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Periodic orbits of planar integrable birational maps.

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Abstract A birational planar map F possessing a rational first integral preserves a foliation of the plane given by algebraic curves which, if F is not globally periodic, is given by a foliation of curves that have generically genus 0 or 1. In the genus 1 case, the group structure of the foliation characterizes the dynamics of any birational map preserving it. We will see how to take advantage of this structure to find periodic orbits of such maps.

1 Introduction

A planar *rational* map $F : \mathscr{U} \to \mathscr{U}$, where $\mathscr{U} \subseteq \mathbb{K}^2$ is an open set and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, is called *birational* if it has a rational inverse F^{-1}. In this paper we will say that a map F is *integrable* if there exists a non-constant function $V : \mathscr{U} \to \mathbb{K}$ such that

$$V(F(x,y)) = V(x,y),$$

which is called a *first integral* or *invariant* of F. If a map F possesses a first integral V then each orbit lies in some level set of V or, in other words, the level sets of V are invariant under F.

Planar birational maps are a classical object of study in algebraic geometry and have been the focus of intense research activity in recent years (see [24] and references therein). The integrable cases appear in many contexts, from algebraic geometry and number theory to mathematical physics. This is the case of the celebrated QRT family of maps introduced in [44, 45] (see also [26]), which contains the well-known McMillan family of maps, and some of the integrable cases studied

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by Gumovski and Mira [30, 40]. Many maps in this family arise as special solutions, termed discrete solitons, of differential-difference equations arising in statistical mechanics. The QRT maps all have a rational first integral.

In this paper, we will consider only those integrable maps that have *rational first integrals*. In fact, all the examples of integrable birational maps that we know have rational first integrals, but as far as we know there is no reason for an integrable birational map to be rationally integrable. In this sense, it is interesting to recall the case given by the composition maps associated to the 5-periodic Lyness recurrences. These maps are birational, and the numerical results show phase portraits compatible with the existence of first integrals, however it has been recently proved that, generically, these maps are not rationally integrable, see [17, Theorem 19] and [20, Theorem 1 and Proposition 3].

Observe that if the first integral is a rational function,

$$V(x,y) = \frac{P(x,y)}{Q(x,y)}, \tag{1}$$

then the map preserves the foliation[⋆] of $\mathscr{U}$ given by the algebraic curves

$$\mathscr{F} = \{P(x,y) - hQ(x,y) = 0, h \in \mathrm{Im}(V)\}. \tag{2}$$

We will assume that P and Q are coprime and, although it is not essential in this paper, that V has *minimal* degree. Recall that the degree of a rational first integral is the greater of the degrees of P and Q. We say that the degree n of V is minimal if any other rational first integral of F has degree at least n. Given a rational first integral one always can find a minimal rational first integral.

In this note, our objective is to show how to take advantage of the algebraic-geometric properties of the invariant foliation $\mathscr{F}$ to study periodic orbits of the birational maps preserving it. Although the techniques explained in this paper have been used to study several birational maps [4, 5, 6, 7, 8, 26, 53] and [54], to illustrate them we will refer only to a particular, but paradigmatic, example: the well-known Lyness family of maps.

Example 1. Lyness's maps are a 1-parametric family of birational maps given by

$$F_a(x,y) = \left(y, \frac{a+y}{x}\right), \tag{3}$$

These maps give the dynamical system associated to recurrence $x_{n+2} = (a+x_{n+1})/x_n$. There is a large recent literature concerning this family, in the Appendix of this paper the reader can find a short account of references and the history of the Lyness recurrences and maps.

Each map F_a has the first integral

⋆ In this paper we say that a map F preserves a foliation of curves $\{C_h\}$ if each curve C_h is invariant under the iterates of F.

$$V(x,y) = \frac{(x+1)(y+1)(x+y+a)}{xy}, \tag{4}$$

so it preserves the foliation given by

$$\mathscr{F} = \{C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\}, h \in \mathrm{Im}(V)\}. \tag{5}$$

The paper is structured as follows: in Section 2 we will recall the notion of genus of an algebraic curve, and we will see that if we are interested in those maps not being globally periodic, then we can consider that the curves in the foliation (2) have genus 0 or 1, see Corollary 3. In Section 3 we restrict our attention to maps having invariant curves with genus 1 (also named elliptic curves). We recall the group structure of these curves and also a result of Jogia, Roberts and Vivaldi (Theorem 4) which relates the dynamics of a particular birational map on an invariant elliptic curve and its group operation. We will take advantage of this result to obtain a description of the periodic orbits in terms of the torsion of the curve (Equation 8). In Section 4, we discuss the global dynamics of birational maps preserving a foliation given by elliptic curves C_h. First we start by introducing and discussing the nature of the rotation number function $\theta(h)$ associated to each curve C_h. Then we see that a typical situation occurs when there is a dense set of curves in phase space filled by p-periodic orbits of all the periods $p \geq p_0 \in \mathbb{N}$, for some integer p_0 which is sometimes computable (see Proposition 9 and Subsection 4.3).

In Section 5, as a straightforward application, we show how to address the problem of finding the curves containing periodic orbits with a prescribed period, by using the characterization of periodic orbits given by the *group law of the curve* (see Equation (8)). We show the main technique by applying it to the Lyness case, as already done in [5].

In Section 6 we will see how the group structure of rational elliptic curves is strongly related to the existence of rational periodic orbits. We will recall Mazur's Theorem and its dynamical implications. We also give some insight on the known results of rational periodic orbits in the Lyness case [5, 29, 54]. This section ends with a digression about why the numerical simulations of the phase portrait of birational maps preserving an elliptic foliation do not show the plethora of periodic orbits that they possess, on the contrary of what happens when general integrable diffeomorphisms are considered.

We end these notes with a comment on the genus 0 case, and with an Appendix giving more information about the Lyness maps and curves.

The aim of the paper is expository, and it is inspired in the papers of Bastien and Rogalski [5] and of Jogia, Roberts and Vivaldi [33]. The reader is invited to read them, as well as their references. Another essential reference is the book of Duistermaat [26] about some algebraic-geometric aspects of QRT maps.

2 A first dynamical result. Restriction to the genus 0 and 1 cases

When studying the dynamics of an integrable map, a first step is to know the topology of the invariant level sets. When the level sets are algebraic curves, the natural way to study them is to consider their extension, and also the extension of the birational maps, to the complex projective space

$$\mathbb{C}\mathbf{P}^2 = \{[x:y:z] \neq [0:0:0], x,y,z \in \mathbb{C}\}/\sim,$$

where $[x_1:y_1:z_1] \sim [x_2:y_2:z_2]$ if and only if $[x_1:y_1:z_1] = \lambda[x_2:y_2:z_2]$ for $\lambda \neq 0$.

In this paper $[x:y:1]$ denotes an affine point, corresponding with the point $(x,y) \in \mathbb{K}^2$ (where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$), and $[x:y:0]$ denotes an infinite point. The infinite points are added to real affine algebraic curves in order to capture the asymptotic directions of possible unbounded components. See Figure 2 for instance.

Any real affine algebraic curve can be extended to $\mathbb{C}\mathbf{P}^2$ by the formal process of *homogenization*. For instance, any Lyness curve

$$C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\} \subset \mathbb{R}^2$$

where $x,y \in \mathbb{R}$ extends to $\mathbb{C}\mathbf{P}^2$ as

$$\widetilde{C_h} := \{(x+z)(y+z)(x+y+az) - hxyz = 0, x,y,z \in \mathbb{C}\}.$$

Notice also that any birational map in $\mathbb{R}^2$ extends formally to a polynomial map in $\mathbb{C}\mathbf{P}^2$. For instance, the Lyness map $F_a(x,y) = (y,(a+x)/y)$ extends formally to

$$\widetilde{F_a}([x:y:z]) = [xy : az^2 + yz : xz],$$

except for the points $[x:0:0]$, $[0:y:0]$ and $[0:-a:1]$ (see also the alternative description given by Equation (7) in Section 3), where $x,y,z \in \mathbb{C}$.

Any algebraic curve $\widetilde{C}$ in $\mathbb{C}\mathbf{P}^2$ is a Riemann surface characterized by its *genus*, [34]. On any irreducible component of a curve in $\mathbb{C}\mathbf{P}^2$, the genus g is related to the *degree* d by the *degree-genus formula*:

$$g = \frac{(d-1)(d-2)}{2} - \sum_{p \in \mathrm{Sing}(C)} \frac{m_p(m_p-1)}{2},$$

where m_p stands for the multiplicity of any possible singular ordinary point. Recall that a singular point is called ordinary when all the tangents at the point are distinct and that, given an irreducible curve, it is always possible to find a birationaly equivalent curve with only ordinary multiple points, so that the above formula gives the genus.

In this paper, we will say that an invariant foliation has *generic genus g* if the genus has constant value g on the irreducible components of $\{P - hQ\}$, except maybe for a finite set of values of $h \in \mathrm{Im}(V)$ for which the genus is lower. This is a common situation. The reader is addressed, for instance, to [43] for a characterization of the singular curves corresponding to a biquadratic foliation that generalizes the classical elliptic QRT foliations. We will assume that in our foliations (2) the genus is generic.

Next, we will see that if one expects to obtain a rich dynamics of a birational map preserving a foliation $\{C_h\}$ where C_h are irreducible curves, then one has to restrict attention to those maps that preserve foliations of generic genus 0 or 1, because any birational map F preserving a foliation of generic genus greater or equal than 2 is a globally periodic map, that is, there exists $p \in \mathbb{N}$ such that $F^p(x,y) = (x,y)$ for all (x,y) where F is defined. This fact is a consequence of the following two classical results.

Theorem 1 (Montgomery, [42]) *Any pointwise periodic homeomorphism in a connected metric space, locally homeomorphic to $\mathbb{R}^n$, is globally periodic.*

The next one is an adaptation to our context of the Hurwitz automorphisms theorem which states that any compact Riemann surface with genus $g > 1$ admits at most $84(g-1)$ conformal automorphisms, that is, homeomorphisms of the surface onto itself which preserve the local structure; see [21, 22]. In our context, Hurwitz's theorem can be stated as follows, [33]:

Theorem 2 (Hurwitz, 1893) *The group of birational maps on a non-singular algebraic curve of genus $g > 1$ is finite, and of order less or equal than $84(g-1)$.*

The above result states that any birational map preserving *a particular* non-singular curve of genus $g \geq 2$ must be periodic (on the curve) with a period bounded by $84(g-1)$.

Corollary 3 ([18]) *A birational map in $\mathscr{U} \subseteq \mathbb{K}^2$ (where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$) preserving a foliation of non-singular curves $\{C_h\} \subseteq \mathscr{U}$ that have generic genus $g > 1$, must be globally periodic.*

Proof. If the foliation $\{C_h\}$ has generic genus $g > 1$ then there exists an open set $\mathscr{V} \subseteq \mathscr{U}$ foliated by curves of genus g. By Hurwitz's Theorem on each of these curves the map must be periodic, so F is pointwise periodic on $\mathscr{V}$. Therefore, by Montgomery's Theorem, F must be globally periodic on the whole $\mathscr{V}$. Since F is rational, and so the global periodicity is characterized by some formal polynomial identities, then it must be periodic on the whole $\mathbb{K}^2$ except at the points where its iterates are not well defined. □

In summary, from a dynamic viewpoint it makes sense to restrict our attention to birational maps preserving foliations of algebraic curves with genus 0 or 1.

3 The elliptic case: dynamics on invariant curves through its group structure

In this paper we will concentrate our attention on those birational maps that preserve a foliation of algebraic curves $\{C_h\}$ of generic genus 1. Recall that a projective algebraic curve of genus 1 is called an *elliptic curve*. Any elliptic curve has an associated group structure ([34, 50, 51]). In this section we will see that in the case that $\{C_h\}$ is generically given by elliptic curves, then the group structure of the elliptic foliation characterizes the dynamics of any birational map preserving it.

First we recall the group structure associated with an elliptic curve $C \in \mathbb{C}\mathbf{P}^2$, the so called *chord-tangent group* law. Given two points P and Q in C we define the addition $P+Q$ in the following way:

1. Select a point $\mathscr{O} \in C$ to be the neutral element of the inner addition.
2. Take the chord passing through P and Q (the tangent line if $P = Q$). It will always intersect C at a unique third point denoted by $P * Q$. This is because the curves of genus 1 are birationally homeomorphic to *smooth cubic* curves, [50, Proposition 3.1].
3. The point $P+Q$ is then defined as $\mathscr{O} * (P * Q)$, see Figure 1.

The curve endowed with this inner addition $(C, +, \mathscr{O})$ is an *abelian group* [51].

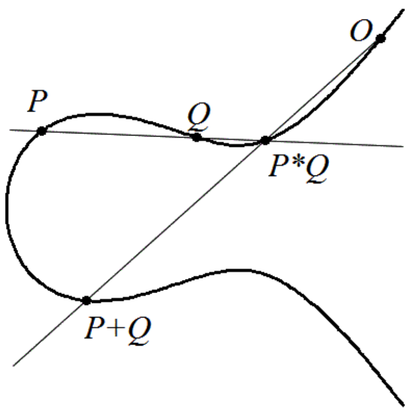

Fig. 1 Group law with an affine neutral element $\mathscr{O}$.

A brief comment on notation: typically algebraic curves are defined on $\mathbb{K}^2$, or on $\mathbb{K}P^2$, where $\mathbb{K}$ is the field of coefficients. In this paper, this field will be mainly $\mathbb{R}$ or $\mathbb{C}$ (or $\mathbb{Q}$ in Section 6). The notation $C(\mathbb{K})$ or $C/\mathbb{K}$ denotes an elliptic curve C which has at least one point $\mathscr{O}$ with coordinates in $\mathbb{K}$. In this paper, unless we explicitly state the contrary, we will assume that C stands for a *real* curve.

The relationship between the dynamics of a birational map preserving an elliptic curve and its group structure is given by the following adaptation of a result of Jogia, Roberts and Vivaldi [33, Theorem 3], that will be referred as the *JRV Theorem* from now on. In [33], the result is stated for birational maps leaving invariant an elliptic curve expressed in a certain Weierstrass normal form (see [34, 50] but especially [51, Section I.3]). This adaptation is immediately obtained by using the isomorphism with this normal form.

Theorem 4 (Jogia, Roberts, Vivaldi, [33]) *Let F be a birational map over a field $\mathbb{K}$, not of characteristic 2 or 3, that preserves an elliptic curve $C(\mathbb{K})$. Then there exists a point $Q \in C(\mathbb{K})$ such that the map can be expressed in terms of the group law $+$ on $C(\mathbb{K})$ as either*

(i) $F_{|C(\mathbb{K})} : P \mapsto P + Q$, *or*
(ii) $F_{|C(\mathbb{K})} : P \mapsto i(P) + Q$, *where i is an automorphism of possible order (period) 2, 4, 3 or 6, and the map F has the same order (period) as i.*

We will give an easier dynamical interpretation of the above result, but first we will illustrate it.

Example 2. The Lyness curves $\widetilde{C_h} := \{(x+z)(y+z)(x+y+az) - hxyz = 0\} \subset \mathbb{C}\mathbf{P}^2$, are elliptic except for $h \in \{0, a-1, h_c^{\pm}\}$, where

$$h_c^{\pm} := \frac{2a^2 + 10a - 1 \pm (4a+1)\sqrt{4a+1}}{2a}. \tag{6}$$

An interesting fact is that for all values of h the curves $\widetilde{C_h}$ contain the infinity points $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$, see Figure 2. An straightforward computation (or a geometrical interpretation) shows that, setting $\mathscr{O} := [1:-1:0]$, for any elliptic level h the map $\widetilde{F_a}([x:y:z]) = [xy : az^2 + yz : xz]$ can be written as:

$$\widetilde{F}_{a|\widetilde{C_h}}([x:y:z]) = [x:y:z] + [1:0:0]. \tag{7}$$

The non-elliptic levels correspond to curves of genus 0, and on those levels the map $\widetilde{F}_{a|\widetilde{C_h}}$ is conjugate to a Möbius transformation, see [29, Section 3.1].

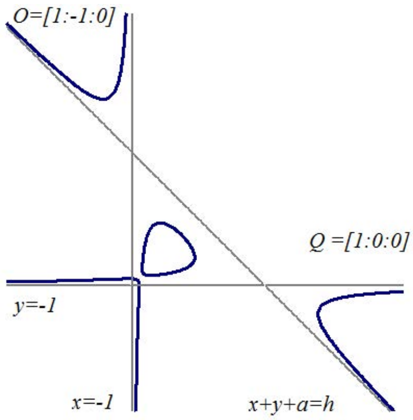

Fig. 2 A typical real Lyness curve $C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\}$, for $h > h_c^+$. Adding the infinite points $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$, the displayed curve is isomorphic to $\mathbb{S}^1 \times \mathbb{Z}/(2)$.

The JRV Theorem (Theorem 4) has the following dynamical interpretations given in Corollaries 5 and 7 below.

Corollary 5 *Let F be a birational map preserving a real foliation of algebraic curves $\{C_h\} \subset \mathscr{U} \in \mathbb{R}^2$ of generic genus 1. Then, on each invariant elliptic curve C_h, either F or F^2 are conjugate to a rotation.*

The above corollary is a direct consequence of Theorem 4 and the following result (a direct consequence of Corollary 2.3.1 in Chapter V.2 of [49]), based on the fact that every real elliptic curve (adding, if necessary, some infinite points in the real projective space) can be seen as either one or two closed simple curves, and that the inner sum can be easily represented as the usual Lie group operation of $\mathbb{S}^1$ or $\mathbb{S}^1 \times \mathbb{Z}/(2)$.

Proposition 6 *There is a continuous isomorphism between any non-singular elliptic curve $(C(\mathbb{R}),+,\mathscr{O})$ and either the Lie group $\mathbb{S}^1 \times \mathbb{Z}/(2) = \{e^{it} : t \in [0,2\pi)\} \times \{1,-1\}$ if $\Delta(C) > 0$, or $\mathbb{S}^1 = \{e^{it} : t \in [0,2\pi)\}$ if $\Delta(C) < 0$, with the operation in $\mathbb{S}^1$ being given by $u \cdot z = uz$, where $\Delta(C)$ is the discriminant of the Weierstrass equation associated to $C(\mathbb{R})$.*

Observe that if F is a birational map preserving an elliptic curve $(C,+,\mathscr{O})$ whose dynamics corresponds to case (ii) of Theorem 4, then all the points in C give rise to periodic orbits. If the dynamics corresponds to case (i), then $F^n_{|C}(P) = P + nQ$, and we observe that P gives rise to a p-periodic orbit if and only if

$$pQ = \mathscr{O}. \tag{8}$$

In other words, in case (i) of Theorem 4 the curve is filled by periodic orbits of F if and only if Q is a *finite order* point of the group $(C,+,\mathscr{O})$, also called a *torsion* point (the torsion of a group G, denoted by $\mathrm{Tor}(G)$ is the set of its finite order elements). The following result characterizes the dynamics of birational maps on particular elliptic curves.

Corollary 7 *Let F be a birational map preserving a real elliptic curve $(C(\mathbb{R}),+,\mathscr{O})$, named C from now on, such that its dynamics is given by $F_{|C}(P) = P + Q$, where $Q \in C$. Then*

(i) If $Q \in \mathrm{Tor}(C)$, then all the orbits in C are periodic.
(ii) If $Q \notin \mathrm{Tor}(C)$, then the orbits of F fill densely the connected components of C.

4 Global dynamics on elliptic foliations.

The JRV Theorem ensures that the action of a birational map on a particular elliptic curve is linear. However, the behavior in the whole phase plane is a little bit more complex. The typical situation occurs when there is a dense set of curves filled by p-periodic orbits of all the periods $p \geq p_0 \in \mathbb{N}$, for some integer p_0. This integer p_0 is sometimes computable if the rotation set $\{\theta(h), h \in \mathrm{Im}(V)\}$ is known. In this section we describe the reason for this behavior and we give an example of how to compute the set of periods of a particular map. We also will see that if a particular subinterval I in the rotation set is known, then it is possible to construct a number P such that the map F contains at least all the periods $p > P$.

4.1 The rotation number function and its nature

4.1.1 Piecewise continuity

From this point we will assume that the invariant foliation of irreducible curves $\{C_h\}$ obtained from (2) is given by *real* curves which are generically elliptic. Also we will assume that F preserves each connected component of the invariant real elliptic curves (on the contrary, we can study F^2). Finally, we will also assume that the action of our birational maps F falls within case (i) of Theorem 4. Under these assumptions, Corollary 5 ensures that on each curve C_h the map F is conjugate to a rotation. So we can consider a *rotation number* $\theta(h)$ associated to each level set h, or equivalently to each curve C_h.

Of course this rotation number function θ can be constant. In this case we say that the map F is *rigid*. For instance, if $a = 1$ then the Lyness map F_1 is globally 5-periodic, thus $\theta(h) = 1/5$ for all $h \in \mathrm{Im}(V)$, where V is given in (4).

If θ is not constant then it is possible to prove that this rotation number function is piecewise continuous. This is because when the irreducible components of (2) are generically elliptic, any birational map F (or F^2) can be thought as a family of homeomorphisms in the circle, which is piecewise continuous in the parameter h. By using the fact that the rotation number function of a continuous family of maps of $\mathbb{S}^1$ (in the $\mathscr{C}^0$ topology) is continuous, and taking into account that, in principle, there could be levels $h \in \mathrm{Im}(V)$ corresponding to curves in the *forbidden set* of F, the piecewise continuity of $\theta(h)$ is achieved.

4.1.2 Piecewise analyticity and existence of Lie Symmetries

In fact, $\theta(h)$ is a piecewise analytic function in the domain $h \in \mathrm{Im}(V)$. This is because if the irreducible curves C_h in (2) are generically elliptic, then it is possible to construct an isomorphism (piecewise analytic in h) between them and a new foliation of Weierstrass curves, so that the corresponding associated map F_W defined on this foliation has the same rotation number function as F. These Weierstrass curves can be parameterized using the Weierstrass $\wp$ function (see [34, 50], and also [26, 53]). Using this parametrization, and the fact that $\wp$ satisfies a certain differential equation, it is always possible to give an integral expression of the rotation number function, from which the piecewise analyticity of it can be deduced. This approach has been introduced in [28] (and later developed in [5]) to study the rotation number function associated to a Lyness map, and has been successfully applied to study the periods of other birational maps in the successive papers of Bastien and Rogalski [4, 6, 7, 8] and others.

An alternative proof of the piecewise analyticity of $\theta(h)$ comes from the fact that if the invariant curves C_h in (2) are generically elliptic, then it is possible to construct a vector field X such that the map F can be seen as the flow of this vector field at certain time $\tau(h)$. Such a vector field is called a *Lie Symmetry* of F. A Lie Symmetry of a map F in $\mathbb{R}^n$ is a *vector field* X such that F maps any orbit of the differential system

$$\dot{\mathbf{x}} = X(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n \tag{9}$$

into another orbit of the system.

The existence of Lie Symmetries is an important issue in the theory of discrete integrability, see for instance [31]. From a dynamical viewpoint this importance is clear in the case of integrable diffeomorphisms. In this case, the dynamics of the maps are in practice one-dimensional, and the existence of a Lie Symmetry whose orbits are preserved by F implies that this one-dimensional dynamics is linear on each orbit. The next results illustrate this fact.

Theorem 8 ([14]) *Let $F : \mathscr{U} \subseteq \mathbb{R}^n \to \mathscr{U}$ be a diffeomorphism having a Lie symmetry X, and let γ be an orbit of X, preserved by F (i.e. $F : \gamma \to \gamma$). Then the dynamics of F restricted to γ is either: (1) conjugate to a rotation with rotation number given by τ/T, where T is the period of γ and τ is defined by the equation $F(p) = \varphi(\tau, p)$, where φ denotes the flow of X; or (2) conjugate to a translation of the line; or (3) it is constant; according to whether γ is homeomorphic to $\mathbb{S}^1$, $\mathbb{R}$, or a point, respectively.*

If F is an integrable map and it possesses a Lie Symmetry X, this vector field is also integrable and shares the same first integral with F. So in our case the curves in (2) would also be integral curves of any possible symmetry X. Since each connected component C_h of any curve in (2) is diffeomorphic to $\mathbb{S}^1$ (adding, possibly some infinite points), it is a periodic orbit of X (or a compactification of X) with period $T(h)$, see also [19]. Hence Theorem 8 guarantees that

$$\theta(h) = \frac{\tau(h)}{T(h)}. \tag{10}$$

The regularity of the rotation number function is, then, a consequence of the regularity of the flow of X.

Again, the existence of the Lie Symmetry of a birational map preserving an elliptic foliation can be proved using the associated Weierstrass foliation associated to curve C_h, see [26, Section 2.6.3]. The Lie Symmetry approach was used to study the rotation number function of the Lyness map and to prove a conjecture about its monotonicity established by Zeeman [54]. This was done by Beukers and Cushman in the relevant paper [10]. This approach has been also applied to study the rotation number function and the set of periods of the extension of the Lyness map in $\mathbb{R}^3$ [13], and also to study a birational integrable map arising in the study of 2-periodic Gumovski-Mira type maps [16].

4.2 An infinite number of periods

Taking into account the above considerations we can prove the following result:

Proposition 9 *A birational map F preserving a generically real elliptic foliation $\{C_h\}$ is either rigid, or there are an infinite number of possible periods and a dense set of curves in the phase space filled with periodic orbits.*

Proof. Let $\mathscr{E} = \{h$ such that the curves C_h are elliptic$\}$. From the JRV Theorem on each curve $C_h \in \mathscr{E}$ the map F (or F^2) is conjugate to a rotation. From the considerations in Section 4.1 (see also [26, Lemma 8.1.5]) the rotation number function $\theta(h)$ is piecewise continuous for $h \in \mathscr{E}$.

Since $\{C_h\}$ is generically elliptic, if F is not rigid, then there exists a nonempty open interval I such that $I \subseteq \{\theta(h), h \in \mathscr{E}\}$. Then for any irreducible fraction $q/p \in I$, there exists a value of $h \in \mathscr{E}$ such that $\theta(h) = q/p$, hence an invariant real elliptic curve C_h which is full of periodic orbits of of minimal period p. □

4.3 Towards a constructive characterization of the set of periods

It is interesting to notice that if a rotation interval I containing some of the values of $\theta(h)$ is known, then it is always possible to compute a value P such that $q/p \in I$ for all $p > P$, hence characterizing at least an infinite number of periods in the set of periods of F. One tool to construct a (non-optimal) number P is the following result.

Lemma 1 ([13]). *Consider an open interval (c,d) with $0 \le c < d$; denote by $p_1 = 2, p_2 = 3, p_3, \ldots, p_n, \ldots$ the set of all the prime numbers, ordered following the usual order. Also consider the following natural numbers:*

- *Let p_{m+1} be the smallest prime number satisfying that $p_{m+1} > \max(3/(d-c), 2)$,*
- *Given any prime number p_n, $1 \le n \le m$, let s_n be the smallest natural number such that $p_n^{s_n} > 4/(d-c)$.*
- *Set $P := p_1^{s_1-1} p_2^{s_2-1} \cdots p_m^{s_m-1}$.*

Then, for any $r > P$ there exists an irreducible fraction q/r such that $q/r \in (c,d)$.

In the Example 3 we will illustrate how to apply the above result and the known facts on the rotation number function to compute effectively some set of minimal periods appearing in a particular Lyness map F_a. Prior to stating this example we recall some basic facts. When $a > 0$ the first integral of F_a, given in (4), has a global minimum in $Q^+ := \{(x,y), x,y > 0\}$, located at the fixed point of F_a, given by (x_c, x_c) where $x_c = (1+\sqrt{1+4a})/2$. This minimum corresponds to the non-elliptic level $h_c = (x_c+1)^3/x_c$. With respect to the rotation number function, it is known that for $a > 0$

$$\theta_c = \lim_{h \to h_c^+} \theta(h) = \frac{1}{2\pi} \arccos\left(\frac{1}{1+\sqrt{1+4a}}\right),$$

see [5, 54]. Also it was conjectured in [54], and proved in [10], that when $0 < a < 1$, then $\theta(h)$ is a strictly increasing continuous function in (h_c, ∞) and strictly decreasing when $1 < a < \infty$. Moreover, in [5] it was proved (strongly using the elliptic nature of the Lyness curves) that

$$\lim_{h_c \to \infty} \theta(h) = \frac{1}{5}.$$

The case $a = 1$ corresponds to the globally 5-periodic case with $\theta(h) \equiv 1/5$.

In summary, the interval

$$I^+ := (\min(\theta_c, 1/5), \max(\theta_c, 1/5))$$

gives the optimal rotation interval for the orbits in Q^+ when $a > 0$ and $a \neq 1$.

Example 3. When $a = 10$, the optimal rotation interval for the orbits in Q^+ is

$$I^+ := \left(\frac{1}{5}, \frac{1}{2\pi} \arccos\left(\frac{1}{1+\sqrt{41}} \right) \right).$$

Using the notation introduced in Lemma 1 we have $m = 27$, and $p_1 = 2$, $s_1 = 7$; $p_2 = 3$, $s_2 = 4$; $p_3 = 5$, $s_3 = 2$; $p_4 = 7$, $s_4 = 2$; $p_5 = 11$, $s_5 = 2$; $p_6 = 13$, $s_6 = 1$; $p_7 = 17$, $s_7 = 1$; ... ; $p_{27} = 103$, $s_{27} = 1$. Hence

$$P := \prod_{n=1}^{27} p_n^{s_n} = 79783116986616878993690973578945928329152944000,$$

and therefore there are r-periodic orbits of F_{10} for any $r > P$. Of course, using a finite algorithm one could check for any $r \leq P$ if there exist an irreducible fraction $q/r \in I^+$, obtaining in principle the forbidden denominators in I^+, but in practice the high value of P makes this observation useless.

Notice however that a computation (that takes 1.03 seconds using Maple 17 code on an Intel Core i5-3210M CPU at 2.50GHz) gives that the numbers $S = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18, 20, 21, 25, 26, 27, 28, 30, 35, 36, 38, 42, 45, 46, 48, 56, 66, 70, 72, 96, 98, 120, 126\}$ are *some* forbidden periods. It can be proved, using an alternative method that this is the *exact* set of forbidden periods, see Remark 2.

Another approach is the one introduced in [5] based on the following result (see also Remark 1 below)

Lemma 2 (Bastien & Rogalski). *Consider an open interval (c,d) with $0 \leq c < d$. Set*

$$f(x) = \frac{dx-1}{\ln(dx-1)} - \frac{cx}{\ln(cx)} \left(1 + \frac{3}{2\ln(cx)} \right) - 1.38402 \frac{\ln(x)}{\ln(\ln(x))} - 1.$$

Then, for any $p \in \mathbb{N}$, $p \geq 17$ such that $f(p) > 0$ there exists $q \in \mathbb{N}$ coprime with p such that $c < q/p < d$.

Proof. Let $\pi(x)$ be the prime-counting function, which gives the number of prime numbers which are less or equal than x. Using Theorem 1 and Corollary 1 of [47] we have that if $x \geq 17$

$$\frac{x}{\ln(x)} \leq \pi(x) \leq \frac{x}{\ln(x)}\left(1+\frac{3}{2\ln(x)}\right). \tag{11}$$

So, given a number $p \in \mathbb{N}$, $p \geq 17$, we can estimate that the number of integer numbers q such that $cp < q < dp$ is at least the the number of prime numbers in this interval which is, using the inequalities (11), at least

$$\frac{dp-1}{\ln(dp-1)} - \frac{cp}{\ln(cp)}\left(1+\frac{3}{2\ln(cp)}\right). \tag{12}$$

Since we are interested in those values of q which are coprime with p, we should subtract from (12) the number of divisors of p, denoted by $\omega(p)$, which is bounded by

$$\omega(p) \leq 1.38402\frac{\ln(p)}{\ln(\ln(p))}, \text{ for } p \geq 3,$$

(see [46, Theorem 11]). So the number of integer numbers q coprime with p in (c,d) is at least

$$\frac{dp-1}{\ln(dp-1)} - \frac{cp}{\ln(cp)}\left(1+\frac{3}{2\ln(cp)}\right) - 1.38402\frac{\ln(p)}{\ln(\ln(p))}.$$

Clearly if the above number is greater than one (i.e. $f(p) > 0$), then p is a possible denominator in (c,d). □

The methodology summarized in Lemma 2 was introduced in [5] (see also [28, Section 6]), to study the set of periods for the whole family of Lyness' maps with $a \geq 0$ in Q^+ [5, Theorem 4] (see also [54, Theorem 9]). The familiar set of periods is 5, 6, 9, 11, 13, 14, 16 17, 19 and all integers ≥ 21 except 42. Other periods appear when negative initial conditions are considered, [29].

Remark 1 (Added in proof). Lemma 2 has been recently improved by Bastien and Rogalski. See [9, Proposition 23], which states that if $0 < c < d < 1/2$ then for every $p > \max\left(e^{2.55|\ln(c)|/(d-c)}, \frac{1}{c}e^{3.82/(d-c)}\right)$ there exists a prime number q with $(p,q) = 1$ such that $q/p \in (c,d)$.

Remark 2. With respect to the particular case studied in Example 3, setting $c = 1/5$ and $d = \arccos(1/(1+\sqrt{41}))/(2\pi)$, and by proving that for all $x > 6\cdot 10^5$ the function $f(x)$ in Lemma 2 is positive, one would get that the set of periods of F_{10} in Q^+ contains all periods greater than $6\cdot 10^5$. An straightforward computation (that takes 1444.23 seconds with the same software and CPU as in Example 3) gives that every integer number $r < 6\cdot 10^5$ such that $r \notin S$ is a possible denominator in I^+. Thus $\mathbb{N}\setminus S$ would be the complete set of periods of F_{10} in Q^+.

5 An application. The locus of periodic orbits

In this section, as a straightforward application of the characterization of the periodic orbits given by Equation (8), we will use it to address the problem of finding the location of the curves having periodic orbits with a prescribed period. Following the aim of the notes, again we will take the Lyness maps as a paradigmatic example. This approach was used in [5] for studying Lyness' maps and, of course, can be used to study other birational maps on elliptic foliations [4].

As mentioned above, the real Lyness curves $C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\} \subset \mathbb{R}^2$ are elliptic curves except for $h \in \{0, a-1, h_c^{\pm}\}$, where $h_c^{\pm}$ are given in (6), and together with the infinity points $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$ are isomorphic to either $\mathbb{S}^1$ or $\mathbb{S}^1 \times \mathbb{Z}/(2)$. Setting $\mathscr{O} := [1:-1:0]$, for all elliptic levels h, the dynamics on each real connected component of the Lyness' map (3) is

$$F_{a|C_h}([x:y:1]) = [x:y:1] + [1:0:0].$$

Hence, the characterization of the p-periodic orbits given by Equation (8), implies a curve C_h will be full of periodic orbits if and only if

$$p \cdot [1:0:0] = [1:-1:0]. \tag{13}$$

Equation (13) gives a naive way for finding the locus of the periodic orbits of a prescribed period in the elliptic levels, obtaining the following result which is, in fact, the summary of well-known ones (see [3, 5] and [54] among other references).

Proposition 10 *Consider the real elliptic Lyness curves $C_{h_p} = \{(x+1)(y+1)(x+y+a) - h_p xy = 0\}$, then the following statements hold:*

(i) The maps F_0 and F_1 are globally periodic with periods 6 and 5 respectively.
(ii) If $a(a-1) \neq 0$ then there are no elliptic curves C_{h_p} with periodic orbits of the Lyness maps F_a with period $p = 1,2,3,4,5$ and 6.
(iii) If $a(a-1) \neq 0$ the elliptic curves C_{h_p} filled with periodic orbits of the Lyness maps F_a with periods $p = 7,8,9,10,11$ and 12 are given by:

$$\begin{aligned}
h_7 &= (a-1)/a,\\
h_8 &= -(a-1)^2/a,\\
h_9 &= (a-1)(a^2-a+1)/a,\\
h_{10} &= (a-1)/(a(a+1)),\\
h_{11} &= (a-1)\left(2a-1 \pm \sqrt{4a^3-4a^2+1}\right)/(2a^2), \text{ for } a > a_*,\\
h_{12} &= (a-1)\left(-a+3 \pm \sqrt{-3a^2+2a+1}\right)/(2a) \text{ for } a \in [-1/3, 1],
\end{aligned}$$

where $a_* \simeq -0.41964$ is the only real root of $4a^3 - 4a^2 + 1$.

Proof. (i) Setting $Q = [1:0:0]$, using the inner addition rules of $(C_h(\mathbb{R}), +, [1:-1:0])$, and using that the infinite points $[1:0:0]$, $[0:1:0]$ and $[1:-1:0]$ are tangent to the asymptotes of C_h given by $y=-1$, $x=-1$ and $x+y+a-h=0$ respectively, some straightforward computations** show that if $a=0$ then

$$\begin{array}{l} Q*Q=[0:-1:1]\Rightarrow 2Q:=[0:-1:1]*\mathscr{O}=[-1:0:1];\\ 2Q*Q=[0:0:1]\Rightarrow 3Q:=[0:0:1]*\mathscr{O}=[0:0:1];\\ 3Q*Q=[-1:0:1]=2Q\Rightarrow 4Q:=2Q*\mathscr{O}=[0:-1:1];\\ 4Q*Q=Q\Rightarrow 5Q:=Q*\mathscr{O}=[0:1:0]\text{ and finally,}\\ 5Q*Q=\mathscr{O}\Rightarrow 6Q:=\mathscr{O}*\mathscr{O}=\mathscr{O}. \end{array}$$

So we reobtain the well-known fact that the Lyness map F_0 is globally 6-periodic.

If $a=1$ then

$$\begin{array}{l} Q*Q=[0:-1:1]\Rightarrow 2Q:=[0:-1:1]*\mathscr{O}=[-1:0:1];\\ 2Q*Q=2Q\Rightarrow 3Q:=2Q*\mathscr{O}=[0:-1:1];\\ 3Q*Q=Q\Rightarrow 4Q:=Q*\mathscr{O}=[0:1:0]\text{ and finally,}\\ 4Q*Q=\mathscr{O}\Rightarrow 5Q:=\mathscr{O}*\mathscr{O}=\mathscr{O}, \end{array}$$

so the map F_1 is globally 5-periodic.

(i) and (iii) Now we assume that $a(a-1)\neq 0$ and we apply formally the addition rules obtaining that: $2Q=[-1:0:1]$; $3Q=[0:-a:1]$;

$$\begin{array}{l} 4Q=\left[-a:\dfrac{ah-a+1}{a-1}:1\right];\,5Q=\left[\dfrac{ah-a+1}{a-1}:\dfrac{-a^2-ah+2a-1}{a(a-1)}:1\right];\\ 6Q=\left[\dfrac{-a^2-ah+2a-1}{a(a-1)}:\dfrac{a^3-2a^2-ah+2a-1}{a(ah-a+1)}:1\right],\\ 7Q=\left[\dfrac{a^3-2a^2-ah+2a-1}{a(ah-a+1)}:-\dfrac{(a-1)^2\left(a^2h+ah-a+1\right)}{(a^2+ah-2a+1)\,(ah-a+1)}:1\right], \end{array}$$

(notice that there is a misprint in the expression of the first component of $7Q$ given in [29]). Hence, from Equation (13) it is easy to see that there are no periodic orbits on the elliptic levels for $p=1,2,3,4$ and, assuming $a(a-1)\neq 0$, for $p=5$ and 6.

Observe that $7Q$ can only be an infinite point if either $a^2+ah-2a+1$ or $ah-a+1=0$. The first case trivially gives that $7Q\neq\mathscr{O}$, and the second case directly gives that $7Q=[a(a-1)^3:-a(a-1)^3:0]=\mathscr{O}$, thus

$$h_7:=\frac{a-1}{a}.$$

To obtain the other periods a simple way is to impose the relations $4Q=-4Q$ (period 8); $4Q=-5Q$ (period 9); $5Q=-5Q$ (period 10); $6Q=-5Q$ (period 11); and $7Q=-5Q$ (period 12). The points $-nQ$ are easily obtained from the points nQ because given a point P in a Lyness curve, it is straightforward to see that $-P$ is just the symmetric point with respect to $y=x$. So $-Q=[0:1:0]$; $-2Q=[0:-1:1]$;

** Notice that the above computations can be done by using that $Q*Q$ is obtained by substituting $y=-1$ into the expression of C_h, and in general $[x_0:y_0:1]*Q$ is obtained by substituting $y=y_0$ at the expression of C_h, and using also that $[x:y:z]*\mathscr{O}=[y:x:z]$ because of the symmetry of C_h with respect to $y=x$.

$-3Q=[-a:0:1]$;

$$-4Q=\left[\frac{ah-a+1}{a-1}:-a:1\right];\ \text{and}$$
$$-5Q=\left[\frac{-a^2-ah+2a-1}{a(a-1)}:\frac{ah-a+1}{a-1}:1\right].$$

To obtain the elliptic curves containing 9-periodic orbits we impose that $4Q=-5Q$, obtaining that $-a=(-a^2-ah+2a-1)/(a(a-1))$. This equality yields

$$h_9:=\frac{(a-1)(a^2-a+1)}{a},$$

To get the elliptic levels with period 12 orbits we impose that $7Q=-5Q$, obtaining

$$\begin{cases}\dfrac{a^3-2a^2-ah+2a-1}{(ah-a+1)a}=-\dfrac{a^2+ah-2a+1}{a(a-1)},\\ -\dfrac{(a-1)^2\left(a^2h+ah-a+1\right)}{(a^2+ah-2a+1)(ah-a+1)}=\dfrac{ah-a+1}{a-1}.\end{cases}$$

From these equations we obtain that h_{12} must be given by the roots of the polynomial $P(a,h)=a^2h^2+a(a-1)(a-3)h+\left(a^2-2a+2\right)(a-1)^2$, which gives the result. The other cases follow similarly. □

Finally, we remark that there are values of a for which the non elliptic levels (which correspond to genus 0 curves) are filled by periodic orbits with periods 1, 2 and 3, [29, Lemma 5]. Finally, there are no Lyness maps with 4-periodic orbits, although some authors consider that this period arises for the case $a=+\infty$, [54].

6 Rational periodic orbits

In this section we will assume that F is a birational map with rational coefficients and with an invariant foliation (2) such that the polynomials $P(x,y)-hQ(x,y)$ are in $\mathbb{Q}[x,y]$, and $h\in\mathbb{Q}$. In this case, it makes sense to study the *rational orbits* of F. That is, orbits such that all the iterates have rational coordinates.

In the previous sections we have seen the relationship between the dynamics of a birational map preserving an elliptic curve $C_h(\mathbb{R})$ and its group structure (Theorem 4 and Corollary 7). Under our new assumptions (F has rational coefficients) and assuming also that $\mathscr{O}$ has rational components, the rational orbits lie in the rational elliptic curves $C_h(\mathbb{Q})$ which is a subgroup of $C_h(\mathbb{R})$. In this case, the structure of each curve $C_h(\mathbb{Q})$ is characterized by the theorems of Mordell and Mazur, summarized below, and it will impose strong restrictions on the set of periods of rational orbits. Indeed, Mordell proved in 1922 that a rational elliptic curve is a finitely-generated abelian group, and in 1978 Mazur gave a description of its torsion term. The following result characterizes, therefore, the group structure of $C_h(\mathbb{Q})$.

Theorem 11 (Mazur, 1978) *If $\mathscr{E}$ is a non-singular cubic, then $(\mathscr{E}(\mathbb{Q}),+)$ is a finitely-generated abelian group*

$$\mathscr{E}(\mathbb{Q}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathrm{Tor}(\mathscr{E})$$

where $\mathrm{Tor}(\mathscr{E})$ *is either the empty set; or* $\mathbb{Z}/p$ *where* p *is either* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, *or* 12*; or* $\mathbb{Z}/2 \oplus \mathbb{Z}/p$ *where* p *is* 2, 4, 6 *or* 8.

Recalling that by Corollary 7, on certain invariant elliptic curve $C_h(\mathbb{Q})$ there will be periodic orbits of a birational map F if and only if the point Q of Theorem 4 is in $\mathrm{Tor}(C_h(\mathbb{Q}))$, we easily get that the only a priori allowed periods for rational orbits are the orders p described by the Mazur's Theorem. Hence:

Corollary 12 *Any birational map preserving $C_h(\mathbb{Q})$ only can have, a priori, rational periodic orbits of periods* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 *or* 12.

6.1 Rational periodic orbits of the Lyness maps

The study of the rational periodic orbits in the case of the Lyness maps and its relationship with the structure of rational points of elliptic curves can be traced back to the first papers that studied these maps from an algebraic geometric point of view [3, 5, 28] and [54]. After these works it was known that for $a > 0$ and considering *positive initial conditions* only the rational periods 5 and 9 where possible. It was known that 5-periodic orbits appear only when $a = 1$. However, the existence of rational 9-periodic orbits was not known, and it was left as an open problem in [5]. Their non-existence was conjectured in [54]. Now, from [29] we know that all the Mazur periods, except 4, appear for rational orbits and $a \in \mathbb{Q}^+ \cup \{0\}$ (but the periods different from 5 and 9 are not located in $\mathbb{Q}^+ \times \mathbb{Q}^+$).

Theorem 13 ([29]) *For any $p \in \{1,2,3,5,6,7,8,9,10,12\}$ there exist values of $a \in \mathbb{Q}^+ \cup \{0\}$ and rational initial conditions (x_0, x_1) giving rise to p-periodic orbits of the Lyness maps F_a. Moreover these values of p are the only possible minimal periods for rational initial conditions and $a \in \mathbb{Q}$.*

With respect to period 9, indeed there are 9-periodic rational orbits of the Lyness maps F_a with $a \in \mathbb{Q}^+$ and initial conditions in $\mathbb{Q}^+ \times \mathbb{Q}^+$. For instance, take $a = 7$ and the initial condition $(3/2, 5/7)$. Furthermore, the next result shows that there are infinitely many positive rational values of the parameter a giving rise to 9-periodic positive rational orbits. We sketch the proof because it is constructive and because again the basic arithmetic on an elliptic curve (different from the Lyness ones) plays an essential role in the construction of the periodic orbits. See [29] for more details.

Theorem 14 ([29]) *There are infinitely many values $a \in \mathbb{Q}^+$ and initial conditions $x_0(a), x_1(a) \in \mathbb{Q}^+$ giving rise to 9-periodic orbits of the Lyness map F_a.*

Proof. Using the characterization of the curve of 9-periodic orbits given in Proposition 10, it is easy to see that the proof of the result will follow if we find infinitely many points $(x,y,a) \in (\mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+) \cap S_a$, where

$$S_a := \{(a;x,y) : a(x+1)(y+1)(x+y+a) \\ -(a-1)(a^2-a+1)xy = 0, x>0, y>0, a>a_*\},$$

and $a_* \simeq 5.41147624$ is the infimum number a such that C_{h_9} has an oval in Q^+.

It can be proved that the points in S_a satisfying $x+y = 23/4$ are in an elliptic curve isomorphic to

$$\mathscr{E} := \left\{ \quad Y^2 = X^3 - \frac{1288423179}{71639296} X + \frac{8775405707427}{303177500672} \right\}.$$

This fact is not obvious. To obtain the expression of the curve $\mathscr{E}$ some changes of variables are needed, and the extra condition $x+y = 23/4$ is imposed on the points of S_a. This condition is motivated by the fact that if $(x,y) \in S_a$ is such that $(x+y, xy) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ and $\Delta := (x+y)^2 - 4xy$ is a perfect square, then $(x,y) \in \mathbb{Q}^+ \times \mathbb{Q}^+$. The condition is found when trying to obtain a suitable expression of Δ that facilitates to find values of a for which Δ is a perfect square. The reader is referred to [29, Proof of Theorem 2] to obtain all the details.

By taking into account Proposition 6, it is easy to see that if we are able to find a rational point $R \in \mathscr{E}$, such that R is not in the torsion of $\mathscr{E}$, then kR gives an infinite number of rational points in $\mathscr{E}$.

Observe that if the point $(x(R), y(R), a(R))$ corresponding to R is in the connected component of C_{h_9} in Q^+, then by recovering the values $(x(kR), y(kR), a(kR)) \in S_a$ corresponding to the points $kR \in \mathscr{E}$, we would get the result.

By using the software MAGMA [12], one can obtain the valid point $R = \left(\frac{18243}{8464}, \frac{81}{184}\right) \in \mathscr{E}$. □

6.2 A digression on numerics

A curious fact is that the numerical plots of the phase portrait of birational maps preserving an elliptic foliation typically contain very few periodic orbits (although sometimes it is possible to find traces of them). Bastien and Rogalski noticed this even when working with symbolic algebra software, [6]:

> If we wish to study possible periods with a computer, it is easier to work with rational numbers. So, we suppose that a is rational, and that the point (u_1, u_0) is rational. With the use of a computer and a program of calculation with fractions, is it possible to see periodic points? Only in few cases!

This is especially significant if one takes into account that Corollary 9 states that if a birational rational map preserving an elliptic foliation is not rigid, then the phase space is densely filled by invariant curves full of periodic orbits of an infinite number of periods. It is commonly thought that Mazur's Theorem (in fact Corollary 12) is the real reason for the lack of periodic orbits in the numerical simulations, but there is not a rigorous proof of this fact.

A priori one could think, however, that the lack of periodic orbits in the numerical simulations should be a consequence of other factors, like the fact that the rotation number is a piecewise analytic function. But this is not the case as Example 4 shows.

Indeed, the piecewise analyticity of the rotation function indicates how far we are from the general situation in the context of diffeomorphisms. For orientable diffeomorphisms of the circle, the persistence of the rotation number is known to hold, [2, Theorem A] and [41] (see also [1]). However there are diffeomorphisms with analytic rotation number for which it is easy to encounter periodic orbits when doing numerical simulations. This is the case, for instance, of proper Poncelet maps [15], such as the one shown in the next example.

Example 4. We consider the planar Poncelet map F associated to the ellipse $\gamma = \{x^2+xy+y^2-1/5=0\}$, and the family of circles $\Gamma(h)=\{x^2+y^2-h=0\}$, for $h > 2/5$ (observe that each curve $\Gamma(h)$ surrounds γ). The map F is the diffeomorphism defined in $\mathscr{U} = \{(x,y)\in\mathbb{R}^2, x^2+y^2>2/5\}$ in the following way: given any $p\in\Gamma(h)$ there are exactly two points q_1,q_2 in γ such that the lines pq_1, pq_2 are tangent to γ. On each curve $\Gamma(h)$ we define $F:\Gamma(h)\to\Gamma(h)$, associated to the pair as

$$F(p)=\overline{pq_1}\cap\Gamma(h),$$

where $p\in\Gamma(h)$, $\overline{pq_1}\cap\Gamma(h)$ is the first point in the set $\{\overline{pq_1}\cap\Gamma(h),\overline{pq_2}\cap\Gamma(h)\}$ that we find when, starting from p, we follow $\Gamma(h)$ counterclockwise (see Figure 3c).

It can be shown that the map has an expression of the form

$$F(x,y)=\left(\frac{-N_1N_2-4N_3\sqrt{\Delta}}{M},\frac{-N_1N_3+4N_2\sqrt{\Delta}}{M}\right),$$

where N_i, M and Δ are large polynomials whose expression can be found in [15]. Observe that F is not a birational map. Additionally, the map F has the first integral $V(x,y)=x^2+y^2$ defined in $\mathscr{U}$, and on each curve $\Gamma(h)=\{V=h\}$ the map is conjugate to a rotation with certain rotation number $\theta(h)$. This rotation function is analytic in the interval $(2/5,\infty)$ because F has the Lie Symmetry given by

$$X(x,y)=\sqrt{(x^2+y^2)(x^2+xy+y^2-1/5)}\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right),$$

so by Theorem 8 the rotation number can be obtained using Equation (10). See [15] for proofs of all the above facts.

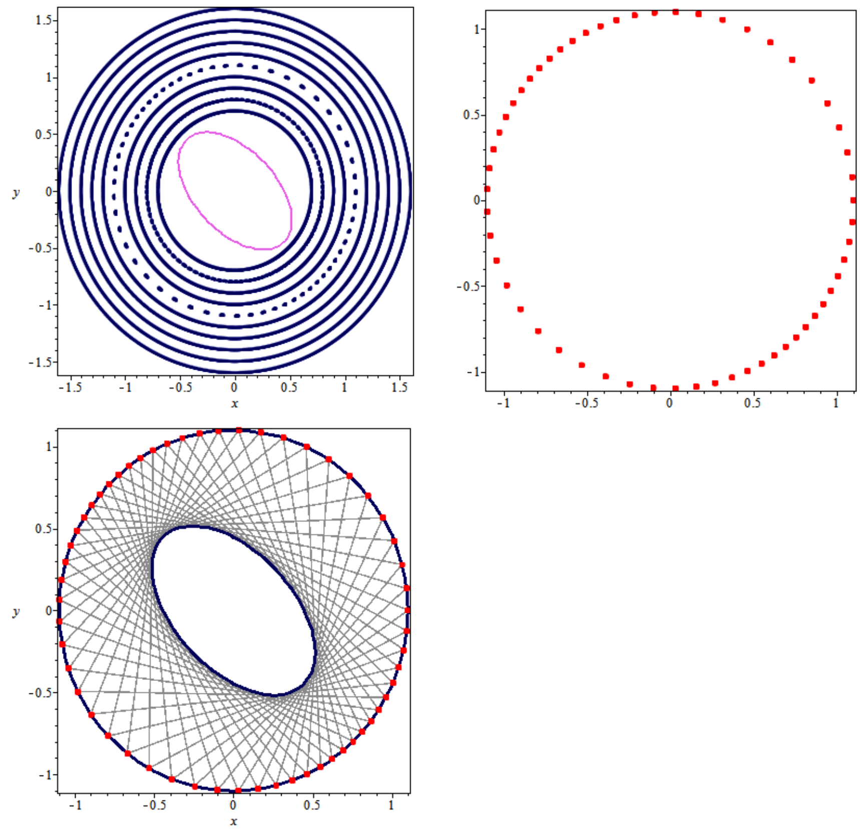

Fig. 3 (a) 5000 iterates of F with the grid of initial conditions $\mathscr{I}$. (b) 19950 iterates with initial condition $(1.1000045,0)$. (c) The Poncelet's construction of the 57-periodic orbit.

Now we show how some evidence of periodic orbits appears when considering a numerical experiment. This particular one is obtained after 5000 iterates by F from each initial condition in the set $\mathscr{I}=\{(x_j,0)=(0.7+(j-1)/10,0),\, j=1,\ldots,10\}$. As can be seen in Figure 3a, there is numerical evidence that the initial condition $(x_5,0)$ is near to a 57-periodic orbit. After some approximations we locate the circle full of 57-periodic orbits close to the curve $x^2+y^2\simeq 1.2100099$ (for instance, by choosing $p_0:=(1.1000045,0)$ and setting $p_n=F^n(p_0)$, we have that after 350 periods $|p_0-p_{19950}|\simeq 0.003$. These iterates are depicted in Figure 3b. In Figure 3c the construction of the 57-periodic orbit via the Poncelet process is shown.

7 Some words on the genus 0 case

With respect to the global dynamics of birational maps preserving a genus 0 invariant foliation (2) no general results are known, although one should expect to have also a large number of curves with periodic orbits of arbitrary period, curves filled with dense solutions, as well as curves with one or two attractive and/or repulsive points. An example of what is expected to be found is given in the first example of [6]. We are now working on this problem together with M. Llorens.

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Appendix. The Lyness map and curves: a bit more than an academic example.

The study of the Lyness map (3) has a long history. It started with the study of the 5-cycle corresponding to the particular case $a = 1$ in the Lyness equation. According to Linero [36],

> It is precisely under this aspect that Lyness cycle appeared: in fact, Gauss obtained it when working in the spherical geometry of the *pentagrama mirificum*, a spherical pentagram formed by five successively orthogonal great-circle arcs. To see its construction and the relation with the 5-cycle, the reader can consult [23]. According to this paper: "This 5-cycle seems to have been transmitted in the form of mathematical gossip for a long time". The 5-cycle receives the name of Lyness cycle because R.C. Lyness accounted for it in a series of papers dealing with the existence of cycles (see [37, 38, 39] and also [32]). Surprisingly, the interest of Lyness was associated neither to dynamical systems nor difference equations, he found the equation while investigating a problem related to the number theory: to obtain three integer numbers such that the sum or the difference of any different pair of them is a square. The first time that the equation is referred to as the "Lyness equation" occurred in 1961, in [48].

The study of the Lyness map has attracted the attention of the dynamical systems community in the last years. Its dynamics is completely understood after the independent work done by Bastien and Rogalski [5] and Zeeman [54], and the work of Beukers and Cushman [10]. See also [3, 26, 28] and [29].

The 5-periodic Lyness map F_1 plays also a structural role when studying the group of symplectic birational transformations of the plane (i.e. birational transformations of $\mathbb{C}^2$ which preserve the differential form $\omega = dx \wedge dy/(xy)$), since this group is generated by compositions of the Lyness map F_1, a scaling, and a map of the form $(x,y) \to (x^a y^b, x^c y^d)$ where a,b,c and d are such that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

This result has been recently proved by Blanc in [11], and it was conjectured by Usnich in [52].

From the algebraic view point, it is also interesting to note that the Lyness curves are a universal normal form for most elliptic curves with a relatively easy form for the addition formula $n \cdot P$. Recall that the addition formula is useful for instance in Elliptic Curve Cryptography [35] and is generally complicated.

Theorem 15 (Lyness' curves normal form, [29]) *The family of elliptic curves* $C_{a,h} = \{(x+1)(y+1)(x+y+a) - hxy = 0\}$ *over any field* $\mathbb{K}$ *(not of characteristic* 2 *or* 3*) together with the points* $\mathscr{O} = [1:-1:0]$ *and* $Q = [1:0:0]$*, is the universal family of elliptic curves with a point of order* n*,* $n \geq 5$ *(including* $n = \infty$*).*

The above result states that for any elliptic curve $\mathscr{E}(\mathbb{K})$ with a point R of order $n \geq 5$, there exists some unique values $a_{(\mathscr{E},R)}, h_{(\mathscr{E},R)} \in \mathbb{K}$ and a unique isomorphism between $\mathscr{E}$ and $C_{a_{(\mathscr{E},R)},h_{(\mathscr{E},R)}}$ sending the neutral element of $\mathscr{E}$ to $\mathscr{O} = [1:-1:0]$ and R to $Q = [1:0:0]$.

This also implies that the known results on elliptic curves with a point of order greater than 4 also holds in Lyness curves. In particular we find the curves with high rank and prescribed torsion given in Dujella's site, [27].

To prove the above result it is only needed to observe that any elliptic curve having a point R that is not a 2 or a 3 torsion point can be brought to the *Tate normal form*

$$Y^2Z + (1-c)XYZ - bYZ^2 = X^3 - bX^2Z.$$

where R is sent to $(0,0)$. But on the other hand the change of variables

$$X = bz, \quad Y = bc(y+z), \quad Z = c(x+y) + (c+1)z$$

and the relations

$$h = -\frac{b}{c^2}, \quad a = \frac{c^2+c-b}{c^2},$$

show that the curves $C_{a,h} = \{(x+z)(y+z)(x+y+az) - hxyz = 0\}$ and the Tate normal form are equivalent. Finally, observe that the case $c = 0$ corresponds to a curve with a 4-torsion point so, as seen in Section 5, it does not correspond to an elliptic Lyness curve.

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