \int_0^\infty k(t;a) \left(1 + \log^+ \frac{1}{t} \right) dt < \infty \cdot$$

To prove the last embedding, we have to proceed as in Proposition 2.1. Let $a \in M(\varphi_{R_+}; \bar{A})$. Then,

$$\int_0^t k(s,a)\, ds \leq \|a\|_{M(\varphi_{R_+};\bar{A})} (1 + \log^+ t) \cdot$$

Then, if we define $\overline{k}(s,a) = k(s,a)\chi_{(0,1)}$ and $\underline{k}(s,a) = k(s,a)\chi_{(1,\infty)}$ we have that

$$\begin{aligned} K(t,a) &\leq \int_0^t \overline{k}(s,a)\, ds + \int_0^t \underline{k}(s,a)\, ds \\ &\leq \int_0^t \overline{k}(s,a) ds + \int_0^t (\underline{k}(s,a) + k(1,a)\chi_{(0,1)}) ds, \end{aligned}$$

and since the last two functions are concave, we can use the K-divisibility theorem (see [2], Theorem 3.2.7) to have that there exist a_0 and a_1 such that $a = a_0 + a_1$,

$$K(t, a_0) \leq \int_0^t \overline{k}(s,a)\, ds,$$

and

$$K(t, a_1) \leq \int_0^t (\underline{k}(s,a) + k(1,a)\chi_{(0,1)})\, ds \cdot$$

Now, if we define $\tilde{A}_0$ as the set of elements in $A_0 + A_1$ such that $\sup_t K(t, a_0) < \infty$ then, using Holmstedt's formula and Theorem 1.5 of [1] (p. 297), we have that $A_0 + A_{\theta,1} = \tilde{A}_0 + A_{\theta,1}$ with equivalent norms and, hence,

$$\begin{aligned}
\|a\|_{A_0+A_{\theta,1}} &\approx \|a\|_{\tilde{A}_0+A_{\theta,1}} \leq \|a_0\|_{\tilde{A}_0} + \|a_1\|_{A_{\theta,1}} \\
&= \sup_t K(t,a_0) + \theta(1-\theta)\int_0^\infty \frac{K(t,a_1)}{t^{1+\theta}}\,dt \\
&\leq \int_0^\infty \overline{k}(s,a)\,ds + \theta(1-\theta)K(1,a)\int_0^1 t^{-\theta}\,dt \\
&\quad +\theta(1-\theta)\int_1^\infty \frac{K(t,a)}{t^{1+\theta}}\,dt \\
&\preceq \int_0^1 k(s,a)\,ds + K(1,a) \\
&\quad +\|a\|_{M(\varphi_{R_+};\bar{A})}\theta(1-\theta)\int_1^\infty \frac{(1+\log^+ t)}{t^{1+\theta}}\,dt \\
&\preceq \frac{1}{\theta}\|a\|_{M(\varphi_{R_+};\bar{A})},
\end{aligned}$$

from which the result follows. □

And, similarly:

Theorem 2.2 *Let $\bar{A}$ be a pair. Then, for every $0 < \theta < 1$,*

$$A_1 \cap \bar{A}_{\theta,\infty} \subset \Lambda(\varphi_{D_-};\bar{A}) \subset A_1,$$

and

$$A_1 \subset M(\varphi_{R_-};\bar{A}) \subset A_1 + \bar{A}_{\theta,1},$$

where the constants of the first and last embeddings are less than or equal to $C/(1-\theta)$.

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