AN ABSTRACT COIFMAN-ROCHBERG-WEISS COMMUTATOR THEOREM

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It is a special pleasure for us to dedicate this paper to you, our dear friend Dan Waterman on the occasion of your 80th birthday. But that is not all.The things we discuss here are intimately related to important work by another dear friend, and so to you too Richard Rochberg, warmest greetings on the occasion of your 65th birthday. We wish both of you many many more wonderful and creative years.

Abstract. We formulate and prove a version of the celebrated Coifman-Rochberg-Weiss commutator theorem for the real method of interpolation

1. Introduction

Commutator estimates play an important role in analysis (cf. [20]). Our starting point in this paper is the celebrated commutator theorem of Coifman-Rochberg-Weiss [5]. Let K be a Calderón-Zygmund operator, and let $b \in BMO(R^n)$. Denote by M_b the operator "multiplication by b", then (cf. [5])

$$\|[K, M_b]f\|_p \leq c\,\|b\|_{BMO}\,\|f\|_p\,, 1 < p < \infty, \tag{1.1}$$

where $[K, M_b]f = K(bf) - bK(f)$. Since each of the operators $f \to K(bf)$ and $f \to bK(f)$ is unbounded on L^p, the cancellation that results of taking their difference is essential for the validity of (1.1).

The Coifman-Rochberg-Weiss commutator theorem has found many applications in the study of PDEs, Jacobians, Harmonic Analysis, and was also the starting point of the Rochberg-Weiss [19] abstract theory of commutators in the setting of scales of interpolation spaces, which itself has had many applications (cf. [12], [13], [18], and the references therein).

It is instructive to review informally one of the proofs of (1.1) provided in [5]. Suppose that $b \in BMO$, and fix $p > 1$. Then it is well known that we can find $\varepsilon > 0$ small enough such that, for all $0 < \alpha < \varepsilon$, $e^{\alpha b}$ and $e^{-\alpha b} \in A_p$ (here A_p is the class of Muckenhoupt weights). Let K be a CZ operator, then K is bounded on the weighted spaces $L^p(e^{\alpha b}), |\alpha| < \varepsilon$. In other words, the family of operators $f \to e^{\alpha b}K(e^{-\alpha b}f)$ is uniformly bounded on L^p for $|\alpha| < \varepsilon$. It follows readily that one can extended these operators to an analytic family of operators $T(z)f = e^{zb}K(e^{-zb}f)$, for $|z| < \varepsilon$, and then show that, $\frac{d}{dz}T(z)f\big|_{z=0} = \frac{1}{2}[K, M_b]f$ is also a bounded operator on L^p. In particular, it follows that, in the statement of the theorem, we can replace CZ operators by operators T with the same weighted norm inequalities, i.e. the result holds for any operator T, such that for all weights in the A_p class of Muckenhoupt, $T : L^p(w) \to L^p(w), 1 < p < \infty$, boundedly.

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The previous argument was the starting point of the Rochberg-Weiss [19] theory of abstract commutator estimates for the complex method of interpolation, later extended to the real method by these authors jointly with Jawerth (cf. [11]). The subject has been intensively developed in the last 30 years (cf. the recent survey by Rochberg [18] and the references therein).

While the Rochberg-Weiss theory, when suitably specialized to weighted L^p spaces, can be used to re-prove the Coifman-Rochberg-Weiss commutator theorem, in this paper we consider a different problem: we give an abstract formulation of the Coifman-Rochberg-Weiss commutator theorem which is valid for interpolation scales themselves. Since we work with the real method, the cancellations will be exploited via integration by parts and a suitable re-interpretation of the relevant BMO condition.

Before we formulate our main result let us recall some basic definitions associated with the real method of interpolation (cf. [3] for more details). Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces. To define the real interpolation spaces[1] $(X_0, X_1)_{\theta,q}$ we start by considering on $X_0 \cap X_1$ the family of norms

$$J(t, x; \bar{X}) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\}, t > 0.$$

Let $\theta \in (0,1), 1 \leq q \leq \infty$. We consider the elements $f \in X_0 + X_1$, that can be represented by

$f = \int_0^\infty u(s)\frac{ds}{s}$ (crucially here the convergence of the integral is in the X_0+X_1 sense),

where $u : (0,\infty) \to X_0 \cap X_1$. We let

$$\Phi_{\theta,q}(g) = \left\{\int_0^\infty \left(s^{-\theta}|g(s)|\right)^q \frac{ds}{s}\right\}^{1/q},$$

$$\bar{X}_{\theta,q} = \{f = \int_0^\infty u(s)\frac{ds}{s} \text{ in } X_0 + X_1 : \Phi_{\theta,q}(J(s, u(s); \bar{X})) < \infty\},$$

$$\|f\|_{\bar{X}_{\theta,q}} = \inf\{\Phi_{\theta,q}(J(s, u(s); \bar{X})) : f = \int_0^\infty u(s)\frac{ds}{s} \text{ in } X_0 + X_1\}.$$

Likewise, if w is a positive function on $(0,\infty)$, we define the corresponding spaces $\bar{X}_{\theta,q,w}$ by means of the use of the function norm

$$\Phi_{\theta,q,w}(g) = \Phi_{\theta,q}(wg).$$

In this setting we consider the nonlinear operator

$$f \to u_f : (0,\infty) \to X_0 \cap X_1,$$

where u_f has been selected so that

$$f = \int_0^\infty u_f(s)\frac{ds}{s} \text{ in } X_0 + X_1, \tag{1.2}$$

and[2]

$$\Phi_{\theta,q}(J(s, u_f(s); \bar{X})) \leq 2\|f\|_{\bar{X}_{\theta,q}}.$$

We then define

$$\Omega f = \Omega_{\bar{X}} f = \int_0^\infty u_f(s)\log s\frac{ds}{s}. \tag{1.3}$$

[1] we shall only consider the $J-$method in this note.

[2] we use 2 for definitiness, obviously can replace 2 by $1+\varepsilon$.

The commutator theorem in this context (cf. [11]) states that if $T:\bar{X}\rightarrow\bar{Y}$ is a bounded linear operator, then the nonlinear operator

$$
\begin{aligned}
[T,\Omega]\,f &= T(\Omega_{\bar{X}}f)-\Omega_{\bar{Y}}(Tf) \\
&= \int_0^\infty (T(u_f(s))-u_{Tf}(s))\log s\frac{ds}{s} \qquad (1.4)
\end{aligned}
$$

is bounded,

$$\|[T,\Omega]\,f\|_{\bar{Y}_{\theta,q}}\leq c\,\|T\|_{\bar{X}\rightarrow\bar{Y}}\,\|f\|_{\bar{X}_{\theta,q}}\,.$$

One possible interpretation of the appearance of the logarithm in formula (1.3) (and hence (1.4)) can be given if we try to imitate the arguments of Coifman-Rochberg-Weiss and bring into the argument analytic functions with suitable cancellations. Indeed, if we represent the elements of $\bar{X}_{\theta_0,q}$ using the normalization $u_{\theta_0 f}(s)=s^{\theta_0}u_f(s)$, then the elements in $\bar{X}_{\theta_0,q}$ can be represented by analytic functions (with appropriate control),

$$F(z)=\int_0^\infty s^{(z-\theta_0)}(u_{\theta_0 f}(s))\frac{ds}{s},\ F(\theta_0)=f.$$

In this setting we have

$$F'(\theta_0)=\Omega f.$$

The crucial point of the cancellation argument is that, while operators represented by derivatives of analytic functions can be unbounded (since we may lose control of the norm estimates), the canonical representation of $[T,\Omega]$

$$G'(\theta_0)=[T,\Omega]\,f,$$

with

$$G(z)=\int_0^\infty s^{(z-\theta_0)}(Tu_{\theta_0 f}(s)-u_{\theta_0 Tf}(s))\frac{ds}{s},$$

exhibits the crucial cancellation

$$
\begin{aligned}
G(\theta_0) &= \int_0^\infty (Tu_{\theta_0 f}(s)-u_{\theta_0 Tf}(s))\frac{ds}{s} \\
&= Tf-Tf \\
&= 0, \qquad (1.5)
\end{aligned}
$$

which allows us to control the norm of $G'(\theta_0)$.

It is, of course, possible to eliminate all references to analytic functions, and formulate the results in terms of representations that exhibit cancellations. From this point of view the "badness" of the commutators is expressed by the fact that their canonical representations have an extra unbounded log factor (cf. (1.4)) which would lead to the weaker estimate

$$[T,\Omega]:\bar{X}_{\theta,q}\rightarrow\bar{Y}_{\theta,q,\frac{1}{(1+|\log s|)}},\ (\text{note that } \bar{Y}_{\theta,q}\subsetneqq\bar{Y}_{\theta,q,\frac{1}{(1+|\log s|)}}).$$

Here is where the cancellation (1.5), now expressed without reference to analytic functions, simply as an integral equal to zero, comes to our rescue and allows us to integrate by parts to find the "better" representation,

$$[T,\Omega]\,f=\int_0^\infty (\int_0^t (Tu_{\theta_0 f}(s)-u_{\theta_0 Tf}(s))\frac{ds}{s})\frac{ds}{s}, \qquad (1.6)$$

which leads to the correct estimate

$$[T,\Omega]:\bar{X}_{\theta,q}\rightarrow\bar{Y}_{\theta,q}.$$

This point of view was developed in [15].

To formulate the Coifman-Rochberg-Weiss theorem in our setting we give a different interpretation to the logarithm that appears in the formulae. First, for a given weight w we introduce the (possibly non linear) operators Ω_w, defined by

$$\Omega_w(f) = \int_0^\infty u_f(s)w(s)\frac{ds}{s}.$$

It follows that for $w \in L^\infty(0,\infty)$, the corresponding Ω_w is (trivially) a bounded operator,

$$\|\Omega_w(f)\|_{\bar{X}_{\theta,q}} \leq c\,\|w\|_{L^\infty}\,\|f\|_{\bar{X}_{\theta,q}}\,,$$

and therefore the corresponding commutators $[T,\Omega_w]$ are also bounded. On the other hand, for the mildly unbounded function $w(s)=\log(s)$, we have $\Omega_w=\Omega$, which is not bounded on $\bar{X}_{\theta,q}$, but for which cancellations imply the boundedness of commutators of the form $[T,\Omega]$. Now, as is well known, the logarithm is a typical example of a function with BMO behavior. Therefore we now ask more generally: for which weights w can we assert that for all bounded linear operators $T:\bar{X}\to\bar{Y}$, we have that $[T,\Omega_w]$ is a bounded operator as well? The answer to this question is what we shall call "the abstract Coifman-Rochberg-Weiss theorem."

Not surprisingly the answer is given in terms of a suitable BMO type space which allows us to control the oscillations of w. Let $Pw(t)=\frac{1}{t}\int_0^t w(s)ds$ and define

$$w^{\#}(t) = Pw(t) - w(t) = \frac{1}{t}\int_0^t w(s)ds - w(t) = \frac{1}{t}\int_0^t (w(s)-w(t))\,ds.$$

Then we consider the following analog[3] of $BMO(R_+)$ introduced in [16]:

$$W = \{w : w^{\#}(t) \in L^\infty(0,\infty)\}, \quad \text{with} \quad \|w\|_W = \|Pw - w\|_{L^\infty}.$$

There is a direct connection between W and the space $L(\infty,\infty)$ of Bennett-DeVore-Sharpley [4]:

$$w \in L(\infty,\infty) \Leftrightarrow w^* \in W,$$

where w^* denotes the non-increasing rearrangement of w. In particular, we note that, as expected, the log has bounded oscillation since

$$(\log t)^{\#} = \frac{1}{t}\int_0^t \log s ds - \log t = -1.$$

It will turn out that W is the correct way to measure oscillation in our context. In particular, we will show below that, when dealing with the commutators $[T,\Omega_w]$, the corresponding "good representation" (cf. (1.6)) is given by

$$[T,\Omega_w]\,f = \int_0^\infty (\int_0^t (Tu_f(s) - u_{Tf}(s))\frac{ds}{s})w^{\#}(s)\frac{ds}{s}.$$

The purpose of this note is to prove the following abstract analog of the Coifman-Rochberg-Weiss commutator theorem

[3]For martingales it can be explicitly shown, by means of selecting appropriate sigma fields (cf. [10]), that W is a BMO martingale space. W has also appeared before in several papers on interpolation theory (cf. [9], [2]).

Theorem 1. *Suppose that $w \in W$, and let $\bar{X}, \bar{Y}$, be Banach pairs. Then, for any bounded linear operator $T : \bar{X} \to \bar{Y}$, the commutator $[T, \Omega_w]$ is bounded, $[T, \Omega_w] : \bar{X}_{\theta,q} \to \bar{Y}_{\theta,q}$, $0 < \theta < 1, 1 \le q \le \infty$, and, moreover,*

$$\|[T, \Omega_w] f\|_{\bar{Y}_{\theta,q}} \le c \|T\|_{\bar{X} \to \bar{Y}} \|w\|_W \|f\|_{\bar{X}_{\theta,q}} .$$

We will also prove higher order versions of this result (cf. [15] and the references therein). Using the strong form of the fundamental lemma (cf. [6]) one can connect the results above with those obtained in [1] for the $K-$method, and, moreover, give explicit instances of these operators.

2. Representation Theorems

As we have indicated in the Introduction, commutator theorems can be formulated as results about special representations of certain elements in interpolation scales. To develop our program explicitly it will be necessary to integrate by parts often, so we start by collecting some elementary calculations that will be useful for that purpose.

Lemma 1. *The operator P is bounded on W.*

Proof.

$$\begin{aligned}(Pw)^{\#}(t) &= \frac{1}{t}\int_0^t Pw(s)ds - Pw(t) \\ &= \frac{1}{t}\int_0^t (Pw(s) - w(s))\, ds + Pw(t) - Pw(t).\end{aligned}$$

Therefore,

$$\left|(Pw)^{\#}(t)\right| \le \|w\|_W .$$

□

Lemma 2. *Let $w \in W$, and let $0 < \theta < 1$. Then*

$$\lim_{t\to 0} t^{\theta} w(t) = \lim_{t\to\infty} t^{-\theta} w(t) = 0.$$

Proof. Write $Pw = w^{\#} + w$, then, since $w^{\#}$ is bounded, $\lim_{t\to 0} t^{\theta} w^{\#}(t) = \lim_{t\to\infty} t^{-\theta} w^{\#}(t) = 0$, and we see that it is enough to show that $\lim_{t\to 0} t^{\theta} Pw(t) = \lim_{t\to\infty} t^{-\theta} Pw(t) = 0$. Now, from $tPw(t) = \int_0^t w(s)ds$, we get $(Pw)'(t) = -\frac{Pw(t)-w(t)}{t}$. Therefore,

$$\begin{aligned}|Pw(t)| &\le |Pw(1)| + \left|\int_t^1 w^{\#}(s)\frac{ds}{s}\right| \\ &\le \|w\|_W (1 + |\log t|).\end{aligned}$$

and the result follows. □

Although we shall not make use of the next result in this section it is convenient to state it here to stress the BMO characteristics of the space W.

Lemma 3. *(i) (cf. [1]) Let $\overline{Q}f(t) = \int_t^1 f(s)\frac{ds}{s}$ then*

$$W = L_\infty + \overline{Q}(L_\infty).$$

(ii) Let $W_1 = \left\{ w : \sup_s |sw'(s)| < \infty \right\}$. Then, $W_1 \subset W$.

(iii)

$$W = L_\infty + W_1.$$

Proof. (i) see [1].

(ii) Suppose that $w \in W_1$. Integrating by parts

$$\frac{1}{x}\int_0^x sw'(s)ds = \frac{1}{x}\; sw(s)|_{s=0}^{s=x} - \frac{1}{x}\int_0^x w(s)ds.$$

It is easy to see (cf. Lemma 2) that $\lim_{x\to 0} sw(s) = 0$, hence

$$\left|w^{\#}(x)\right| = \left|P(sw'(s))(x)\right|.$$

Consequently, since P is bounded on L^∞, it follows that $w^{\#} \in L^\infty$ and therefore $w \in W$.

(iii) Suppose that $w \in W$. Since $\left|t\,(Pw)'\right| = \left|w^{\#}(t)\right|$, it follows that $Pw \in W_1$. The desired decomposition is therefore given by

$$w = \underbrace{(w - Pw)}_{L^\infty} + \underbrace{Pw}_{W_1}.$$

□

The next result gives the representation theorem that we need to prove Theorem 1.

Theorem 2. *Let $\overline{H} = (H_0, H_1)$ be a Banach pair, and suppose that $w \in W$. Suppose that an element $f \in H_0 + H_1$ can be represented as*

$$f = \int_0^\infty u(s)w(s)\frac{ds}{s},$$

with

$$\int_0^\infty u(s)\frac{ds}{s} = 0,\ \Phi_{\theta,q}(J(t, u(t); \overline{H})) < \infty.$$

Then,

$$f \in \overline{H}_{\theta,q},$$

and, moreover,

$$\|f\|_{\overline{H}_{\theta,q}} \le c_{\theta,q}\,\|w\|_W\,\Phi_{\theta,q}(J(t, u(t); \overline{H})).$$

Proof. Write

$$\begin{aligned} f &= \int_0^\infty u(s)w(s)\frac{ds}{s} \\ &= \int_0^\infty u(s)(w(s) - Pw(s))\frac{ds}{s} + \int_0^\infty u(s)Pw(s)\frac{ds}{s} \\ &= I_1 + I_2. \end{aligned}$$

It is plain that

$$\|I_1\|_{\overline{H}_{\theta,q}} \le \|w\|_W\,\Phi_{\theta,q}(J(t, u(t); \overline{H})).$$

It remains to estimate I_2. We integrate by parts:

$$I_2 = Pw(t)\int_0^t u(s)\frac{ds}{s}\Big]_0^\infty - \int_0^\infty \left(\int_0^t u(s)\frac{ds}{s}\right)(w(t) - Pw(t))\frac{dt}{t}.$$

The integrated term vanishes. Suppose first that $q > 1$. We can write

(2.1)

$$\left\|\int_0^t u(s)\frac{ds}{s}\right\|_{H_0} \leq \int_0^t J(s,u(s))\frac{ds}{s} \leq \left(\int_0^t \left(\frac{J(s,u(s))}{s^\theta}\right)^q \frac{ds}{s}\right)^{1/q} \left(\int_0^t s^{\theta q'}\frac{ds}{s}\right)^{1/q'}$$
$$\leq c_{\theta,q}\Phi_{\theta,q}(J(t,u(t);\overline{H}))t^\theta.$$

By Lemma 1 $Pw \in W$ and therefore we may apply Lemma 2 to conclude that

$$\lim_{t\to 0}|Pw(t)|\left\|\int_0^t u(s)\frac{ds}{s}\right\|_{H_0} \leq c_{\theta,q}\lim_{t\to 0}\Phi_{\theta,q}(J(t,u(t);\overline{H}))t^\theta\,|Pw(t)|$$
$$= 0.$$

Likewise, using the cancelation condition

$$\int_0^t u(s)\frac{ds}{s} = -\int_t^\infty u(s)\frac{ds}{s}, \tag{2.2}$$

we have that

$$\left\|\int_t^\infty u(s)\frac{ds}{s}\right\|_{H_1} \leq c\Phi_{\theta,q}(J(t,u(t);\overline{H}))t^{-\theta},$$

and once again we can apply Lemma 2 and find that

$$\lim_{t\to\infty}|Pw(t)|\left\|\int_t^\infty u(s)\frac{ds}{s}\right\|_{H_1} = 0.$$

The case $q = 1$ is simpler. For example, instead of using Holder's inequality in (2.1) we write

$$\int_0^t J(s,u(s))\frac{ds}{s} = \frac{t^\theta}{t^\theta}\int_0^t J(s,u(s))\frac{ds}{s} \leq t^\theta\int_0^t \frac{J(s,u(s))}{s^\theta}\frac{ds}{s}.$$

It remains to estimate the $\overline{H}_{\theta,q}$ norm of $I_2 = \int_0^\infty \left(\int_0^t u(s)\frac{ds}{s}\right)(w(t) - Pw(t))\frac{dt}{t}$. By definition,

$$\|I_2\|_{\overline{H}_{\theta,q}} \leq \Phi_{\theta,q}(J(t)), \tag{2.3}$$

where

$$J(t) = J(t, \left(\int_0^t u(s)\frac{ds}{s}\right)(w(t) - Pw(t)); \overline{H})$$
$$\leq \|w\|_W\left(\left\|\int_0^t u(s)\frac{ds}{s}\right\|_{H_0} + t\left\|\int_0^t u(s)\frac{ds}{s}\right\|_{H_1}\right).$$

The first term on the right hand side can be estimated directly by Minkowski's inequality

$$\left\|\int_0^t u(s)\frac{ds}{s}\right\|_{H_0} \leq \int_0^t J(s,u(s);\overline{H})\frac{ds}{s},$$

while for the second we argue that, by (2.2),

$$t\left\|\int_0^t u(s)\frac{ds}{s}\right\|_{H_1} = t\left\|\int_t^\infty u(s)\frac{ds}{s}\right\|_{H_1}$$
$$\leq t\int_t^\infty J(s,u(s);\overline{H})\frac{ds}{s^2}.$$

Altogether, we arrive at

$$J(t) \leq \|w\|_W \left(\int_0^t J(s, u(s); \overline{H}) \frac{ds}{s} + t \int_t^\infty J(s, u(s); \overline{H}) \frac{ds}{s^2} \right).$$

Therefore, applying the $\Phi_{\theta,q}$ norm on both sides of the previous inequality and then using Hardy's inequalities to estimate the right hand side, we get

$$\Phi_{\theta,q}(J(t)) \leq c_{\theta,q} \|w\|_W \, \Phi_{\theta,q} \left(J(t, u(t); \overline{H}) \right). \tag{2.4}$$

Combining (2.4) and (2.3)

$$\|I_2\|_{\overline{H}_{\theta,q}} \leq c_{\theta,q} \|w\|_W \, \Phi_{\theta,q}(J(t)),$$

and collecting the estimates for I_1 and I_2 we finally obtain

$$\|f\|_{\overline{H}_{\theta,q}} \leq c_{\theta,q} \|w\|_W \, \Phi_{\theta,q}(J(t, u(t); \overline{H}))$$

as we wished to show. □

We are now ready for the proof of Theorem 1.

Proof. Suppose that T is a given bounded linear operator $T : \bar{X} \to \bar{Y}$, and let $w \in W$. Let $\tilde{u}(t) = ((u_{Tf}(t) - T(u_f(t)))$. Then

$$[T, \Omega_w] \, f = \int_0^\infty \tilde{u}(t) w(t) \frac{dt}{t}$$

with

$$\Phi_{\theta,q}(J(t, \tilde{u}(t); \bar{Y}) \leq c \|T\|_{\bar{X} \to \bar{Y}} \|f\|_{\overline{X}_{\theta,q}}.$$

Since, moreover,

$$\int_0^\infty \tilde{u}(t) \frac{dt}{t} = 0,$$

we can apply theorem 2 to conclude that

$$\|[T, \Omega_w] \, f\|_{\overline{Y}_{\theta,q}} \leq c \|w\|_W \|T\|_{\bar{X} \to \bar{Y}} \|f\|_{\overline{X}_{\theta,q}},$$

as we wished to show. □

3. Higher order cancellations

We adapt the analysis of [15] to handle higher order cancellations. The corresponding higher order commutator theorems that follow will be stated and proved in the next section.

Theorem 3. *Let $\overline{H}$ be a Banach pair, and let $w \in W$. Suppose that f admits a representation*

$$f = \int_0^\infty u(s) \left(Pw(s) \right)^2 \frac{ds}{s},$$

with

$$\int_0^\infty u(s) \frac{ds}{s} = 0, \quad \int_0^\infty u(s) Pw(s) \frac{ds}{s} = 0; \quad \Phi_{\theta,q}(J(t, u(t); \overline{H})) < \infty$$

then,

$$f \in \overline{H}_{\theta,q},$$

and, moreover,

$$\|f\|_{\overline{H}_{\theta,q}} \leq c \|w\|_W^2 \, \Phi_{\theta,q}(J(t, u(t); \overline{H})).$$

Proof. We will integrate by parts repeatedly. We start writing

$$f=\int_0^\infty u(t)\left(Pw(t)\right)^2\frac{dt}{t}=\int_0^\infty Pw(t)d\left(\int_0^t u(s)Pw(s)\frac{ds}{s}\right).$$

Then,

$$f=Pw(t)\int_0^t u(s)Pw(s)\frac{ds}{s}\Big]_0^\infty-\int_0^\infty\left(\int_0^t u(s)Pw(s)\frac{ds}{s}\right)(w(t)-Pw(t))\frac{dt}{t},$$

we will show below that the integrated term vanishes, then

$$f=-\int_0^\infty\left(\int_0^t u(s)Pw(s)\frac{ds}{s}\right)(w(t)-Pw(t))\frac{dt}{t}. \tag{3.1}$$

Now we consider the inner integral and integrate by parts

$$\int_0^t u(s)Pw(s)\frac{ds}{s}=\int_0^t Pw(s)d\left(\int_0^s u(r)\frac{dr}{r}\right),$$

using the fact that (cf. the proof of Theorem 2) $\lim\limits_{t\to0}Pw(t)\int_0^t u(s)\frac{ds}{s}=0$, we get

$$\int_0^t u(s)Pw(s)\frac{ds}{s}=Pw(t)\int_0^t u(s)\frac{ds}{s}-\int_0^t\left(\int_0^r u(s)\frac{ds}{s}\right)(w(r)-Pw(r))\,\frac{dr}{r}.$$

Inserting this result back in (3.1) we find that

$$\begin{aligned}f&=-\int_0^\infty\left(\int_0^t u(s)Pw(s)\frac{ds}{s}\right)(w(t)-Pw(t))\frac{dt}{t}\\&=\int_0^\infty\left(Pw(t)\int_0^t u(s)\frac{ds}{s}\right)w^{\#}(t)\frac{dt}{t}+\int_0^\infty\left(\int_0^t\left(\int_0^r u(s)\frac{ds}{s}\right)w^{\#}(r)\frac{dr}{r}\right)w^{\#}(t)\frac{dt}{t}\\&=I_0+I_1.\end{aligned}$$

Integrating by parts I_0 we get

$$\begin{aligned}I_0=&\;Pw(t)\int_0^t\left(w^{\#}(r)\int_0^r u(s)\frac{ds}{s}\right)\frac{dr}{r}\Big|_0^\infty\\&+\int_0^\infty\left(\int_0^t\left(\int_0^r u(s)\frac{ds}{s}\right)w^{\#}(r)\frac{dr}{r}\right)w^{\#}(t)\frac{dt}{t},\end{aligned}$$

where once again the integrated term vanishes. Hence,

$$I_0=I_1.$$

Therefore, if we let $U(t)=2\left(\int_0^t\left(\int_0^r u(s)\frac{ds}{s}\right)w^{\#}(r)\frac{dr}{r}\right)w^{\#}(t)$, f can be represented by

$$f=\int_0^\infty U(t)\frac{dt}{t}.$$

Now we estimate the corresponding $J-$functional, $J(t)=J(t,U(t);\bar{H})$, by

$$\begin{aligned}&2\left\|w\right\|_W\left(\left\|\int_0^t\left(\int_0^r u(s)\frac{ds}{s}\right)w^{\#}(r)\frac{dr}{r}\right\|_{H_0}+t\left\|\int_0^t\left(\int_0^r u(s)\frac{ds}{s}\right)w^{\#}(r)\frac{dr}{r}\right\|_{H_1}\right)\\&=2\left\|w\right\|_W(C_0+tC_1).\end{aligned}$$

We readily see that C_0 is majorized by

$$C_0\le\|w\|_W\int_0^t\left(\int_0^r J(s,u(s);\overline{H})\frac{ds}{s}\right)\frac{dr}{r}=\|w\|_W\int_0^t J(r,u(r);\overline{H})\ln\frac{t}{r}\frac{dr}{r}.$$

To handle C_1 we work with the integral inside the norm H_1 by first using $\int_0^r u(s)\frac{ds}{s} = -\int_r^\infty u(s)\frac{ds}{s}$ and then changing the order of integration. We find that

$$C_1 = \left\| \lim_{\alpha\to 0} C(\alpha) \right\|_{H_1},$$

where $C(\alpha) = \int_0^t \int_\alpha^s w^{\#}(r)\frac{dr}{r}u(s)\frac{ds}{s} + \int_t^\infty \int_\alpha^t w^{\#}(r)\frac{dr}{r}u(s)\frac{ds}{s}$. We compute $C(\alpha)$ using the formula $(Pw)'(t) = -\frac{w^{\#}(t)}{t}$, and we get

$$C(\alpha) = Pw(\alpha)\int_0^t u(s)\frac{ds}{s} - \int_0^t Pw(s)u(s)\frac{ds}{s} + Pw(\alpha)\int_t^\infty u(s)\frac{ds}{s} - Pw(t)\int_t^\infty u(s)\frac{ds}{s}.$$

Now by the cancellation conditions:

$$\int_0^\infty u(s)\frac{ds}{s} = 0 \Longrightarrow Pw(\alpha)\int_0^t u(s)\frac{ds}{s} = -Pw(\alpha)\int_t^\infty u(s)\frac{ds}{s},$$

and

$$\int_0^\infty u(s)Pw(s)\frac{ds}{s} = 0 \Longrightarrow \int_0^t u(s)Pw(s)\frac{ds}{s} = -\int_t^\infty u(s)Pw(s)\frac{ds}{s},$$

we have

$$\begin{aligned} C(\alpha) &= \int_t^\infty u(s)[Pw(s) - Pw(t)]\frac{ds}{s} \\ &= \int_t^\infty u(s)\int_t^s w^{\#}(r)\frac{dr}{r}\frac{ds}{s}. \end{aligned}$$

All in all it follows that,

$$\begin{aligned} C_1 &\le \|w\|_W \int_t^\infty \|u(s)\|_{H_1} \ln\frac{s}{t}\frac{ds}{s} \\ &\le \|w\|_W \int_t^\infty J(s, u(s); \overline{H}) \ln\frac{s}{t}\frac{ds}{s^2}. \end{aligned}$$

Summarizing,

$$J(t) \le 2\|w\|_W^2 \left(\int_0^t J(r, u(r); \overline{H}) \ln\frac{t}{r}\frac{dr}{r} + t\int_t^\infty J(r, u(r); \overline{H}) \ln\frac{r}{t}\frac{dr}{r^2} \right).$$

Applying the $\Phi_{\theta,q}$ norm and Hardy's inequalities (twice) we finally obtain

$$\begin{aligned} \|f\|_{\overline{H}_{\theta,q}} &\le c\Phi_{\theta,q}\left(J(t, u(t); \overline{H})\right) \\ &\le c\|w\|_W^2 \Phi_{\theta,q}\left(J(t, u(t); \overline{H})\right). \end{aligned}$$

To conclude the proof it remains to verify that the integrated terms we have collected along the way effectively vanish. More precisely, it remains to prove that

$$\lim_{t\to\xi} Pw(t)\int_0^t u(s)Pw(s)\frac{ds}{s} = 0, \text{ for } \xi = 0, \infty, \tag{3.2}$$

and

$$\lim_{t\to\xi} Pw(t)\int_0^t \left((w(r) - Pw(r))\int_0^r u(s)\frac{ds}{s} \right)\frac{dr}{r} = 0, \text{ for } \xi = 0, \infty. \tag{3.3}$$

To handle these limits we shall assume that $q > 1$, the case $q = 1$ is easier (cf. the proof of Theorem 2 above). We start with (3.2):

$$\left\| \int_0^t u(s)Pw(s)\frac{ds}{s} \right\|_{H_0} \leq \int_0^t J(s,u(s);\overline{H})\,|Pw(s)|\,\frac{ds}{s}$$

$$\leq \left(\int_0^t \left(\frac{J(s,u(s);\overline{H})}{s^\theta} \right)^q \frac{ds}{s} \right)^{1/q} \left(\int_0^t \left(s^\theta\,|Pw(s)| \right)^{q'} \frac{ds}{s} \right)^{1/q'}$$

$$\leq \left(\Phi_{\theta,q}(J(s,u(s);\overline{H})) \right) c \left(\int_0^t \left(s^\theta\,|w(s)| \right)^{q'} \frac{ds}{s} \right)^{1/q'} \quad \text{(by Hardy's inequality)}$$

Let $\widetilde{\theta} > 0$ be such that $\theta - \widetilde{\theta} > 0$. Since $w \in W \Rightarrow Pw \in W$ (cf. Lemma 1), therefore, by Lemma 2, we have

$$\left| t^{\widetilde{\theta}} Pw(t) \right| \leq 1 \quad \text{(if } t \text{ suff. close to 0).}$$

Thus, for small t,

$$\left(\int_0^t \left(s^\theta\,|Pw(s)| \right)^{q'} \frac{ds}{s} \right)^{1/q'} \leq \left(\int_0^t \left(s^{\theta-\widetilde{\theta}} \right)^{q'} \frac{ds}{s} \right)^{1/q'} \leq ct^{\theta-\widetilde{\theta}},$$

and

$$\lim_{t\to 0} \left\| Pw(t) \int_0^t u(s)w(s)\frac{ds}{s} \right\|_{H_0} \leq \lim_{t\to 0} ct^{\theta-\widetilde{\theta}}\,|Pw(t)| = 0.$$

The corresponding limit when $t \to \infty$ can be handled by the same argument if we first use the cancellation property $\int_0^t u(s)Pw(s)\frac{ds}{s} = -\int_t^\infty u(s)Pw(s)\frac{ds}{s}$ and then apply the H_1 norm.

To see (3.3) we note that

$$|Pw(t)| \left\| \int_0^t (w(r) - Pw(r)) \int_0^r u(s)\frac{ds}{s}\frac{dr}{r} \right\|_{H_0}$$

$$\leq \|w\|_W\,|Pw(t)| \int_0^t J(s,u(s);\overline{H}) \ln\frac{t}{s}\frac{ds}{s}$$

$$\leq \|w\|_W\,|Pw(t)|\,t^\theta \left(\Phi_{\theta,q}(J(s,u(s);\overline{H})) \right) t^{-\theta} \left(\int_0^t \left(s^\theta \ln\frac{t}{s} \right)^{q'} \frac{ds}{s} \right)^{1/q'}.$$

Now, the term on the right hand side converges to zero when $t \to 0$ by Lemma 2 and the fact that near zero, $t^{-\theta}\left(\int_0^t \left(s^\theta \ln\frac{t}{s} \right)^{q'} \frac{ds}{s} \right)^{1/q'} \leq t^{-\theta}\left(\int_0^t \left(s^\theta \frac{s}{t} \right)^{q'} \frac{ds}{s} \right)^{1/q'} \leq ct^{-\theta}t^{-1}t^{1+\theta}$. Again the case $t \to \infty$ is reduced to the case $t \to 0$ by a familiar argument using cancellations. □

Corollary 1. *Let $\overline{H}$ be a Banach pair, and let $w \in W$. Suppose that*

$$f = \int_0^\infty u(s)\,(w(s))^2\,\frac{ds}{s},$$

with

$$\int_0^\infty u(s)\frac{ds}{s} = 0,\ \int_0^\infty u(s)w(s)\frac{ds}{s} = 0,\ \int_0^\infty u(s)Pw(s)\frac{ds}{s} = 0; \tag{3.4}$$

and

$$\Phi_{\theta,q}(J(t,u(t);\overline{H})) < \infty.$$

Then,

$$f \in \overline{H}_{\theta,q},$$

and, moreover,

$$\|f\|_{\overline{H}_{\theta,q}} \leq c\,\|w\|_W^2\,\Phi_{\theta,q}(J(t,u(t);\overline{H})).$$

Proof. Write

$$\int_0^\infty u(s)\,(w(s))^2\,\frac{ds}{s} = \int_0^\infty u(s)\,(w(s)-Pw(s))\,w(s)\frac{ds}{s} + \int_0^\infty u(s)w(s)Pw(s)\frac{ds}{s}.$$

Since $w(t)Pw(t) = \frac{(w(t))^2+(Pw(t))^2-(w(t)-Pw(t))^2}{2}$, we have

$$\int_0^\infty u(s)\,(w(s))^2\,\frac{ds}{s} = 2\int_0^\infty u(s)\,(w(s)-Pw(s))\,w(s)\frac{ds}{s} - \int_0^\infty u(s)\,(w(s)-Pw(s))^2\,\frac{ds}{s} + \int_0^\infty u(s)\,(Pw(s))^2\,\frac{ds}{s}$$

We now show how to control each of these terms. Let $\tilde{u}(s) = u(s)\,(w(s)-Pw(s))$, by the cancellation conditions (3.4) it follows that $\int_0^\infty \tilde{u}(s)\frac{ds}{s} = 0$. Therefore we can apply Theorem 2 to conclude that $\int_0^\infty \tilde{u}(s)w(s)\frac{ds}{s} \in \overline{H}_{\theta,q}$. It follows that

$$\left\|2\int_0^\infty u(s)\,(w(s)-Pw(s))\,w(s)\frac{ds}{s}\right\|_{\overline{H}_{\theta,q}} \leq c\,\|w\|_W^2\,\Phi_{\theta,q}(J(t,u(t);\overline{H})).$$

The second term is also under control since $(w(s)-Pw(s))^2$ is bounded. Finally we may apply Theorem 3 to control the remaining term. □

Theorem 4. *Let $\overline{H}$ be a Banach pair, and let $w_0, w_1 \in W$. Suppose that*

$$f = \int_0^\infty u(s)w_0(s)w_1(s)\frac{ds}{s},$$

with

$$\int_0^\infty u(s)\frac{ds}{s} = 0, \quad \int_0^\infty u(s)w_j(s)\frac{ds}{s} = 0, \quad \int_0^\infty u(s)Pw_j(s)\frac{ds}{s} = 0 \quad (j=0,1)$$

and

$$\Phi_{\theta,q}(J(t,u(t);\overline{H})) < \infty.$$

Then,

$$f \in \overline{H}_{\theta,q}$$

and, moreover,

$$\|f\|_{\overline{H}_{\theta,q}} \leq c\max\{\|w_0\|_W, \|w_1\|_W\}^2\Phi_{\theta,q}(J(t,u(t);\overline{H})).$$

Proof. Write

$$w_0(s)w_1(s) = \frac{(w_0(s)+w_1(s))^2 - w_0(s)^2 - w_1(s)^2}{2}$$

and apply Corollary 1. □

For $n > 2$ we proceed by induction and we obtain

Theorem 5. *Let $\overline{H}$ be a Banach pair, and let $w \in W$.*

(i) Suppose that

$$f = \int_0^\infty u(s) \left(Pw(s)\right)^n \frac{ds}{s},$$

with

$$\int_0^\infty u(s)w(s)^k \frac{ds}{s} = 0, \quad \int_0^\infty u(s)Pw(s)^k \frac{ds}{s} = 0, \quad (k = 0, \cdots, n-1);$$

and

$$\Phi_{\theta,q}(J(t, u(t); \overline{H})) < \infty.$$

Then,

$$f \in \overline{H}_{\theta,q},$$

and, moreover,

$$\|f\|_{\overline{H}_{\theta,q}} \leq c\Phi_{\theta,q}(J(t, u(t); \overline{H})).$$

(ii) If

$$f = \int_0^\infty u(s) \left(w(s)\right)^n \frac{ds}{s},$$

with

$$\int_0^\infty u(s)w(s)^k \frac{ds}{s} = 0, \quad \int_0^\infty u(s)Pw(s)^k \frac{ds}{s} = 0, \quad \int_0^\infty u(s)w(s)^{n-k}Pw(s)^k \frac{ds}{s} = 0, \quad ,(k = 0, \cdots, n-1);$$

and

$$\Phi_{\theta,q}(J(t, u(t); \overline{H})) < \infty,$$

then,

$$f \in \overline{H}_{\theta,q},$$

and, moreover,

$$\|a\|_{\overline{H}_{\theta,q}} \leq c \left\|w\right\|_W^n \Phi_{\theta,q}(J(t, u(t); \overline{H})).$$

Remark 1. *In the classical case (cf.* [15], *theorem 3) $w(t) = \ln t$, and therefore $Pw(t) = \ln t - 1$. Consequently the conditions*

$$\int_0^\infty u(s)Pw(s)^k \frac{ds}{s} = 0, \int_0^\infty u(s)w(s)^{n-k}Pw(s)^k \frac{ds}{s} = 0, \ (k = 0, \cdots, n-1);$$

actually follow from

$$\int_0^\infty u(s) \left(w(s)\right)^k \frac{ds}{s} = 0, \ (k = 0, \cdots, n-1).$$

4. Higher order commutators

We consider higher order commutators defined as follows (cf. [15], [1], [18]). Let $\bar{X}$ and $\bar{Y}$ be Banach pairs, and let $T : \bar{X} \to \bar{Y}$ be a bounded linear operator. Given a nearly optimal representation (cf. 1.2 above)

$$f = \int_0^\infty u_f(s) \frac{ds}{s}$$

we let

$$\Omega_{n,w} f = \frac{1}{n!} \int_0^\infty u_f(s)(w(s))^n \frac{ds}{s}, \ n = 0, 1, ...$$

and form the commutators

$$C_{n,w}f=\begin{cases} Tf & ,\ n=0\\ [T,\Omega_{1,w}]\,f & ,\ n=1\\ [T,\Omega_{2,w}]\,f-\Omega_{1,w}(C_{1,w}f) & ,\ n=2\\ \dots\dots\dots\dots & \\ [T,\Omega_{n,w}]\,f-\Omega_{1,w}(C_{n-1,w}f)-\cdots\Omega_{n-1,w}(C_{1,w}f) & \end{cases}$$

Observe that the commutators $[T,\Omega_{n,w}]$ alone are not bounded and we need to form more complicated expressions like $C_{n,w}$ in order to produce the necessary cancellations. Moreover, since the operations $\Omega_{j,w}$ are not linear, simple minded iterations of the form $\Omega_{1,w}\,[T,\Omega_{1,w}]-[T,\Omega_{1,w}]\,\Omega_{1,w}$, *etc*, cannot be treated directly using Theorem 1.

Theorem 6. *Suppose that $w\in W$. Then the commutators $C_{n,w}$ are bounded, $C_{n,w}$: $\bar{X}_{\theta,q}\to\bar{Y}_{\theta,q}$, $0<\theta<1, 1\leq q\leq\infty$, and, moreover, for each instance $g=w$, or $g=Pw$, we have*

$$\|C_{n,w}f\|_{\bar{Y}_{\theta,q}}\leq c\,\|T\|_{\bar{X}\to\bar{Y}}\,\|w\|_{W}^{n}\,\|f\|_{\bar{X}_{\theta,q}}\,.$$

Proof. We only consider in detail the case $n=2$. Writing $w=(w-Pw)+Pw$, we see that we only need to deal with the commutator $C_{2,Pw}$. Let

$$u(s)=T(u_f(s))-u_{T(f)}(s)$$

then

$$C_{2,Pw}(Tf)=\frac{1}{2}\int_0^\infty u(t)(Pw(t))^2\frac{dt}{t}-\int_0^\infty \widetilde{u}(t)Pw(t)\frac{dt}{t},$$

with

$$\int_0^\infty \widetilde{u}(t)\frac{dt}{t}=\int_0^\infty u(t)Pw(t)\frac{dt}{t};\int_0^\infty u(t)\frac{dt}{t}=0,$$

and

$$\Phi_{\theta,q}(J(t,\widetilde{u}(t),\overline{X}))\leq c\,\|w\|_W\,\|f\|_{\bar{X}_{\theta,q}}$$

$$\Phi_{\theta,q}(J(t,u(t),\overline{X}))\leq c\,\|w\|_W\,\|f\|_{\bar{X}_{\theta,q}}$$

Since

$$\begin{aligned}\frac{1}{2}\int_0^\infty u(t)(Pw(t))^2\frac{dt}{t}&=\frac{1}{2}\int_0^\infty (Pw(t))^2 d\left(\int_0^t u(s)\frac{ds}{s}\right)\\ &=\int_0^\infty\left(\int_0^t u(s)\frac{ds}{s}\right)Pw(t)w^{\#}(t)\frac{dt}{t},\end{aligned}$$

it follows that if we let

$$v(t)=(\int_0^t u(s)\frac{ds}{s})w^{\#}(t)$$

then

$$C_{2,Pw}(Tf)=\int_0^\infty (v(t)-\widetilde{u}(t))Pw(t)\frac{dt}{t},$$

and

$$\int_0^\infty (v(t)-\widetilde{u}(t))\frac{dt}{t}=0.$$

then theorem 2 implies that

$$\begin{aligned}\|C_{2,Pw}(Tf)\|_{\bar{Y}_{\theta,q}} &\leq c\,\|w\|_W\;\Phi_{\theta,q}(J(t,u(t);\bar{X}))+c\Phi_{\theta,q}(J(t,\widetilde{u}(t);\bar{X}))\\ &\leq c\,\|w\|_W^2\,\|f\|_{\bar{X}_{\theta,q}}.\end{aligned}$$

as we wished to show. □

5. Comparison with earlier results and some questions

This paper was originally conceived in 1999-2000, when the first named author spent one year in the Tropics. So publication was delayed somewhat and in the mean time several papers on the subject have appeared. In particular, [17] has similar statements framed in terms of weights of the form

$$w(t)=\phi(\log t),\ \text{with } \phi \text{ Lipchitz.} \tag{5.1}$$

One recognizes that these weights are included in our theory since for w of the form (5.1) we have (cf. Lemma 3 above)

$$\|w\|_{W_1}=\sup|tw'(t)|=\|\phi'\|_\infty<\infty.$$

There is also a connection with [1] (a longer version of this paper was originally circulated in 1996 (cf. [2])). These papers emphasize the connection between weighted norm inequalities, commutators and BMO type conditions using the $K-$method, and BMO conditions are formulated in terms of properties of weights. Recall that for the $K-$method of interpolation we define the corresponding Ω operations by

$$\Omega^K f=\int_0^1 x_0(t)\frac{dt}{t}-\int_1^\infty x_1(t)\frac{dt}{t},$$

or, more generally, by

$$\Omega^K_w f=\int_0^1 x_0(t)w(t)\frac{dt}{t}-\int_1^\infty x_1(t)w(t)\frac{dt}{t},$$

where

$$f=x_0(t)+x_1(t),\ \text{and}\ \|x_0(t)\|_{H_0}+t\,\|x_1(t)\|_{H_0}\leq cK(t,f;\bar{H}).$$

Using the strong form of the fundamental lemma of interpolation theory (cf. [6]) we can arrange to have $f=\int_0^\infty u_f(s)\frac{ds}{s}$, and

$$x_0(t)=\int_0^t u_f(s)\frac{ds}{s},x_1(t)=\int_t^\infty u_f(s)\frac{ds}{s}.$$

It formally follows that

$$\Omega^K_w f=-\Omega_{Gw}f,$$

where

$$Gw(s)=\int_1^s w(r)\frac{dr}{r}.$$

In particular, if $w=1$, then $Gw(s)=\log s$. Also note that

$$\sup_s|s(Gw)'(s)|=\|w\|_\infty.$$

Now a brief attempt to informally connect our work with Dan Waterman's classical Fourier analysis. One source of inspiration for the formulation of some of the results in this paper comes from the Littlewood-Paley theory, framed in terms of semigroups, e.g. as developed in Stein [21]. In the abstract theory of Stein [21] (cf.

[21] pag 121) the relevant semigroups are represented, using the spectral theorem, by

$$T^t = \int_0^\infty e^{-\lambda t} dE(\lambda),$$

and one considers (multiplier) operators of the form

$$T_w f = \int_0^\infty e^{-\lambda t} w(t) dE(\lambda) f,$$

with $w \in L^\infty$. The conclusion is that the operator $T_{(Lw)'}$ is bounded on $L^p, 1 < p < \infty$, where

$$Lw(\lambda) = \int_0^\infty e^{-\lambda t} w(t) dt.$$

We hope to come back to explore this subject elsewhere.

We conclude with a few suggestions for future explorations on related topics.

T1. One can formulate iterations of the operation # (cf. [9]) and ask for its relevance in the study of higher order commutators.

T2. Despite several results (cf. [7], [2], [18]) one feels that the duality theory associated to the Ω operators is still not well developed. In particular, in [2] a predual H of the space W is constructed but the consequences have not been explored.

T3. Incidentally we note that the duality theory for the interpolation spaces introduced in [8] has not been studied.

T4. Compactness is a natural issue that has not been considered so far in abstract theory of commutators. For example, it is an important known result that commutators of CZO and functions in VMO generate compact operators on L^p (cf. [22]). We believe that the framework proposed in this paper could be useful to formulate the corresponding abstract result. In particular, one can define an appropriate analog of VMO...

T5. In connection with T3 and T4 it would be of interest to study compactness (weak compactness) in the abstract setting of [14] using the ideas in this paper.

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