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# Isoperimetric Hardy type and Poincaré inequalities on metric spaces

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**Abstract** We give a general construction of manifolds for which Hardy type operators characterize Poincaré inequalities. We also show a class of spaces where this property fails. As an application we extend recent results of E. Milman to our setting.

## 1 Introduction

While working on sharp Sobolev-Poincaré inequalities in the classical Euclidean setting (cf. [17]) as well as the Gaussian setting (cf. [14]), we observed that the symmetrization methods we were developing could be readily extended to the more general setting of metric spaces (cf. [6], [14], [15], [16]). However, in the metric setting we found that we could not always decide if the results we had obtained were "sharp" or best possible.

Indeed, generally speaking, the methods that we use to show sharpness require the construction of special rearrangements and thus our spaces need to exhibit sufficient symmetries. In fact, in all the examples where we know how to prove sharpness, the "winning" rearrangements are those that are somehow connected with the solution of the underlying isoperimetric problems (e.g. the symmetric decreasing rearrangements in the Euclidean case, which are associated with balls (cf. [17]), while in the Gaussian case one uses special rearrangements associated with half spaces (cf. [8], [14]) and, likewise, in the

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more general model cases of log concave measures (cf. [5], [7] and the more recent [1], [15], [16]). In particular, these special rearrangements allow us to show that there exist “special symmetrizations that do not increase the norm of the gradient”, i.e. that a suitable version of the Pólya-Szegö principle holds.

In preparation for a systematic study we observed that, in all the model cases we could treat, a key role was played by the boundedness of certain Hardy operators, which we termed “isoperimetric Hardy operators”. This led us to isolate the concept of “isoperimetric Hardy type spaces”. This property can be formulated in very general metric spaces and can be applied, if we have estimates on the isoperimetric profiles. By formulating the problem in this fashion, while we may lose information about best constants, we gain the possibility of obtaining positive results that would be hard to obtain by other methods.

In this note we continue this program and we address the question: which metric spaces are of “Hardy isoperimetric type”? On the positive side we show, using ideas of Ros [26], how to construct a class of metric spaces of “Hardy isoperimetric type” that contains all the model cases mentioned before. Therefore this construction provides us with a large class of spaces where our inequalities are sharp.

As another application we continue the discussion of the connection between our results and the recent work of E. Milman ([23], [22], [24]), who has shown the equivalence, under convexity assumptions, of certain estimates for isoperimetric profiles. In [16] we extended and simplified E. Milman’s results to the setting of metric spaces of isoperimetric Hardy type. The construction presented in this paper thus gives a general concrete class of model spaces where Milman’s equivalences hold.

Finally on the negative side we also construct spaces that do not satisfy the “isoperimetric Hardy type condition.”

In the spirit of this book we now comment briefly on the influence of Maz’ya’s work in our development. Underlying the equivalences of Theorem 0.1 below are two deep insights due to Maz’ya: Maz’ya’s fundamental result showing the equivalence between the Gagliardo-Nirenberg inequality and the isoperimetric inequality (cf. [18] and also [9]), and Maz’ya’s technique of showing self improvement of Sobolev’s inequality via smooth cut-offs (cf. [20]). Indeed, one of the themes of Theorem 0.1 is to develop the explicit connection of these two ideas using pointwise symmetrization inequalities (“symmetrization by truncation” cf. [17], [14]). Another theme of our method is that we formulate our inequalities incorporating directly geometric information, an idea that one can also already find in Maz’ya’s fundamental work characterizing Sobolev inequalities in rough domains [20] as well as in Maz’ya’s method characterizing Sobolev-Poincaré inequalities via isocapacitory inequalities[1] (cf. [21]).

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[1] A detailed discussion of the connection between Maz’ya’s isocapacitory inequalities and symmetrization inequalities is given in [16].

As this outline shows, and we hope the rest of the paper proves, our methods owe a great deal to the pioneering work of Professor Maz'ya and we are grateful and honored by the opportunity to contribute to this book.

## 2 Background

Let $(\Omega, d, \mu)$ be a metric probability space equipped with a separable Borel probability measure $\mu$. Let $A \subset \Omega$ be a Borel set, then the boundary measure or **Minkowski content** of $A$ is by definition

$$\mu^{+}(A) = \liminf_{h\to 0} \frac{\mu\left(A_h\right) - \mu\left(A\right)}{h},$$

where $A_h = \{x \in \Omega : d(x, y) < h\}$ denotes the $h$-neighborhood of $A$.

The **isoperimetric profile** $I_{(\Omega,d,\mu)}$ is defined as the pointwise maximal function $I_{(\Omega,d,\mu)} : [0, 1] \to [0, \infty)$ such that

$$\mu^{+}(A) \geqslant I_{(\Omega,d,\mu)}(\mu(A)),$$

holds for all Borel sets $A$.

**Condition:** We shall assume throughout that our metric spaces have isoperimetric profile functions $I_{(\Omega,d,\mu)}$ which are: continuous, concave, increasing on $(0, 1/2)$, symmetric about the point $1/2$, and vanish at cero[2].

A continuous function $I : [0, 1] \to [0, \infty)$ , with $I(0) = 0$, concave, increasing on $(0, 1/2)$ and symmetric about the point $1/2$, and such that

$$I \geqslant I_{(\Omega,d,\mu)},$$

will be called an **isoperimetric estimator** on $(\Omega, d, \mu)$.

For measurable functions $u : \Omega \to \mathbb{R}$, the distribution function of $u$ is given by

$$\lambda_u(t) = \mu\{x \in \Omega : |u(x)| > t\} \qquad (t > 0).$$

The **decreasing rearrangement** $u^*$ of $u$ is defined, as usual, by

$$u^*_\mu(s) = \inf\{t \geqslant 0 : \lambda_u(t) \leqslant s\} \qquad (t \in (0, \mu(\Omega)]),$$

and we let

$$u^{**}_\mu(t) = \frac{1}{t}\int_0^t u^*_\mu(s)ds.$$

[2] For a large class of examples where these assumptions are satisfied we refer to [6] [23], and the references therein.

Given a a locally Lipschitz real function, $f$ defined on $(\Omega, d)$ (we shall write in what follows $f \in Lip(\Omega)$), the **modulus of the gradient** of $f$ is defined, by

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0} \frac{|f(x)-f(y)|}{d(x,y)},$$

and zero at isolated points[3].

A Banach function space $X = X(\Omega)$ on $(\Omega, d, \mu)$ is called a rearrangement-invariant (r.i.) space, if $g \in X$ implies that all $\mu-$measurable functions $f$ with the same rearrangement function with respect to the measure $\mu$, i.e. such that $f_\mu^* = g_\mu^*$, also belong to $X$, and, moreover, $\|f\|_X = \|g\|_X$. An r.i. space $X(\Omega)$ can be represented by a r.i. space $\overline{X} = \overline{X}(0,1)$ on the interval $(0,1)$, with Lebesgue measure, such that

$$\|f\|_X = \|f_\mu^*\|_{\overline{X}},$$

for every $f \in X$. Typical examples of r.i. spaces are the $L^p$-spaces, Lorentz spaces and Orlicz spaces. For more information we refer to [4].

In our recent work on symmetrization of Sobolev inequalities we showed the following general theorem (cf. [15], [16] and the references therein)

**Theorem 0.1.** *Let $I : [0,1] \to [0,\infty)$ be an isoperimetric estimator on $(\Omega, d, \mu)$. The following statements hold and are in fact equivalent:*

*1. Isoperimetric inequality:*

$$\forall A \subset \Omega, \text{ Borel set, } \mu^+(A) \geqslant I(\mu(A)).$$

*2. Ledoux's inequality:*

$$\forall f \in Lip(\Omega), \quad \int_0^\infty I(\lambda_f(s))ds \leqslant \int_\Omega |\nabla f(x)|\, d\mu(x).$$

*3. Maz'ya's inequality[4]:*

$$\forall f \in Lip(\Omega), \ (-f_\mu^*)'(s)I(s) \leqslant \frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}} |\nabla f(x)|\, d\mu(x).$$

*4. Pólya-Szegö's inequality*

$$\forall f \in Lip(\Omega), \quad \int_0^t ((-f_\mu^*)'(.)I(.))^*(s)ds \leqslant \int_0^t |\nabla f|_\mu^*(s)ds.$$

*(The second rearrangement on the left hand side is with respect to the Lebesgue measure).*

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[3] In fact it is enough in order to define $|\nabla f|$ that $f$ will be Lipschitz on every ball in $(\Omega, d)$ cf. [6, pp. 184,189] for more details.

[4] See [19], one can also find this inequality in [27], [28].

*5. Oscillation inequality:*

$$\forall f \in Lip(\Omega),\ (f_{\mu}^{**}(t) - f_{\mu}^{*}(t)) \leqslant \frac{t}{I(t)} \left|\nabla f\right|_{\mu}^{**}(t). \tag{2.1}$$

Given any rearrangement invariant space $X(\Omega)$, it follows readily from (2.1) that for all $f \in Lip(\Omega)$, we have

$$\|f\|_{LS(X)} := \left\| \left(f_{\mu}^{**}(t) - f_{\mu}^{*}(t)\right) \frac{I(t)}{t} \right\|_{\overline{X}} \preceq \|\nabla f\|_{X}.$$

One salient characteristic of these spaces is that they explicitly incorporate in their definition the isoperimetric profiles associated with the geometry in question and thus they can automatically select the correct optimal spaces for different geometries (for more on this see [14], [15], [16]). While the $LS(X)$ spaces are not necessarily normed, often they are equivalent to normed spaces (cf. [25]), and, in the classical cases, lead to optimal Sobolev-Poincaré inequalities and embeddings (cf. [17], [13], [14] as well as [3], [2], [29] and the references therein).

## 3 Hardy isoperimetric type

Let $Q_I$ be the operator defined on measurable functions on $(0,1)$ by

$$Q_I f(t) = \int_t^1 f(s) \frac{ds}{I(s)},$$

where $I$ is an isoperimetric estimator. We consider the possibility of completely characterizing Poincaré inequalities in terms of the of the boundedness of $Q_I$ as an operator from $\overline{X}$ to $\overline{Y}$.

In order to motivate what follows we recall the following result, obtained in [15], [16], for classical settings see [14].

**Theorem 0.2.** *Let $X, Y$ be two r.i. spaces on $\Omega$. Suppose that there exists a constant $c = c(X,Y)$ such that for every positive function $f \in \overline{X}$, with $suppf \subset (0,1/2)$,*

$$\|Q_I f(t)\|_{\overline{Y}} \leqslant c \left\|f_{\mu}^{*}\right\|_{\overline{X}}.$$

*Then, for all $g \in Lip(\Omega)$*[5]

$$\left\| g - \int_{\Omega} g d\mu \right\|_{Y} \preceq \|\nabla g\|_{X}. \tag{3.1}$$

[5] We note for future use that Poincaré inequalities can be equivalently formulated replacing $\int_{\Omega} g d\mu$ by a median value $m$ of $g$, i.e. $\mu(g \geqslant m) \geqslant 1/2$ and $\mu(g \leqslant m) \geqslant 1/2$.

*Furthermore, if the space* $\overline{X}$ *is such that* $\left\| f_\mu^* \right\|_{\overline{X}} \simeq \left\| f_\mu^{**} \right\|_{\overline{X}}$, *then,*

$$\|f\|_Y \preceq \|f\|_{LS(X)} + \|f\|_{L^1}.$$

*In fact,* $LS(X)$ *is an optimal space in the sense that if (3.1) holds, then for all* $g \in Lip(\Omega)$ *we have*

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| g - \int_\Omega g d\mu \right\|_{LS(X)} \preceq \|\nabla g\|_X.$$

We give a simple, but non trivial example, that illustrates how the preceeding developments allow us to transplant Sobolev-Poincaré inequalities to the metric setting.

*Example 0.1.* Suppose that $(\Omega, \mu)$ has an isoperimetric estimator

$$I(s) \simeq s^{1-1/n}, \quad (0 < s < 1/2).$$

It follows that on functions supported on $(0, 1/2)$,

$$Q_I f(t) \simeq \int_t^{1/2} s^{1/n} f(t) \frac{ds}{s}.$$

Since the conditions for the boundedness $Q_I$ on r.i. spaces are well understood, we can transplant the classical Sobolev inequalities to $(\Omega, \mu)$. Furthermore, we note that in the borderline case $q = n$, the corresponding result using the optimal $LS(L^n)$ spaces is sharper than the classical Sobolev theorems (cf. [2]).

As mentioned before it is known that the converse to Theorem 0.2 is true in a number of important classical cases, in other words the operator $Q_I$ in those cases gives a complete characterization of the Poincaré inequalities (for the most recent results cf. [15], [16]).

This led us to introduce the following condition

**Definition 0.1.** We shall say that a probability metric space $(\Omega, d, \mu)$ is of isoperimetric Hardy type if for any given isoperimetric estimator $I$, the following are equivalent for all r.i. spaces $X = X(\Omega)$, $Y = Y(\Omega)$ :
1. There exists $c = c(X, Y)$ such that

$$\forall f \in Lip(\Omega), \quad \left\| f - \int_\Omega f d\mu \right\|_Y \leqslant c \|\nabla f\|_X. \tag{3.2}$$

2. There exists $c = c(X, Y)$ such that

$$\|Q_I f\|_{\overline{Y}} \preceq \|f\|_{\overline{X}}, \quad f \in \overline{X}, \text{ with } supp(f) \subset (0, 1/2).$$

## 4 Model Riemannian manifolds

In this section we construct a class of spaces of isoperimetric Hardy type spaces that includes the $n-$sphere $\mathbb{S}^n$, $(\mathbb{R}^n, \gamma_n)$ ($\mathbb{R}^n$ with Gaussian measure) and symmetric log-concave probability measures on $\mathbb{R}$.

We follow a construction of Ros (cf. [26]). Let $M_0$ a complete smooth oriented $n_0$-dimensional Riemannian manifold with distance $d$. An absolutely continuous probability measure $\mu_0$ w.r. to $dV$ in $M_0$, will be called a **model measure**, if there exists a continuous family (in the sense of the Hausdorff distance on compact subsets) $\mathcal{D} = \{D^t : 0 \leqslant t \leqslant 1\}$ of closed subsets of $M_0$ satisfying the following conditions:

1. $\mu_0(D^t) = t$ and $D^s \subset D^t$, for $0 \leqslant s < t \leqslant 1$,
2. $D^t$ is a smooth isoperimetric domain of $\mu_0$ and $I_{\mu_0}(t) = \mu_0^+(D^t)$ is positive and smooth for $0 < t < 1$, where $I_{\mu_0}$ denotes the isoperimetric profile of $M_0$,
3. The $r$-enlargement of $D^t$, defined by $(D^t)_r = \{x \in M_0 : d(x, D^t) \leqslant r\}$ verifies $(D^t)_r = D^s$ for some $s = s(t, r)$, $0 \leqslant t \leqslant 1$, and
4. $D^1 = M_0$ and $D^0$ is either a point or the empty set.

**Theorem 0.3.** *Let $(M_0, d)$ be an $n_0$-dimensional Riemannian manifold endowed with a model measure $\mu_0$. Then $(M_0, d)$ is of isoperimetric Hardy type.*

*Proof.* Consider the function defined by

$$\begin{aligned} p : M_0 &\to [0,1] \\ x \in \partial D^t &\to t. \end{aligned}$$

Let $x, y \in M_0$ be such that $0 < p(y) < p(x)$. Let $D \in \mathcal{D}$ such that $y \notin D$. Consider the function $h(r) = \mu_0(D_r)$, which is continuous and smooth for $0 < h(r) < 1$ and, in this range (see [26]),

$$h'(r) = I_{\mu_0}(h(r)). \tag{4.1}$$

From the definition of $p$, it follows that $p(x) = h(d(x, D))$ and $p(y) = h(d(y, D))$. Since $d(x, D) - d(y, D) \leqslant d(x, y)$, we see that

$$\frac{p(x) - p(y)}{d(x, y)} \leqslant \frac{h(d(x, D)) - h(d(y, D))}{d(x, D) - d(y, D)} \leqslant \sup_s h'(s)$$

i.e. $p \in Lip(M_0)$,and $|\nabla p(x)| = \limsup_{y \to x} \left| \frac{p(x)-p(y)}{d(x,y)} \right|$ is finite, it follows that $|\nabla p(x)|$ exists a.e. w.r. to $dV$ (cf. [6, page 2]) and hence a.e. w.r. $\mu_0$. Let us now compute $|\nabla p|$. Given $x \in M_0$ such that $p(x) = t < 1$, let $D \in \mathcal{D}$ so that $x \notin D$, and as before consider the function $h(r) = \mu_0(D_r)$. Let $z(x) \in M_0$ be such that

$$d(x, D) = d(x, z(x)).$$

Select $y_n$ on the geodesic that joints $z(x)$ and $x$ such that $y_n \to x$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{p(x)-p(y_n)}{d(x,y_n)}\right| &= \lim_{n\to\infty}\left|\frac{h(d(x,D))-h(d(y_n,D))}{d(x,D)-d(y_n,D)}\right| = h'(d(x,D)) \quad (4.2)\\ &= I_{\mu_0}(h(d(x,D))) \quad \text{(by (4.1))}\\ &= I_{\mu_0}(p(x)).\end{aligned}$$

Let $f \in \overline{X}$, be a positive function with supp$f \subset (0,1/2)$, and define

$$F(x) = \int_{p(x)}^{1} f(s)\frac{ds}{I_{\mu_0}(s)}.$$

Obviously, $F \in Lip(M_0)$ and by (4.2),

$$|\nabla F(x)| = f(p(x))\frac{1}{I_{\mu_0}(p(x))}\,|\nabla p(x)| = f(p(x)), \text{ a.e..}$$

We claim that the map $p : (M_0, \mu_0) \to ([0,1], ds)$ is a measure-preserving transformation. To prove this claim we need to see that for any measurable subset $R \subset [0,1]$,

$$\mu_0\left(p^{-1}(R)\right) = \int_R ds. \quad (4.3)$$

It is enough to see (4.3) for a closed interval. Let $[a,b] \subset [0,1]$ $(0 \leqslant a < b \leqslant 1)$ then

$$\mu_0\left(p^{-1}([a,b])\right) = \mu_0(D^b) - \mu_0(D^a) = b - a,$$

Using this claim (see [4, Proposition 7.2, page 80]) then a.e. we have

$$|F|_{\mu_0}^{*}(s) = \int_t^1 f(s)\frac{ds}{I_{\mu_0}(s)} \quad \text{and} \quad |\nabla F|_{\mu_0}^{*}(s) = f^{*}(s).$$

Obviously, condition (3.2) is equivalent to

$$\|u - m\|_Y \preceq \|\nabla u\|_X,$$

where $m$ is a median[6] of $f$, now since $\mu_0(F = 0) \geqslant 1/2$, 0 is a median of $F$, and from

$$\|F - 0\|_Y \preceq \|\nabla F\|_X$$

we obtain

$$\left\|\int_t^1 f(s)\frac{ds}{I_{\mu_0}(s)}\right\|_{\overline{Y}} \preceq \|f\|_{\overline{X}}$$

as we wished to show. □

[6] i.e. $\mu_0(f \geqslant m) \geqslant 1/2$ and $\mu_0(f \leqslant m) \geqslant 1/2$.

## 5 E. Milman's equivalence theorems

In the next result we have a list of progressively weaker statements that nevertheless have been shown by E. Milman (cf. [23], [22], [24]) to be equivalent under certain convexity assumptions. Likewise E. Milman also has formulated similar results in the context of Orlicz spaces.

In [16] we have simplified and extended Milman's results to the context of metric spaces with Hardy isoperimetric type, as well as considering general r.i. spaces.

**Theorem 0.4.** *Let $(\Omega, d, \mu)$ be a space of Hardy isoperimetric type. Then the following statements are equivalent*
*(E1) Cheeger's inequality*

$$\exists C > 0 \ s.t. \quad I_{(\Omega,d,\mu)} \geqslant Ct, \quad t \in (0, 1/2].$$

*(E2) Poincaré inequality*

$$\exists P > 0 \ s.t. \quad \|f - m\|_{L^2(\Omega)} \leqslant P \|f\|_{L^2(\Omega)} .$$

*(E3) Exponential concentration: for all $f \in Lip(\Omega)$ with $\|f\|_{Lip(\Omega)} \leqslant 1$,*

$$\exists c_1, c_2 > 0 \ s.t. \quad \mu\{|f - m| > t\} \leqslant c_1 e^{-c_2 t}, \quad t \in (0, 1).$$

*(E4) First moment inequality: for all $f \in Lip(\Omega)$ with $\|f\|_{Lip(\Omega)} \leqslant 1$,*

$$\exists F > 0 \ s.t. \quad \|f - m\|_{L^1(\Omega)} \leqslant F.$$

**Theorem 0.5.** *Let $(\Omega, d, \mu)$ be a space of isoperimetric Hardy type. Let $1 \leqslant q \leqslant \infty$, and let $N$ be a Young's function such that $\frac{N(t)^{1/q}}{t}$ is non-decreasing and there exists $\alpha > \max\{\frac{1}{q} - \frac{1}{2}, 0\}$, such that $\frac{N(t^\alpha)}{t}$ non-increasing. Then the following statements are equivalent,*
*(E5) $(L_N, L^q)$ Poincaré inequality holds*

$$\exists P > 0 \ s.t. \quad \|f - m\|_{L_N(\Omega)} \leqslant P \|f\|_{L^q(\Omega)} .$$

*(E6) Any isoperimetric profile estimator $I$ satisfies: there exists a constant $c > 0$ such that*

$$I(t) \geqslant c \frac{t^{1-1/q}}{N^{-1}(1/t)}, \quad t \in (0, 1/2].$$

The construction of the previous section thus provides a class of spaces were the previous theorems apply.

## 6 Some spaces that are not of isoperimetric Hardy type

In this section we show that, unfortunately, not all metric spaces are of isoperimetric Hardy type.

Let $I : [0,1] \to [0,\infty)$ be concave, continuous, increasing on $(0,1/2)$, symmetric about the point $1/2$, and such that such that $I(0)=0$. Let $0 \leqslant \beta \leqslant 1$. We shall say that $I$ has $\beta-$asymptotic behavior if the limit $\lim_{s\to 0^+}\frac{I(s)}{s^{1-\beta}}$ exists and lies on $(0,\infty)$.

**Theorem 0.6.** *Suppose that $I$ is of $\beta-$asymptotic behavior. Then:*

*(i) Given $0<\beta<1/2$, there is a metric space $(\Omega_0,d,\mu)$ with $I(s) \simeq I_{(\Omega_0,d_0,\mu_0)}(s)$, and a pair of r.i. spaces $X,Y$ on $\Omega_0$, and a constant $c=c(X,Y)$ such that*

$$\left\| g-\int_{\Omega_0} g d\mu_0 \right\|_Y \leqslant c\left\|\nabla g\right\|_X, \quad g\in Lip(\Omega_0),$$

*but $Q_I:\overline{X}\to\overline{Y}$ is not bounded.*

*(ii) Given $0<\beta<1$, there is a metric space $(\Omega_1,d_1,\mu_1)$ such that*

$$I(s)\simeq I_{(\Omega_1,d_1,\mu_1)}(s)$$

*and $(\Omega_1,d,\mu)$ is of isoperimetric Hardy type.*

*Proof.* (i) (see [13] for a more general result) Let $1<\alpha<2$, and let $\Omega$ be an $\alpha-$John domain on $\mathbb{R}^2$, $(|\Omega|=1)$. Then (cf. [11]),

$$I_\Omega(s)\simeq s^{\alpha/2}=s^{1-(1-\alpha/2)}, \quad 0\leqslant s\leqslant 1/2.$$

Let $t>1$ be such that $\alpha>t-1$, and let $r=\frac{2t}{\alpha+(1-t)}$. Note that $1<t<r$. Then (cf. [12])

$$\left\| g-\int_{\Omega} g \right\|_{L^r} \preceq \left\|\nabla g\right\|_{L^t}.$$

In this case the operator $Q_{I_\Omega}$ is given by

$$Q_{I_\Omega}f(t)=\int_t^1 u^{-\alpha/2}f(u)du.$$

$Q_{I_\Omega}$ is not bounded from $L^t$ to $L^r$. Indeed, the boundedness of $Q_{I_\Omega}$ can be reformulated as a weighted norm inequality for the operator $g\to\int_x^1 g(u)du$, namely

$$\left\| \int_x^1 g(u)du \right\|_{L^r} \leqslant c\left\| g(x)x^{\alpha/2}\right\|_{L^t}. \tag{6.1}$$

It is well known that (6.1) holds iff (cf. [20])

$$\sup_{a>0}\left(\int_0^a 1\right)^{1/r}\left(\int_a^1\left(u^{\alpha t/2}\right)^{\frac{-1}{t-1}}du\right)^{\frac{t-1}{t}}<\infty. \tag{6.2}$$

Now, since $\alpha < 2$, it follows that $\frac{-\alpha t}{2(t-1)} + 1 < 0$, and for $a$ near zero we have

$$\left(\int_0^a 1\right)^{1/r} \left(\int_a^1 \left(u^{\alpha t/2}\right)^{\frac{-1}{t-1}} du\right)^{\frac{t-1}{t}} \simeq a^{1/r} \left(a^{\frac{-\alpha t+2(t-1)}{2(t-1)}} - 1\right)^{\frac{t-1}{t}}$$
$$\simeq a^{\frac{(1-t)(\alpha-1)}{2t}}.$$

Consequently, since $\frac{(1-t)(\alpha-1)}{2t} < 0$, (6.2) cannot hold.

(ii) We shall follow Gallot's method (see [10]) in order to build $(\Omega_1, d_1, \mu_1)$. Let

$$B(r) = \int_r^1 \frac{ds}{I(s)}, \quad 0 \leqslant r \leqslant 1.$$

Since $I$ is of $\beta-$asymptotic behavior we see that $L = B(0) < \infty$. Since $B$ is decreasing it has an inverse which we denote by $A$. Consider the revolution surface $M = (0, L) \times \mathbb{S}^1$ (compactified by adjoining the points $\{0\} \times \mathbb{S}^1$ and $\{L\} \times \mathbb{S}^1$)) provided with the Riemannian metric

$$g = dr^2 + I(A(r))^2 d\theta^2,$$

where $\theta \in \mathbb{S}^1$ and $d\theta^2$ is the canonical Riemannian metric on $\left(\mathbb{S}^1, can\right)$. Notice that $I(A(0)) = I(A(L)) = 0$. We denote the volume of $(M, g)$ by $Vol_M$, and multiplying the metric $g$ by a constant, we can and will assume without loss that $Vol_M(M) = 1$. Let us denote by $I_M$ the isoperimetric profile of $(M, g, Vol_M)$, then (cf. [10, Appendix A.1.]), we can find a constant $c$, depending only on $I$, such that

$$cI(s) \leqslant I_M(s) \leqslant I(s).$$

Let $X, Y$ be two r.i. spaces on $M$, such that.

$$\left\| g - \int_M g dVol_M \right\|_Y \preceq \|\nabla g\|_X, \quad g \in Lip(M).$$

Let $f$ be a positive Lebesgue measurable function on $(0,1)$ with $suppf \subset (0, 1/2)$. Define

$$u(r, \theta_1, \theta_2) = \int_{A(r)}^1 f(s) \frac{ds}{I(s)}, \quad (r, \theta_1, \theta_2) \in M.$$

It is plain that $u$ is a Lipschitz function on $M$ such that $Vol_M\{u = 0\} \geqslant 1/2$. Hence 0 is a median of $u$.

On the other hand, recall that (cf. [10, Page 57]) for any domain of revolution $\Omega(\lambda) = (0, \lambda) \times \mathbb{S}^1 \subset M$ we have that

$$Vol_M^+(\partial\Omega(\lambda)) = I\left(Vol_M\left(\Omega(\lambda)\right)\right).$$

In other words,

$$A'(r) = I(A(r)).$$

Therefore,

$$|\nabla u(r,\theta_1,\theta_2)| = \left|\frac{\partial}{\partial r}u(r)\right| = \left|-f(A(r))\frac{A'(r)}{I(A(r))}\right| = f(A(r)).$$

Now, since

$$u^*_{Vol_M}(t) = \int_t^1 f(s)\frac{ds}{I(s)} \quad \text{and} \quad |\nabla u|^*_{Vol_M}(t) = f^*(t),$$

from

$$\|u-0\|_Y \preceq \|\nabla u\|_X$$

we deduce that

$$\left\|\int_t^1 f(s)\frac{ds}{I(s)}\right\|_Y \preceq \|f\|_X\,,$$

as we wished to show. □

By Theorem 0.6 the verification of a Sobolev-Poincaré inequality cannot be reduced, in general, to establish the boundedness of the associated isoperimetric Hardy operator. However, if the profiles are of $\beta-$asymptotic behavior, we have the following weaker positive result:

**Theorem 0.7.** *Let $I$ be of $\beta-$asymptotic behavior ($0<\beta<1$). Let $\mathcal{M}_I$ be the set of metric probability spaces $(\Omega,d,\mu)$ such that*

$$I_{(\Omega,d,\mu)} \geqslant I.$$

*Let $\overline{X}$, $\overline{Y}$ be two r.i. spaces on $[0,1]$. Then, the following statements are equivalent*

*1.*

$$\inf_{(\Omega,d,\mu)\in\mathcal{M}_h}\ \inf_{g\in Lip(\Omega)} \frac{\left\||\nabla g|^*_\mu\right\|_{\overline{X}}}{\left\|\left(g-\int_\Omega g d\mu\right)^*_\mu\right\|_{\overline{Y}}} = c > 0$$

*2.*

$$Q_I : \overline{X} \to \overline{Y} \text{ is bounded.} \tag{6.3}$$

*Proof.* $1\to 2$) Given $I$ of $\beta-$asymptotic behavior, consider the revolution surface $M$ constructed in part (ii) of the previous Theorem. Since $M\in\mathcal{M}_I$, by hypothesis:

$$\left\|\left(g-\int_\Omega g d\mu\right)^*_\mu\right\|_{\overline{Y}} \leqslant c\left\|(\nabla g)^*_\mu\right\|_{\overline{X}}\ , \quad g\in Lip(M),$$

which is equivalent to (6.3) since $M$ is of Hardy isoperimetric type.

$2 \to 1$) Is a direct consequence of Theorem 0.2. □

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## References

1. Barthe, F., Cattiaux, P., Roberto, C.: Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. Rev. Mat. Iberoam. **22**, 993-1067 (2006)
2. Bastero. J., Milman, M., Ruiz, F.: A note on $L(\infty, q)$ spaces and Sobolev embeddings. Indiana Univ. Math. J. **52**, 1215–1230 (2003)
3. Bennett, C., DeVore, R., Sharpley, R.: Weak $L^\infty$ and $BMO$. Ann. of Math. **113**, 601-611 (1981)
4. Bennett, C., Sharpley, R.: Interpolation of Operators, Academic Press, Boston, (1988)
5. Bobkov, S.G.: Extremal properties of half-spaces for log-concave distributions. Ann. Probab. **24** , 35-48 (1996)
6. Bobkov, S.G., Houdré, C.: Some connections between isoperimetric and Sobolev-type inequalities. Mem. Amer. Math. Soc. **129**, no. 616 (1997)
7. Borell, C.: Intrinsic bounds on some real-valued stationary random functions. Lecture Notes in Math. **1153**, 72-95 (1985)
8. Ehrhard, A.: Symétrisation dans le space de Gauss. Math. Scand. **53**, 281-301 (1983)
9. Federer, H,: Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, **153**, Springer-Verlag, (1969)
10. Gallot, S.: Inégalités isopérimétriques et analytiques sur les variétés riemanniennes. Astérisque. **163–164**, 31-91 (1988)
11. Hajlasz, P., Koskela, P.: Isoperimetric inequalities and Imbedding theorems in irregular domains. J. London Math. Soc. **58**, 425-450 (1998)
12. Kilpeläinen, T., Maly, J.: Sobolev inequalities on sets with irregular boundaries. Z. Anal. Anwendungen. **19**, 369-380 (2000)
13. Martín J., Milman, M.: Self improving Sobolev-Poincare inequalities, truncation and symmetrization. Potential Anal. **29**, 391-408 (2008)
14. Martín J., Milman, M.: Isoperimetry and Symmetrization for Logarithmic Sobolev inequalities. J. Funct. Anal. **256**, 149–178 (2009)
15. Martín J., Milman, M.: Isoperimetry and symmetrization for Sobolev spaces on metric spaces. Comptes Rendus Math. **347**, 627–630 (2009)
16. Martín J., Milman, M.: Pointwise Symmetrization Inequalities for Sobolev functions and applications. Preprint (2009)
17. Martín, J., Milman, M., Pustylnik, E.: Sobolev inequalities: Symmetrization and self-improvement via truncation. J. Funct. Anal. **252**, 677-695 (2007)
18. Maz'ya, V.G.: Classes of regions and imbedding theorems for function spaces. Soviet Math. Dokl. **1**, 882-885 (1960)
19. Maz'ya, V.G.: On weak solutions of the Dirichlet and Neumann problems. Trans. Moscow Math. Soc. **20**, 135-172 (1969)
20. Maz'ya, V.G.: Sobolev Spaces. Springer-Verlag, Berlin–Tokyo (1985)
21. Maz'ya, V.G.: Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev type imbeddings. J. Funct. Anal. **224**, 408-430 (2005)

22. Milman, E.: Concentration and isoperimetry are equivalent assuming curvature lower bound. C. R. Math. Acad. Sci. Paris. **347**, 73-76 (2009)
23. Milman, E.: On the role of Convexity in Isoperimetry, Spectral-Gap and Concentration. Invent. Math. **177**, 1-43 (2009)
24. Milman, E.: On the role of convexity in functional and isoperimetric inequalities. Proc. London Math. Soc. **99**, 32-66 (2009)
25. Pustylnik, E.: A rearrangement-invariant function set that appears in optimal Sobolev embeddings. J. Math. Anal. Appl. **344**, 788-798 (2008)
26. Ros, A.: The isoperimetric problem. In: Global Theory of Minimal Surfaces. Clay Math. Proc., vol. 2, pp. 175-209, Am. Math. Soc., Providence (2005)
27. Talenti, G,: Elliptic Equations and Rearrangements, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3**, 697-718 (1976)
28. Talenti, G,: Inequalities in rearrangement-invariant function spaces, in: Nonlinear Analysis, Function Spaces and Applications, vol. 5, Prometheus, Prague, 177-230 (1995)
29. Tartar, L.: Imbedding theorems of Sobolev spaces into Lorentz spaces. Boll. Unione Mat. Ital. Sez B Artic. Ric. Mat. **8**, 479-500 (1998)