A NOTE ON SOBOLEV INEQUALITIES AND LIMITS OF LORENTZ SPACES

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To Michael Cwikel with friendship and admiration

ABSTRACT. Motivated by the theory of Sobolev embeddings we shall present a new way to obtain L^∞ estimates by means of taking limits of Lorentz spaces (*extrapolation*). Although our result is independent from the theory of embeddings we thought it would be worthwhile to present rather succinctly the issues that motivated us. We refer to other papers in these proceedings for more complete and detailed accounts of the relevant theory of embeddings.

1. INTRODUCTION

Suppose that $f \in C_0^1(\mathbb{R})$. Using the fundamental theorem of calculus we write

$$f(x) = \int_{-\infty}^{x} f'(s)ds$$

from which it follows that

$$(1.1) \qquad \|f\|_\infty \leq \|f'\|_1.$$

This is in some sense the prototype of the famous Sobolev inequalities. In dimension $n > 1$ the previous argument fails, and in fact, it is no longer possible to deduce that f is bounded if we know that $\nabla f \in L^n$. Instead we have the following family of Sobolev inequalities: for all $f \in C_0^\infty(\mathbb{R}^n), 1 \leq p < n, \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$,

$$(1.2) \qquad \|f\|_{p^*} \leq c_{p,n} \|\nabla f\|_p,$$

with $c_{p,n} \to \infty$ when $p \to n$. In the limiting case $p = n$ we have the following imperfect replacement of (1.1)

$$(1.3) \qquad W_0^{1,n}(\Omega) \subset e^{L^{n'}(\Omega)},$$

for domains with finite measure see [17], for the general case see [13]. While (1.3) is an important result in its own right and is, in a certain sense, a natural result[1], it is not the best possible. To understand the reason for this we recall that the sharpest version of (1.2) involves the Lorentz spaces $L(p^*, p)$ (see (1.9) below). Indeed, the

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[1](1.3) follows from (1.2) by taking a suitable limit (i.e. by *extrapolation*); and when $n = 1$, we have $n' = \infty$, so that in one dimension we recover (1.1) from (1.3).

following inequality holds (cf. O'Neil [12]): for all $f \in C_0^\infty(\mathbb{R}^n), 1 \leq p < n, \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, we have

$$\|f\|_{L(p^*,p)} \leq C_{p,n} \|\nabla f\|_p \,, \tag{1.4}$$

where again[2] $C_{p,n} \to \infty$ as $p \to n$. Observe that (1.4) improves on (1.2). Indeed, while the degree of integrability of the gradient is the same in both inequalities, the integrability of f is improved since

$$L(p^*, p) \subsetneq L^{p^*}.$$

Building on this idea Brezis-Wainger [4] and Hansson [5], showed that for bounded domains we have that for all $f \in C_0^\infty(\Omega)$

$$\|f\|_{H_n(\Omega)} \leq c_n \|\nabla f\|_{L^n(\Omega)} \,, \tag{1.5}$$

where, for $q \geq 1, H_q(\Omega)$ is the generalized Lorentz space defined by

$$H_q(\Omega) = \left\{ f : \|f\|_{H_q(\Omega)} = \left\{ \int_0^{|\Omega|} \left(\frac{f^{**}(s)}{1 + \ln \frac{|\Omega|}{s}} \right)^q \frac{ds}{s} \right\}^{1/q} < \infty \right\}. \tag{1.6}$$

They show that, moreover, among the class of rearrangement invariant spaces (1.5) is best possible. It is important to remark here that results of this type had already been anticipated by Maz'ya [9] using his theory of capacities, which also allows the consideration of domains with infinite measure.

In view of this picture, and in analogy with the exponential integrability results discussed above, one would like to understand the limits (extrapolation) of Lorentz spaces. At first one is led to consider spaces of the form $e^{L(p,q)}$ (cf. [4] and the references therein) but here we face the same objections as in previous discussion: the underlying inequalities with these norms are not the sharpest.

A different insight as to what could be the correct limiting spaces for the Sobolev embedding theorem was developed by Bastero-Milman-Ruiz [1]. The following sharpening of (1.4) was obtained in [1]: for all $f \in C_0^\infty(\mathbb{R}^n)$, $1 \leq p \leq n$, $1 \leq q \leq \infty$,

$$\left\{ \int_0^\infty [(f^{**}(t) - f^*(t))t^{1/p-1/n}]^q \frac{dt}{t} \right\}^{1/q} \leq c_n \|\nabla f\|_{L(p,q)} \,. \tag{1.7}$$

The "norm" that appears on the left hand side of (1.7) is in fact equivalent to the usual $L(p,q)$ norms if the parameters are kept in the traditional range used in the definition of Lorentz spaces. More precisely, for $1 < p < \infty, 0 < q \leq \infty$, the following functionals are equivalent on functions such that $f^{**}(\infty) = 0$ (in particular if $f \in C_0^\infty(\mathbb{R}^n)$)

$$\left\{ \int_0^\infty [(f^{**}(t) - f^*(t))t^{1/p}]^q \frac{dt}{t} \right\}^{1/q} \simeq \left\{ \int_0^\infty [(f^{**}(t)t^{1/p}]^q \frac{dt}{t} \right\}^{1/q} \tag{1.8}$$

$$\simeq \left\{ \int_0^\infty [f^*(t)t^{1/p}]^q \frac{dt}{t} \right\}^{1/q} ,$$

where as usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\, f$, while the symbol $f \preceq g$ means that $f \leq c\, g$.

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[2]The precise value of the constants now depends also on the definition of the norms we use for the Lorentz spaces, we will discuss this shortly.

In the case $p = 1$ the equivalence between the middle and the last terms fails, while in the case $p = \infty$ the last two terms are finite only when $f = 0$, in fact we have

$$\left\{ \int_0^\infty [f^*(t)]^q \frac{dt}{t} \right\}^{1/q} < \infty \Rightarrow f = 0.$$

We are thus led to define[3]

$$L(1,q) = \left\{ f : \|f\|_{L(1,q)} = \left\{ \int_0^\infty f^*(t)^q t^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}, \ 0 < q \le \infty.$$

(1.9)

$$L(p,q) = \left\{ f : \|f\|_{L(p,q)} = \left\{ \int_0^\infty f^{**}(t)^q t^{q/p} \frac{dt}{t} \right\}^{1/q} < \infty \right\}, \ 1 < p < \infty, 0 < q \le \infty,$$

and

$$L(\infty,q) = \left\{ f : \|f\|_{L(\infty,q)} = \left\{ \int_0^\infty [(f^{**}(t) - f^*(t))]^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}, \ 0 < q \le \infty.$$

It is easy to see that $L(\infty, q)$ is not trivial; and moreover with this notation (1.7) takes the form

$$\|f\|_{L(p^*,q)} \le c_n \|\nabla f\|_{L(p,q)}, \ f \in C_0^\infty(\mathbb{R}^n), 1 \le p \le n, \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, 1 \le q \le \infty.$$

The previous result is optimal[4] and holds without restrictions on the measure or smoothness of the domain (cf. [11]): for any r.i. space $X(\Omega)$, $1 \le p \le n$,

$$W_0^{1,p}(\Omega) \subset X(\Omega) \Rightarrow L(p^*,p) \subset X(\Omega).$$

This gives a nice formulation to the limiting case of the Hardy-Littlewood-O'Neil program.

The purpose of this note is to discuss a simple extrapolation theorem that is connected with the construction of the $L(\infty, q)$ spaces. To motivate our discussion we consider again (1.1) and we ask: what is the correct n dimensional version of this result? In other words: what are the minimal integrability conditions we need to impose on the gradient of f to guarantee that f is bounded?

It has been known for a long time that the correct analog of (1.1) in dimension n involves the Lorentz space[5] $L(n, 1)$ (cf. [4], [15]). To see this recall that using the Riesz transforms $\{R_j\}_{j=1}^n$ we can write for $f \in C_0^\infty(\mathbb{R}^n)$, say, (cf. [14])

$$f(x) = c \sum_{j=1}^n \int_{R^n} \left(R_j(\frac{\partial f}{\partial x_j}) \right)(y) |x - y|^{1-n} \, dy,$$

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[3]The $L(\infty, \infty)$ spaces had been introduced earlier by Bennett-DeVore-Sharpley [2] who showed that $L(\infty, \infty)$ is the rearrangement invariant hull of BMO.

[4]There is no contradiction with the fact that (1.5) is best possible among the class of r.i. spaces since $L(\infty, q)$ does not have a linear structure.

[5]Note that when $n = 1$, $L^{1,1} = L^1$.

where c is a fixed constant. Therefore,

$$(1.10) \qquad \|f\|_\infty \leq c \sum_{j=1}^n \left\| R_j(\frac{\partial f}{\partial x_j}) \right\|_{L(n.1)} \left\| \left\| |x-y|^{1-n} \right\|_{L(n',\infty)} \right\|_\infty$$

$$\leq c \left(\sup_{j=1,..n} \|R_j\|_{L(n,1)\to L(n,1)} \right) \|\nabla f\|_{L(n,1)}.$$

As observed in [1] one can also obtain (1.10) from (1.7). Indeed, (1.7) for $p = n$, $q = 1$, gives

$$(1.11) \qquad \int_0^\infty (f^{**}(s) - f^*(s))\frac{ds}{s} \leq c_n \|\nabla f\|_{L(n,1)}, \ f \in C_0^\infty(\mathbb{R}^n).$$

Now, since $\frac{d}{ds}(-f^{**}(s)) = \frac{f^{**}(s)-f^*(s)}{s}$, we see, by the fundamental theorem of calculus, that (1.11) implies that

$$(1.12) \qquad \|f\|_\infty = f^{**}(0) \leq c_n \|\nabla f\|_{L(n,1)}, \ f \in C_0^\infty(\mathbb{R}^n),$$

as we wished to show.

The approach of [1] can be generalized considerably. For example, [8] shows that for a Lip domain Ω, with $|\Omega| < \infty$, we have

$$(f - \int_\Omega f)^{**}(t) - (f - \int_\Omega f)^*(t) \leq c_n t^{1/n}(\nabla f)^{**}(t), \ f \in C^1(\Omega),$$

which leads to

$$\left\| f - \int_\Omega f \right\|_{L(p^*,q)} \leq c_n \|\nabla f\|_{L(p,q)}, \ 1 \leq p \leq n, \ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \ 1 \leq q \leq \infty.$$

This note was motivated by our desire to see how much one can obtain by taking careful limits in (1.4). At first we thought that there was a chance that one could prove (1.7) by taking limits in (1.4). While this is not the case we can, however, show that (1.11), and thus (1.12), follow from (1.4) by extrapolation. We feel there are some insights that one gains going through the exercise since the analysis could be useful in certain contexts where some of the other approaches are not available. In particular, we formulate the result as an extrapolation theorem that can be applied in other contexts.

In the remaining part of this long Introduction we indicate our proof of (1.12) by extrapolation from (1.4). We take as our point of departure (1.4). Then by Talenti [16] we can write: for all $f \in C_0^\infty(\mathbb{R}^n)$, $1 \leq p < n$, $1 \leq q \leq p$,

$$(1.13) \qquad \left\{ \int_0^\infty [f^*(t)t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q} \leq c_n(p,q)p^* \left\{ \int_0^\infty [|\nabla f|^*(t)t^{1/p}]^q \frac{dt}{t} \right\}^{1/q},$$

where $c_n(p,q)$ remains bounded as $p \to n$. We shall prove below (cf. Theorem 1) that

$$(1.14)$$
$$\left\{ \int_0^\infty [(f^{**}(t) - f^*(t))t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q} \preceq \left(\frac{1}{p^*} \right)^{1/q} \frac{p^*}{p^* - 1} \left\{ \int_0^\infty [f^*(t)t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q}.$$

Combining the estimates (1.13) and (1.14) we get that for all $f \in C_0^\infty(\mathbb{R}^n)$, $1 \leq p < n$, $1 \leq q \leq p$,

$$\left\{ \int_0^\infty [(f^{**}(t) - f^*(t))t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q} \preceq \left(\frac{1}{p^*} \right)^{1/q} \frac{(p^*)^2}{p^* - 1} \left\{ \int_0^\infty [|\nabla f|^*(t)t^{1/p}]^q \frac{dt}{t} \right\}^{1/q}.$$

In particular, if we fix $q = 1$, and we let $p \to n$, then $p^* \to \infty$, and we arrive at

$$\|f\|_\infty = \int_0^\infty (f^{**}(t) - f^*(t)) \frac{dt}{t} \preceq \|\nabla f\|_{L(n,1)}.$$

As the reader can see ultimately everything depends on a careful analysis of the constants of equivalence in (1.8). More precisely, the inequalities available in the literature[6]:

$$\left\{ \int_0^\infty [(f^{**}(t) - f^*(t)) t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q} \leq \left\{ \int_0^\infty [f^{**}(t) t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q}$$

$$\leq \frac{p^*}{p^* - 1} \left\{ \int_0^\infty [f^*(t) t^{1/p^*}]^q \frac{dt}{t} \right\}^{1/q},$$

are of no use for us here and we need to work a little bit harder to prove (1.14).

Finally, although we shall not focus on interpolation theory in this note, it is appropriate to briefly mention here that, from a slightly more general point of view, our discussion corresponds to the study of norm equivalences between different real interpolation constructions associated with the $K-$functional and Gagliardo diagrams. In fact, these more general results, without consideration of the sharpness of the constants, were obtained many years ago by Holmstedt [6], and the subject was later taken up by Jawerth-Milman [7]. From this point of view in the computation of the K-functional

$$K(t, f; X_0, X_1) = \inf_{f = f_0 + f_1, \ f_i \in X_i} \{ \|f_0(t)\|_{X_0} + t \|f_1(t)\|_{X_1} \},$$

an optimal decomposition satisfies (cf. [6], [7])

$$\|f_0(t)\|_{X_0} \simeq K(t, f; X_0, X_1) - t K'(t, f, X_0, X_1)$$
$$\|f_1(t)\|_{X_1} \simeq K'(t, f; X_0, X_1).$$

One can then consider spaces that control each of these "coordinates" (think in terms of the Gagliardo diagram, see for example [3, page 39])

$$(X_0, X_1)_{\theta,q}^{(0)}$$
$$= \left\{ f \in X_0 + X_1 : \|f\|_{(X_0, X_1)_{\theta,q}^{(0)}} = \left\{ \int_0^\infty (t^{-\theta} (K(t, f; X_0, X_1) - t K'(t, f; X_0, X_1))^q \frac{dt}{t} \right\}^{1/q} < \infty \right\},$$

$$(X_0, X_1)_{\theta,q}^{(1)} = \left\{ f \in X_0 + X_1 : \|f\|_{(X_0, X_1)_{\theta,q}^{(1)}} = \left\{ \int_0^\infty (t^{-\theta} t K'(t, f; X_0, X_1))^q \frac{dt}{t} \right\}^{1/q} < \infty \right\},$$

and compare them to the classical spaces of Lions-Peetre

$$(X_0, X_1)_{\theta,q} = \left\{ f \in X_0 + X_1 : \|f\|_{(X_0, X_1)_{\theta,q}} = \left\{ \int_0^\infty (t^{-\theta} K(t, f; X_0, X_1))^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}.$$

It turns out that when $\theta \in (0, 1), q \in (0, \infty]$, it makes no difference which space we consider [6]

$$(X_0, X_1)_{\theta,q}^{(0)} = (X_0, X_1)_{\theta,q}^{(1)} = (X_0, X_1)_{\theta,q},$$

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[6]The first inequality is trivial since $f^{**}(t) - f^*(t) \leq f^{**}(t)$. The second inequality follows from Hardy's inequality.

but we have to pay a price in the constants of equivalence. Our results for Lorentz norms can be formulated immediately in this context. Conversely note that when $(X_0, X_1) = (L^1, L^\infty)$ we have

$$K(t, f; L^1, L^\infty) - tK'(t, f; L^1, L^\infty) = t(f^{**}(t) - f^*(t))$$
$$K'(t, f; L^1, L^\infty) = f^*(t),$$

and we see that (1.8) is a special case of real interpolation theory.

2. A SHARP INEQUALITY AND AN EXTRAPOLATION THEOREM

We start with the following observation. The trivial inequality $f^{**}(t) - f^*(t) \leq f^{**}(t)$, implies that for $1 < p < \infty, 0 < q \leq \infty$, we have

$$(2.1) \qquad \left\{ \int_0^\infty [(f^{**}(s) - f^*(s))s^{1/p}]^q \frac{ds}{s} \right\}^{1/q} \leq \left\{ \int_0^\infty [f^{**}(s)s^{1/p}]^q \frac{ds}{s} \right\}^{1/q}.$$

But computations with characteristic functions suggest that a stronger inequality holds. In fact we have

$$\left\{ \int_0^\infty [(\chi_E^{**}(s) - \chi_E^*(s))s^{1/p}]^q \frac{ds}{s} \right\}^{1/q} = \left(\frac{p}{p-1} \right)^{1/q} q^{-1/q} |E|^{\frac{1}{p}}.$$

while

$$\left\{ \int_0^\infty [\chi_E^{**}(s)s^{1/p}]^q \frac{ds}{s} \right\}^{1/q} = \left(\frac{p}{p-1} \right)^{1/q} p^{1/q} q^{-1/q} |E|^{\frac{1}{p}}.$$

This computation suggested the following sharpening of (2.1).

Theorem 1. *Suppose that $f^{**}(\infty) = 0$, then for $0 < q \leq \infty$, $1 < p < \infty$,*

$$(2.2) \quad \left\{ \int_0^\infty [(f^{**}(s) - f^*(s))s^{1/p}]^q \frac{ds}{s} \right\}^{1/q} \leq \left(\frac{1}{p} \right)^{1/q} \left\{ \int_0^\infty [f^{**}(s)s^{1/p}]^q \frac{ds}{s} \right\}^{1/q}.$$

Proof. We shall integrate by parts. So we first prove (2.2) assuming that f is bounded and has finite support. We can also assume that $q < \infty$, since the inequality is trivially true when $q = \infty$. Note that under these assumptions it follows that $f^{**}(t) \leq \|f\|_1 t^{-1}$.

By the fundamental theorem of calculus

$$(2.3) \qquad\qquad f^{**}(t) = \int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s}.$$

Then

$$\int_0^\infty f^{**}(t)^q t^{q/p} \frac{dt}{t} = \int_0^\infty \left[\int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s} \right]^q t^{q/p} \frac{dt}{t}$$
$$= \left(\frac{p}{q} \right) \int_0^\infty \left[\int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s} \right]^q dt^{q/p}$$
$$= \left(\frac{p}{q} \right) t^{q/p} \left[\int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s} \right]^q \Big|_0^\infty$$
$$+ \left(\frac{p}{q} \right) \int_0^\infty t^{q/p} q \left[\int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s} \right]^{q-1} (f^{**}(t) - f^*(t)) \frac{dt}{t}.$$

The integrated term vanishes on account of our assumptions on f. In fact

$$\lim_{t\to 0}\int_t^\infty (f^{**}(s) - f^*(s))\frac{ds}{s} = \lim_{t\to 0}\int_t^\infty (-f^{**}(s))'ds = -f^{**}(\infty) + f^{**}(0) = \|f\|_\infty\,,$$

while the term at ∞ vanishes on account of the fact that $f^{**}(t)^q t^{q/p} \leq t^{-q+q/p}\|f\|_1^q$.

Therefore,

$$\int_0^\infty f^{**}(t)^q t^{q/p}\frac{dt}{t} = \frac{p}{q}\int_0^\infty t^{q/p}q\left[\int_t^\infty (f^{**}(s) - f^*(s))\frac{ds}{s}\right]^{q-1}(f^{**}(t) - f^*(t))\frac{dt}{t}$$

$$= p\int_0^\infty t^{q/p}(f^{**}(t)^{q-1}(f^{**}(t) - f^*(t))\frac{dt}{t} \text{ (by (2.3))}$$

$$\geq p\int_0^\infty t^{q/p}(f^{**}(t) - f^*(t))^{q-1}(f^{**}(t) - f^*(t))\frac{dt}{t}$$

$$= p\int_0^\infty t^{q/p}(f^{**}(t) - f^*(t))^q\frac{dt}{t},$$

and the result follows.

To prove the inequality (2.2) in general we use an approximation argument. We may assume without loss of generality that $f \geq 0$. To remove the restriction that f is bounded we proceed as follows. Let $f_k(x) = f^*(1/k)\chi_{\{f>f^*(1/k)\}} + f(x)\chi_{\{f\leq f^*(1/k)\}}$, then $f_k \uparrow f$ a.e., moreover $f_k^* \uparrow f^*$, $f_k^{**} \uparrow f^{**}$ and $f_k^{**} - f_k^* \to f^{**} - f^*$. Thus, by the monotone convergence theorem and Fatou's lemma,

$$\int_0^\infty f^{**}(t)^q t^{q/p}\frac{dt}{t} = \lim_k \int_0^\infty f_k^{**}(t)^q t^{q/p}\frac{dt}{t} \geq p\lim_k \int_0^\infty t^{q/p}(f_k^{**}(t) - f_k^*(t))^q\frac{dt}{t}$$

$$\geq p\int_0^\infty t^{q/p}(f^{**}(t) - f^*(t))^q\frac{dt}{t}.$$

The restriction that f has finite support can be removed in similar fashion. □

Remark 1. *For a related inequality see* [10].

We now use Theorem 1 to derive the following type of extrapolation theorem

Theorem 2. *Let* $\{X_{pq}\}_{1<p\leq p_0}$ *be a scale of Banach spaces such that for* $f \in X = \bigcap X_{pq}$ *we have* $\|f\|_{X_{p_0 q}} = \lim_{p\to p_0}\|f\|_{X_{pq}}$. *We let*

$$L^{p,q} = \left\{f : \|f\|_{L^{p,q}} = \left\{\int_0^\infty f^*(t)^q t^{q/p}\frac{dt}{t}\right\}^{1/q} < \infty\right\},$$

and let T *be an operator (not necessarily linear) such that*

$$T : X_{pq} \to L^{\frac{pc}{p_0-p},q}, \ p < p_0, c > 0,$$

with

$$\|T\|_{X_{pq}\to L^{\frac{pc}{p_0-p},q}} \preceq (p_0 - p)^{-1/q}, \ as \ p \to p_0.$$

Then,

$$\|Tf\|_{L(\infty,q)} \preceq \|f\|_{X_{p_0 q}}, \ f \in X.$$

Proof. Combining Hardy's inequality

$$(2.4) \qquad \left\{\int_0^\infty [(Tf)^{**}(t)t^{1/p}]^q\frac{dt}{t}\right\}^{1/q} \leq \frac{p}{p-1}\left\{\int_0^\infty [(Tf)^*(t)t^{1/p}]^q\frac{dt}{t}\right\}^{1/q}.$$

with

$$\|Tf\|_{L^{\frac{pc}{p_0-p},q}} \le c(p_0-p)^{-1/q} \|f\|_{X_{pq}},$$

we have (as $p \to p_0$)

$$\left\{ \int_0^\infty [(Tf)^{**}(t)t^{\frac{p_0-p}{pc}}]^q \frac{dt}{t} \right\}^{1/q} \preceq (p_0-p)^{-1/q} \|f\|_{X_{pq}}.$$

But according to Theorem 1 we have

$$\left(\frac{p_0-p}{pc} \right)^{1/q} \left\{ \int_0^\infty [(Tf)^{**}(t)t^{\frac{p_0-p}{pc}}]^q \frac{dt}{t} \right\}^{1/q}$$

$$\ge \left(\frac{p_0-p}{pc} \right)^{1/q} \left(\frac{p_0-p}{pc} \right)^{-1/q} \left\{ \int_0^\infty \left[((Tf)^{**}(s) - (Tf)^{*}(s))s^{\frac{p_0-p}{pc}} \right]^q \frac{ds}{s} \right\}^{1/q}$$

$$= \left\{ \int_0^\infty \left[((Tf)^{**}(s) - (Tf)^{*}(s))s^{\frac{p_0-p}{pc}} \right]^q \frac{ds}{s} \right\}^{1/q}.$$

Combining these inequalities it follows that for $p < p_0$

$$\left\{ \int_0^\infty \left[((Tf)^{**}(s) - (Tf)^{*}(s))s^{\frac{p_0-p}{pc}} \right]^q \frac{ds}{s} \right\}^{1/q} \preceq \|f\|_{X_{pq}}.$$

We may now let $p \to p_0$ and obtain the desired result. $\qquad\square$

Corollary 1. *Let $\{X_{pq}\}_{1<p\le p_0}$ be a scale of Banach spaces such that for $f \in X = \bigcap X_{pq}$ we have $\|f\|_{X_{p_0 q}} = \lim_{p\to p_0} \|f\|_{X_{pq}}$, and let T be an operator (not necessarily linear) such that*

$$T : X_{pq} \to L^{\frac{pc}{p_0-p},q}(\Omega), \ p < p_0$$

with

$$\|T\|_{X_{pq} \to L^{\frac{pc}{p_0-p},q}} \preceq (p_0-p)^{-1/q},$$

and $|\Omega| < \infty$.

  Then,

$$\|Tf\|_{Hq} \preceq \|f\|_{X_{p_0 q}}, \ f \in X,$$

where H_q was defined in (1.6) above.

Proof. This follows directly from Theorem 2 and the fact that $L(\infty,q) \subset H_q$, with (cf. [1])

$$\|f\|_{Hq} \preceq \|f\|_{L(\infty,q)}.$$

$$\square$$

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