This is the accepted versión of : Montolio, P. and Rifà i Coma, J. "Hadamard $Z_2Z_4Q_8$-codes : rank and kernel" in Coding Theory and Applications, vol. 3, 2015 , p. 273-279 (CIM Series in Mathematical Sciencies). The final publication is available at Springer via 10.1007/978-3-319-17296-5_29

Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. Rank and Kernel

P. Montolio and J. Rifà

Abstract Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes are Hadamard binary codes coming from a subgroup of the direct product of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Q_8 groups, where Q_8 is the quaternionic group. We characterize Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes as a quotient of a semidirect product of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and we show that all these codes can be represented in a standard form, from a set of generators. On the other hand, we show that there exist Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes with any given pair of allowable parameters for the rank and dimension of the kernel.

Key words: Dimension of the kernel, error-correcting codes, Hadamard codes, rank, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes.

1 Introduction

Non-linear codes with a group structure (like $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes) have received a great deal of attention since [4]. The codes in this paper can be characterized as the image of a subgroup, by a suitable Gray map, of the direct product of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Q_8, the quaternionic group of order 8 [8, 2].

Hadamard matrices with a subjacent algebraic structure have been deeply studied, as well as the links with other topics in algebraic combinatorics or applications [5]. We quote just a few papers about this subject [6, 3, 1], where we can find beautiful equivalences between Hadamard groups, 2-cocyclic matrices and relative difference sets. On the other hand, from a coding theory point of view, it is desirable that

P. Montolio
Computing, Multimedia and Telecommunication Studies, Universitat Oberta de Catalunya, e-mail: pma@uoc.edu

J. Rifà
Department of Information and Communications Engineering, Universitat Autònoma de Barcelona, e-mail: josep.rifa@uab.cat

the algebraic structures we are dealing with preserves the Hamming distance. This is the case, for example, of the $\mathbb{Z}_2\mathbb{Z}_4$-linear codes which has been intensively studied during the last years [4]. More generally, the propelinear codes and, specially those which are translation invariant, are particularly interesting because the subjacent group structure has the property that both, left and right product, preserve the Hamming distance. Translation invariant propelinear codes has been characterized as the image of a subgroup by a suitable Gray map of a direct product of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and Q_8 [8].

In this paper we analyze codes that have both properties, being Hadamard and $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. These codes were previously studied and classified [2] in five shapes. The aim of this paper is to go further. First of all by giving an standard form for a set of generators of the code, depending on the parameters, which helps to understand of the characteristics of each shape and then by putting the focus in an exact computation of the values of the rank and dimension of the kernel. One of the main results of this paper is to characterize the $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes as a quotient of a semidirect product of Hadamard $\mathbb{Z}2\mathbb{Z}_4$-linear codes. The second main result is to construct, using the above characterization, Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes whose values for the rank and dimension of the kernel are any allowable pair previously chosen.

The structure of the paper is as follows. Section 2 introduces the notation and preliminary concepts; Section 3 shows the standard form of generators that allows to represent any Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code in a unique way, this section finishes with two important theorems which characterizes a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code as a quotient of a semidirect product of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes (Theorems 1 and 2). Finally, in Section 4 we give the constructions of $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes fulfilling the requirements for the prefixed values of the dimension of the kernel and rank. We finish the last section with a couple of examples about the constructions and achievement of codes with each allowable pair of values for the rank and dimension of the kernel.

2 Preliminaries

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ denote the binary field and the ring of integers modulo 4, respectively. Any non-empty subset of $\mathbb{Z}_2^n$ is called a binary code and a linear subspace of $\mathbb{Z}_2^n$ is called a *binary linear code* or a $\mathbb{Z}_2$-*linear code*. Let $\mathrm{wt}(v)$ denote the *Hamming weight* of a vector $v \in \mathbb{Z}_2^n$ (i.e., the number of its nonzero components), and let $d(v,u) = \mathrm{wt}(v+u)$, the *Hamming distance* between two vectors $v, u \in \mathbb{Z}_2^n$.

Let Q_8 be the *quaternionic group* on eight elements. The following equalities provides a presentation and the list of elements of Q_8:

$$Q_8 = \langle \mathbf{a}, \mathbf{b} : \mathbf{a}^4 = \mathbf{a}^2\mathbf{b}^2 = \mathbf{1}, \mathbf{bab}^{-1} = \mathbf{a}^{-1} \rangle = \{\mathbf{1}, \mathbf{a}, \mathbf{a}^2, \mathbf{a}^3, \mathbf{b}, \mathbf{ab}, \mathbf{a}^2\mathbf{b}, \mathbf{a}^3\mathbf{b}\}.$$

Given three non-negative integers k_1, k_2 and k_3, denote as $\mathscr{G}$ the group $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$. Any element of $\mathscr{G}$ can be represented as a vector where the first k_1 components belong to $\mathbb{Z}_2$, the next k_2 components belong to $\mathbb{Z}_4$ and the last k_3 components belong to Q_8.

We use the multiplicative notation for $\mathscr{G}$ and we denote by $\mathbf{e}$ the identity element of the group and by $\mathbf{u}$ the element with all components of order two. Hence, $\mathbf{e} = (0,\overset{k_1+k_2}{\ldots},0,\mathbf{1},\overset{k_3}{\ldots},\mathbf{1})$ and $\mathbf{u} = (1,\overset{k_1}{\ldots},1,2,\overset{k_2}{\ldots},2,\mathbf{a}^2,\overset{k_3}{\ldots},\mathbf{a}^2)$.

We call *Gray map* the function Φ:

$$\Phi:\ \mathbb{Z}_2^{k_1}\times\mathbb{Z}_4^{k_2}\times Q_8^{k_3}\ \longrightarrow\ \mathbb{Z}_2^{k_1+2k_2+4k_3},$$

acting componentwise in such a way that over the binary part is the identity, over the quaternary part acts as the usual Gray map, so $0\to(00)$, $1\to(01)$, $2\to(11)$, $3\to(10)$ and over the quaternionic part acts in the following way [2]: $\mathbf{1}\to(0,0,0,0)$, $\mathbf{b}\to(0,1,1,0)$, $\mathbf{a}\to(0,1,0,1)$, $\mathbf{ab}\to(1,1,0,0)$, $\mathbf{a}^2\to(1,1,1,1)$, $\mathbf{a}^2\mathbf{b}\to(1,0,0,1)$, $\mathbf{a}^3\to(1,0,1,0)$, $\mathbf{a}^3\mathbf{b}\to(0,0,1,1)$.

Note that $\Phi(\mathbf{e})$ is the all-zeros vector and $\Phi(\mathbf{u})$ is the all-ones vector.

Let $\mathscr{C}$ be a subgroup of $\mathbb{Z}_2^{k_1}\times\mathbb{Z}_4^{k_2}\times Q_8^{k_3}$. Binary codes $C=\Phi(\mathscr{C})$ are called $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. In the specific case $k_3=0$, code C is called $\mathbb{Z}_2\mathbb{Z}_4$-linear. In this last case, note that $\mathscr{C}$ is isomorphic to $=\mathbb{Z}_2^{\gamma}\times\mathbb{Z}_4^{\delta}\subset\mathbb{Z}_2^{k_1}\times\mathbb{Z}_4^{k_2}$. We will say that $\mathscr{C}$ is of type $2^{\gamma}4^{\delta}$ [4].

We are interested in Hadamard binary codes $C=\Phi(\mathscr{C})$ where $\mathscr{C}$ is a subgroup of $\mathscr{G}=\mathbb{Z}_2^{k_1}\times\mathbb{Z}_4^{k_2}\times Q_8^{k_3}$ of length $n=2^m$. All through the paper we are assuming it.

The *kernel* of a binary code C of length n is $K(C)=\{z\in\mathbb{Z}_2^n : C+z=C\}$. The dimension of $K(C)$ is denoted by $k(C)$ or simply k. The *rank* of a binary code C is the dimension of the linear span of C. It is denoted by $r(C)$ or simply r.

A *Hadamard matrix* of order n is a matrix of size $n\times n$ with entries ±1, such that $HH^T=nI$. Any two rows (columns) of a Hadamard matrix agree in precisely $n/2$ components. If $n>2$ then any three rows (columns) agree in precisely $n/4$ components. Thus, if $n>2$ and there is a Hadamard matrix of order n then n is multiple of 4.

Two *Hadamard matrices* are *equivalent* if one can be obtained from the other by permuting rows and/or columns and multiplying rows and/or columns by -1. With the last operations we can change the first row and column of H into $+1$'s and we obtain an equivalent Hadamard matrix which is called normalized. If $+1$'s are replaced by 0's and -1's by 1's, the initial Hadamard matrix is changed into a (binary) Hadamard matrix and, from now on, we will refer to it when we deal with Hadamard matrices. The binary code consisting of the rows of a (binary) Hadamard matrix and their complements is called a (binary) *Hadamard code*, which is of length n, with $2n$ codewords, and minimum distance $n/2$.

Let $C=\Phi(\mathscr{C})$ be a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code of length 2^m. Set $|T(\mathscr{C})|=2^{\sigma}$, $|Z(\mathscr{C})/T(\mathscr{C})|=2^{\delta}$ and $|\mathscr{C}/Z(\mathscr{C})|=2^{\rho}$, where $T(\mathscr{C})$ is the subgroup of elements of order two, $Z(\mathscr{C})$ is the center of $\mathscr{C}$. Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes were studied in [2] and classified in five different shapes based on the parameters σ,δ,ρ.

There are two important tools that has been used in the technical proofs of the statements throughout this paper, the commutator and the swapper.

Two elements a and b of $\mathscr{C}$ commutes if and only if $ab = ba$. As an extension of this concept, the *commutator* of a and b is defined as the element $[a,b]$ such that $ab = [a,b]ba$. Note that all commutators belong to $T(\mathscr{C})$ and any element of $T(\mathscr{C})$ commutes with all elements of $\mathscr{C}$.

We say that two elements a and b of $\mathscr{C}$ *swap* if and only if $\Phi(ab) = \Phi(a)+\Phi(b)$. As an extension of this concept, define the *swapper* of a and b as the element $(a\colon b)$ such that $\Phi((a\colon b)ab) = \Phi(a)+\Phi(b)$. Note that all swappers belong to $T(\mathscr{G})$ but they can be out of $\mathscr{C}$. In other words, for any element a of $\mathscr{C}$ we have $\Phi(a) \in K(C)$ if and only if $(a\colon b) \in \mathscr{C}$, for every $b \in \mathscr{C}$. Moreover, the linear span of C can be seen as $\Phi(\langle \mathscr{C} \cup S(\mathscr{C})\rangle)$, where $\langle \mathscr{C} \cup S(\mathscr{C})\rangle$ is the group generated by $\mathscr{C}$ and $S(\mathscr{C})$, the set of swappers of the elements in $\mathscr{C}$.

3 The standard form for the generator set of a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code

In this section, starting from a given a $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code we construct a standard generator set, which allow to characterize it.

Proposition 1. *Let $\mathscr{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $C = \Phi(\mathscr{C})$ is a Hadamard code. We can always construct a standard set of generators $\{x_1,\dots,x_\sigma;$ $r_1,\dots,r_\tau;$ s_1,s_υ $\}$ of $\mathscr{C}$ such that:*

- *The elements x_i are of order two and generate $T(\mathscr{C})$.*
- *The elements r_i are of order four and commute with each other, $[r_i,r_j] = \mathbf{e}$ for every $1 \le i,j \le \tau$. When $\mathbf{u} \in \langle r_1 \dots r_\tau\rangle$ we will take $\mathbf{u} = r_1^2$ and we have $r_1^2 = \mathbf{u} \notin \langle r_2^2 ... r_\tau^2\rangle$.*
- *The cardinal υ of the set $\{s_1,s_\upsilon\}$ is in $\{0,1,2\}$ and when $\upsilon = 2$ we have $s_1^2 = \mathbf{u} \neq s_2^2$, and $[s_1,s_2] = \mathbf{e}$. Moreover, when $r_1^2 = s_1^2 = \mathbf{u}$ then $[r_1,s_1] = \mathbf{u}$.*
- *Any element $c \in \mathscr{C}$ can be written in a unique way as*

$$c = \prod_{i=1}^{\sigma} x_i^{a_i} \prod_{j=1}^{\tau} r_j^{b_j} \prod_{k=1}^{\upsilon} s_k^{c_k}, \text{ where } a_i,b_j,c_k \in \{0,1\}.$$

The next theorem shows that a subgroup $\mathscr{C}$, such that $\phi(\mathscr{C})$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code, has an abelian maximal subgroup $\mathscr{A}$ which is normal in $\mathscr{C}$ and $\mathscr{C}/\mathscr{A}$ is an abelian group of order 2^a, for $a \in \{0,1,2\}$.

Theorem 1. *Let $\mathscr{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $\phi(\mathscr{C}) = C$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code. Then $\mathscr{C}$ has an abelian maximal subgroup $\mathscr{A}$ which is normal in $\mathscr{C}$ and $|\mathscr{C}/\mathscr{A}| \in \{1,2,4\}$. Futher, $\mathscr{C}$ may be expressed as a quotient of a semidirect product of $\mathscr{A}$.*

The next result characterizes the maximal abelian subgroup $\mathscr{A}$ and, since Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are well known [7], it will make possible the construction of all Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes.

Theorem 2. *Let $\mathscr{C}$ be a subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$ such that $\phi(\mathscr{C}) = C$ is a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code and $\mathscr{A}$ the abelian maximal subgroup in $\mathscr{C}$. Then $\phi(\mathscr{A})$ can be described as a duplication of a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code when $\upsilon = 1$ or as a quadruplication of a Hadamard $\mathbb{Z}_4$-linear code, if $\upsilon = 2$.*

Depending on the values of the parameters σ, τ, υ the Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes are classified in several shapes, as we can see in Table 1. In fact there are two big classes of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. Despite all codes contains the all one vector $\mathbf{u}$, there are codes where there exist an element r_1 such that $r_1^2 = \mathbf{u}$ (codes of shape 1^*, 2, 4^* and 5) and there are codes where $\mathbf{u}$ is not the square of any other element (codes of shape 1, 3 and 4). We will define the new parameter $\overline{\tau} = \tau - 1$ in the first case ($r_1^2 = \mathbf{u}$) and $\overline{\tau} = \tau$ in the second case ($r_1^2 \neq \mathbf{u}$). The existence conditions for Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes easily come from Theorem 2 and [7], where it was stated the existence conditions for Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

Table 1 summarizes what we have done in this section.

Table 1 Existence conditions and parameters k_1, k_2, k_3 depending on the shape of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes of length $n = 2^m$, where $m = \sigma + \tau + \upsilon - 1$. For all starred shapes $r_1^2 = \mathbf{u}, \overline{\tau} = \tau - 1$ and for all non-starred shapes $r_1^2 \neq \mathbf{u}, \overline{\tau} = \tau$.

	$\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times Q_8^{k_3}$				
shape	k_1	k_2	k_3	$\mathscr{C}$	existence
1^*	0	$2^{\sigma+\tau-2}$	0	$\mathscr{A}$	$\forall \tau \leq \lfloor \frac{m+1}{2} \rfloor$; $\sigma = m - \tau + 1$
1	$2^{\sigma-1}$	$(2^\tau - 1)2^{\sigma-2}$	0	$\mathscr{A}$	$\forall \tau \leq \lfloor \frac{m}{2} \rfloor$; $\sigma = m - \tau + 1$
2	0	0	$2^{\sigma+\tau-2}$	$\mathscr{A} \rtimes \mathbb{Z}_4 / (\mathbf{u}, s_1^2)$	$\forall \tau \leq \lfloor \frac{m}{2} \rfloor$; $\sigma = m - \tau$
3	0	$2^{\sigma-1}$	$(2^\tau - 1)2^{\sigma-2}$	$\mathscr{A} \rtimes \mathbb{Z}_4 / (\mathbf{u}, s_1^2)$	$\forall \tau \leq \lfloor \frac{m-1}{2} \rfloor$; $\sigma = m - \tau$
4	$2^{\sigma-1}$	0	$2^{\sigma-3}$	$\mathscr{A} \rtimes \mathbb{Z}_4 / (r_1^2, s_1^2)$	m even; $\tau = 1$; $\sigma = \frac{m}{2} + 1$
4^*	0	2^σ	$2^{\sigma-1}$	$\mathscr{A} \rtimes \mathbb{Z}_4 / (r_2, s_1^2)$	m even; $\tau = 2$; $\sigma = \frac{m}{2} - 1$
5	0	0	$2^{\sigma+1}$	$\mathscr{A} \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_4) / (r_1^2, s_1^2)(r_2^2, s_2^2)$	$\tau = 2$; $\sigma = m - 3$

4 Construction of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes

In this section we deal with the construction of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes with any allowable pair of values for the rank and the dimension of the kernel. We do not include all the constructions but, as a summary, we include Theorem 3, where it is described what are the allowable parameters for the dimension of the kernel and, for each one of these values, it is said what is the range of values for the rank. For each one of the possible pair of allowable values for the dimension of the kernel and rank, we construct a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code fulfilling it. As an illustration of the constructions we include two examples at the end of the section.

Let C a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code of length 2^m; let $T(\mathscr{C})$ be the subgroup of elements in $\mathscr{C}$ of order two; let $\mathscr{A}(\mathscr{C}) = \langle x_1,\dots,x_\sigma, r_1,\dots,r_\tau\rangle$ and let $\mathscr{R}(\mathscr{C})$ be defined by

$$\begin{cases} \mathscr{R}(\mathscr{C}) = \langle x_1 \dots x_\sigma, r_2 \dots r_\tau\rangle; & \text{if } r_1^2 = \mathbf{u} \\ \mathscr{R}(\mathscr{C}) = \mathscr{A}(\mathscr{C}); & \text{if } r_1^2 \neq \mathbf{u} \end{cases}$$

Theorem 3. *Let C a Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code of length 2^m and $|T(\mathscr{C})| = 2^\sigma$; $|\mathscr{C}/\mathscr{A}(\mathscr{C})| = 2^\upsilon$; $|\mathscr{A}(\mathscr{C})/T(\mathscr{C})| = 2^\tau$; $|\mathscr{R}(\mathscr{C})/T(\mathscr{C})| = 2^{\overline{\tau}}$; $|\mathscr{C}/T(\mathscr{C})| = 2^{\tau+\upsilon}$ and $m+1 = \sigma+\tau+\upsilon$. Then the rank r and the dimension of the kernel k of C satisfy the following conditions.*

1. *The values of the dimension of the kernel are $1 \neq m+1-k \in \{0,4,\tau-1,\tau,\tau+1\}$. The specific case $m+1-k=0$ is obtained in codes where $\overline{\tau} \leq 1$ or in codes of shape 5. The specific case $m+1-k=4$ is obtained in codes of shape 5.*
2. *a) If $m+1-k=0$ then $r-(m+1)=0$,*
   b) If $m+1-k=4$ and $\upsilon=2$ then $r-(m+1)=2$,
   c) If $m+1-k=\tau-1\geq 2$ then $r-(m+1)=\binom{\tau-1}{2}$,
   d) If $m+1-k=\tau\geq 2$ then $r-(m+1)=\binom{\tau}{2}$,
   e) If $m+1-k=\tau+1$ and $\overline{\tau} <= 1$ then $r-(m+1)=\tau$.
   f) If $m+1-k=\tau+1$ and $\overline{\tau}=\tau-1\geq 2$ then $r-(m+1)\in\{\binom{\tau-1}{2},\dots\binom{\tau}{2}+1\}$.
   g) If $m+1-k=\tau+1$ and $\overline{\tau}=\tau\geq 2$ then $r-(m+1)\in\{\binom{\tau}{2}+1,\dots\binom{\tau+1}{2}\}$.

Example 1. The following example shows constructions of codes of length $n=2^m=2^6=64$, with $\tau=3\geq\overline{\tau}=2\geq 2$, $\upsilon=1$ and $\sigma=3$. The resulting codes are of shape 2 and, before the Gray map, subgroups of Q_8^{16}.

$$\begin{aligned} r_1 &= (\mathbf{a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a\ a}) \\ r_2 &= (\mathbf{a\ a\ a^3\ a^3\ a\ a\ a^3\ a^3\ 1\ 1\ a^2\ a^2\ 1\ 1\ a^2\ a^2}) \\ r_3 &= (\mathbf{a\ a^3\ a\ a^3\ 1\ a^2\ 1\ a^2\ a\ a^3\ a\ a^3\ 1\ a^2\ 1\ a^2}) \end{aligned}$$

The codes with all possible pairs of values rank,dimension of the kernel are generated by r_1,r_2,r_3 and s_1. We show the vector s_1 and the values of the pair rank, dimension of the kernel.

$$\begin{array}{l|l} s_1 = (\mathbf{b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b}) & (k,r)=(5,8) \\ s_1 = (\mathbf{b\ b\ b\ ab\ b\ b\ b\ ab\ b\ b\ b\ ab\ b\ b\ b\ ab}) & (k,r)=(3,11) \\ s_1 = (\mathbf{b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ ab\ b\ b\ b\ ab}) & (k,r)=(3,10) \\ s_1 = (\mathbf{b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ ab} & (k,r)=(3,9) \\ s_1 = (\mathbf{b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ b\ ab\ ab\ ab\ ab}) & (k,r)=(3,8) \end{array}$$

Example 2. The following example shows constructions of codes of length 64, with $\tau=\overline{\tau}=2$, $\upsilon=1$ and $\sigma=4$. The resulting codes are of shape 3 and, before the Gray map, subgroups of $\mathbb{Z}_4^8 Q_8^{12}$. All possible pairs of rank and dimension of the kernel are presented.

Take the following vectors in $\mathbb{Z}_4^8 Q_8^{16}$:

$$r_1 = (0\ 2\ 0\ 2\ 0\ 2\ 0\ 2\ \mathbf{1}\ \mathbf{a}^2\ \mathbf{a}\ \ \mathbf{a}\ \ \mathbf{a}\ \ \mathbf{a}\ \mathbf{1}\ \mathbf{a}^2\ \mathbf{a}\ \ \mathbf{a}\ \ \mathbf{a}\ \ \mathbf{a})$$
$$r_2 = (0\ 0\ 2\ 2\ 0\ 0\ 2\ 2\ \mathbf{a}\ \ \mathbf{a}\ \ \mathbf{1}\ \mathbf{a}^2\ \mathbf{a}\ \mathbf{a}^3\ \mathbf{a}\ \ \mathbf{a}\ \ \mathbf{1}\ \mathbf{a}^2\ \mathbf{a}\ \mathbf{a}^3)$$

The codes with all possible pairs of values rank, dimension of the kernel are generated by r_1, r_2 and s_1. We show the vector s_1 and the values of the pair rank, dimension of the kernel.

$$\begin{array}{l|l} s_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}) & (k,r) = (5,8) \\ s_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{ab}) & (k,r) = (4,10) \\ s_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{ab}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}\ \mathbf{b}) & (k,r) = (4,9) \end{array}$$

Acknowledgements This work has been partially supported by the Spanish MICINN grant TIN2013-40524-P and the Catalan AGAUR grant 2014SGR-691.

References

1. de Launey, W., Flannery, D.L., Horadam, K.J.: Cocyclic Hadamard matrices and difference sets. Discrete Applied Mathematics **102**(1-2), 47–61 (2000). doi:http://dx.doi.org/10.1016/S0166-218X(99)00230-9
2. del Río, Á., Rifà, J.: Families of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. IEEE Transactions on Information Theory **59**(8), 5140–5151 (2013). doi:http://dx.doi.org/10.1109/TIT.2013.2258373
3. Flannery, D.: Cocyclic Hadamard matrices and Hadamard groups are equivalent. Journal of Algebra **192**(2), 749 – 779 (1997). doi:http://dx.doi.org/10.1006/jabr.1996.6949
4. Hammons, A.R., Kumar, P.V., Calderbank, A.R., Sloane, N.J.A., Solé, P.: The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals, and related codes. IEEE Transactions on Information Theory **40**(2), 301–319 (1994). doi:http://dx.doi.org/10.1109/18.312154
5. Horadam, K.J.: Hadamard matrices and their applications: Progress 2007-2010. Cryptography and Communications **2**(2), 129–154 (2010). doi:http://dx.doi.org/10.1007/s12095-010-0032-0
6. Ito, N.: On Hadamard groups. Journal of Algebra **168**(3), 981 – 987 (1994). doi:http://dx.doi.org/10.1006/jabr.1994.1266
7. Phelps, K.T., Rifà, J., Villanueva, M.: On the additive ($\mathbb{Z}_4$-linear and non-$\mathbb{Z}_4$-linear) Hadamard codes: rank and kernel. IEEE Transactions on Information Theory **52**(1), 316–319 (2006). doi:http://dx.doi.org/10.1109/TIT.2005.860464
8. Rifà, J., Pujol, J.: Translation-invariant propelinear codes. IEEE Transactions on Information Theory **43**(2), 590–598 (1997). doi:http://dx.doi.org/10.1109/18.556115