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The averaging theory for periodic orbits, and a brief summary on the 16–Hilbert problem

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Preface

The method of averaging is a classical tool that allows to study the dynamics of the nonlinear *differential systems* under periodic forcing. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [45]. Important practical and theoretical contributions to the averaging theory were made in the 1930's by Bogoliubov and Krylov [12], in 1945 by Bogoliubov [11], and by Bogoliubov and Mitropolsky [13] (English version 1961). For a more modern exposition of the averaging theory see the book of Sanders, Verhulst and Murdock [107].

Every orbit of a differential system is homeomorphic either to a point, or to a circle, or to a straight line. In the first case it is called a *singular point* or an *equilibrium point* and in the second case it is called a *periodic orbit* . The third case does not have a name. These notes are dedicated to study analytically the periodic orbits of a given differential system.

We consider differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 R(t,\mathbf{x},\varepsilon), \tag{1}$$

with $\mathbf{x}$ in some open subset D of $\mathbb{R}^n$, $F_i\colon \mathbb{R}\times D \to \mathbb{R}^n$ of class C^2 for $i = 1, 2$, $R\colon \mathbb{R}\times D\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ of class C^2 with $\varepsilon_0 > 0$ small, the functions F_i and R are T–periodic in the variable t. Here the dot denotes derivative with respect to the time t.

In general to obtain analytically periodic solutions of a differential system is a very difficult problem, many times a problem impossible to solve. As we shall see when we can apply the averaging theory this difficult problem for the differential systems (1) is reduced to find the zeros of a nonlinear function of dimension at most n, i.e. now the problem has the same difficulty of the problem of finding the singular or equilibrium points of a differential system.

An important problem for studying the periodic solutions of the differential systems of the form

$$\dot{\mathbf{x}} = F(t,\mathbf{x}), \quad \text{or} \quad \dot{\mathbf{x}} = F(\mathbf{x}), \tag{2}$$

using the averaging theory is to transform them in systems written in the *normal form of the averaging theory* , i.e. as a system (1). Note that systems (2), in general, are not periodic in the independent variable t and do not have any small parameter ε. So we must find changes of variables which allow to write the differential systems (2) into the form (1) where F_0 eventually can be zero.

These notes are divided in three chapters. Chapter 1 is dedicated to the averaging theory of first order, we present in it three main results for studying the periodic solutions of the differential systems, see Theorems 1.1.1, 1.3.1 and 1.5.1. We do four applications of Theorems 1.1.1, namely to van der Pol equation, to the Liénard differential system, to study the zero–Hopf bifurcation in $\mathbb{R}^n$, and to a class of Hamiltonian systems. We present three applications of Theorem 1.3.1, in the first we study the Hopf bifurcation of the Michelson system, in the second the periodic solutions of a third–order differential equation, and in the third we analyze the periodic solutions of the Vallis system which models "El Niño" phenomenon. Finally we do an application of Theorem 1.5.1 to a class of Duffing differential equation.

In Chapter 2 we present the averaging theory for studying the periodic solutions of a differential system in $\mathbb{R}^n$ at any order in the small parameter. This theory is developed using the weaker assumptions. This is the more theoretical chapter of this work.

In Chapter 3 we provide some applications of the averaging theory of order higher than one. Thus using the averaging theory of second order we study the periodic solutions of the Hénon–Heiles Hamiltonian, and using the averaging theory of third order we study first the limit cycles of the quadratic polynomial differential systems, and of the linear with cubic homogeneous nonlinearities polynomial differential systems; and finally we analyze the periodic solutions of the generalized Liénard polynomial differential equations.

In the last Chapter 4 we do a brief summary on some of the known results on the second part of the 16–th Hilbert problem, which essentially are on the number and configurations of limit cycles of the polynomial differential systems in the plane in function of the degree of these systems. As we shall see we almost do not know nothing about the the number of limit cycles, and we know something more on the possible topological configurations of these limit cycles.

Chapter 1

Introduction. The classical theory

1.1 A first order averaging method for periodic orbits

We consider the differential system

$$\dot{\mathbf{x}} = \varepsilon F(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon), \tag{1.1}$$

with $\mathbf{x} \in D \subset \mathbb{R}^n$, D a bounded domain, and $t \geq 0$. Moreover we assume that $F(t, \mathbf{x})$ and $R(t, \mathbf{x}, \varepsilon)$ are T–periodic in t.

The *averaged system* associated to system (1.1) is defined by

$$\dot{\mathbf{y}} = \varepsilon f^0(\mathbf{y}), \tag{1.2}$$

where

$$f^0(\mathbf{y}) = \frac{1}{T} \int_0^T F(s, \mathbf{y}) ds. \tag{1.3}$$

The next theorem says under which conditions the singular points of the averaged system (1.2) provide T–periodic orbits of system (1.1). The proof presented here comes from [119].

Theorem 1.1.1. *We consider system* (1.1) *and assume that the vector functions F, R, $D_{\mathbf{x}}F$, $D^2_{\mathbf{x}}F$ and $D_{\mathbf{x}}R$ are continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times D$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Moreover, we suppose that F and R are T–periodic in t, with T independent of ε.*

(a) *If $p \in D$ is a singular point of the averaged system* (1.2) *such that*

$$\det(D_{\mathbf{x}} f^0(p)) \neq 0, \tag{1.4}$$

then for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ of system (1.1) *such that $\mathbf{x}(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *If the singular point* $\mathbf{y} = p$ *of the averaged system* (1.2) *has all its eigenvalues with negative real part then, for* $|\varepsilon| > 0$ *sufficiently small, the corresponding periodic solution* $\mathbf{x}(t, \varepsilon)$ *of system* (1.1) *is asymptotically stable, and if one of the eigenvalues has positive real part* $\mathbf{x}(t, \varepsilon)$ *is unstable.*

Theorem 1.1.1 is proved in section 1.6, before its proof we shall present some applications of it in section 1.2.

For each $\mathbf{z} \in D$ we denote by $\mathbf{x}(\cdot, \mathbf{z}, \varepsilon)$ the solution of (1.1) with the initial condition $\mathbf{x}(0, \mathbf{z}, \varepsilon) = z$. We consider also the function $\zeta : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ defined by

$$\zeta(\mathbf{z}, \varepsilon) = \int_0^T \left[\varepsilon F(t, \mathbf{x}(t, \mathbf{z}, \varepsilon)) + \varepsilon^2 R(t, \mathbf{x}(t, \mathbf{z}, \varepsilon), \varepsilon)\right] dt. \tag{1.5}$$

From (1.1) it follows for every $\mathbf{z} \in D$ that

$$\zeta(\mathbf{z}, \varepsilon) = \mathbf{x}(T, \mathbf{z}, \varepsilon) - \mathbf{x}(0, \mathbf{z}, \varepsilon). \tag{1.6}$$

The function ζ can be written in the form

$$\zeta(\mathbf{z}, \varepsilon) = \varepsilon f^0(\mathbf{z}) + O(\varepsilon^2), \tag{1.7}$$

where f^0 is given by (1.3). Moreove,r under the assumptions of Theorem 1.1.1 the solution $\mathbf{x}(t, \varepsilon)$, for $|\varepsilon|$ sufficiently small, satisfies that $\mathbf{z}_\varepsilon = \mathbf{x}(0, \varepsilon)$ tends to be an isolated zero of $\zeta(\cdot, \varepsilon)$ when $\varepsilon \to 0$. Of course, due to (1.6) the function ζ is a *displacement function* for system (1.1), and its fixed points are initial conditions for the T–periodic solutions of system (1.1).

1.2 Four applications

We recall that a *limit cycle* of a differential system is a periodic orbit isolated in the set of all periodic orbits of the system.

1.2.1 The van der Pol differential equation

Consider the *van der Pol differential equation*

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x},$$

which can be written as the differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon(1 - x^2)y. \end{aligned} \tag{1.8}$$

In polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, this system becomes

$$\begin{aligned} \dot{r} &= \varepsilon r(1 - r^2\cos^2\theta)\sin^2\theta, \\ \dot{\theta} &= -1 + \varepsilon\cos\theta(1 - r^2\cos^2\theta)\sin\theta, \end{aligned}$$

or equivalently

$$\frac{dr}{d\theta} = -\varepsilon r(1 - r^2 \cos^2 \theta) \sin^2 \theta + O(\varepsilon^2).$$

Note that the previous differential system is in the normal form (1.1) for applying the averaging theory described in Theorem 1.1.1 if we take $\mathbf{x} = r$, $t = \theta$, $T = 2\pi$ and $F(t, \mathbf{x}) = -r(1 - r^2 \cos^2 \theta) \sin^2 \theta$.

From (1.3) we get that

$$f^0(r) = -\frac{1}{2\pi} \int_0^{2\pi} r(1 - r^2 \cos^2 \theta) \sin^2 \theta d\theta = \frac{1}{8} r(r^2 - 4).$$

The unique positive root of $f^0(r)$ is $r = 2$. Since $(df^0/dr)(2) = 1$, by statement (a) of Theorem 1.1.1, it follows that system (1.8) has for $|\varepsilon| \neq 0$ sufficiently small a limit cycle bifurcating from the periodic orbit of radius 2 of the unperturbed system (1.8) with $\varepsilon = 0$. Moreover since $(df^0/dr)(2) = 1 > 0$, by statement (b) of Theorem 1.1.1, this limit cycle is unstable.

1.2.2 The Liénard differential system

The following result is due to Lins, de Melo and Pugh [71]. Here we provide an easy and shorter proof with respect to the initial proof given by the mentioned authors.

Proposition 1.2.1. *The Liénard differential systems of the form*

$$\begin{aligned} \dot{x} &= y - \varepsilon(a_1 x + \cdots + a_n x^n), \\ \dot{y} &= -x, \end{aligned}$$

with ε sufficiently small and $a_n \neq 0$ have at most $[(n-1)/2]$ limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, and there are examples with exactly $[(n-1)/2]$ limit cycles. Here $[\cdot]$ denotes the integer part function.

Proof. We write system

$$\dot{x} = y - \varepsilon(a_1 x + \cdots + a_n x^n), \qquad \dot{y} = -x,$$

in polar coordinates (r, θ) where $x = r \cos \theta$, $y = r \sin \theta$, and we obtain

$$\begin{aligned} \dot{r} &= -\varepsilon \sum_{k=1}^{n} a_k r^k \cos^{k+1} \theta, \\ \dot{\theta} &= -1 + \varepsilon \sin \theta \sum_{k=1}^{n} a_k r^{k-1} \cos^k \theta, \end{aligned}$$

or equivalently

$$\frac{dr}{d\theta} = -\varepsilon \sum_{k=1}^{n} a_k r^k \cos^{k+1} \theta + O(\varepsilon^2).$$

Again taking $\mathbf{x} = r$, $t = \theta$, $T = 2\pi$ and $F(t, \mathbf{x}) = -\sum_{k=1}^{n} a_k r^k \cos^{k+1}\theta$, the previous differential system is in the normal form (1.1) for applying the averaging theory described in Theorem 1.1.1.

We have that

$$f^0(r) = -\frac{1}{2\pi}\sum_{k=1}^{n} a_k r^k \int_0^{2\pi} \cos^{k+1}\theta\, d\theta = -\frac{\varepsilon}{2\pi}\sum_{\substack{k=1\\ k\,\text{odd}}}^{n} a_k b_k r^k = p(r),$$

where $b_k = \int_0^{2\pi} \cos^{k+1}\theta\, d\theta \neq 0$ if k is odd, and $b_k = 0$ if k is even. Now we apply Theorem 1.1.1, since the polynomial $p(r)$ has at most $[(n-1)/2]$ positive roots, and we can choose the coefficients a_k with k odd in such a way that $p(r)$ has exactly $[(n-1)/2]$ simple positive roots, the proposition follows. □

1.2.3 Zero–Hopf bifurcation in $\mathbb{R}^n$

In this example we study a zero–Hopf bifurcation of C^3 differential systems in $\mathbb{R}^n$ with $n \geq 3$. The results on this example come from Llibre and Zhang [83].

We assume that these systems have a singularity at the origin, whose linear part has eigenvalues $\varepsilon a \pm bi$ with $b \neq 0$ and εc_k for $k = 3, \ldots, n$, where ε is a small parameter. Since the eigenvalues of the linearization at the origin when $\varepsilon = 0$ are $\pm bi \neq 0$ and 0 with multiplicity $n-2$, if an infinitesimal periodic orbit bifurcates from the origin when $\varepsilon = 0$ we call such a kind of bifurcation a *zero–Hopf bifurcation*. Such systems can be written into the form

$$\begin{aligned}
\dot{x} &= \varepsilon a x - b y + \sum_{i_1+\ldots+i_n=2} a_{i_1\ldots i_n} x^{i_1} y^{i_2} z_3^{i_3} \ldots z_n^{i_n} + \mathcal{A},\\
\dot{y} &= b x + \varepsilon a y + \sum_{i_1+\ldots+i_n=2} b_{i_1\ldots i_n} x^{i_1} y^{i_2} z_3^{i_3} \ldots z_n^{i_n} + \mathcal{B},\\
\dot{z}_k &= \varepsilon c_k z_k + \sum_{i_1+\ldots+i_n=2} c^{(k)}_{i_1\ldots i_n} x^{i_1} y^{i_2} z_3^{i_3} \ldots z_n^{i_n} + \mathcal{C}_k, \quad k = 3, \ldots, n
\end{aligned} \tag{1.9}$$

where $a_{i_1\ldots i_n}$, $b_{i_1\ldots i_n}$, $c^{(k)}_{i_1\ldots i_n}$, a, b and c_k are real parameters, $ab \neq 0$, and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_k$ are the Lagrange expression of the error function of third order in the expansion of the functions of the system in Taylor series.

Theorem 1.2.2. *There exist C^3 systems (1.9) for which $l \in \{0, 1, \ldots, 2^{n-3}\}$ limit cycles bifurcate from the origin at $\varepsilon = 0$, i.e. for ε sufficiently small the system has exactly l limit cycles in a neighborhood of the origin and these limit cycles tend to the origin when $\varepsilon \searrow 0$.*

As far as we know in Theorem 1.2.2 was the first time that it is proved that the number of limit cycles that can bifurcate in a Hopf bifurcation increases exponentially with the dimension of the space. We recall that a *Hopf bifurcation* takes place when one or several limit cycles bifurcate from an equilibrium point.

From the proof of Theorem 1.2.2 it follows immediately the next result.

Corollary 1.2.3. *There exist quadratic polynomial differential systems* (1.9) (*i.e. with* $\mathcal{A} = \mathcal{B} = \mathcal{C}_k = 0$) *for which* $l \in \{0, 1, \ldots, 2^{n-3}\}$ *limit cycles bifurcate from the origin at* $\varepsilon = 0$, *i.e. for* ε *sufficiently small the system has exactly* l *limit cycles in a neighborhood of the origin and these limit cycles tend to the origin when* $\varepsilon \searrow 0$.

Proof of Theorem 1.2.2. Doing the cylindrical change of coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z_i = z_i, \quad i = 3, \ldots, n, \tag{1.10}$$

in the region $r > 0$ system (1.9) becomes

$$\begin{aligned}
\dot{r} &= \varepsilon a r + \sum_{i_1+\ldots+i_n=2} (a_{i_1\ldots i_n}\cos\theta + b_{i_1\ldots i_n}\sin\theta)(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3),\\
\dot{\theta} &= \frac{1}{r}\left[br + \sum_{i_1+\ldots+i_n=2} (b_{i_1\ldots i_n}\cos\theta - a_{i_1\ldots i_n}\sin\theta)(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3)\right],\\
\dot{z}_k &= \varepsilon c_k z_k + \sum_{i_1+\ldots+i_n=2} c^{(k)}_{i_1\ldots i_n}(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3), \quad k = 3, \ldots, n,
\end{aligned} \tag{1.11}$$

where $O(3) = O_3(r, z_3, \ldots, z_n)$.

As usual $\mathbb{Z}_+$ denotes the set of all non–negative integers. Taking $a_{00e_{ij}} = b_{00e_{ij}} = 0$ where $e_{ij} \in \mathbb{Z}_+^{n-2}$ has the sum of the entries equal to 2, it is easy to show that in a suitable small neighborhood of $(r, z_3, \ldots, z_n) = (0, 0, \ldots, 0)$ we have $\dot{\theta} \neq 0$. Then choosing θ as the new independent variable system (1.11) in a neighborhood of $(r, z_3, \ldots, z_n) = (0, 0, \ldots, 0)$ becomes

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{r\left(\varepsilon a r + \sum\limits_{i_1+\ldots+i_n=2} (a_{i_1\ldots i_n}\cos\theta + b_{i_1\ldots i_n}\sin\theta)(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3)\right)}{br + \sum\limits_{i_1+\ldots+i_n=2} (b_{i_1\ldots i_n}\cos\theta - a_{i_1\ldots i_n}\sin\theta)(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3)},\\
\frac{dz_k}{d\theta} &= \frac{r\left(\varepsilon c_k z_k + \sum\limits_{i_1+\ldots+i_n=2} c^{(k)}_{i_1\ldots i_n}(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3)\right)}{br + \sum\limits_{i_1+\ldots+i_n=2} (b_{i_1\ldots i_n}\cos\theta - a_{i_1\ldots i_n}\sin\theta)(r\cos\theta)^{i_1}(r\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} + O(3)},
\end{aligned} \tag{1.12}$$

for $k = 3, \ldots, n$. We note that this system is 2π periodic in the variable θ.

In order to write system (1.12) in the normal form of the averaging theory we rescale the variables

$$(r, z_3, \ldots, z_n) = (\rho\varepsilon, \eta_3\varepsilon, \ldots, \eta_n\varepsilon). \tag{1.13}$$

Then system (1.12) becomes

$$\begin{aligned} \frac{d\rho}{d\theta} &= \varepsilon f_1(\theta,\rho,\eta_3,\ldots,\eta_n) + \varepsilon^2 g_1(\theta,\rho,\eta_3,\ldots,\eta_n,\varepsilon), \\ \frac{d\eta_k}{d\theta} &= \varepsilon f_k(\theta,\rho,\eta_3,\ldots,\eta_n) + \varepsilon^2 g_k(\theta,\rho,\eta_3,\ldots,\eta_n,\varepsilon), \quad k=3,\ldots,n, \end{aligned} \tag{1.14}$$

where

$$f_1 = \frac{1}{b}\left(a\rho + \sum_{i_1+\ldots+i_n=2} (a_{i_1\ldots i_n}\cos\theta + b_{i_1\ldots i_n}\sin\theta)(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} \right),$$

$$f_k = \frac{1}{b}\left(c\eta_k + \sum_{i_1+\ldots+i_n=2} c^{(k)}_{i_1\ldots i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2} z_3^{i_3}\ldots z_n^{i_n} \right).$$

We note that system (1.14) has the form of the normal form (1.1) of the averaging theory with $\mathbf{x}=(\rho,\eta_3,\ldots,\eta_n)$, $t=\theta$, $F(\theta,\rho,\eta_3,\ldots,\eta_n)=(f_1(\theta,\rho,\eta_3,\ldots,\eta_n),$ $f_3(\theta,\rho,\eta_3,\ldots,\eta_n),\ldots,f_n(\theta,\rho,\eta_3,\ldots,\eta_n))$ and $T=2\pi$. The averaged system of (1.14) is

$$\dot y = \varepsilon f^0(y), \qquad y = (\rho,\eta_3,\ldots,\eta_n)\in\Omega, \tag{1.15}$$

where Ω is a suitable neighborhood of the origin $(\rho,\eta_3,\ldots,\eta_n)=(0,0,\ldots,0)$, and

$$f^0(y) = (f_1^0(y), f_3^0(y),\ldots,f_n^0(y)),$$

with

$$f_i^0(y) = \frac{1}{2\pi}\int_0^{2\pi} f_i(\theta,\rho,\eta_3,\ldots,\eta_n)d\theta, \qquad i=1,3,\ldots,n.$$

After some calculations we have that

$$f_1^0 = \frac{1}{2b}\rho\left(2a + \sum_{j=3}^n (a_{10e_j} + b_{01e_j})\eta_j\right),$$

$$f_k^0 = \frac{1}{2b}\left(2c_k\eta_k + \left(c^{(k)}_{20\mathbf{0}_{n-2}} + c^{(k)}_{02\mathbf{0}_{n-2}}\right)\rho^2 + 2\sum_{3\le i\le j\le n} c^{(k)}_{00e_{ij}}\eta_i\eta_j\right), \quad k=3,\ldots,n,$$

where $e_j\in\mathbb{Z}_+^{n-2}$ is the unit vector with the jth entry equal to 1, and $e_{ij}\in\mathbb{Z}_+^{n-2}$ has the sum of the ith and jth entries equal to 2 and the other equal to 0.

Now we shall apply Theorem 1.1.1 for studying the limit cycles of system (1.14). Note that these limits after the rescaling (1.13) will become infinitesimal limit cycles for system (1.12), which will tend to origin when $\varepsilon \searrow 0$, consequently they will be bifurcated limit cycles of the Hopf bifurcation of system (1.12) at the origin.

From Theorem 1.1.1 for studying the limit cycles of system (1.14) we only need to compute the non–degenerate singularities of system (1.15). Since the transformation from the cartesian coordinates $(r,z_3,\ldots,z_n)$ to the cylindrical ones

$(\rho, \eta_3, \ldots, \eta_n)$ is not a diffeomorphism at $\rho = 0$, we deal with the zeros having the coordinate $\rho > 0$ of the averaged function f^0. So we need to compute the roots of the algebraic equations

$$\begin{aligned}
&2a + \sum_{j=3}^{n} (a_{10e_j} + b_{01e_j})\eta_j = 0, \\
&2c_k\eta_k + \left(c^{(k)}_{2\mathbf{0}\mathbf{0}_{n-2}} + c^{(k)}_{0\mathbf{2}\mathbf{0}_{n-2}}\right)\rho^2 + 2\sum_{3\le i\le j\le n} c^{(k)}_{00e_{ij}}\eta_i\eta_j = 0, \quad k = 3, \ldots, n.
\end{aligned} \tag{1.16}$$

Since the coefficients of system (1.16) are independent and arbitrary. In order to simplify the notation we write system (1.16) as

$$a + \sum_{j=3}^{n} a_j\eta_j = 0, \quad c_0^{(k)}\rho^2 + c_k\eta_k + \sum_{3\le i\le j\le n} c_{ij}^{(k)}\eta_i\eta_j = 0, \quad k = 3, \ldots, n, \tag{1.17}$$

where $a_j, c_0^{(k)}, c_k$ and $c_{ij}^{(k)}$ are arbitrary constants.

Denote by $\mathcal{C}$ the set of algebraic systems of form (1.17). We claim that there is a system belonging to $\mathcal{C}$ which has exactly 2^{n-3} simple roots. The claim can be verified by the example:

$$a + a_3\eta_3 = 0, \tag{1.18}$$

$$c_0^{(3)}\rho^2 + c_3\eta_3 + \sum_{3\le i\le j\le n} c_{ij}^{(3)}\eta_i\eta_j = 0, \tag{1.19}$$

$$c_k\eta_k + \sum_{3\le i\le j\le k} c_{ij}^{(k)}\eta_i\eta_j = 0, \qquad k = 4, \ldots, n, \tag{1.20}$$

with all the coefficients non–zero. Equations (1.20) can be treated as quadratic algebraic equations in η_k. Substituting the unique solution η_{30} of η_3 in (1.18) into (1.20) with $k = 4$, then this last equation has exactly two different solutions η_{41} and η_{42} for η_4 choosing conveniently c_4. Introducing the two solutions (η_{30}, η_{4i}), $i = 1, 2$, into (1.20) with $k = 5$ and choosing conveniently the values of the coefficients of equation (1.20) with $k = 5$ and $(\eta_3, \eta_4) = (\eta_{30}, \eta_{4i})$ we get two different solutions η_{5i1} and η_{5i2} of η_5 for each i. Moreover playing with the coefficients of the equations, the four solutions $(\eta_{30}, \eta_{4i}, \eta_{5ij})$ for $i, j = 1, 2$, are distinct. By induction we can prove that for suitable choice of the coefficients equations (1.18) and (1.20) have 2^{n-3} different roots $(\eta_3, \ldots, \eta_n)$. Since $\eta_3 = \eta_{30}$ is fixed, for any given $c_{ij}^{(3)}$ there exist values of c_3 and $c_0^{(3)}$ such that equation (1.19) has a positive solution ρ for each of the 2^{n-3} solutions $(\eta_3, \ldots, \eta_n)$ of (1.18) and (1.20). Since the 2^{n-3} solutions are different, and the number of the solutions of (1.18)-(1.20) is the maximum that the equations can have (by the Bezout Theorem, see for instance [110]), it follows that every solution is simple, and consequently the determinant of the Jacobian of the system evaluated at it is not zero. This proves the claim.

Using the same arguments which allowed to prove the claim, we also can prove that we can choose the coefficients of the previous system in order that it has $0, 1, \ldots, 2^{n-3} - 1$ simple real solutions.

Taking the averaged system (1.15) with f^0 having the convenient coefficients as in (1.18)-(1.20), the averaged system (1.15) has exactly $k \in \{0, 1, \ldots, 2^{n-3}\}$ singularities with the components $\rho > 0$. Moreover the determinants of the Jacobian matrix $\partial f^0/\partial y$ at these singularities do not vanish, because all the singularities are simple. In short. by Theorem 1.1.1 we get that there are systems of form (1.9) which have $k \in \{0, 1, \ldots, 2^{n-3}\}$ limit cycles. This proves the theorem. □

1.2.4 An application to Hamiltonian systems

The results of this subsection come from the paper of Guirao, Llibre and Vera [50].

We consider the following class of Hamiltonians in the action–angle variables

$$\mathcal{H}(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n) = \mathcal{H}_0(I_1) + \varepsilon \mathcal{H}_1(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n), \tag{1.21}$$

where ε is a small parameter. For more details on the action–angle variables see for instance [3].

As usual the *Poisson bracket* of the functions $f(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$ and $g(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$ is

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial I_i} - \frac{\partial f}{\partial I_i} \frac{\partial g}{\partial \theta_i} \right).$$

The next result provides sufficient conditions for computing periodic orbits of the Hamiltonian system associated to the Hamiltonian (1.21).

Theorem 1.2.4. *We define*

$$\langle \mathcal{H}_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n) d\theta_1,$$

and we consider the differential system

$$\begin{aligned} \frac{dI_i}{d\theta_1} &= \varepsilon \frac{\{I_i, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))} = \varepsilon f_{i-1}(I_2, \ldots, I_n, \theta_2, \ldots, \theta_n) \quad i = 2, ..., n, \\ \frac{d\theta_i}{d\theta_1} &= \varepsilon \frac{\{\theta_i, \langle \mathcal{H}_1 \rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))} = \varepsilon f_{i+n-2}(I_2, \ldots, I_n, \theta_2, \ldots, \theta_n) \quad i = 2, \ldots, n, \end{aligned} \tag{1.22}$$

restricted to the energy level $\mathcal{H} = h^*$ with $h^* \in \mathbb{R}$. The value h^* is such that the function $\mathcal{H}_0^{-1}$ in a neighborhood of h^* is a diffeomorphism. System (1.22) is a Hamiltonian system with Hamiltonian $\varepsilon \langle \mathcal{H}_1 \rangle$. If $\varepsilon \neq 0$ is sufficiently small then *for every equilibrium point* $p = (I_2^0, \ldots, I_n^0, \theta_2^0, \ldots, \theta_n^0)$ *of system* (1.22) *satisfying that*

$$\det\left(\frac{\partial(f_1, \ldots, f_{2n-2})}{\partial(I_2, \ldots, I_n, \theta_2, \ldots, \theta_n)}\bigg|_{(I_2,\ldots,I_n,\theta_2,\ldots,\theta_n)=(I_2^0,\ldots,I_n^0,\theta_2^0,\ldots,\theta_n^0)} \right) \neq 0,$$

there exists a 2π*–periodic solution* $\gamma_\varepsilon(\theta, \ldots, I_n(\theta_1, \varepsilon), \theta_2(\theta_1, \varepsilon), \ldots, \theta_n(\theta_1, \varepsilon))$ *of the Hamiltonian system associated to the Hamiltonian* (1.21) *taking as independent variable the angle* θ_1 *such that* $\gamma_\varepsilon(0) \to (\mathcal{H}_0^{-1}(h^*), I_2^0, \ldots, I_n^0, \theta_2^0, \ \ldots, \theta_n^0)$ *when* $\varepsilon \to 0$. *The stability or instability of the periodic solution* $\gamma_\varepsilon(\theta_1)$ *is given by the stability or instability of the equilibrium point* p *of system* (1.22). *In fact, the equilibrium point* p *has the stability behavior of the Poincaré map associated to the periodic solution* $\gamma_\varepsilon(\theta_1)$.

Now we clarify some of the notations used in the statement of Theorem 1.2.4. We have that the function $\mathcal{H}_0$ is only function of the variable I_1, i.e. $\mathcal{H}_0 \colon J \to \mathbb{R}$ where J is an open subset of $\mathbb{R}$ (the domain of definition of $\mathcal{H}_0$), and consequently $\mathcal{H}_0(I_1) \in \mathbb{R}$. Therefore $\mathcal{H}_0'$ means derivative with respect to the variable I_1.

The differential system (2) is defined on the energy level $\mathcal{H}(I_1,\ldots, I_n, \theta_1,\ldots, \theta_n) = h^*$ with $h^* \in \mathbb{R}$, and we assume that the value h^* is such that the function $\mathcal{H}_0^{-1}$ in a neighborhood of h^* is a diffeomorphism. Therefore the expression $\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))$ is well defined.

On the other hand, every periodic solution of a differential system has defined in its neighborhood a return map F usually called the Poincaré map. The periodic solution provides a fixed point of the map F. The stability or instability of this fixed point for the map F is what we call the stability behavior of the Poincaré map associated to the periodic solution in the statement of Theorem 1.2.4. For more details on the Poincaré map see for instance [105].

Theorem 1.2.4 will be proved later on.

The next objective of the present work is to study the periodic orbits of the Hamiltonian system with the *perturbed Keplerian Hamiltonian* of the form

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + P_2^2 + P_3^2\right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + \varepsilon \mathcal{P}_1(Q_1^2 + Q_2^2, Q_3). \tag{1.23}$$

Note that the perturbation is symmetric with respect to the Q_3–axis. It is easy to check that the third component $K = Q_1P_2 - Q_2P_1$ of the angular momentum is a first integral of the Hamiltonian system associated to the Hamiltonian (1.23). We use this second first integral to simplify the analysis of the given axially symmetric Keplerian perturbed system.

In the following we use the *Delaunay variables* for studying easily the periodic orbits of the Hamiltonian system associated to the Hamiltonian (1.23), see [31, 97] for more details on the Delaunay variables. Thus, in Delaunay variables the Hamiltonian (1.23) has the form

$$\mathcal{H} = -\frac{1}{2L^2} + \varepsilon \mathcal{P}(l, g, k, L, G, K) = -\frac{1}{2L^2} + \varepsilon \mathcal{P}(l, g, L, G, K), \tag{1.24}$$

where l is the *mean anomaly*, g is the *argument of the perigee* of the unperturbed elliptic orbit measured in the invariant plane, k is the *longitude of the node*, L is the *square root of the semi-major axis* of the unperturbed elliptic orbit, G is the *modulus of the total angular momentum* and K is the *third component of the angular momentum*. Moreover, $\mathcal{P}$ is the perturbation obtained from the perturbation $\mathcal{P}_1$ using the transformation to Delaunay variables, namely

$$\begin{aligned} Q_1 &= r\left(\cos(f+g)\cos k - c\sin(f+g)\sin k\right), \\ Q_2 &= r\left(\cos(f+g)\sin k + c\sin(f+g)\cos k\right), \\ Q_3 &= rs\sin(f+g), \end{aligned} \tag{1.25}$$

with

$$c = \frac{K}{G}, \qquad s^2 = 1 - \frac{K^2}{G^2}.$$

The *true anomaly* f and the *eccentric anomaly* E are auxiliary quantities defined by the relations

$$\sqrt{1-e^2} = \frac{G}{L}, \qquad r = a(1 - e\cos E), \qquad l = E - e\sin E.$$

$$\sin f = \frac{a\sqrt{1-e^2}\sin E}{r}, \qquad \cos f = \frac{a(\cos E - e)}{r},$$

where e is the eccentricity of the unperturbed elliptic orbit.

Note that the angular variable k is a cyclic variable for the Hamiltonian (1.24), and consenquently K is a first integral of the Hamiltonian system as we already knew.

The family of Hamiltonians (1.24) is a particular subclass of the Hamiltonians (1.21) with $\mathcal{H}_1 = \mathcal{P}$. We denote by $\langle \mathcal{P} \rangle$ the averaged map of $\mathcal{P}$ with respect to the mean anomaly l, i.e.,

$$\langle \mathcal{P} \rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}(l, g, L, G, K)dl = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}(E - e\sin E, g, L, G, K)(1 - e\cos E)dE.$$

We remark that the map $\langle \mathcal{P} \rangle$ only depends on the angle g and the three action variables L, G, K. We claim that $\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*)) = (-2h^*)^{3/2}$. Indeed $\mathcal{H}_0(L) = -1/(2L^2) = h^*$, so $\mathcal{H}_0^{-1}(h^*) = (-2h^*)^{1/2}$. Since $\mathcal{H}_0'(L) = 1/L^3$, the claim follows.

We also have from the definition of Poisson parenthesis that

$$\begin{aligned}
\{G, \langle\mathcal{P}\rangle\} &= -\frac{\partial G}{\partial G}\frac{\partial \langle\mathcal{P}\rangle}{\partial g} = -\frac{\partial \langle\mathcal{P}\rangle}{\partial g},\\
\{g, \langle\mathcal{P}\rangle\} &= \frac{\partial g}{\partial g}\frac{\partial \langle\mathcal{P}\rangle}{\partial G} = \frac{\partial \langle\mathcal{P}\rangle}{\partial G},\\
\{k, \langle\mathcal{P}\rangle\} &= \frac{\partial k}{\partial k}\frac{\partial \langle\mathcal{P}\rangle}{\partial K} = \frac{\partial \langle\mathcal{P}\rangle}{\partial K}.
\end{aligned}$$

Then, by Theorem 1.2.4 at the energy level $\mathcal{H} = h^*$ with $h^* < 0$ (because $\mathcal{H}_0(L) = -1/(2L^2)$) and with angular momentum $K = k^*$, the differential system (1.22) with respect to the mean anomaly l is

$$\begin{aligned}
\frac{dG}{dl} &= \varepsilon\frac{\{G, \langle\mathcal{P}\rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))} = -\varepsilon(-2h^*)^{3/2}\frac{\partial \langle\mathcal{P}\rangle}{\partial g} = -\varepsilon f_1(g, G, K),\\
\frac{dg}{dl} &= \varepsilon\frac{\{g, \langle\mathcal{P}\rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))} = \varepsilon(-2h^*)^{3/2}\frac{\partial \langle\mathcal{P}\rangle}{\partial G} = \varepsilon f_2(g, G, K),\\
\frac{dk}{dl} &= \varepsilon\frac{\{k, \langle\mathcal{P}\rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))} = \varepsilon(-2h^*)^{3/2}\frac{\partial \langle\mathcal{P}\rangle}{\partial K} = \varepsilon f_3(g, G, K).
\end{aligned} \tag{1.26}$$

Note that we do not write the differential equation $dK/dt = 0$ because we are working in the invariant set $\mathcal{H} = h^*$ and $K = k^*$.

Now we are ready to state a corollary of Theorem 1.2.4 which provides sufficient conditions for the existence and the kind of stability of the periodic orbits in the perturbed Kepler problems with axial symmetry.

Corollary 1.2.5. *System* (1.26) *is the Hamiltonian system taking as independent variable the mean anomaly l of the Hamiltonian* (1.23) *written in Delaunay variables on the fixed energy level $\mathcal{H} = h^* < 0$ and on the fixed third component of the angular momentum $K = k^*$. If $\varepsilon \neq 0$ is sufficiently small then for every solution $p = (g_0, G_0, k^*)$ of the system $f_i(g, G, K) = 0$ for $i = 1, 2, 3$ satisfying that*

$$\det\left(\left.\frac{\partial(f_1, f_2, f_3)}{\partial(g, G, K)}\right|_{(g,G,K)=(g_0,G_0,k^*)}\right) \neq 0, \tag{1.27}$$

and all $k_0 \in [0, 2\pi)$ there exists a 2π–periodic solution $\gamma_\varepsilon(l) = (g(l,\varepsilon), k(l,\varepsilon), L(l,\varepsilon), G(l,\varepsilon), K(l,\varepsilon) = k^*)$ such that $\gamma_\varepsilon(0) \to (g_0, k_0, \sqrt{-2h^*}, G_0, k^*)$ when $\varepsilon \to 0$. The stability or instability of the periodic solution $\gamma_\varepsilon(l)$ is given by the stability or instability of the equilibrium point p of system (1.26)*. In fact, the equilibrium point p has the stability behavior of the Poincaré map associated to the periodic solution $\gamma_\varepsilon(l)$.*

We remark that the fact that we have a periodic solution for every $k_0 \in [0, 2\pi)$ with the same initial conditions for all the other variables, means that we really have a 2–dimensional torus foliated by periodic solutions.

There are many articles studying the periodic orbits of different perturbed Keplerian problems, see for instance [54, 61, 108] and the papers quoted therein.

In what follows we shall study the spatial generalized van der Waals Hamiltonian system modeling the dynamical symmetries of the perturbed hydrogen atom.

The generalized *van der Waals Hamiltonian* system was proposed in the paper [5] via the following Hamiltonian with $\beta \in \mathbb{R}$

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + P_2^2 + P_3^2\right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + \varepsilon\left(Q_1^2 + Q_2^2 + \beta^2 Q_3^2\right). \tag{1.28}$$

Note that this Hamiltonian is of the form (1.23). For more references on this Hamiltonian system see the ones quoted in [49].

Theorem 1.2.6. *On every energy level $\mathcal{H} = h^* < 0$ and for the third component of the angular momentum $K = k^*$, the spatial van der Waals Hamiltonian system associated to the Hamiltonian* (1.28) *for $\varepsilon \neq 0$ sufficiently small has:*

(a) *For $K = k^* = 0$ two 2π–periodic solution $\gamma_\varepsilon^{\pm}(l) = (g(l,\varepsilon), k(l,\varepsilon)), L(l,\varepsilon), G(l,\varepsilon), K(l,\varepsilon))$ such that*

$$\gamma_\varepsilon^{\pm}(l)(0) \to \left(\pm\frac{1}{2}\arccos\left(\frac{3(\beta^2+1)}{5(\beta^2-1)}\right), k_0, \frac{1}{\sqrt{-2h^*}}, \frac{1}{\sqrt{-2h^*}}, 0\right) \text{ when } \varepsilon \to 0,$$

for each $k_0 \in [0, 2\pi)$ if $\beta \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty)$. These periodic orbits have a stable manifold of dimension 2 *and an unstable of dimension* 1 *if $\beta \in (-1/2, 1/2)$, and have a stable manifold of dimension* 1 *and an unstable of dimension* 2 *if $\beta \in (-\infty, -2) \cup (2, \infty)$. Consequently these periodic orbits are unstable.*

(b) *For $K = k^* \neq 0$ four 2π–periodic solutions $\gamma_\varepsilon^{\pm,\pm}(l) = (g(l,\varepsilon), k(l,\varepsilon)), L(l,\varepsilon), G(l,\varepsilon), K(l,\varepsilon))$ such that*

$$\gamma_\varepsilon^{\pm,\pm}(0) \to \left(\pm\frac{\pi}{2}, k_0, \frac{1}{\sqrt{-2h^*}}, \frac{1}{2}\sqrt{\frac{5}{-2h^*}}, \pm\frac{1}{4}\sqrt{\frac{5(1-4\beta^2)}{-2h^*(1-\beta^2)}}\right) \text{ when } \varepsilon \to 0,$$

for each $k_0 \in [0, 2\pi)$ if $\beta \in (-1, -1/2) \cup (1/2, 1)$.

Theorem 1.2.6 is proved later on.

The result of statement (a) of Theorem 1.2.6 was already obtained using cylindrical coordinates in [49].

The stability or instability of the four periodic orbits of statement (b) of Theorem 1.2.6 can be determined analyzing the eigenvalues of the corresponding

Jacobian matrices, but since the expression of these eigenvalues are huge and depend on the two parameters h^* and β, this study is a long task that we do not do here.

We remark that when $(\beta^2-1)(\beta^2-4)(\beta^2-1/4)=0$, i.e. for the values that the averaging theory for finding periodic orbits do not provide any information, it is known that for those values of β the van der Waals Hamiltonian system is integrable, see [44]. Therefore, the averaging method when cannot be applied for finding periodic orbits provides a suspicion that for such values of the parameter the system could be integrable.

The Hamiltonian system associated to the Hamiltonian (1.21) can be written as

$$\begin{aligned}\frac{dI_i}{dt} &= \varepsilon\{I_i,\mathcal{H}_1\} = -\varepsilon\frac{\partial \mathcal{H}_1}{\partial \theta_i} \qquad i=1,\ldots,n,\\ \frac{d\theta_i}{dt} &= \varepsilon\{\theta_i,\mathcal{H}_1\} = \varepsilon\frac{\partial \mathcal{H}_1}{\partial I_i} \qquad i=2,\ldots,n,\\ \frac{d\theta_1}{dt} &= \mathcal{H}_0'(I_1)+\varepsilon\{\theta_1,\mathcal{H}_1\} = \mathcal{H}_0'(I_1)+\varepsilon\frac{\partial \mathcal{H}_1}{\partial I_1}.\end{aligned} \tag{1.29}$$

Lemma 1.2.7. *Taking as new independent variable the variable* θ_1 *we have in the fixed energy level* $\mathcal{H}=h^*<0$ *that the differential system* (1.29) *becomes*

$$\begin{aligned}\frac{dI_i}{d\theta_1} &= \varepsilon\frac{\{I_i,\mathcal{H}_1\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))}+O(\varepsilon^2),\ i=2,\ldots,n\\ \frac{d\theta_i}{d\theta_1} &= \varepsilon\frac{\{\theta_i,\mathcal{H}_1\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))}+O(\varepsilon^2),\ i=2,\ldots,n\end{aligned} \tag{1.30}$$

with $I_1=\mathcal{H}_0^{-1}(h^*)+O(\varepsilon)$ *if* $\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))\neq 0$.

Proof. Taking as new independent variable θ_1, the equations (1.29) become

$$\begin{aligned}\frac{dI_i}{d\theta_1} &= \frac{\varepsilon\{I_i,\mathcal{H}_1\}}{\mathcal{H}_0'(I_1)+\varepsilon\{\theta_1,\mathcal{H}_1\}} = \varepsilon\frac{\{I_i,\mathcal{H}_1\}}{\mathcal{H}_0'(I_1)}+O(\varepsilon^2) \quad i=1,\ldots,n,\\ \frac{d\theta_i}{d\theta_1} &= \frac{\varepsilon\{\theta_i,\mathcal{H}_1\}}{\mathcal{H}_0'(I_1)+\varepsilon\{\theta_1,\mathcal{H}_1\}} = \varepsilon\frac{\{\theta_i,\mathcal{H}_1\}}{\mathcal{H}_0'(I_1)}+O(\varepsilon^2) \quad i=2,\ldots,n.\end{aligned}$$

Fixing the energy level of $\mathcal{H}=h^*<0$ we obtain $h^*=\mathcal{H}_0(I_1)+\varepsilon\mathcal{H}_1(I_1,\ldots,I_n,\theta_1,\ldots,\theta_n)$. Using the Implicit Function Theorem and the fact that $\mathcal{H}_0'(\mathcal{H}_0^{-1}(h^*))\neq 0$, for ε sufficiently small, we get $I_1=\mathcal{H}_0^{-1}(h^*)+O(\varepsilon)$, and the equations are reduced to (1.30). □

Proof of Theorem 1.2.4. The averaged system in the angle θ_1 obtained from (1.30) is

$$
\begin{aligned}
\frac{dI_i}{d\theta_1} &= -\frac{1}{2\pi}\frac{\varepsilon}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h))}\int_0^{2\pi}\frac{\partial \mathcal{H}_1}{\partial \theta_i}d\theta_1 \quad i=2,\dots,n,\\
\frac{d\theta_i}{d\theta_1} &= \frac{1}{2\pi}\varepsilon\frac{\{\theta_i,\mathcal{H}_1\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h))}\int_0^{2\pi}\frac{\partial \mathcal{H}_1}{\partial I_i}d\theta_1 \quad i=2,\dots,n.
\end{aligned} \tag{1.31}
$$

Since

$$
\begin{aligned}
\frac{\partial\langle\mathcal{H}_1\rangle}{\partial\theta_i} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial \mathcal{H}_1}{\partial \theta_i}d\theta_1 \quad i=2,\dots,n,\\
\frac{\partial\langle\mathcal{H}_1\rangle}{\partial I_i} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial \mathcal{H}_1}{\partial I_i}d\theta_1 \quad i=2,\dots,n,
\end{aligned}
$$

the differential system (1.31) becomes

$$
\begin{aligned}
\frac{dI_i}{d\theta_1} &= -\frac{\varepsilon}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h))}\frac{\partial\langle\mathcal{H}_1\rangle}{\partial\theta_i} = \varepsilon\frac{\{I_i,\langle\mathcal{H}_1\rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h))} \quad i=2,\dots,n,\\
\frac{d\theta_i}{d\theta_1} &= \frac{\varepsilon}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h))}\frac{\partial\langle\mathcal{H}_1\rangle}{\partial I_i} = \varepsilon\frac{\{\theta_i,\langle\mathcal{H}_1\rangle\}}{\mathcal{H}_0'(\mathcal{H}_0^{-1}(h))} \quad i=2,\dots,n,
\end{aligned}
$$

which coincides with the system (1.22).

Once we have obtained the averaged system (1.22) it is immediate to check that it satisfies the assumptions of Theorem 1.1.1, then applying the conclusions of this theorem to the averaged system (1.22) the rest of the statement of Theorem 1.2.4 follows immediately. □

Proof of Theorem 1.2.6. For the generalized van der Waals Hamiltonian system the function $\mathcal{P}(E,g,h,G,K)$ is equal to

$$
\begin{aligned}
&\frac{\left(\beta^2G^2+G^2+K^2-K^2\beta^2\right)(e\cos E-1)^2L^4}{2G^2}-\\
&\frac{L^4(G^2-K^2)(\beta^2-1)(e-\cos E)^2\cos^2 g}{2G^2}+\\
&\frac{L^4(G^2-K^2)(\beta^2-1)(e-\cos E)^2\sin^2 g}{2G^2}\\
&-\frac{2L^3(G^2-K^2)(\beta^2-1)(e-\cos E)\cos g\sin E\sin g}{G}\\
&+\frac{1}{2}L^2(G^2-K^2)(\beta^2-1)\cos^2 g\sin^2 E\\
&-\frac{1}{2}L^2(G^2-K^2)(\beta^2-1)\sin^2 E\sin^2 g.
\end{aligned}
$$

Its averaged function with respect to the mean anomaly is

$$\langle \mathcal{P} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(E, g, h, G, K)(1 - e\cos E)dE = \frac{B}{4G^2},$$

where $B = L^2\big(5(G^2 - K^2)(G^2 - L^2)(\beta^2 - 1)\cos(2g) - (3G^2 - 5L^2)(G^2 + K^2 + (G^2 - K^2)\beta^2)\big)$.

The equations (1.26) are the averaged equations of the Hamiltonian system with Hamiltonian (1.28)

$$\begin{aligned}
\frac{dG}{dl} &= \varepsilon \frac{5(1 + 2h^* G^2)(G^2 - K^2)(\beta^2 - 1)\sin(2g)}{2G^2\sqrt{-2h^*}} = -\varepsilon f_1(g, G, K),\\
\frac{dg}{dl} &= -\varepsilon \frac{C}{2G^3\sqrt{-2h^*}} = \varepsilon f_2(g, G, K),\\
\frac{dk}{dl} &= \varepsilon \frac{K(\beta^2 - 1)(-5 - 6h^* G^2 + 5(1 + 2h^* G^2)\cos(2g))}{2G^2\sqrt{-2h^*}} = \varepsilon f_3(g, G, K),
\end{aligned}$$

where $C = 5K^2(\beta^2 - 1) + 6h^* G^4(\beta^2 + 1) - 5(2h^* G^4 + K^2)(\beta^2 - 1)\cos(2g)$ here $L = 1/\sqrt{-2h^*} + O(\varepsilon)$. The equilibrium solutions (g_0, G_0, k^*) of this averaged system satisfying (1.27) give rise to periodic orbits of the Hamiltonian system with Hamiltonian (1.28) for each $\mathcal{H} = h^* < 0$ and $K = k^*$, see Theorem 1.1.1. These equilibria (g_0, G_0, k^*) are

$$\left(\pm\frac{1}{2}\arccos\left(\frac{3(\beta^2+1)}{5(\beta^2-1)}\right), \frac{1}{\sqrt{-2h^*}}, 0\right), \left(\pm\frac{\pi}{2}, \frac{1}{2}\sqrt{\frac{5}{-2h^*}}, \pm\frac{1}{4}\sqrt{\frac{5(1-4\beta^2)}{-2h^*(1-\beta^2)}}\right).$$

The first two equilibria exist if $3(\beta^2 + 1)/(5(\beta^2 - 1)) \in [-1, 1]$, i.e. if $\beta \in (-\infty, -2] \cup [-1/2, 1/2] \cup [2, \infty)$.

The Jacobian (1.27) of the first equilibrium is equal to $J = 16\sqrt{-2h^*}(\beta^2 - 1)(\beta^2 - 4)(\beta^2 - 1/4)$. So each of these equilibria when $\beta \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty)$ provides one periodic orbit of the Hamiltonian system with Hamiltonian (1.28) for each $\mathcal{H} = h^* < 0$ and $K = k^* = 0$. Since $k^* = 0$ these periodic orbits bifurcate from an elliptic orbit ($g_0 \neq 0$) of the Kepler problem living in the plane of motion of the two bodies of the Kepler problem. Moreover, since the eigenvalues of the Jacobian matrix at these equilibra are $\pm 2\sqrt{(\beta^2 - 4)(4\beta^2 - 1)}$ and $\sqrt{-2h^*}(\beta^2 - 1)$, these periodic orbits have a stable manifold of dimension 2 and an unstable of dimension 1 if $\beta \in (-1/2, 1/2)$, and have a stable manifold of dimension 1 and an unstable of dimension 2 if $\beta \in (-\infty, -2) \cup (2, \infty)$. This proves statement (a) of the theorem.

The last four equilibria exist if $\beta \in (-1, -1/2] \cup [1/2, 1)$ and have Jacobian equal to $J = -15\sqrt{-2h^*}(\beta^2 - 1)(4\beta^2 - 1)$. So, for each value of $k \in [0, 2\pi)$ these four equilibria when $\beta \in (-1, -1/2) \cup (1/2, 1)$ provide four periodic orbits of the Hamiltonian system with Hamiltonian (1.28) for each $\mathcal{H} = h^* < 0$ and $K = k^* = \pm\frac{1}{4}\sqrt{\frac{5(1-4\beta^2)}{-2h^*(1-\beta^2)}} \neq 0$. Since $k^* \neq 0$ these periodic orbits bifurcate from elliptic orbits ($g_0 \neq 0$) of the Kepler problem which are not in the plane of motion defined by the two bodies. This proves statement (b) of the theorem. □

1.3 Other first order averaging method for periodic orbits

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon), \tag{1.32}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times D \to \mathbb{R}^n$ and $R\colon \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\mathbf{x}' = F_0(t, \mathbf{x}), \tag{1.33}$$

has a submanifold of periodic solutions.

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (1.33) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}. \tag{1.34}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (1.34), and by $\xi\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

The next result goes back to Malkin [95] and Roseau [105]. Here we shall present the shorter proof given in [17].

Theorem 1.3.1. *Let $V \subset \mathbb{R}^k$ be open and bounded, and let $\beta_0\colon\ \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)),\ \ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (1.33) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (1.34) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the right up corner the $k \times (n-k)$ zero matrix, and in the right down corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F}\colon \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\right). \tag{1.35}$$

If there exists $a \in V$ *with* $\mathcal{F}(a) = 0$ *and* $\det\left(\left(d\mathcal{F}/d\alpha\right)(a)\right) \neq 0$*, then there is a* T*–periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system* (1.32) *such that* $\mathbf{x}(0,\varepsilon) \to \mathbf{z}_a$ *as* $\varepsilon \to 0$*.*

Theorem 1.3.1 is proved in section 1.7. In the next section we provide some applications of this theorem.

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic, where $\mathbf{x}(t,\mathbf{z},0)$ denotes the solution of the unperturbed system (1.33) with $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (1.32); i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z},0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Corollary 1.3.2 (**Perturbations of an isochronous set**)**.** *We assume that there exists an open and bounded set* V *with* $\mathrm{Cl}(V) \subset \Omega$ *such that for each* $\mathbf{z} \in \mathrm{Cl}(V)$*, the solution* $\mathbf{x}(t,\mathbf{z})$ *is* T*–periodic, then we consider the function* $\mathcal{F}\colon \mathrm{Cl}(V) \to \mathbb{R}^n$

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t,\mathbf{z}) F_1(t,\mathbf{x}(t,\mathbf{z}))dt. \qquad (1.36)$$

If there exists $a \in V$ *with* $\mathcal{F}(a) = 0$ *and* $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(a)\right) \neq 0$*, then there exists a* T*–periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system* (1.32) *such that* $\mathbf{x}(0,\varepsilon) \to a$ *as* $\varepsilon \to 0$*.*

Proof. It follows immediately from Theorem 1.3.1 taking $k = n$. □

1.4 Three applications

In this section we shall do three applications of Theorem 1.3.1 and of its Corollary 1.3.2.

1.4.1 The Hopf bifurcation of the Michelson system

The *Michelson system*

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = c^2 - y - \frac{x^2}{2}, \qquad (1.37)$$

with $(x,y,z) \in \mathbb{R}^3$ and the parameter $c \geq 0$, was introduced by Michelson [98] in the study of the travelling wave solutions of the Kuramoto–Sivashinsky equation. It is well known that system (1.37) is reversible with respect to the involution $R(x,y,z) = (-x,y,-z)$ and is volume–preserving under the flow of the system. It is easy to check that system (1.37) has two finite singularities $S_1 = (-\sqrt{2}c,0,0)$ and $S_2 = (\sqrt{2}c,0,0)$ for $c > 0$, which are both saddle–foci. The former has a 2–dimensional stable manifold and the latter has a 2–dimensional unstable manifold.

For $c > 0$ small numerical experiments (see for instance Kent and Elgin [66]) and asymptotic expansions in sinus series (see Michelson [98] in 1986 and

Webster and Elgin [120] in 2003) revealed the existence of a zero–Hopf bifurcation at the origin for $c = 0$. But their results do not provide an analytic proof on the existence of such zero–Hopf bifurcation. By a *zero–Hopf bifurcation* we mean that when $c = 0$ the Michelson system has the origin as a singularity having eigenvalues $0, \pm i$, and when $c > 0$ sufficiently small the Michelson system has a periodic orbit which tends to the origin when c tends to zero. The analytic proof of this zero–Hopf bifurcation has been proved in [84] by Llibre and Zang. Now we state this result and reproduce its proof.

Theorem 1.4.1. *For $c \geq 0$ sufficiently small the Michelson system* (1.37) *has a zero–Hopf bifurcation at the origin for $c = 0$. Moreover the bifurcated periodic orbit satisfies $x(t) = -2c\cos t + o(c)$, $y(t) = 2c\sin t + o(c)$ and $z(t) = 2c\cot t + o(c)$ for $c > 0$ sufficiently small.*

Proof. For any $\varepsilon \neq 0$ we take the change of variables $x = \varepsilon\overline{x}$, $y = \varepsilon\overline{y}$, $z = \varepsilon\overline{z}$ and $c = \varepsilon d$, then the Michelson system (1.37) becomes

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -y + \varepsilon d^2 - \varepsilon\frac{1}{2}x^2, \tag{1.38}$$

where we still use x, y, z instead of $\overline{x}, \overline{y}, \overline{z}$. Now doing the change of variables $x = x$, $y = r\sin\theta$ and $z = r\cos\theta$, system (1.38) goes over to

$$\dot{x} = r\sin\theta, \quad \dot{r} = \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta, \quad \dot{\theta} = 1 - \frac{\varepsilon}{2r}(2d^2 - x^2)\sin\theta. \tag{1.39}$$

This system can be written as

$$\begin{aligned} \frac{dx}{d\theta} &= r\sin\theta + \frac{\varepsilon}{2}(2d^2 - x^2)\sin^2\theta + \varepsilon^2 f_1(\theta, r, \varepsilon), \\ \frac{dr}{d\theta} &= \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta + \varepsilon^2 f_2(\theta, r, \varepsilon), \end{aligned} \tag{1.40}$$

where f_1 and f_2 are analytic functions in their variables.

For arbitrary $(x_0, r_0) \neq (0, 0)$, system $(1.40)_{\varepsilon=0}$ has the 2π–periodic solution

$$x(\theta) = r_0 + x_0 - r_0\cos\theta, \quad r(\theta) = r_0, \tag{1.41}$$

such that $x(0) = x_0$ and $r(0) = r_0$. It is easy to see that the first variational equation of $(1.40)_{\varepsilon=0}$ along the solution (1.41) is

$$\begin{pmatrix} \dfrac{dy_1}{d\theta} & \dfrac{dy_2}{d\theta} \\ \dfrac{dy_3}{d\theta} & \dfrac{dy_4}{d\theta} \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}.$$

It has the fundamental solution matrix

$$M = \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}, \tag{1.42}$$

which is independent of the initial condition (x_0, r_0). Applying Corollary 1.3.2 to the differential system (1.40) we have that

$$\mathcal{F}(x_0, r_0) = \frac{1}{2}\int_0^{2\pi} M^{-1} \left(\begin{array}{c} (2d^2 - x^2)\sin^2\theta \\ (2d^2 - x^2)\cos\theta \end{array} \right)\Bigg|_{(1.41)} d\theta.$$

Then $\mathcal{F}(x_0, r_0) = (g_1(x_0, r_0),\, g_2(x_0, r_0))$ with

$$g_1(x_0, r_0) = \frac{1}{4}\left(4d^2 - 5r_0^2 - 6r_0x_0 - 2x_0^2\right), \quad g_2(x_0, r_0) = \frac{1}{2}r_0(x_0 + r_0).$$

We can check that $\mathcal{F} = 0$ has a unique non–trivial solution $x_0 = -2d$ and $r_0 = 2d$, and that $\det D\mathcal{F}(x_0, r_0)|_{x_0=-2d, r_0=2d} = d^2$. Hence by Corollary 1.3.2 it follows that for any given $d > 0$ and for $|\varepsilon| > 0$ sufficiently small system (1.40) has a periodic orbit $(x(\theta, \varepsilon),\, r(\theta, \varepsilon))$ of period 2π, such that $(x(0, \varepsilon), r(0, \varepsilon)) \to (-2d, 2d)$ as $\varepsilon \to 0$. We note that the eigenvalues of $D\mathcal{F}(x_0, r_0)|_{x_0=-2d, r_0=2d}$ are $\pm di$. This shows that the periodic orbit is linearly stable.

Going back to system (1.37) we get that for $c > 0$ sufficiently small the Michelson system has a periodic orbit of period close to 2π given by $x(t) = -2c\cos t + o(c)$, $y(t) = 2c\sin t + o(c)$ and $z(t) = 2c\cos t + o(c)$. We think that this periodic orbit is symmetric with respect to the involution R, but we do not have a proof of it. □

1.4.2 A third–order differential equation

Using Theorem 1.3.1 in the next result we present a third–order differential equation having as many limit cycles as we want.

Proposition 1.4.2. *We consider the third–order differential equation*

$$\dddot{x} - \ddot{x} + \dot{x} - x = \varepsilon \cos(x + t). \tag{1.43}$$

Then for all positive integer m there is $\varepsilon_m > 0$ such that if $\varepsilon \in [-\varepsilon_m, \varepsilon_m] \setminus \{0\}$ the differential equation (1.43) *has at least m limit cycles.*

Proof. If $y = \dot{x}$ and $z = \ddot{x}$, then system (1.43) can be written as

$$\begin{array}{l} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = x - y + z + \varepsilon\cos(x + t) = x - y + z + \varepsilon F(t, x, y, z). \end{array} \tag{1.44}$$

The origin $(0, 0, 0)$ is the unique singular point of system (1.44) when $\varepsilon = 0$. The eigenvalues of the linearized system at this singular point are $\pm i$ and 1. By the linear invertible transformation $(X, Y, Z)^T = C(x, y, z)^T$, where

$$C = \left(\begin{array}{ccc} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{array} \right),$$

we transform the differential system (1.44) in another such that its linear part is the real Jordan normal form of the linear part of system (1.44) with $\varepsilon = 0$, i.e.

$$\begin{aligned} \dot{X} &= -Y, \\ \dot{Y} &= X + \varepsilon \tilde{F}(X, Y, Z, t), \\ \dot{Z} &= Z + \varepsilon \tilde{F}(X, Y, Z, t), \end{aligned} \tag{1.45}$$

where

$$\tilde{F}(X, Y, Z, t) = F\left(\frac{X - Y + Z}{2}, \frac{-X - Y + Z}{2}, \frac{-X + Y + Z}{2}, t\right).$$

Using the notation introduced in (1.32) we have that $\mathbf{x} = (X, Y, Z)$, $F_0(\mathbf{x}, t) = (-Y, X, Z)$, $F_1(\mathbf{x}, t) = (0, \tilde{F}, \tilde{F})$ and $F_2(\mathbf{x}, t) = 0$. Let $\mathbf{x}(t; X_0, Y_0, Z_0, \varepsilon)$ be the solution of system (1.45) such that $\mathbf{x}(0; X_0, Y_0, Z_0, \varepsilon) = (X_0, Y_0, Z_0)$. Clearly the unperturbed system (1.45) with $\varepsilon = 0$ has a linear center at the origin in the (X, Y)–plane, which is an invariant plane under the flow of the unperturbed system, and the periodic solution $\mathbf{x}(t; X_0, Y_0, 0, 0) = (X(t), Y(t), Z(t))$ is

$$X(t) = X_0 \cos t - Y_0 \sin t, \quad Y(t) = Y_0 \cos t + X_0 \sin t, \quad Z(t) = 0. \tag{1.46}$$

Note that all these periodic orbits have period 2π.

For our system the V and the α of Theorem 1.3.1 are $V = \{(X, Y, 0) : 0 < X^2 + Y^2 < \rho\}$ for some arbitrary $\rho > 0$ and $\alpha = (X_0, Y_0) \in V$.

The fundamental matrix solution $M(t)$ of the variational equation of the unperturbed system $(1.45)_{\varepsilon=0}$ with respect to the periodic orbits (1.46) satisfying that $M(0)$ is the identity matrix is

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

We remark that it is independent of the initial condition $(X_0, Y_0, 0)$. Moreover an easy computation shows that

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi} \end{pmatrix}.$$

In short we have shown that all the assumptions of Theorem 1.3.1 hold. Hence we shall study the zeros $\alpha = (X_0, Y_0) \in V$ of the two components of the function $\mathcal{F}(\alpha)$ given in (1.35). More precisely we have $\mathcal{F}(\alpha) = (\mathcal{F}_1(\alpha), \mathcal{F}_2(\alpha))$ where

$$\mathcal{F}_1(\alpha) = \int_0^{2\pi} \sin t \tilde{F}(\mathbf{x}(t; X_0, Y_0, 0, 0), t) dt$$

$$= \int_0^{2\pi} \sin t F\left(\frac{X(t)-Y(t)}{2}, -\frac{X(t)+Y(t)}{2}, \frac{-X(t)+Y(t)}{2}, t\right) dt,$$

$$\mathcal{F}_2(\alpha) = \int_0^{2\pi} \cos t \tilde{F}(\mathbf{x}(t; X_0, Y_0, 0, 0), t) dt$$

$$= \int_0^{2\pi} \cos t F\left(\frac{X(t)-Y(t)}{2}, -\frac{X(t)+Y(t)}{2}, \frac{-X(t)+Y(t)}{2}, t\right) dt,$$

where $X(t), Y(t)$ are given by (1.46).

First we consider the third–order differential equation (1.43). For this equation we have that

$$f_1(X_0, Y_0) = \int_0^{2\pi} \sin t \cos\left(t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t)}{2}\right) dt,$$

$$f_2(X_0, Y_0) = \int_0^{2\pi} \cos t \cos\left(t + \frac{(X_0 - Y_0)\cos t - (X_0 + Y_0)\sin t)}{2}\right) dt.$$

To simplify the computation of these two previous integrals we do the change of variables $(X_0, Y_0) \longmapsto (r, s)$ given by

$$X_0 - Y_0 = 2r\cos s, \quad X_0 + Y_0 = -2r\sin s, \tag{1.47}$$

where $r > 0$ and $s \in [0, 2\pi)$. From now on and until the end of the paper we write $f_1(r, s)$ instead of

$$f_1(X_0, Y_0) = f_1\big(r(\cos s - \sin s), -r(\cos s + \sin s)\big).$$

Similarly for $f_2(r, s)$.

We compute the two previous integrals and we get

$$\begin{aligned} f_1(r, s) &= -\pi J_2(r)\sin 2s, \\ f_2(r, s) &= 2\pi\left(\frac{1}{r}J_1(r) - J_2(r)\cos^2 s\right), \end{aligned} \tag{1.48}$$

where J_1 and J_2 are the *first* and *second Bessel functions of first kind* . For more details on the Bessel functions see [4]. These computations become easier with the help of an algebraic manipulation as Mathematica or Maple.

Using the asymptotic expressions of the Bessel functions of first kind it follows that Bessel functions $J_1(r)$ and $J_2(r)$ have different zeros. Hence $f_i(r, s) = 0$ for $i = 1, 2$ imply that either $s \in \{0, \pi/2, \pi, 3\pi/2\}$. Therefore we have to study the zeros of

$$f_2(r, 0) = f_2(r, \pi) = 2\pi\left(\frac{1}{r}J_1(r) - J_2(r)\right), \tag{1.49}$$

$$f_2(r, \pi/2) = f_2(r, 3\pi/2) = \frac{2\pi}{r}J_1(r). \tag{1.50}$$

We claim that function (1.49) has also infinite zeros for $r \in (0, \infty)$. Note that if ρ is sufficiently large, and we choose $r < \rho$ also sufficiently large, then

$$J_n(r) \approx \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for} \quad n = 1, 2,$$

are asymptotic estimations, see [4]. Considering (1.49) for r sufficiently large we obtain that

$$\begin{aligned} f_2(r, 0) &\approx \frac{2}{r}\sqrt{\frac{2\pi}{r}} \left(\cos\left(r - \frac{3\pi}{4}\right) + r\cos\left(r - \frac{\pi}{4}\right)\right) \\ &= \frac{2}{r}\sqrt{\frac{\pi}{r}}((r-1)\cos r + (r+1)\sin r). \end{aligned}$$

The above function has infinite zeros because the equation

$$\tan r = \frac{1 - r}{r + 1}$$

has infinitely many solutions.

For every zero $r_0 > 0$ of the function (1.49) we have two zeros of system (1.48), namely $(r, s) = (r_0, 0)$ and $(r, s) = (r_0, \pi)$.

We have from (1.48) that

$$\begin{aligned} \left|\frac{\partial(f_1, f_2)}{\partial(r, s)}\right|_{(r,s)=(r_0,0)} &= \frac{4\pi^2(J_0(r_0)r_0 - 2J_1(r_0))(J_0(r_0)r_0 + (r_0^2 - 2)J_1(r_0))}{r_0^3} \\ &= \frac{4\pi^2}{r_0} J_2(r_0)(J_1(r_0)r_0 - J_2(r_0)), \end{aligned} \tag{1.51}$$

where we have used several relation between the Bessel functions of first kind, see [4]. Clearly it is impossible that (1.49) and (1.51) are equal to zero at the same time. Therefore by Theorem 1.1.1 there is a periodic orbit of system (1.43) for each $(r_0, 0)$, that is for each value of $(X_0, Y_0) = (r_0, -r_0)$.

In an analogous way there is a periodic orbit of system (1.43) for each (r_0, π), that is for each value of $(X_0, Y_0) = (-r_0, r_0)$. In fact, the periodic orbit with this initial conditions and the previous one with initial conditions $(X_0, Y_0) = (r_0, -r_0)$ are the same.

Similarly since $J_1(r)$ has infinitely many zeros (see [4]), the function (1.50) has infinitely many positive zeros r_1. Every one of these zeros provides two solutions of system (1.48), namely $(r, s) = (r_1, \pi/2)$ and $(r, s) = (r_1, 3\pi/2)$.

Moreover we have from (1.48) that

$$\left|\frac{\partial(f_1, f_2)}{\partial(r, s)}\right|_{(r,s)=(r_1,\pi/2)} = \frac{4\pi^2}{r_1} J_2^2(r_1) \neq 0. \tag{1.52}$$

Therefore by Theorem 1.1.1 there is a periodic orbit of system (1.43) for each $(r_1, \pi/2)$, that is for each value of $(X_0, Y_0) = (-r_1, -r_1)$.

In an analogous way there is a periodic orbit of system (1.43) for each $(r_1, 3\pi/2)$, that is for each value of $(X_0, Y_0) = (r_1, r_1)$. In fact, the periodic orbit with this initial conditions and the previous one with initial conditions $(X_0, Y_0) = (-r_1, -r_1)$ are the same.

Taking the radius ρ of the disc $V = \{(X_0, Y_0, 0) : 0 < X^2 + Y^2 < \rho\}$ in the proof of Theorem 1.3.1 conveniently large we include in it as many zeros of the system $f_1(X_0, Y_0) = f_2(X_0, Y_0) = 0$ as we want, so from Theorem 1.3.1, Proposition 1.4.2 follows. □

1.4.3 The Vallis system (El Niño phenomenon)

The results of this section come from the paper of Euzébio and Llibre [43].

The Vallis system, introduced by Vallis [118] in 1988, is a periodic non–autonomous 3–dimensional system that models the atmosphere dynamics in the tropics over the Pacific Ocean, related to the yearly oscillations of precipitation, temperature and wind force. Denoting by x the wind force, by y the difference of near–surface water temperatures of the east and west parts of the Pacific Ocean, and by z the average near–surface water temperature, the Vallis system is

$$\begin{aligned} \frac{dx}{dt} &= -ax + by + au(t), \\ \frac{dy}{dt} &= -y + xz, \\ \frac{dz}{dt} &= -z - xy + 1, \end{aligned} \tag{1.53}$$

where $u(t)$ is some C^1 $T-$periodic function that describes the wind force under seasonal motions of air masses, and the parameters a and b are positive.

Although this model neglects some effects like Earth's rotation, pressure field and wave phenomena, it provides a correct description of the observed processes and recovers many of the observed properties of El Niño. The properties of El Niño phenomena are studied analytically in [115] and [118]. More precisely, in [118] it is shown that taking $u \equiv 0$, it is possible to observe the presence of chaos by considering $a = 3$ and $b = 102$. Later on, in [115] it is proved that exists a chaotic attractor for system (1.53) after a Hopf bifurcation. This chaotic motion can be easily understanding if we observe that there exist a strong similarity between system (1.53) and Lorenz system, which becomes more clear under the replacement of z by $z + 1$ in system (1.53).

Now we shall provide sufficient conditions in order that system (1.53) has periodic orbits, and additionally we characterize the stability of these periodic orbits. As far as we know, the study of the periodic orbits in the non–autonomous

Vallis system has not been considered in the literature, with the exception of the Hopf bifurcation studied in [115].

We define
$$I = \int_0^T u(s)ds.$$
Now we state our main results.

Theorem 1.4.3. *For $I \neq 0$ and $a \neq b$ the Vallis system* (1.53) *has a T–periodic solution $(x(t), y(t), z(t))$ such that*
$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1\right),$$
Moreover this periodic orbit is stable if $a > b$ and unstable if $a < b$.

We do not know the reliability of the Vallis model approximating the Niño phenomenon, but it seems that for the moment this is one of the best models for studying the Niño phenomenon. Accepting this reliability we can said the following.

The stable periodic solution provided by Theorem 1 says that the Niño phenomenon exhibits a periodic behavior if the T-periodic function $u(t)$ and the parameters a and b of the system satisfy that $I \neq 0$ and $a > b$. Moreover Theorem 1 states that this periodic solution lives near the point
$$(x, y, z) = \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1\right).$$

Since the periodic solutions found in Theorems 3, 4 and 5 are also stable, we can provide a similar physical interpretation for them as we have done for the periodic solution of Theorem 1.

Theorem 1.4.4. *For $I \neq 0$ the Vallis system* (1.53) *has a T–periodic solution $(x(t), y(t), z(t))$ such that*
$$(x(t), y(t), z(t)) \approx \left(-\frac{aI}{Tb}, -\frac{aI}{Tb}, 1\right),$$
Moreover this periodic orbit is always unstable.

Theorem 1.4.5. *For $I \neq 0$ the Vallis system* (1.53) *has a T–periodic solution $(x(t), y(t), z(t))$ such that*
$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, \frac{I}{T}, 1\right),$$
Moreover this periodic orbit is always stable.

Theorem 1.4.6. *For $I \neq 0$ the Vallis system (1.53) has a T–periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, 0, 1\right),$$

Moreover this periodic orbit is always stable.

In what follows we consider the function

$$J(t) = \int_0^t u(s)ds.$$

and note that $J(T) = I$. So we have the following result.

Theorem 1.4.7. *Consider $I = 0$ and $J(t) \neq 0$ if $0 < t < T$. Then the Vallis system (1.53) has a T–periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{a}{T}\int_0^T J(s)ds, 0, 1\right),$$

Moreover this periodic orbit is always stable.

Proof of the results

The tool for proving our results will be the averaging theory. This theory applies to periodic non–autonomous differential systems depending on a small parameter ε. Since the Vallis system already is a T–periodic non–autonomous differential system, in order to apply to it the averaging theory described in section 3 we need to introduce in such system a small parameter. This is reached doing convenient rescalings in the variables (x, y, z), in the parameters (a, b) and in the function $u(t)$. Playing with different rescalings we shall obtain different result on the periodic solutions of the Vallis system. More precisely, in order to study the periodic solutions of the differential system (1.53), we start doing a rescaling of the variables (x, y, z), of the function $u(t)$, and of the parameters a and b, as follows

$$\begin{array}{lll} x = \varepsilon^{m_1} X, & y = \varepsilon^{m_2} Y, & z = \varepsilon^{m_3} Z, \\ u(t) = \varepsilon^{n_1} U(t), & a = \varepsilon^{n_2} A, & b = \varepsilon^{n_3} B, \end{array} \tag{1.54}$$

where ε always is positive and sufficiently small, and m_i and n_j are non–negative integers, for all $i, j = 1, 2, 3$. Then in the new variables (X, Y, Z) system (1.53) writes

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2+m_3} XZ, \\ \frac{dZ}{dt} &= -Z - \varepsilon^{m_1+m_2-m_3} XY + \varepsilon^{-m_3}. \end{aligned} \tag{1.55}$$

Consequently, in order to have non–negative powers of ε we must impose the conditions

$$m_3 = 0 \quad \text{and} \quad 0 \le m_2 \le m_1 \le L, \tag{1.56}$$

where $L = \min\{m_2 + n_3, n_1 + n_2\}$. So system (1.55) becomes

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2} XZ, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_1+m_2} XY. \end{aligned} \tag{1.57}$$

Our aim is to find periodic solutions of system (1.57) for some special values of m_i, n_j, $i, j = 1, 2, 3$, and after we go back through the rescaling (1.54) to guarantee the existence of periodic solutions in system (1.53). In what follows we consider the case where n_2 and n_3 are positives and $m_2 = m_1 < n_1 + n_2$. These conditions lead to the proofs of Theorems 1.4.3, 1.4.4 and 1.4.5. For this reason we present these proofs together in order to avoid repetitive arguments. Moreover, in what follows we consider

$$K = \int_0^T U(s)ds.$$

Proofs of Theorems 1, 2 *and* 3*:* We start considering system (1.57) with n_2 and n_3 positive and $m_2 = m_1 < n_1 + n_2$. So we have

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{n_3} BY + \varepsilon^{-m_1+n_1+n_2} AU(t), \\ \frac{dY}{dt} &= -Y + XZ, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{2m_1} XY. \end{aligned} \tag{1.58}$$

Now we apply the averaging method to the differential system (1.58). Using the notation of section 1.5 we have $\mathbf{x} = (X, Y, Z)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ -Y + XZ \\ 1 - Z \end{pmatrix}. \tag{1.59}$$

We start considering the system

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}). \tag{1.60}$$

Its solution $\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t), Z(t))$ such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0, Z_0)$ is

$$\begin{aligned} X(t) &= X_0, \\ Y(t) &= (1 - e^{-t}(1+t))X_0 + e^{-t}Y_0 + e^{-t}tX_0Z_0, \\ Z(t) &= 1 - e^{-t} + e^{-t}Z_0. \end{aligned}$$

In order that $\mathbf{x}(t,\mathbf{z},0)$ is a periodic solution we must choose $Y_0 = X_0$ and $Z_0 = 1$. This implies that for every point of the straight line $X = Y$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X,Y,Z,t) \in \mathbb{R}^3 \times \mathbb{S}^1$. Here and in what follows $\mathbb{S}^1$ is the interval $[0,T]$ identifying T with 0.

Observe that, using the notation of section 1.5, we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(X_0) = (X_0, 1)$, and consequently $\mathcal{M}$ is an one–dimensional manifold given by $\mathcal{M} = \{(X_0, X_0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$. The fundamental matrix $M_{\mathbf{z}}(t)$ of (1.60), satisfying that $M_{\mathbf{z}}(0)$ is the identity of $\mathbb{R}^3$, is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - \cosh t + \sinh t & e^{-t} & e^{-t}tX_0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

and its inverse matrix $M_{\mathbf{z}}^{-1}(t)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{t} & e^{t} & -e^{t}tX_0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Since the matrix $M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}(T)$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^{T} & e^{T}TX_0 \\ 0 & 1 - e^{T} \end{pmatrix},$$

with $\det(\Delta) = (1 - e^{T})^2 \neq 0$ because $T \neq 0$, we can apply the averaging theory described in section 1.5.

Let F be the vector field of system (1.58) minus F_0 given in (1.59). Then the components of the function $M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z},0))$ are

$$\begin{aligned} g_1(X_0,t) &= -\varepsilon^{n_2}AX_0 + \varepsilon^{n_3}BX_0 + \varepsilon^{-m_1+n_1+n_2}AU(t), \\ g_2(X_0,t) &= \varepsilon^{2m_1}e^{t}tX_0^3 + (1 - e^{t})g_1(X_0,t), \\ g_3(X_0,t) &= -\varepsilon^{2m_1}e^{t}X_0^2. \end{aligned}$$

In order to apply averaging theory of first order we need to consider only terms up to order ε. Analysing the expressions of g_1, g_2 and g_3 we note that these terms depend on the values of m_1 and n_j, for each $j = 1, 2, 3$. In fact, we just need to study the integral of g_1 because $k = 1$. Moreover studying the function g_1 we observe that the only possibility to obtain an isolated zero of the function

$$f_1(X_0) = \int_0^T g_1(X_0, t)dt$$

is assuming that $n_1 + n_2 - m_1 = 1$. Otherwise, the only solution of $f_1(X_0) = 0$ is $X_0 = 0$ which correspond to the equilibrium point $(X_0, Y_0, Z_0) = (0, 0, 1)$ of system (1.60). The same occurs if n_2 and n_3 are greater than 1 simultaneously. This analysis reduces the existence of possible periodic solutions to the following cases:

(p_1) $n_2 = 1$ and $n_3 = 1$;

(p_2) $n_2 > 1$ and $n_3 = 1$;

(p_3) $n_2 = 1$ and $n_3 > 1$.

In the case (p_1) we have $M_{\mathbf{z}}^{-1}(t)F_1(t, \mathbf{x}(t, \mathbf{z}, 0)) = -AX_0 + BX_0 + AU(t)$, and then

$$f_1(X_0) = (-A + B)TX_0 + AK.$$

Consequently, if $A \neq B$, then $f_1(X_0) = 0$ implies

$$X_0 = \frac{AK}{T(A - B)}.$$

So, by Theorem 1.3.1, system (1.58) has a periodic solution $(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon))$ such that

$$(X(0, \varepsilon), Y(0, \varepsilon), Z(0, \varepsilon)) \longrightarrow (X_0, Y_0, Z_0) = \left(\frac{AK}{T(A - B)}, \frac{AK}{T(A - B)}, 1\right)$$

when $\varepsilon \to 0$. Note that the point (X_0, Y_0, Z_0) is an equilibrium point of system (1.58). Then, if we take $n_1 = n_2 = n_3 = 1$ and going back through the rescaling (1.54) of the variables and parameters, we obtain that the periodic solution of system (1.58) becomes the periodic solution $(x(t), y(t), z(t))$ of system (1.53) satisfying that

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a - b)}, \frac{aI}{T(a - b)}, 1\right).$$

Indeed, we observe that

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(a\varepsilon^{-1})(I\varepsilon^{-1})}{T\varepsilon^{-1}(a - b)} = \frac{aI}{T(a - b)}.$$

Moreover, we note that $f_1'(x_0) = \varepsilon f_1'(X_0) = -a + b \neq 0$, so the periodic orbit corresponding to x_0 is stable if $a > b$, and unstable otherwise. So this completes the proof of Theorem 1.4.3.

Analogously the function f_1 in the cases (p_2) and (p_3) is

$$f_1(X_0) = TBX_0 + AK \qquad \text{and} \qquad f_1(X_0) = -TAX_0 + AK,$$

respectively. In the first case the condition $f_1(X_0) = 0$ implies

$$X_0 = -\frac{AK}{TB}.$$

Now we observe that we have $n_2 > 1$ and $n_3 = 1$. So, going back through the rescaling we obtain

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(-a\varepsilon^{-n_2})(I\varepsilon^{-n_1})}{Tb\varepsilon^{-1}} = -\frac{aI}{Tb\varepsilon^{n_1+n_2-2}}.$$

and consequently, choosing $n_1 = 0$ and $n_2 = 2$, we get $x_0 = -aI/(Tb)$. Note also that $f_1'(x_0) = Tb > 0$, then the periodic orbit corresponding to x_0 is always unstable. Thus Theorem 1.4.4 is proved.

Finally, in the case (p_3), $f_1(X_0) = 0$ implies $X_0 = K/T$. So, taking $n_1 = 1$ and going back through the rescaling, we have $x_0 = \varepsilon X_0 = \varepsilon I/(T\varepsilon) = I/T$. Additionally, we have that $f_1'(x_0) = -Ta < 0$. Therefore the periodic solution that comes from x_0 is always stable. This proves Theorem 1.4.5. □

Proof of Theorem 1.4.6: As in the proofs of Theorems 1, 2 and 3 we start considering a more general case in the powers of ε in (1.57) taking $n_2 > 0$ and $m_2 < m_1 < L$. In this case the function $F_0(t, \mathbf{x})$ of system (1.32) is

$$F_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ Y \\ 1 - Z \end{pmatrix}. \tag{1.61}$$

Then the solution $\mathbf{x}(t, \mathbf{z}, 0)$ of system (1.33) satisfying $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$ is

$$(X(t), Y(t), Z(t)) = (X_0, e^{-t}Y_0, 1 - e^{-t} + e^{-t}Z_0).$$

This solution is periodic if $Y_0 = 0$ and $Z_0 = 1$. Then for every point of the straight line $Y = 0$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. We observe that using the notation of section 1.5 we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(\alpha) = (0, 1)$. Consequently $\mathcal{M}$ is an one–dimensional manifold given by $\mathcal{M} = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$.

The fundamental matrix $M_{\mathbf{z}}(t)$ of (1.34) with F_0 given by (1.61) satisfying $M_{\mathbf{z}}(0) = Id_3$ and its inverse $M_{\mathbf{z}}^{-1}(t)$ are given by

$$M_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad M_{\mathbf{z}}^{-1}(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Since the matrix $M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}T$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$, we can apply the averaging theory described in section 1.5. Again using the notations introduced in the proofs of Theorems 1, 2 and 3, since $k = 1$ we will look only to the integral of the first coordinate of $\mathcal{F} = (f_1, f_2, f_3)$. In this case we have

$$g_1(X_0, Y_0, Z_0, t) = -\varepsilon^{n_2} AX_0 + \varepsilon^{-m_1+n_1+n_2} AU(t).$$

Comparing this function g_1 with the same function obtained in the proof of Theorems 1, 2 and 3, it is easy to see that this case correspond to the case (p_3) of the mentioned theorems. Then, in order to have periodic solutions, we need to choose $n_2 = 1$ and $n_1 + n_2 - m_1 = 1$. So, following the steps of the proof of case (p_3) by choosing $n_1 = 1$ and coming back through the rescaling (1.54) to system (1.53), Theorem 1.4.6 is proved. □

Proof of Theorem 5*:* We start considering system (1.57) with $n_3 = 2$, $n_2 > 0$, $m_1 = n_1 + n_2$ and $m_2 < m_1 < m_2 + n_3$. With these conditions system (1.57) becomes

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{m_2 - n_1 - n_2 + n_3} BY + AU(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{-m_2 + n_1 + n_2} XZ, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_2 + n_1 + n_2} XY. \end{aligned} \tag{1.62}$$

Again we will use the averaging theory described in section 1.5. So considering $\mathbf{x} = (X, Y, Z)^T$ we obtain

$$F_0(t, \mathbf{x}) = \begin{pmatrix} AU(t) \\ -Y \\ 1 - Z \end{pmatrix}. \tag{1.63}$$

Now we note that the solution $\mathbf{x}(t,\mathbf{z},0) = (X(t), Y(t), Z(t))$ such that $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (X_0, Y_0, Z_0)$ of the system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) \tag{1.64}$$

is

$$X(t) = X_0 + \int_0^t AU(s)ds, \quad Y(t) = e^{-t}Y_0, \quad Z(t) = 1 - e^{-t} + e^{-t}Z_0.$$

Since $I = 0$ and $J(t) \neq 0$ for $0 < t < T$, in order that $\mathbf{x}(t,\mathbf{z},0)$ is a periodic solution we need to fix $Y_0 = 0$ and $Z_0 = 1$. This implies that for every point in a neighbourhood of X_0 in the straight line $Y = 0$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$.

Following the notation of section 1.5, we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(X_0) = (0,1)$. Hence $\mathcal{M}$ is an one–dimensional manifold $\mathcal{M} = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$ and the fundamental matrix $M_{\mathbf{z}}(t)$ of (1.64) satisfying that $M_{\mathbf{z}}(0)$ is the identity of $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}.$$

It is easy to see that the matrix $M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}(T)$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$. Then the hypotheses of Theorem 1.3.1 are satisfied. Now the components of the function $M_{\mathbf{z}}^{-1}(t)F(t,\mathbf{x}(t,\mathbf{z},\mathbf{0}))$ are

$$\begin{aligned} g_1(X_0,t) &= -\varepsilon^{n_2} A\left(X_0 + \int_0^t AU(s)ds\right) + AU(t), \\ g_2(X_0,t) &= \varepsilon^{-m_2+n_1+n_2}\left(X_0 + \int_0^t AU(s)ds\right)e^t, \\ g_3(X_0,t) &= 0. \end{aligned}$$

Taking n_1 and n_2 equal to one and observing that $k = 1$ and $n = 3$, we are interested only in the first component of the function $F_1 = (F_{11}, F_{12}, F_{13})$ described in section 1.5. Indeed, applying the averaging theory we must study the zeros of the first component of the function

$$\mathcal{F}(X_0) = (f_1(X_0), f_2(X_0), f_3(X_0)) = \int_0^T M_{\mathbf{z}}^{-1}(t,\mathbf{z})F_{11}(t,\mathbf{x}(t,\mathbf{z}))dt.$$

Since

$$F_{11} = -A\left(X_0 + \int_0^t AU(s)ds\right),$$

then

$$\begin{aligned} f_1(X_0) &= \int_0^T -A\left(X_0 + \int_0^t AU(s)ds\right)dt \\ &= -ATX_0 - A^2\int_0^T\left(\int_0^t U(s)ds\right)ds. \end{aligned}$$

Therefore, from $f_1(X_0) = 0$ we obtain

$$X_0 = -\frac{A}{T}\int_0^T\left(\int_0^t U(s)ds\right)ds \neq 0.$$

So, using rescaling (1.54) we get

$$x_0 = \varepsilon^2 X_0 = -\varepsilon^2\frac{a\varepsilon^{-1}}{\varepsilon T}\int_0^T J(s)ds = -\frac{a}{T}\int_0^T J(s)ds.$$

Moreover, since $f_1'(x_0) = -a/T < 0$, because a and ε are positive, the $T-$periodic orbit detected by the averaging theory is always stable. This ends the proof. □

1.5 Another first order averaging method for periodic orbits

The next result proved in [80] extends the result of Theorem 1.3.1 to the case $n = 2m$ and when the matrix Δ_α of the statement of Theorem 1.3.1 is the zero matrix. Here $\xi^\perp\colon \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection of $\mathbb{R}^n$ onto its second set of m coordinates; i.e. $\xi^\perp(x_1, \dots, x_n) = (x_{m+1}, \dots, x_n)$.

Theorem 1.5.1. *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0\colon \mathrm{Cl}(V) \to \mathbb{R}^m$ be a $\mathcal{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \,|\, \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^{2m}$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of $(1.32)_{\varepsilon=0}$ is T–periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (1.1) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$*

(i) *has in the upper right corner the $m \times m$ matrix Ω_α with $\det(\Omega_\alpha) \neq 0$, and*

(ii) *has in the lower right corner the $m \times m$ zero matrix.*

Consider the function $\mathcal{G}\colon \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathcal{G}(\alpha) = \xi^\perp\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t)F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\right). \tag{1.65}$$

Suppose that there is $\alpha_0 \in V$ *with* $\mathcal{G}(\alpha_0) = 0$*, then the following statements hold for* $\varepsilon \neq 0$ *sufficiently small. If* $\det((\partial \mathcal{G}/\partial \alpha)(\alpha_0)) \neq 0$*, then there is a unique* T*-periodic solution* $\mathbf{x}(t, \varepsilon)$ *of system* (1.32) *such that* $\mathbf{x}(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0})$ *as* $\varepsilon \to 0$.

Theorem 1.5.1 is proved in section 1.8. In the next section we provide some applications of this theorem.

1.5.1 A class of Duffing differential equations

Many different classes of Duffing differential equations have been studied by different authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability, bifurcation,... See for instance the survey of J. Mawhin [96] and for the articles [35, 99].

In this section we shall study the class of Duffing differential equations of the form

$$x'' + cx' + a(t)x + b(t)x^3 = h(t), \tag{1.66}$$

where $c > 0$ is a constant, and $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. These differential equations were studied by Chen and Li in the papers [22, 21]. Their results were improved in [8] by Benterki and Llibre, we present a part of these improvements here as an application of Theorem 1.5.1.

Instead of working with the Duffing differential equation (1.66) we shall work with the equivalent differential system

$$\begin{aligned} x' &= y, \\ y' &= -cy - a(t)x - b(t)x^3 + h(t). \end{aligned} \tag{1.67}$$

Theorem 1.5.2. *For every simple real root of the polynomial*

$$q(x_0) = -\left(\int_0^T b(s)\,ds\right) x_0^3 - \left(\int_0^T a(s)\,ds\right) x_0 + \int_0^T h(s)\,ds.$$

the differential system (1.67) *has a periodic solution* $(x(t), y(t))$ *such that* $(x(0), y(0))$ *is close to* $(x_0, 0)$.

Proof. We start doing a rescaling of the variables (x, y), of the functions $a(t)$, $b(t)$ and $h(t)$ and of the parameter c as follows

$$\begin{aligned} x &= \varepsilon X, & y &= \varepsilon^2 Y, \\ c &= \varepsilon C, & a(t) &= \varepsilon A(t), \\ b(t) &= \varepsilon^{-1} B(t), & h(t) &= \varepsilon^2 H(t). \end{aligned} \tag{1.68}$$

Then system (1.67) becomes

$$\begin{aligned} \dot{X} &= \varepsilon Y, \\ \dot{Y} &= -\varepsilon C Y - A(t)X - B(t)X^3 + H(t), \end{aligned} \tag{1.69}$$

We shall apply the averaging Theorem 1.5.1 to system (1.69) and we shall obtain Theorem 1.5.2. In what follows we shall use the notation of Theorem 1.5.1. Thus $\mathbf{x} = (X, Y)^T$ and

$$F_0(t, \mathbf{x}) = \begin{pmatrix} 0 \\ -A(t)X - B(t)X^3 + H(t) \end{pmatrix},$$
$$F_1(t, \mathbf{x}) = \begin{pmatrix} Y \\ -CY \end{pmatrix},$$
$$F_2(t, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The differential system (1.69) with $\varepsilon = 0$ has the solution $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t))^T$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0)^T$, where

$$X(t) = X_0,$$
$$Y(t) = Y_0 + \int_0^t \big(-B(s)X_0^3 - A(s)X_0 + H(s)\big)\, ds.$$

In order that $\mathbf{x}(t, \mathbf{z}, 0)$ be a periodic solution X_0 must satisfy

$$\int_0^T \big(-B(s)X_0^3 - A(s)X_0 + H(s)\big)\, ds = 0, \tag{1.70}$$

and Y_0 is arbitrary. Therefore we get

$$\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) = \left(Y_0, \bar{X}_0\right),$$

where $\bar{X}_0$ is a real root of the cubic polynomial (1.70). In short the unperturbed system (i.e. system (1.69) with $\varepsilon = 0$) has at most three families of periodic solutions because Y_0 is arbitrary and $\bar{X}_0$ is a real root of the cubic polynomial (1.70). Therefore, using the notation of Theorem 1.5.1, we have $n = 2$ and $m = 1$ for each one of these possible families of periodic solutions.

We compute the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ associated to the first variational system (1.34) associated to the vector field $(\dot{Y}, \dot{X})$ given by (1.69) with $\varepsilon = 0$, and such that $M_{\mathbf{z}_\alpha}(0) = \text{Id of } \mathbb{R}^2$, and we obtain

$$M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & -\displaystyle\int_0^t \big(3B(s)X_0^2 + A(s)\big)\, ds \\ 0 & 1 \end{pmatrix}.$$

The matrix

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T) = \begin{pmatrix} 0 & -\displaystyle\int_0^T \big(3B(s)X_0^2 + A(s)\big)\, ds \\ 0 & 0 \end{pmatrix}$$

has a non–zero 1×1 matrix in the upper right corner if the real root $\bar{X}_0$ of the cubic polynomial (1.70) is simple, and a zero 1×1 matrix in its lower right corner. Therefore the assumptions of Theorem 1.5.1 hold, then by applying this theorem we study the periodic solutions which can be prolonged from the unperturbed differential system to the perturbed one. Since for our differential system we have $\xi^{\perp}(Y, X) = X$, then we must compute the function $\mathcal{G}(\alpha) = \mathcal{G}(Y_0)$ given in (1.2), i.e.

$$\mathcal{G}(Y_0) = \xi^{\perp}\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t)F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha, 0))dt\right) = -\int_0^T CY_0 = -CTY_0.$$

Theorem 1.5.1 says that for every simple real root $Y_0 = 0$ of the polynomial $\mathcal{G}(Y_0)$ the differential system (1.69) with $\varepsilon \neq 0$ sufficiently small has a periodic solution $(X(t), Y(t))$ such that $(X(0), Y(0))$ tends to $(\bar{X}_0, 0)$ when $\varepsilon \to 0$, being $\bar{X}_0$ a simple real root of the cubic polynomial (1.70).

Now it is easy to check that the cubic polynomial (1.70) after the change of variables (1.68), i.e.

$$X = \frac{x}{\varepsilon}, \quad Y = \frac{y}{\varepsilon^2}, \quad H(t) = \frac{h(t)}{\varepsilon^2}, \quad B(s) = \varepsilon b(s), \quad A(s) = \frac{a(s)}{\varepsilon}.$$

becomes the polynomial $q(x_0)$. Hence the theorem is proved. □

1.6 Proof of Theorem 1.1.1

Proof of statement (a) *of Theorem 1.1.1.* The assumptions guarantee the existence and uniqueness of the solutions of the initial valued problems (1.1) and (1.2) on the time–scale $1/\varepsilon$. We introduce

$$u(t, \mathbf{x}) = \int_0^t [F(s, \mathbf{x}) - f^0(\mathbf{x})]ds. \tag{1.71}$$

Since we have subtracted the average of $f(s, \mathbf{x})$ in the integrand, the integral is bounded, i.e.

$$||u(t, \mathbf{x})|| \leq 2MT, \quad t \geq 0, \quad \mathbf{x} \in D.$$

We now introduce a transformation near the identity

$$\mathbf{x}(t) = \mathbf{z}(t) + \varepsilon u(t, \mathbf{z}(t)). \tag{1.72}$$

This transformation will be used for simplifying equation (1.1).

Differentiation of (3.15) and substitution in (1.1) yields

$$\begin{aligned}\dot{\mathbf{x}} &= \dot{\mathbf{z}} + \varepsilon\frac{\partial}{\partial t}u(t, \mathbf{z}) + \varepsilon\frac{\partial}{\partial \mathbf{z}}u(t, \mathbf{z})\dot{\mathbf{z}} \\ &= \varepsilon F(t, \mathbf{z} + \varepsilon u(t, \mathbf{z})) + \varepsilon^2 R(t, \mathbf{z} + \varepsilon u(t, \mathbf{z}), \varepsilon).\end{aligned}$$

Using (3.14) we write this equation in the form

$$\left(I + \varepsilon \frac{\partial}{\partial \mathbf{z}} u(t, \mathbf{z})\right) \dot{\mathbf{z}} = \varepsilon f^0(\mathbf{z}) + S,$$

with I the $n \times n$ identity matrix and where

$$S = \varepsilon F(t, \mathbf{z} + \varepsilon u(t, \mathbf{z})) - \varepsilon F(t, \mathbf{z}) + \varepsilon^2 R(t, \mathbf{z} + \varepsilon u(t, \mathbf{z}), \varepsilon).$$

Since $\partial u / \partial \mathbf{z}$ is uniformly bounded (as u) we can invert to obtain

$$\left(I + \varepsilon \frac{\partial}{\partial \mathbf{z}} u(t, \mathbf{z})\right)^{-1} = I - \varepsilon \frac{\partial}{\partial \mathbf{z}} u(t, \mathbf{z}) + O(\varepsilon^2), \quad t \geq 0, \quad \mathbf{z} \in D. \tag{1.73}$$

From the Lipschitz continuity of $F(t, \mathbf{z})$ we have

$$||F(t, \mathbf{z} + \varepsilon u(t, \mathbf{z})) - F(t, \mathbf{z})|| \leq L\varepsilon ||u(t, \mathbf{z})|| \leq L\varepsilon 2MT,$$

where L is the Lispchitz constant. Due to the boundedness of R it follows that for some positive constant C, independent of ε, we have

$$||S|| \leq \varepsilon^2 C, \quad t \geq 0, \quad \mathbf{z} \in D. \tag{1.74}$$

From (1.73) and (1.74) we get for $\mathbf{z}$ that

$$\dot{\mathbf{z}} = \varepsilon f^0(\mathbf{z}) + S - \varepsilon^2 \frac{\partial u}{\partial \mathbf{z}} f^0(\mathbf{z}) + O(\varepsilon^3), \quad \mathbf{z}(0) = \mathbf{x}(0). \tag{1.75}$$

As $S = O(\varepsilon^2)$ by introducing the time–like variable $\tau = \varepsilon t$, we obtain that the solution of

$$\frac{d\mathbf{y}}{d\tau} = f^0(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{z}(0)$$

approximates the solution of (1.75) with error $O(\varepsilon)$ on the time–scale 1 in τ, i.e. on the time–scale $1/\varepsilon$ in t. Due to the near identity transformation (3.15) we obtain that

$$\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon) \tag{1.76}$$

in the time–scale $1/\varepsilon$.

Now we shall impose the periodicity condition after which we can apply the Implicit Function Theorem. We transform $\mathbf{x} \to \mathbf{z}$ with the near identity transformation (3.15), then the equation for $\mathbf{z}$ becomes

$$\dot{\mathbf{z}} = \varepsilon f^0(\mathbf{z}) + \varepsilon^2 S(t, \mathbf{z}, \varepsilon). \tag{1.77}$$

Due to the choice of $u(t, \mathbf{z}(t))$, a T–periodic solution $\mathbf{z}(t)$ produces a T–periodic solution $\mathbf{x}(t)$. For S we have the expression

$$S(t, \mathbf{z}, \varepsilon) = \frac{\partial F}{\partial \mathbf{z}}(t, \mathbf{z}) u(t, \mathbf{z}) - \frac{\partial u}{\partial \mathbf{z}}(t, \mathbf{z}) f^0(\mathbf{z}) + R(t, \mathbf{z}, 0) + O(\varepsilon).$$

This expression is T–periodic in t and continuously differentiable with respect to $\mathbf{z}$.

Equation (1.77) is equivalent with the integral equation

$$\mathbf{z}(t) = \mathbf{z}(0) + \varepsilon \int_0^t f^0(\mathbf{z}(s))ds + \varepsilon^2 \int_0^t S(s, \mathbf{z}(s), \varepsilon)ds.$$

The solution $\mathbf{z}(t)$ is T–periodic if $\mathbf{z}(t+T) = \mathbf{z}(t)$ for all $t \geq 0$ which leads to the equation

$$h(\mathbf{z}(0), \varepsilon) = \int_0^T f^0(\mathbf{z}(s))ds + \varepsilon \int_0^T S(s, \mathbf{z}(s), \varepsilon)ds = 0. \tag{1.78}$$

Note that this is a short–hand notation. The righthand side of equation (1.78) does not depend on $\mathbf{z}(0)$ explicitly. But the solutions depend continuously on the initial values and so the dependence on $\mathbf{z}(0)$ is implicitly by the bijection $\mathbf{z}(0) \to \mathbf{z}(x)$.

It is clear that $h(p, 0) = 0$. If ε is in a neighborhood of $\varepsilon = 0$, then equation (1.78) has a unique solution $\mathbf{x}(t, \varepsilon) = \mathbf{z}(t, \varepsilon)$ because of the assumption on the Jacobian determinant (1.4). If $\varepsilon \to 0$ then $\mathbf{z}(0, \varepsilon) \to p$. This completes the proof of statement (a). □

For proving statement (b) of Theorem 1.1.1 we need some preliminary results. The first result is the Gronwall's inequality.

Lemma 1.6.1. *Let a be a positive constant. Assume that $t \in [t_0, t_0 + a]$ and*

$$\varphi(t) \leq \delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2, \tag{1.79}$$

where $\psi(t) \leq 0$ and $\varphi(t) \leq 0$ are continuous functions, and $\delta_i > 0$ for $i = 1, 2$. Then

$$\varphi(t) \leq \delta_2 e^{\delta_1 \int_{t_0}^t \psi(s)ds}.$$

Proof. From (1.79) we get

$$\frac{\varphi(t)}{\delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2} \leq 1.$$

Multiplying by $\delta_1\psi(t)$ and integrating we obtain

$$\int_{t_0}^t \frac{\delta_1\psi(s)\varphi(s)}{\delta_1 \int_{t_0}^s \psi(r)\varphi(r)dr + \delta_2} ds \leq \delta_1 \int_{t_0}^t \psi(s)ds,$$

therefore

$$\log\left(\delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2\right) - \log\delta_2 \leq \delta_1 \int_{t_0}^t \psi(s)ds.$$

Hence

$$\delta_1 \int_{t_0}^t \psi(s)\varphi(s)ds + \delta_2 \leq \delta_2 e^{\delta_1 \int_{t_0}^t \psi(s)ds}.$$

From (1.79) the lemma follows. □

We consider the linear differential system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1.80}$$

where A is a constant $n \times n$ matrix. The *eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of system* (1.80) are the zeros of the *characteristic polynomial* $\det(A - \lambda Id)$.

If the eigenvalues λ_k are different with eigenvectors e_k for $k = 1, \ldots, n$, then

$$e_k e^{\lambda_k t} \quad \text{for } k = 1, \ldots, n,$$

are n independent solutions of the system (1.80).

Assume now that not all eigenvalues are different, thus suppose that the eigenvalue λ has multiplicity $m > 1$. Then λ generates m independent solutions of system (1.80) of the form

$$P_0 e^{\lambda t}, P_1(t)e^{\lambda t}, \ldots, P_{m-1}(t)e^{\lambda t},$$

where $P_i(t)$ for $i = 0, 1, \ldots, m-1$ are polynomial vectors of degree at most i.

With n independent solutions $x_1(t), \ldots, x_n(t)$ of system (1.80) we form a matrix

$$\Phi(t) = (x_1(t), \ldots, x_n(t)),$$

called a *fundamental matrix* of system (1.80). Every solution $\mathbf{x}(t)$ of system (1.80) can be written $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector. Moreover the solution $\mathbf{x}(t)$ such that $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0. \tag{1.81}$$

Usually we choose the fundamental matrix $\Phi(t)$ in such away that $\Phi(t_0) = Id$. From (1.81) and the explicit form of the independent solutions of system (1.80) it follows easily the next result.

Proposition 1.6.2. *We consider the linear differential system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is a constant $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the following statements hold.*

(a) *If $\text{Re}\lambda_k < 0$ for $k = 1, \ldots, n$, then for each solution $\mathbf{x}(t)$ such that $\mathbf{x}(t_0) = \mathbf{x}_0$ there exist two positive constants C and μ satisfying*

$$||\mathbf{x}(t)|| \leq C||\mathbf{x}_0||e^{-\mu t} \quad \textit{and} \quad \lim_{t \to \infty} \mathbf{x}(t) = 0.$$

(b) *If $\text{Re}\lambda_k \leq 0$ for $k = 1, \ldots, n$ and the eigenvalues with $\text{Re}\lambda_k = 0$ are different, then the solution $\mathbf{x}(t)$ is bounded for $t \geq t_0$. More precisely*

$$||\mathbf{x}(t)|| \leq C||\mathbf{x}_0|| \quad \textit{with } C > 0.$$

(c) *If there exists an eigenvalue* λ_k *with* $\mathrm{Re}\lambda_k > 0$*, then in each neighborhood of* $\mathbf{x} = 0$ *there are solutions* $\mathbf{x}(t)$ *such that*

$$\lim_{t\to\infty} ||\mathbf{x}(t)|| = \infty.$$

Under the assumptions of statement (a) of Proposition 1.6.2 the solution $\mathbf{x} = 0$ is called *asymptotically stable*. Under the assumptions of statement (b) the solution $\mathbf{x} = 0$ is called *Liapunov stable*. Finally, nder the assumptions of statement (c) the solution $\mathbf{x} = 0$ is called *unstable*.

The next result is also known as the Poincaré–Liapunov Theorem.

Theorem 1.6.3. *Consider the differential system*

$$\dot{\mathbf{x}} = A\mathbf{x} + B(t)\mathbf{x} + f(t,\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1.82}$$

where $t \in \mathbb{R}$*,* A *is a constant* $n \times n$ *matrix having all its eigenvalues with negative real part,* $B(t)$ *is a continuous* $n \times n$ *matrix such that* $\lim_{t\to\infty} ||B(t)|| = 0$*. The function* $f(t,\mathbf{x})$ *is continuous in* t *and* $\mathbf{x}$*, and Lipschitz in* $\mathbf{x}$ *in a neighborhood of* $\mathbf{x} = 0$*. If*

$$\lim_{||\mathbf{x}||\to 0} \frac{f(t,\mathbf{x})}{||\mathbf{x}||} = 0 \quad \textit{uniformly in } t,$$

then there exists a positive constants C*,* t_0*,*δ *and* μ *such that* $||\mathbf{x}_0|| \le \delta$ *implies*

$$||\mathbf{x}(t)|| \le C||\mathbf{x}_0||e^{-\mu(t-t_0)} \quad \textit{for } t \ge t_0.$$

The solution $\mathbf{x} = 0$ *is asymptotically stable and the attraction is exponential in a* δ*–neighborhood of* $\mathbf{x} = 0$*.*

Proof. By Proposition 1.6.2 we have an estimate for the fundamental matrix of the differential system

$$\dot{\Phi} = A\Phi, \qquad \Phi(t_0) = Id.$$

Since all the eigenvalues of the matrix A have negative real part, there exist positive constants C and μ_0 such that

$$||\Phi(t)|| \le Ce^{-\mu_0(t-t_0)}, \qquad \text{for } t \ge t_0.$$

From the assumptions on f and B for $\delta_0 > 0$ sufficiently small there exist a constant $b(\delta_0)$ such that if $||\mathbf{x}|| \le \delta_0$ then

$$||f(t,\mathbf{x})|| \le b(\delta_0)||\mathbf{x}|| \quad \text{for } t \ge t_0,$$

and if t_0 is sufficiently large

$$||B(t)|| \le b(\delta_0), \qquad \text{for } t \ge t_0.$$

The existence and uniqueness Theorem states that in a neighborhood of $\mathbf{x} = 0$ the solution of the initial problem (1.82), exists for $t_0 \leq t \leq t_1$. It can be shown that this solution is defined for all $t \geq t_0$.

We claim that the initial problem (1.82) is equivalent to the integral equation

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t - s + t_0)[B(s)\mathbf{x}(s) + f(s, \mathbf{x}(s))]ds. \tag{1.83}$$

Now we prove the claim. The fundamental matrix $\Phi(t)$ of the differential system $\dot{\mathbf{x}} = A\mathbf{x}$ can be written as $\Phi(t) = e^{A(t-t_0)}$. We substitute $\mathbf{x} = \Phi(t)\mathbf{z}$ into the differential system (1.82) and obtain

$$\frac{d\Phi(t)}{dt}\mathbf{z} + \Phi(t)\dot{\mathbf{z}} = A\Phi(t)\mathbf{z} + B(t)\Phi(t)\mathbf{z} + f(t, \Phi(t)\mathbf{z}).$$

Since $d\Phi(t)/dt = A\Phi(t)$ we get

$$\dot{\mathbf{z}} = \Phi(t)^{-1}B(t)\Phi(t)\mathbf{z} + \Phi(t)^{-1}f(t, \Phi(t)\mathbf{z}).$$

Integrating this expression between t_0 and t and multiplying by $\Phi(t)$ we get the integral equation (1.83). So the claim is proved.

Using the estimates for Φ, B and f we have

$$\begin{aligned} ||\mathbf{x}(t)|| &\leq ||\Phi(t)||||\mathbf{x}_0|| + \int_{t_0}^{t} [||\Phi(t - s + t_0)||||B(s)||||\mathbf{x}(s)|| + ||f(s, \mathbf{x}(s))||]\, ds \\ &\leq Ce^{-\mu_0(t-t_0)}||\mathbf{x}_0|| + \int_{t_0}^{t} Ce^{-\mu_0(t-s)}2b||\mathbf{x}(s)||ds \end{aligned}$$

for $t_0 \leq t \leq t_2 \leq t_1$. Therefore

$$e^{\mu_0(t-t_0)}||\mathbf{x}(t)|| \leq C||\mathbf{x}_0|| + \int_{t_0}^{t} Ce^{-\mu_0(s-t_0)}2b||\mathbf{x}(s)||ds,$$

for $t_0 \leq t \leq t_2$ where t_2 is determined by the condition $||\mathbf{x}|| \leq \delta_0$. Using now the Gronwall's inequality (Lemma 1.6.1 with $\phi(s) = 2Cb$) we obtain

$$e^{-\mu_0(s-t_0)}||\mathbf{x}(t)|| \leq C||\mathbf{x}_0||e^{2Cb(t-t_0)},$$

or

$$||\mathbf{x}(t)|| \leq C||\mathbf{x}_0||e^{(2Cb-\mu_0)(t-t_0)}.$$

If δ and consequently b are sufficiently small, we have that $\mu = \mu_0 - 2Cb$ is positive, and the inequality of the statement of the theorem follows for $t \in [t_0, t_2]$.

Finally if we choose $||\mathbf{x}_0||$ such that $||\mathbf{x}_0|| \leq \delta_0$, then $||\mathbf{x}(t)||$ decreases, consequently the solution $\mathbf{x} = 0$ is asymptotically stable and the attraction is exponential in a δ–neighborhood of $\mathbf{x} = 0$. □

Now we shall consider linear differential systems of the form

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \tag{1.84}$$

where $A(t)$ is a continuous T–periodic $n \times n$ matrix, i.e. $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. For these systems we can define again a *fundamental matrix* putting in each column of this matrix an independent solution of the system (1.84).

The next result usually called the *Floquet Theorem* says that the fundamental matrix of system (1.84) can be written as a product of a T–periodic matrix and a non–periodic matrix in general.

Theorem 1.6.4. *Consider the linear differential system* (1.84) *with $A(t)$ a continuous T–periodic $n \times n$ matrix. Then each fundamental matrix $\Phi(t)$ of system* (1.84) *can be written as the product of two $n \times n$ matrices*

$$\Phi(t) = P(t)e^{Bt},$$

where $P(t)$ is T–periodic and B is a constant matrix.

Proof. Since $\Phi(t)$ is a fundamental matrix of system (1.84), $\Phi(t+T)$ is also a fundamental matrix. Indeed, define $\tau = t + T$, then

$$\frac{d\mathbf{x}}{d\tau} = A(\tau - T)\mathbf{x} = A(\tau)\mathbf{x}.$$

Therefore $\Phi(\tau)$ is also a fundamental matrix.

The fundamental matrices $\Phi(t)$ and $\Phi(t+T)$ are linearly dependent, i.e. there exists a non–singular matrix C such that $\Phi(t+T) = \Phi(t)C$. Let B be a constant matrix such that $C = e^{BT}$. We claim that the matrix $\Phi(t)e^{-Bt}$ is T–periodic. Write $\Phi(t)e^{-Bt} = P(t)$. Then

$$P(t+T) = \Phi(t+T)e^{-B(t+T)} = \Phi(t)Ce^{-BT}e^{-Bt} = \Phi(t)e^{-Bt} = P(t).$$

This completes the proof of the theorem. □

Remark 1.6.5. *The matrix C introduced in the proof of Theorem 1.6.4 is called the monodromy matrix of system* (1.84). *The eigenvalues ρ_k of the matrix C are called the characteristic multipliers. Each complex number λ_k such that $\rho_k = e^{\lambda_k T}$ is called a characteristic exponent. The characteristic multipliers are determined uniquely. We can choose the exponents λ_k that they coincide with the eigenvalues of the matrix B.*

Proposition 1.6.6. *Consider the differential system*

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + f(t, \mathbf{x}), \tag{1.85}$$

in $\mathbb{R}^n$ with $A(t)$ a T–periodic continuous matrix, $f(t,\mathbf{x})$ continuous in $t \in \mathbb{R}$ and in $\mathbf{x}$ in a neighborhood of $\mathbf{x} = 0$. Assume that

$$\lim_{||\mathbf{x}|| \to 0} \frac{f(t,\mathbf{x})}{||\mathbf{x}||} = 0 \quad \textit{uniformly in } t.$$

If the real parts of the characteristic exponents of the linear periodic differential system

$$\dot{\mathbf{y}} = A(t)\mathbf{y}, \tag{1.86}$$

are negative, the solution $\mathbf{x} = 0$ *of system* (1.85) *is asymptotically stable.*

Proof. By remark 1.6.5 and Theorem 1.6.4 we use the change of variables $\mathbf{x} = M(t)\mathbf{z}$ being $M(t)$ the periodic fundamental matrix solution of the system (1.86). Then the differential system (1.85) becomes

$$\dot{\mathbf{z}} = B\mathbf{z} + M(t)^{-1}f(t, M(t)\mathbf{z}).$$

The constant matrix B has all its eigenvalues with negative real part. The solution $\mathbf{z}$ of the previous system satisfies the assumptions of the Theorem 1.6.3 from which the proposition follows. □

Proposition 1.6.7. *Consider the differential system*

$$\dot{\mathbf{x}} = A\mathbf{x} + B(t)\mathbf{x} + f(t, \mathbf{x}) \quad \text{with } t \geq t_0, \tag{1.87}$$

in $\mathbb{R}^n$ *where* A *is a constant* $n \times n$ *matrix having at least one eigenvalue with positive real part,* $B(t)$ *is a continuous* $n \times n$ *matrix such that* $\lim_{t\to\infty} ||B(t)|| = 0$. *The function* $f(t, \mathbf{x})$ *is continuous in* t *and* $\mathbf{x}$, *and Lipschitz in* $\mathbf{x}$ *in a neighborhood of* $\mathbf{x} = 0$. *If*

$$\lim_{||\mathbf{x}||\to 0} \frac{f(t, \mathbf{x})}{||\mathbf{x}||} = 0 \quad \text{uniformly in } t,$$

then the solution $\mathbf{x} = 0$ *is unstable.*

Proof. Doing the change of variables $\mathbf{x} = S\mathbf{y}$ where S is a non–singular constant $n \times n$ matrix the system (1.87) becomes

$$\dot{\mathbf{y}} = S^{-1}AS\mathbf{y} + S^{-1}B(t)S\mathbf{y} + S^{-1}f(t, S\mathbf{y}). \tag{1.88}$$

While the solution $\mathbf{x}(t)$ is real, in general the solution $\mathbf{y}(t)$ will be complex. The instability for the solution $\mathbf{y} = 0$ of system (1.88) implies the instability for the solution $\mathbf{x} = 0$ of system (1.87). We assume that the matrix S can be take in such a way that the matrix $S^{-1}AS$ is diagonal, otherwise the proof is similar, or see chapter 13.1 of [29].

Assume that

$$\text{Re}(\lambda_i) \geq \sigma > 0 \text{ for } i = 1, \ldots, k \quad \text{and} \quad \text{Re}(\lambda_i) \leq 0 \quad \text{for} \quad i = k+1, \ldots, n.$$

Let

$$R^2 = \sum_{i=1}^{k} |y_i|^2 \quad \text{and} \quad r^2 = \sum_{i=k+1}^{n} |y_i|^2.$$

From system (1.88) we shall compute the derivatives of R^2 and r^2 with respect to t. First we have

$$\begin{aligned}\frac{d|y_i|^2}{dt} &= \frac{d(y_i\overline{y}_i)}{dt} = \dot{y}_i\overline{y}_i + y_i\dot{\overline{y}}_i \\ &= 2\text{Re}\lambda_i|y_i|^2 + (S^{-1}B(t)S\mathbf{y})\overline{y}_i + y_i(S^{-1}B(t)S\mathbf{y})_i \\ &\quad + (S^{-1}f(t,S\mathbf{y})_i\overline{y}_i + y_i(S^{-1}f(t,S\mathbf{y})_i.\end{aligned}$$

We can choose $\varepsilon > 0$, δ_0 and δ such that for $t \geq t_0$ and $||\mathbf{y}|| \leq \delta$ we have

$$|S^{-1}B(t)S\mathbf{y}|_i \leq \varepsilon|y_i|, \qquad |(S^{-1}f(t,S\mathbf{y})_i| \leq \varepsilon|y_i|.$$

Therefore

$$\frac{1}{2}\frac{d(R^2-r^2)}{dt} \geq \sum_{i=1}^{k}(\text{Re}\lambda_i - \varepsilon)|y_i|^2 - \sum_{i=k+1}^{n}(\text{Re}\lambda_i + \varepsilon)|y_i|^2.$$

Taking $0 < \varepsilon \leq \sigma/2$ we obtain

$$\text{Re}\lambda_i - \varepsilon \geq \sigma - \varepsilon \geq \varepsilon \text{ for } i = 1,\ldots,k, \quad \text{Re}\lambda_i + \varepsilon \geq \varepsilon \text{ for } i = k+1,\ldots,n.$$

Then

$$\frac{1}{2}\frac{d(R^2-r^2)}{dt} \geq \varepsilon(R^2-r^2) \quad \text{for } t \geq t_0 \text{ and } ||\mathbf{y}|| \leq \delta. \tag{1.89}$$

Taking the initial conditions in such a way that $(R^2-r^2)_{t=t_0} = k > 0$, from (1.89) we get that

$$||\mathbf{y}||^2 \geq R^2 - r^2 \geq ke^{2\varepsilon(t-t_0)}.$$

Hence this solution leaves the ball $||\mathbf{y}|| \leq \delta$. Consequently the solution $\mathbf{y} = 0$ is unstable. □

Proof of statement (b) of Theorem 1.1.1. We linearize equation (1.1) in a neighborhood of the periodic solution $\mathbf{x}(t,\varepsilon)$. After translating $\mathbf{x} = \mathbf{z} + \mathbf{x}(t,\varepsilon)$, expanding with respect to $\mathbf{z}$, omitting the nonlinear terms and renaming the dependent variable again by $\mathbf{x}$, we get the linear differential equation with T–periodic coefficients

$$\dot{\mathbf{x}} = \varepsilon A(t,\varepsilon)\mathbf{x}, \tag{1.90}$$

where

$$A(t,\varepsilon) = \frac{\partial}{\partial \mathbf{x}}[F(t,\mathbf{x}) + \varepsilon R(t,\mathbf{x},\varepsilon)]_{\mathbf{x}=\mathbf{x}_\varepsilon(t)}.$$

We define the T–periodic matrix

$$B(t) = \frac{\partial F}{\partial \mathbf{x}}(t,p),$$

and from statement (a) we have $\lim_{\varepsilon\to 0} A(t,\varepsilon) = B(t)$. We also define the matrices

$$B^0 = \frac{1}{T}\int_0^T B(t)dt \quad \text{and} \quad C(t) = \int_0^T [B(s) - B^0]ds.$$

Note that B^0 is the matrix of the linearized averaging system. The matrix $C(t)$ is T–periodic and its average is zero. The near–identity transformation $\mathbf{x} \to \mathbf{y}$ defined by $\mathbf{y} = (I - \varepsilon C(t))\mathbf{x}$ provides

$$\begin{aligned} \dot{\mathbf{y}} &= -\varepsilon\dot{C}(t)\mathbf{x} + (I - \varepsilon C(t))\dot{\mathbf{x}} \\ &= -\varepsilon B(t)\mathbf{x} + \varepsilon B^0\mathbf{x} + (I - \varepsilon C(t))\varepsilon A(t,\varepsilon)\mathbf{x} \\ &= [\varepsilon B^0 + \varepsilon(A(t,\varepsilon) - B(t)) - \varepsilon^2 C(t)]A(t,\varepsilon)](I - \varepsilon C(t))^{-1}\mathbf{y} \\ &= \varepsilon B^0\mathbf{y} + \varepsilon(A(t,\varepsilon) - B(t))\mathbf{y} + \varepsilon^2 S(t,\varepsilon)\mathbf{y}. \end{aligned} \tag{1.91}$$

The function $S(t,\varepsilon)$ is T–periodic and bounded. We note that $A(t,\varepsilon) - B(t) \to 0$ when $\varepsilon \to 0$, and also that the characteristic exponents of differential system (1.91) depend continuously on the small parameter ε. Therefore, for ε sufficiently small, the sign of the real parts of the characteristic exponents is equal to the sign of the real parts of the eigenvalues of the matrix B^0. The same conclusion holds, using the near–identity transformation, for the characteristic exponents of differential system (1.90).

Applying now Proposition 1.6.6 we obtain the stability of the periodic solution in the case of negative real parts. If at least one real part is positive, the Floquet transformation and the application of Proposition 1.6.7 provides the instability of the periodic solution. □

1.7 Proof of Theorem 1.3.1

Proof of Theorem 1.3.1. We consider the function $f\colon D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$, given by

$$f(z,\varepsilon) = x(T,z,\varepsilon) - z. \tag{1.92}$$

Then, every $(z_\varepsilon, \varepsilon)$ such that

$$f(z_\varepsilon, \varepsilon) = 0 \tag{1.93}$$

provides the periodic solution $x(\cdot, z_\varepsilon, \varepsilon)$ of (1.32).

We need to study the zeros of the function (1.92), or, equivalently, of

$$g(z,\varepsilon) = Y^{-1}(T,z)f(z,\varepsilon).$$

We have that $g(z_\alpha, 0) = 0$, because $x(\cdot, z_\alpha, 0)$ is T–periodic, and we shall prove that

$$G_\alpha = \frac{dg}{dz}(z_\alpha, 0) = Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T). \tag{1.94}$$

For this we need to know $(\partial x/\partial z)(\cdot, z, 0)$. Since it is the matrix solution of (1.34) with $(\partial x/\partial z)(0, z, 0) = I_n$, we have that $(\partial x/\partial z)(t, z, 0) = Y(t, z)Y^{-1}(0, z)$. Moreover,

$$\frac{df}{dz}(z, 0) = \frac{\partial x}{\partial z}(T, z, 0) - I_n = Y(T, z)Y^{-1}(0, z) - I_n$$

and

$$\frac{dg}{dz}(z, 0) = Y^{-1}(0, z) - Y^{-1}(T, z) + \left(\frac{\partial Y^{-1}}{\partial z_1}(T, z)f(z, 0), \ldots, \frac{\partial Y^{-1}}{\partial z_n}(T, z)f(z, 0)\right),$$

which, for $z_\alpha \in \mathcal{Z}$, reduces to (1.94).

We have

$$\frac{\partial g}{\partial \varepsilon}(z, 0) = Y^{-1}(T, z)\frac{\partial x}{\partial \varepsilon}(T, z, 0).$$

The function $(\partial x/\partial \varepsilon)(\cdot, z, 0)$ is the unique solution of the initial value problem

$$y' = D_x F_0(t, x(t, z, 0))y + F_1(t, x(t, z, 0)), \quad y(0) = 0.$$

Hence

$$\frac{\partial x}{\partial \varepsilon}(t, z, 0) = Y(t, z)\int_0^t Y^{-1}(s, z)F_1(s, x(s, z, 0))ds.$$

Now we have

$$\frac{\partial g}{\partial \varepsilon}(z, 0) = \int_0^T Y^{-1}(s, z)F_1(s, x(s, z, 0))ds,$$

Hence

$$\frac{\partial (\pi g)}{\partial \varepsilon}(z_\alpha, 0) = f_1(\alpha),$$

where f_1 is given by (1.35). Applying Theorem 2.1, there exists $\alpha_\varepsilon \in V$ such that $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and, further, $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$, which assures that $\varphi(\cdot, \varepsilon) = x(\cdot, z_{\alpha_\varepsilon}, \varepsilon)$ is a T–periodic solution of (1.32). □

1.8 Proof of Theorem 1.5.1

Since the result of Theorem 1.5.1 is analogous to the result of Theorem 1.3.1, their proofs are similar.

Proof of Theorem 1.5.1. Since $\mathcal{Z}$ is a compact set and $\mathbf{x}(t, \mathbf{z}_\alpha)$ is T-periodic for each $\mathbf{z}_\alpha \in \mathcal{Z}$, there is an open neighborhood D of $\mathcal{Z}$ in Ω and $0 < \varepsilon_1 \leq \varepsilon_0$ such that any solution $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ of (1.32) with initial conditions in $D \times (-\varepsilon_1, \varepsilon_1)$ is well defined in $[0, T]$. We consider the function $L\colon D \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^{2m}$, $(\mathbf{z}, \varepsilon) \mapsto \mathbf{x}(T, \mathbf{z}, \varepsilon) - \mathbf{z}$. If $(\bar{\mathbf{z}}, \bar{\varepsilon}) \in D \times (-\varepsilon_1, \varepsilon_1)$ is such that $L(\bar{\mathbf{z}}, \bar{\varepsilon}) = 0$, then $\mathbf{x}(t, \bar{\mathbf{z}}, \bar{\varepsilon})$ is a T-periodic solution of $(1.32)_{\varepsilon=\bar{\varepsilon}}$. Clearly the converse is true. Hence the problem of finding T-periodic orbits of (1.32) close to the periodic orbits with initial conditions in $\mathcal{Z}$ is reduced to find the zeros of $L(\mathbf{x}, \varepsilon)$.

The sets of zeros of $L(\mathbf{z},\varepsilon)$ and $\widetilde{L}(\mathbf{z},\varepsilon) = M_{\mathbf{z}}^{-1}(T)L(\mathbf{z},\varepsilon)$ are the same, since $M_{\mathbf{z}}(T)$ is a fundamental matrix. Moreover following the proof of Theorem 1.3.1 we can compute that

$$D_{\mathbf{z}}\widetilde{L}(\mathbf{z},\varepsilon)=\left(M_{\mathbf{z}}^{-1}(0)-M_{\mathbf{z}}^{-1}(T)\right)+D_{\mathbf{z}}\left(\int_0^T M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z},0))dt\right)\varepsilon+O(\varepsilon^2). \tag{1.95}$$

We note that $\widetilde{L}^{-1}(0) = (\xi^{\perp}\circ\widetilde{L})^{-1}(0)\cap(\xi\circ\widetilde{L})^{-1}(0)$. From (1.95) we obtain $D_{\mathbf{z}}\widetilde{L}(\mathbf{z}_\alpha,0) = M_{\mathbf{z}_\alpha}^{-1}(0)-M_{\mathbf{z}_\alpha}^{-1}(T)$. If we write $\mathbf{z}\in\mathbb{R}^{2m}$ as $\mathbf{z}=(u,v)$ with $u,v\in\mathbb{R}^m$, then $D_v(\xi\circ\widetilde{L})(\mathbf{z}_\alpha,0)$ is the upper right corner of $M_{\mathbf{z}}^{-1}(0)–M_{\mathbf{z}}^{-1}(T)$. Then from (a) we can apply the Implicit Function Theorem, thus it follows that there exist an open neighborhood $U\times(-\varepsilon_2,\varepsilon_2)$ of $\mathrm{Cl}(V)$ in $\xi(D)\times(-\varepsilon_1,\varepsilon_1)$, an open neighborhood $\mathcal{O}$ of $\beta_0(\mathrm{Cl}(V))$ in $\mathbb{R}^m$ and a unique $\mathcal{C}^k$ function $\beta(\alpha,\varepsilon)\colon U\times(-\varepsilon_2,\varepsilon_2)\to\mathcal{O}$ such that $(\xi\circ\widetilde{L})^{-1}(0)\cap(U\times\mathcal{O}\times(-\varepsilon_2,\varepsilon_2))$ is exactly the graphic of $\beta(\alpha,\varepsilon)$. Now if we define the function $\delta\colon U\times(-\varepsilon_2,\varepsilon_2)\to\mathbb{R}$ as $\delta(\alpha,\varepsilon)=(\xi^{\perp}\circ\widetilde{L})(\alpha,\beta(\alpha,\varepsilon),\varepsilon)$, then δ is a function of class $\mathcal{C}^k$ and $\widetilde{L}^{-1}(0)\cap(U\times\mathcal{O}\times(-\varepsilon_2,\varepsilon_2))=\{(\alpha,\beta(\alpha,\varepsilon),\varepsilon)\,|\,(\alpha,\varepsilon)\in\delta^{-1}(0)\}$. Therefore for describing the set $\widetilde{L}^{-1}(0)$ in an open neighborhood of $\mathcal{Z}$ in $\mathbb{R}^n\times(-\varepsilon_0,\varepsilon_0)$, it is sufficient to describe $\delta^{-1}(0)$ in an open neighborhood of $\mathrm{Cl}(V)$ in $\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$.

Since $M_{\mathbf{z}_\alpha}^{-1}(0)-M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the lower right corner the $m\times m$ zero matrix and $\delta(\alpha,0)=0$ in $V\times(-\varepsilon_2,\varepsilon_2)$, the function $\delta(\alpha,\varepsilon)$ can be written as $\delta(\alpha,\varepsilon)=\varepsilon\mathcal{G}(\alpha)+\varepsilon^2\widetilde{\mathcal{G}}(\alpha,\varepsilon)$ in $V\times(-\varepsilon_2,\varepsilon_2)$, where $\mathcal{G}(\alpha)$ is the function given in (1.65), see [17]. In addition if $\widetilde{\delta}(\alpha,\varepsilon)=\mathcal{G}(\alpha)+\varepsilon\widetilde{\mathcal{G}}(\alpha,\varepsilon)$, then $\delta^{-1}(0)=\widetilde{\delta}^{-1}(0)$.

If there is $\alpha_0\in V$ such that $\widetilde{\delta}(\alpha_0,0)=\mathcal{G}(\alpha_0)=0$ and $\det((\partial\mathcal{G}/\partial\alpha)(\alpha_0))\neq 0$, then from the Implicit Function Theorem there exist $\varepsilon_3>0$ small, an open neighborhood V_0 of α_0 in V and a unique function of class $\mathcal{C}^k$ $\alpha(\varepsilon)\colon(-\varepsilon_3,\varepsilon_3)\to V_0$ such that $\widetilde{\delta}^{-1}(0)\cap(V_0\times(-\varepsilon_3,\varepsilon_3))$ is the graphic of $\alpha(\varepsilon)$, which also represents the set $\delta^{-1}(0)\cap(V_0\times(-\varepsilon_3,\varepsilon_3))$. This completes the proof of the theorem. □

Chapter 2

Averaging theory of arbitrary order and dimension for finding periodic solutions

In this chapter we shall study the periodic solutions of the systems of the form

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{2.1}$$

where $F_i \colon \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 0, 1, 2, \cdots, k$, and $R \colon \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are locally Lipschitz functions, and T–periodic in the first variable, being D an open subset of $\mathbb{R}^n$; eventually F_0 can be the zero constant function.

The classical works using the averaging theory for studying the periodic solutions of a differential system (2.1) usually only provide this theory up to first ($k = 1$) or second order ($k = 2$) in the small parameter ε, moreover these theories assume differentiability of the functions F_i and R up to class $\mathcal{C}^2$ or $\mathcal{C}^3$, respectively. Recently in [19] this averaging theory for computing periodic solutions was developed up to second order in dimension n, and up to third order ($k = 3$) in dimension 1, only using that the functions F_i and R are locally Lipschitz. Also in the recent work [48] the averaging theory for computing periodic solutions was developed to an arbitrary order k in ε for analytical differential equations in dimension 1.

In this chapter we shall develop the averaging theory for studying the periodic solutions of a differential system (2.1) up to arbitrary order k in dimension n, with zero or non–zero F_0, and where the functions F_i and R are only locally Lipschitz. In fact this chapter is based in the results of the paper [76] by Llibre, Novaes and Teixeira.

An example that qualitative new phenomena can be found only when considering higher order analysis is the following. Consider arbitrary polynomial perturbations

$$\begin{aligned} \dot{x} &= -y + \sum_{j\geq 1} \varepsilon^j f_j(x,y), \\ \dot{y} &= x + \sum_{j\geq 1} \varepsilon^j g_j(x,y), \end{aligned} \tag{2.2}$$

of the harmonic oscillator, where ε is a small parameter. In this differential system the polynomials f_j and g_j are of degree n in the variables x and y and the system is analytic in the variables x, y and ε. Then in [48] (see also Iliev [56]) it is proved that system (2.2) for $\varepsilon \neq 0$ sufficiently small has no more than $[s(n-1)/2]$ periodic solutions bifurcating from the periodic solutions of the linear center $\dot{x} = -y$, $\dot{y} = x$, using the averaging theory up to order s, and this bound can be reached. Here $[x]$ denotes the integer part function of the real number x. So, to take into account higher order averaging theory can improve qualitatively and quantitatively the results on the periodic solutions.

In short, the goal of this chapter is to extend the averaging theory for computing the periodic solutions of a differential system in n variables (2.1) up to an arbitrary order k in ε for locally Lipschitz differential systems, using the Brouwer degree.

2.1 Statement of the main results

We are interested in studying the existence of periodic orbits of general differential systems expressed by

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{2.3}$$

where $F_i\colon \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 1, 2, \cdots, k$, and $R\colon \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, and T–periodic in the first variable, being D an open subset of $\mathbb{R}^n$.

In order to state our main results we introduce some notation. Let L be a positive integer, let $x = (x_1, \ldots, x_n) \in D$, $t \in \mathbb{R}$ and $y_j = (y_{j1}, \ldots, y_{jn}) \in \mathbb{R}^n$ for $j = 1, \ldots, L$. Given $F\colon \mathbb{R} \times D \to \mathbb{R}^n$ a sufficiently smooth function, for each $(t,x) \in \mathbb{R} \times D$ we denote by $\partial^L F(t,x)$ a symmetric L–multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^n$, which we denote as $\bigodot_{j=1}^{L} y_j \in \mathbb{R}^{nL}$. The definition of this L–multilinear map is

$$\partial^L F(t,x) \bigodot_{j=1}^{L} y_j = \sum_{i_1,\ldots,i_L=1}^{n} \frac{\partial^L F(t,x)}{\partial x_{i_1} \cdots \partial x_{i_L}} y_{1i_1} \cdots y_{Li_L}. \tag{2.4}$$

We define ∂^0 as the identity functional. Given a positive integer b and a vector $y \in \mathbb{R}^n$ we also denote $y^b = \bigodot_{i=1}^{b} y \in \mathbb{R}^{nb}$.

Remark 2.1.1. *The L–multilinear map defined in* (2.4) *is the L^{th} Fréchet derivative of the function $F(t,x)$ with respect to the variable x. Indeed, fixed $t \in \mathbb{R}$, if we consider the function $F_t\colon D \to \mathbb{R}^n$ such that $F_t(x) = F(t,x)$, then $\partial^L F(t,x) = F_t^{(L)}(x) = \partial^L/\partial x^L F(t,x)$.*

Example 2.1.2. *To illustrate the above notation* (2.4) *we consider a smooth function $F\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. So for $x = (x_1, x_2)$ and $y^1 = (y_1^1, y_2^1)$ we have*

$$\partial F(t,x) y^1 = \frac{\partial F}{\partial x_1}(t,x) y_1^1 + \frac{\partial F}{\partial x_2}(t,x) y_2^1.$$

Now, for $y^1 = (y_1^1, y_2^1)$ and $y^2 = (y_1^2, y_2^2)$ we have

$$\begin{aligned} \partial^2 F(t,x)(y^1, y^2) = & \frac{\partial^2 F(t,x)}{\partial x_1 \partial x_1} y_1^1 y_1^2 + \frac{\partial^2 F(t,x)}{\partial x_1 \partial x_2} y_1^1 y_2^2 \\ & + \frac{\partial^2 F(t,x)}{\partial x_2 \partial x_1} y_2^1 y_1^2 + \frac{\partial^2 F(t,x)}{\partial x_2 \partial x_2} y_2^1 y_2^2. \end{aligned}$$

Observe that for each $(t,x) \in \mathbb{R} \times D$, $\partial F(t,x)$ is a linear map in $\mathbb{R}^2$ and $\partial^2 F(t,x)$ is a bilinear map in $\mathbb{R}^2 \times \mathbb{R}^2$.

Let $\varphi(\cdot, z)\colon [0, t_z] \to \mathbb{R}^n$ be the solution of the unperturbed system,

$$x'(t) = F_0(t,x) \tag{2.5}$$

such that $\varphi(0,z) = z$.

For $i = 1, 2, \ldots, k$, we define the *Averaged Function* $f_i\colon D \to \mathbb{R}^n$ of order i as

$$f_i(z) = \frac{y_i(T,z)}{i!}, \tag{2.6}$$

where $y_i\colon \mathbb{R} \times D \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k-1$, are defined recurrently by the following integral equation

$$\begin{aligned} y_i(t,z) = i! \int_0^t \Bigg(& F_i\big(s, \varphi(s,z)\big) \\ & + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_{i-l}\big(s, \varphi(s,z)\big) \bigodot_{j=1}^{l} y_j(s,z)^{b_j} \Bigg) ds, \end{aligned} \tag{2.7}$$

where S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + l b_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

In Section 2.3 we compute the sets S_l for $l = 1, 2, 3, 4, 5$. Furthermore, we make explicit the functions $f_k(z)$ up to $k = 5$ when $F_0 = 0$, and up to $k = 4$ when $F_0 \neq 0$.

Related to the averaging functions (2.6) there exist two cases of (2.3), essentially different, that must be treated separately. Namely, when $F_0 = 0$ and when $F_0 \neq 0$. It can be seen in the following remarks.

Remark 2.1.3. *If $F_0 = 0$, then $\varphi(t,z) = z$ for each $t \in \mathbb{R}$. So*

$$y_1(t,z) = \int_0^t F_1(t,z)ds, \quad \text{and} \quad f_1(t,z) = \int_0^T F_1(t,z)dt$$

as usual in averaging theory (see for instance [8]).

Remark 2.1.4. *If $F_0 \neq 0$, then*

$$y_1(t,z) = \int_0^t F_1(s,\varphi(s,z)) + \partial F_0(s,\varphi(s,z))\, y_1(s,z)ds. \tag{2.8}$$

The integral equation (2.8) is equivalent to the following Cauchy Problem

$$\dot{u}(t) = F_1(t,\varphi(t,z)) + \partial F_0(t,\varphi(t,z))\, u \quad \text{and} \quad u(0) = 0, \tag{2.9}$$

i.e, $y_1(t,z) = u(t)$. If we denote

$$\eta(t,z) = \int_0^t \partial F_0(s,\varphi(s,z))ds \tag{2.10}$$

so

$$y_1(t,z) = e^{\eta(t,z)} \int_0^t e^{-\eta(s,z)} F_1(s,\varphi(s,z))ds \tag{2.11}$$

and

$$f_1(z) = \int_0^T e^{-\eta(t,z)} F_1(t,\varphi(t,z))dt.$$

Moreover, each $y_i(t,z)$ is obtained similarly from a Cauchy problem. The formulae are given explicitly in section 2.3.

In the following, we state our main results: Theorem 2.1.5 when $F_0 = 0$, and Theorem 2.1.6 when $F_0 \neq 0$. The Brouwer degree d_B, which is defined in Appendix B, is used.

Theorem 2.1.5. *Suppose that $F_0 = 0$. In addition, for the functions of (2.3), we assume the following conditions.*

(i) *For each $t \in \mathbb{R}$, $F_i(t,\cdot) \in \mathcal{C}^{k-i}$ for $i = 1,2,\cdots,k$; $\partial^{k-i}F_i$ is locally Lipschitz in the second variable for $i = 1,2,\cdots,k$; and R is continuous and locally Lipschitz in the second variable.*

(ii) *Assume that $f_i = 0$ for $i = 1,2,\ldots,r-1$ and $f_r \neq 0$ with $r \in \{1,2,\ldots,k\}$ (here we are taking $f_0 = 0$). Moreover, suppose that for some $a \in D$ with $f_r(a) = 0$, there exists a neighborhood $V \subset D$ of a such that $f_r(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$, and that $d_B(f_r(z),V,0) \neq 0$.*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*–periodic solution* $x(\cdot, \varepsilon)$ *of* (2.3) *such that* $x(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$*.*

Theorem 2.1.6. *Suppose that* $F_0 \neq 0$*. In addition, for the functions of* (2.3)*, we assume the following conditions.*

(j) *There exists an open subset* W *of* D *such that for any* $z \in \overline{W}$*,* $\varphi(t, z)$ *is* T*–periodic in the variable* t*.*

(jj) *For each* $t \in \mathbb{R}$*,* $F_i(t, \cdot) \in \mathcal{C}^{k-i}$ *for* $i = 0, 1, 2, \cdots, k$*;* $\partial^{k-i} F_i$ *is locally Lipschitz in the second variable for* $i = 0, 1, 2, \cdots, k$*; and* R *is continuous and locally Lipschitz in the second variable.*

(jjj) *Assume that* $f_i = 0$ *for* $i = 1, 2, \ldots, r-1$ *and* $f_r \neq 0$ *with* $r \in \{1, 2, \ldots, k\}$*. Moreover, suppose that for some* $a \in W$ *with* $f_r(a) = 0$*, there exists a neighborhood* $V \subset W$ *of* a *such that* $f_r(z) \neq 0$ *for all* $z \in \overline{V} \setminus \{a\}$*, and that* $d_B(f_r(z), V, 0) \neq 0$*.*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*–periodic solution* $x(\cdot, \varepsilon)$ *of* (2.3) *such that* $x(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$*.*

Theorems 2.1.5 and 2.1.6 are proved in section 2.2.

Remark 2.1.7. *When* f_i *for* $i = 1, 2, \ldots, k$ *(defined in* (2.6)*) are* $\mathcal{C}^1$ *functions the hypotheses (ii) and (jjj) become:*

(k) *Assume that* $f_i = 0$ *for* $i = 1, 2 \ldots, r-1$ *and* $f_r \neq 0$ *with* $r \in \{1, 2, \ldots, k\}$*. Moreover, suppose that for some* $a \in W$ *with* $f_r(a) = 0$ *we have that* $f_r'(a) \neq 0$*.*

In this case, instead Brouwer degree theory, the Implicit Function Theorem could be used to prove Theorems 2.1.5 and 2.1.6.

We emphasize that our main contribution to the advanced averaging theory is based on Theorems 2.1.5 and 2.1.6. In fact, we provide conditions on the regularity of the functions, weaker than those given in [48].

2.2 Proofs of Theorems 2.1.5 and 2.1.6

Let $g\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ be a function defined on a small interval $(-\varepsilon_0, \varepsilon_0)$. We say that $g(\varepsilon) = \mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1 > 0$ and $M > 0$ such that $||g(\varepsilon)|| \leq M|\varepsilon^\ell|$ for $-\varepsilon_1 < \varepsilon < \varepsilon_1$. The symbol $\mathcal{O}$ is one of the *Landau's symbol* (see for instance [107]).

To prove Theorems 2.1.5 and 2.1.6 we need the following lemma.

Lemma 2.2.1 (Fundamental Lemma)**.** *Under the assumptions of Theorems 2.1.5 or 2.1.6 let* $x(\cdot, z, \varepsilon)\colon [0, t_z] \to \mathbb{R}^n$ *be the solution of* (2.3) *with* $x(0, z, \varepsilon) = z$*. If* $t_z = T$*, then*

$$x(t, z, \varepsilon) = \varphi(t, z) + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t, z)}{i!} + \varepsilon^{k+1}\mathcal{O}(1),$$

where $y_i(t, z)$ *for* $i = 1, 2, \ldots, k$ *are defined in* (2.7)*.*

Proof of Lemma 2.2.1. By continuity of the solution $x(t,z,\varepsilon)$ and by compactness of the set $[0,T]\times\overline{V}\times[-\varepsilon_1,\varepsilon_1]$, there exits a compact subset K of D such that $x(t,z,\varepsilon)\in K$ for all $t\in[0,T]$, $z\in\overline{V}$ and $\varepsilon\in[-\varepsilon_1,\varepsilon_1]$. Now, by the continuity of the function R, $|R(s,x(s,z,\varepsilon),\varepsilon)|\leq\max\{|R(t,x,\varepsilon)|,(t,x,\varepsilon)\in[0,T]\times K\times[-\varepsilon_1,\varepsilon_1]\}=N$. Then

$$\left|\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds\right|\leq\int_0^T|R(s,x(s,z,\varepsilon),\varepsilon)|\,ds=TN,$$

which implies that

$$\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds=\mathcal{O}(1). \tag{2.12}$$

Related to the functions $x(t,z,\varepsilon)$ and $\varphi(t,z)$ we have the followings equalities

$$\begin{aligned} x(t,z,\varepsilon)&=z+\sum_{i=0}^{k}\varepsilon^i\int_0^t F_i(s,x(s,z,\varepsilon))ds+\mathcal{O}(\varepsilon^{k+1}),\quad\text{and}\\ \varphi(t,z)&=z+\int_0^t F_0(s,\varphi(s,z))ds. \end{aligned} \tag{2.13}$$

Moreover $x(t,z,\varepsilon)=\varphi(t,z)+\mathcal{O}(\varepsilon)$. Indeed, F_0 is locally Lipschitz in the second variable, so from the compactness of the set $[0,T]\times\overline{V}\times[-\varepsilon_0,\varepsilon_0]$ and from (2.13) it follows

$$\begin{aligned} |x(t,z,\varepsilon)-\varphi(t,z)|\leq&\int_0^t|F_0(s,x(s,z,\varepsilon))-F_0(s,\varphi(s,z))|ds\\ &+|\varepsilon|\int_0^t|F_1(s,x(s,z,\varepsilon))|ds+\mathcal{O}(\varepsilon^2)\\ \leq&\,|\varepsilon|M+\int_0^t L_0|x(s,z,\varepsilon)-\varphi(s,z)|ds<|\varepsilon|Me^{TL_0}. \end{aligned}$$

Here L_0 is the Lipschitz constant of F_0 on the compact K. The first and second inequality was obtained similarly to (2.12). The last inequality is a consequence of Gronwall Lemma (see, for example, Lemma 1.3.1 of [107]).

In order to prove the present lemma we need the following claim.

Claim. For some positive integer m let $G\colon\mathbb{R}\times D\to\mathbb{R}^n$ be a $\mathcal{C}^m$ function. Then

$$\begin{aligned} &G(t,x(t,z,\varepsilon))=\\ &=\int_0^1\lambda_1^{m-1}\int_0^1\lambda_2^{m-2}\cdots\int_0^1\lambda_{m-1}\int_0^1\Big[\partial^m G\big(t,\ell_m\circ\ell_{m-1}\circ\cdots\circ\ell_1(x(t,z,\varepsilon))\big)\\ &\qquad-\partial^m G(t,\varphi(t,z))\Big]d\lambda_m d\lambda_{m-1}\cdots d\lambda_1\cdot(x(t,z,\varepsilon)-\varphi(t,z))^m\\ &\quad+\sum_{L=0}^{m}\partial^L G(t,\varphi(t,z))\frac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!}, \end{aligned}$$

where $\ell_i(v)=\lambda_i v+(1-\lambda_i)\varphi(t,z)$ for $v\in\mathbb{R}^n$.

We shall prove this claim using the principle of finite induction on m.

For $m = 1$, $G \in \mathcal{C}^1$. Let $\updownarrow_1(\lambda_1) = G\big(t, \ell_1(x(t,z,\varepsilon))\big)$. So

$$\begin{aligned} G(t,x(t,z,\varepsilon)) =& G(t,\varphi(t,z)) + \updownarrow_1(1) - \updownarrow_1(0) = G(t,\varphi(t,z)) + \int_0^1 \updownarrow_1'(\lambda_1)d\lambda_1 \\ =& G(t,\varphi(t,z)) + \int_0^1 \partial G(t,\ell_1(x(t,z,\varepsilon)))d\lambda_1 \cdot (x(t,z,\varepsilon)-\varphi(t,z)) \\ =& \int_0^1 \Big[\partial G(t,\ell_1(x(t,z,\varepsilon))) - \partial G(t,\varphi(t,z))\Big] d\lambda_1 \cdot (x(t,z,\varepsilon)-\varphi(t,z)) \\ & G(t,\varphi(t,z)) + \partial G(t,\varphi(t,z))(x(t,z,\varepsilon)-\varphi(t,z)). \end{aligned}$$

Given an integer $\overline{k} > 1$ we assume as the *inductive hypothesis* $(I1)$ that the claim is true for $m = \overline{k} - 1$.

Now for $m = \overline{k}$, $G \in \mathcal{C}^{\overline{k}} \subset \mathcal{C}^{\overline{k}-1}$. So from inductive hypothesis $(I1)$,

$$\begin{aligned} G(t,x(t,z,\varepsilon)) = & \int_0^1 \lambda_1^{\overline{k}-2} \int_0^1 \lambda_2^{\overline{k}-3} \cdots \int_0^1 \lambda_{\overline{k}-2} \int_0^1 \Big[\partial^{\overline{k}-1} G\big(t, \ell_{\overline{k}-1} \circ \ell_{\overline{k}-2} \circ \cdots \\ & \circ\, \ell_1(x(t,z,\varepsilon))\big) - \partial^{\overline{k}-1} G(t,\varphi(t,z))\Big] d\lambda_{\overline{k}-1} d\lambda_{\overline{k}-2} \cdots d\lambda_1 \\ & \cdot (x(t,z,\varepsilon)-\varphi(t,z))^{\overline{k}-1} \\ & + \sum_{L=0}^{\overline{k}-1} \partial^L G(t,\varphi(t,z)) \frac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!}. \end{aligned} \tag{2.14}$$

Let $\updownarrow(\lambda_{\overline{k}}) = \partial^{\overline{k}-1} G\big(t, \ell_{\overline{k}} \circ \ell_{\overline{k}-1} \circ \cdots \circ\, \ell_1(x(t,z,\varepsilon))\big)$. So

$$\begin{aligned} \int_0^1 \updownarrow'(\lambda_{\overline{k}}) d\lambda_{\overline{k}} &= \updownarrow(1) - \updownarrow(0) \\ &= \partial^{\overline{k}-1} G\big(t, \ell_{\overline{k}-1} \circ \ell_{\overline{k}-2} \circ \cdots \circ\, \ell_1(x(t,z,\varepsilon))\big) - \partial^m G(t,\varphi(t,z)). \end{aligned} \tag{2.15}$$

The derivative of $\updownarrow(\lambda_{\overline{k}})$ can be easily obtained as

$$\updownarrow'(\lambda_{\overline{k}}) = \lambda_{\overline{k}-1}\lambda_{\overline{k}-2}\cdots\lambda_1 \partial^{\overline{k}} G\big(t, \ell_{\overline{k}} \circ \ell_{\overline{k}-1} \circ \cdots \circ\, \ell_1(x(t,z,\varepsilon))\big)(x(t,z,\varepsilon)-\varphi(t,z)).$$

So

$$\begin{aligned} \int_0^1 \updownarrow'(\lambda_{\overline{k}}) d\lambda_{\overline{k}} = & \lambda_{\overline{k}-1}\lambda_{\overline{k}-2}\cdots\lambda_1 \int_0^1 \Big[\partial^{\overline{k}} G\big(t, \ell_{\overline{k}} \circ \ell_{\overline{k}-1} \circ \cdots \circ\, \ell_1(x(t,z,\varepsilon))\big) \\ & - \partial^{\overline{k}} G(t,\varphi(t,z))\Big] d\lambda_{\overline{k}} \cdot (x(t,z,\varepsilon)-\varphi(t,z)) \\ & + \lambda_{\overline{k}-1}\lambda_{\overline{k}-2}\cdots\lambda_1 \partial^{\overline{k}} G(t,\varphi(t,z))(x(t,z,\varepsilon)-\varphi(t,z)). \end{aligned} \tag{2.16}$$

Hence, from (2.14) and (2.16) we conclude that

$$
\begin{aligned}
&G(t,x(t,z,\varepsilon)) = \\
&= \int_0^1 \lambda_1^{\overline{k}-1}\int_0^1 \lambda_2^{\overline{k}-2}\cdots\int_0^1 \lambda_{\overline{k}-1}\int_0^1 \Big[\partial^{\overline{k}}G\big(t,\ell_{\overline{k}}\circ\ell_{\overline{k}-1}\circ\cdots\circ\ell_1(x(t,z,\varepsilon))\big) \\
&\quad - \partial^{\overline{k}}G(t,\varphi(t,z))\Big]d\lambda_{\overline{k}}d\lambda_{\overline{k}-1}\cdots d\lambda_1\cdot(x(t,z,\varepsilon)-\varphi(t,z))^{\overline{k}} \\
&\quad + \sum_{L=0}^{\overline{k}}\partial^L G(t,\varphi(t,z))\frac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!}.
\end{aligned}
$$

This completes the proof of the claim.

Given a non–negative integer m, we note that for a $\mathcal{C}^m$ function G such that $\partial^m G$ is locally Lipschitz in the second variable, the claim implies the following equality

$$
G(t,x(t,z,\varepsilon)) = \textstyle\sum_{L=0}^{m}\partial^L G(t,\varphi(t,z))\dfrac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!} + \mathcal{O}(\varepsilon^{m+1}). \quad (2.17)
$$

Indeed, for $m=0$ G is a continuous function locally Lipschitz in the second variable, so

$$
|G(t,x(t,z,\varepsilon)) - G(t,\varphi(t,z))| \leq \quad L_G|x(t,z,\varepsilon)-\varphi(t,z)| < |\varepsilon|L_G M e^{TL_0}.
$$

Here L_G is the Lipschitz constant of the function G on the compact K. Thus

$$
G(t,x(t,z,\varepsilon)) = G(t,\varphi(t,z)) + \mathcal{O}(\varepsilon).
$$

Moreover for $m \geq 1$ the claim implies (2.17) in an similar way to (2.12).

Again we shall use the principle of finite induction, now on k, to prove the present lemma.

For $k=1$, $F_0 \in \mathcal{C}^1$ and the functions ∂F_0 and F_1 are locally Lipschitz in the second variable. Thus from (2.17), taking $G = F_0$ and $G = F_1$, we obtain

$$
\begin{aligned}
&F_0(t,x(t,z,\varepsilon)) = F_0(t,\varphi(t,z)) + \partial F_0(t,\varphi(t,z))(x(t,z,\varepsilon)-\varphi(t,z)) + \mathcal{O}(\varepsilon^2) \quad \text{and} \\
&F_1(t,x(t,z,\varepsilon)) = F_1(t,\varphi(t,z)) + \mathcal{O}(\varepsilon),
\end{aligned} \quad (2.18)
$$

respectively. From (2.13) and (2.18) we compute

$$
\frac{d}{dt}\left(x(t,z,\varepsilon)-\varphi(t,z)\right) = \partial F_0(t,\varphi(t,z))\left(x(t,z,\varepsilon)-\varphi(t,z)\right) + \varepsilon F_1(t,\varphi(t,z)) + \mathcal{O}(\varepsilon^2). \quad (2.19)
$$

Solving the linear differential equation (2.18) with respect to $x(t,z,\varepsilon)-\varphi(t,z)$ for the initial condition $x(0,z,\varepsilon)-\varphi(0,z,\varepsilon)=0$ and comparing the solution with (2.11) we conclude that

$$
x(t,z,\varepsilon) = \varphi(t,z) + \varepsilon y_1(t,z) + \mathcal{O}(\varepsilon^2).
$$

Given an integer $\overline{k}$ we assume as the *inductive hypothesis* ($I2$) that the lemma is true for $k = \overline{k} - 1$.

Now for $k = \overline{k}$, $F_i = \mathcal{C}^{\overline{k}-i}$ for $i = 0, 1, \ldots, \overline{k}$ and $\partial^{\overline{k}-i} F_i$ is locally Lipschitz in the second variable for $i = 0, 1, \ldots, \overline{k}$. So from (2.17)

$$F_i(t, x(t, z, \varepsilon)) = \sum_{L=0}^{\overline{k}-i} \partial^L F_i(t, \varphi(t, z)) \frac{(x(t, z, \varepsilon) - \varphi(t, z))^L}{L!} + \mathcal{O}(\varepsilon^{\overline{k}-i+1}), \quad (2.20)$$

for $i = 0, 1, \ldots, \overline{k}$.

Applying the inductive hypothesis ($I2$) in (2.20) we get

$$\begin{aligned} F_i(t, x(t, z, \varepsilon)) = {} & F_1(t, \varphi(t, z)) \\ & + \sum_{L=1}^{\overline{k}-i} \partial^L F_i(t, \varphi(t, z)) \left(\sum_{i=1}^{\overline{k}-i-L+1} \varepsilon^i \frac{y_i(t, z)}{i!} \right)^L + \mathcal{O}(\varepsilon^{\overline{k}-i+1}) \end{aligned} \quad (2.21)$$

for $i = 1, 2, \ldots, \overline{k}$. Now using the *Multinomial Theorem* (see for instance [51], p. 186) in (2.21) we obtain

$$\begin{aligned} F_i(t, x(t, z, \varepsilon)) = {} & F_i\left(t, \varphi(t, z)\right) \\ & + \sum_{L=1}^{\overline{k}-i} \sum_{l=L}^{\overline{k}-i} \sum_{S_{l,L}^{\overline{k}-1}} \frac{\varepsilon^l}{b_1!\, b_2! 2!^{b_2} \cdots b_{\overline{k}-1}! (\overline{k}-1)!^{b_{\overline{k}-1}}} \partial^L F_i\left(t, \varphi(t, z)\right) \bigodot_{j=1}^{\overline{k}-1} y_j(t, z)^{b_j} \\ & + \mathcal{O}(\varepsilon^{\overline{k}-i+1}), \end{aligned}$$

for $i = 1, 2, \ldots, \overline{k}$. Here $S_{l,L}^n$ is the set of all n–tuples of non–negative integers $(b_1, b_2, \ldots, b_n)$ satisfying $b_1 + 2b_2 + \cdots + nb_n = l$ and $b_1 + b_2 + \cdots + b_n = L$. We note that if $n > l$ then $b_{l+1} = b_{l+2} = \cdots = b_n = 0$. Hence

$$\begin{aligned} F_i(t, x(t, z, \varepsilon)) = {} & F_i\left(t, \varphi(t, z)\right) \\ & + \sum_{L=1}^{\overline{k}-i} \sum_{l=L}^{\overline{k}-i} \sum_{S_{l,L}^{l}} \frac{\varepsilon^l}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_i\left(t, \varphi(t, z)\right) \bigodot_{j=1}^{l} y_j(t, z)^{b_j} \\ & + \mathcal{O}(\varepsilon^{\overline{k}-i+1}). \end{aligned} \quad (2.22)$$

for $i = 1, 2, \ldots, \overline{k}$, because $\overline{k} - i \geq l$

Finally, doing a change of indexes in (2.22) and observing that $\cup_{L=1}^{l} S_{l,L}^{l} = S_l$, we may write

$$
\begin{aligned}
F_i(t,x(t,z,\varepsilon)) &= F_i\left(t,\varphi(t,z)\right)\\
&+\sum_{l=1}^{\overline{k}-i}\varepsilon^l\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_i\left(t,\varphi(t,z)\right)\bigodot_{j=1}^{l} y_j(t,z)^{b_j}\\
&+\mathcal{O}(\varepsilon^{\overline{k}-i+1}),
\end{aligned}
\tag{2.23}
$$

for $i = 1, 2, \ldots, \overline{k}$.

Following the above steps we also obtain

$$
\begin{aligned}
F_0(t,x(t,z,\varepsilon)) &= F_0\left(t,\varphi(t,z)\right)+\partial F_0(t,\varphi(t,z))(x(t,z,\varepsilon)-\varphi(t,z))\\
&+\sum_{i=1}^{\overline{k}}\varepsilon^i\Bigg[\sum_{S_i}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_i!i!^{b_r}}\partial^L F_0\left(t,\varphi(t,z)\right)\bigodot_{j=1}^{i} y_j(t,z)^{b_j}\\
&-\partial F_0(t,\varphi(t,z))\frac{y_i(t,z)}{i!}\Bigg]+\mathcal{O}(\varepsilon^{\overline{k}+1}).
\end{aligned}
\tag{2.24}
$$

Now from (2.13) we compute

$$
\begin{aligned}
\frac{d}{dt}\left(x(t,z,\varepsilon)-\varphi(t,z)\right) &= F_0(t,x(t,z,\varepsilon))\\
&-F_0(t,\varphi(t,z))+\sum_{i=1}^{\overline{k}}\varepsilon^i F_i(t,x(t,z,\varepsilon))+\mathcal{O}(\varepsilon^{\overline{k}+1}).
\end{aligned}
\tag{2.25}
$$

Proceeding with a change of index we obtain from (2.23) that

$$
\begin{aligned}
\sum_{i=1}^{\overline{k}}\varepsilon^i F_i(t,x(t,z,\varepsilon)) &= \sum_{i=1}^{\overline{k}}\varepsilon^i\sum_{l=0}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}\left(t,\varphi(t,z)\right)\\
&\bigodot_{j=1}^{l} y_j(t,z)^{b_j}+\mathcal{O}(\varepsilon^{\overline{k}+1}).
\end{aligned}
\tag{2.26}
$$

Substituting (2.24) and (2.26) in (2.25) we conclude that

$$
\begin{aligned}
\frac{d}{dt}\left(x(t,z,\varepsilon)-\varphi(t,z)\right) &= \partial F_0(t,\varphi(t,z))\left(x(t,z,\varepsilon)-\varphi(t,z)\right)\\
&+\sum_{i=1}^{\overline{k}}\varepsilon^i\Bigg[\sum_{l=0}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}\left(t,\varphi(t,z)\right)\\
&\bigodot_{j=1}^{l} y_j(s,z)^{b_j}-\partial F_0(t,\varphi(t,z))\frac{y_i(t,z)}{i!}\Bigg]+\mathcal{O}(\varepsilon^{\overline{k}+1}).
\end{aligned}
\tag{2.27}
$$

Solving the linear differential equation (2.27) with respect to $x(t,z,\varepsilon)-\varphi(t,z)$ for the initial condition $x(0,z,\varepsilon)-\varphi(0,z)=0$ we obtain

$$x(t,z,\varepsilon)=\varphi(t,z)+\sum_{i=1}^{\overline{k}}\varepsilon^i\frac{Y_i(t,z)}{i!}+\mathcal{O}(\varepsilon^{\overline{k}+1}),$$

where

$$\begin{aligned}
&Y_i(t,z)=\\
&=e^{\eta(t,z)}\int_0^t e^{-\eta(s,z)}\Bigg[\sum_{l=0}^{i}\sum_{S_l}\frac{i!}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}\left(s,\varphi(s,z)\right)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}\\
&\qquad -\partial F_0(s,\varphi(s,z))y_i(s,z)\Bigg]ds.
\end{aligned}$$

The function $\eta(t,z)$ was defined in (2.10). Hence

$$\begin{aligned}
\frac{d}{dt}Y_i(t,z)&=\partial F_0(t,\varphi(t,z))Y_i(t,z)\\
&+\sum_{l=0}^{i}\sum_{S_l}\frac{i!}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}\left(t,\varphi(t,z)\right)\bigodot_{j=1}^{l} y_j(t,z)^{b_j}\\
&-\partial F_0(t,\varphi(t,z))y_i(t,z)ds.
\end{aligned}$$

Computing the derivative of the function $y_i(t,z)$ we conclude that the functions $y_i(t,z)$ and $Y_i(t,z)$ are defined by the same differential equation. Since $Y_i(0,z)=y_i(0,z)=0$ it follows that $Y_r(t,z)\equiv y_r(t,z)$ for every $i=1,2,\dots,\overline{k}$. So we have concluded the induction, which completes the proof of the lemma. □

In few words the proof of Theorem 2.1.5 is an application of the Brouwer degree (see Appendix B) to the approximated solution given by Lemma 2.2.1.

Proof of Theorem 2.1.5. Let $x(\cdot,z,\varepsilon)$ be a solution of (2.3) such that $x(0,z,\varepsilon)=z$. For each $z\in\overline{V}$, there exists $\varepsilon_1>0$ such that if $\varepsilon\in[-\varepsilon_1,\varepsilon_1]$ then $x(\cdot,z,\varepsilon)$ is defined in $[0,T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [107]), $x(\cdot,z,\varepsilon)$ is defined for all $0\le t\le \inf\,(T,d/M(\varepsilon))$, where

$$M(\varepsilon)\ge\left|\sum_{i=1}^{k}\varepsilon^iF_i(t,x)+\varepsilon^{k+1}R(t,x,\varepsilon)\right|$$

for all $t\in[0,T]$, for each x with $|x-z|<d$ and for every $z\in\overline{V}$. When ε is sufficiently small we can take $d/M(\varepsilon)$ sufficiently large in order that $\inf\,(T,d/M(\varepsilon))=T$ for all $z\in\overline{V}$.

We denote

$$\varepsilon f(z,\varepsilon)=x(T,z,\varepsilon)-z.$$

From Lemma 2.2.1 and equation (2.12) we have that

$$f(z,\varepsilon) = f_1(z) + \varepsilon f_2(z) + \varepsilon^2 f_3(z) + \cdots + \varepsilon^{k-1} f_k(z) + \varepsilon^k \mathcal{O}(1),$$

where the function f_i is the one defined in (2.6) for $i = 1, 2, \cdots, k$. From the assumption (ii) of the theorem we have that

$$f(z,\varepsilon) = \varepsilon^{r-1} f_r(z) + \cdots + \varepsilon^{k-1} f_k(z) + \varepsilon^k \mathcal{O}(1),$$

Clearly $x(\cdot, z, \varepsilon)$ is a T–periodic solution if and only if $f(z,\varepsilon) = 0$, because $x(t, z, \varepsilon)$ is defined for all $t \in [0, T]$.

From the Brouwer degree theory (see Lemma 2.6.3 of the appendix B) and hypothesis (ii) we have for $|\varepsilon| > 0$ sufficiently small that

$$d_B\left(f_r(z), V, 0\right) = d_B\left(f(z,\varepsilon), V, 0\right) \neq 0.$$

Hence, by item (i) of Theorem 2.6.1 (see Appendix B), $0 \in f(V, \varepsilon)$ for $|\varepsilon| > 0$ sufficiently small, i.e, there exists $a_\varepsilon \in V$ such that $f(a_\varepsilon, \varepsilon) = 0$.

Therefore, for $|\varepsilon| > 0$ sufficiently small, $x(t, a_\varepsilon, \varepsilon)$ is a periodic solution of (2.3). Clearly we can choose a_ε such that $a_\varepsilon \to a$ when $\varepsilon \to 0$, because $f(z,\varepsilon) \neq 0$ in $V \setminus \{a\}$. This completes the proof of the theorem. □

For proving Theorem 2.1.6 we also need the following lemma.

Lemma 2.2.2. *Let $w(\cdot, z, \varepsilon)\colon [0, \check{t}_z] \to \mathbb{R}^n$ be the solution of the system*

$$w'(t) = \sum_{i=1}^{k} \varepsilon^i \left([D_2\varphi(t,w)]^{-1} F_i(t, \varphi(t,w)) \right) + \varepsilon^{k+1} [D_2\varphi(t,w)]^{-1} R(t, \varphi(t,w), \varepsilon), \tag{2.28}$$

such that $w(0, z, \varepsilon) = z$. Then $\psi(\cdot, z, \varepsilon)\colon [0, \tilde{t}_z] \to \mathbb{R}^n$ defined as $\psi(t, z, \varepsilon) = \varphi\left(t, w(t, z, \varepsilon)\right)$ is the solution of (2.3) *such that $\psi(0, z, \varepsilon) = z$.*

Proof. Given $z \in D$, let $M(t) = D_2\varphi(t, z)$. The result about differentiable dependence on initial conditions implies that the function $M(t)$ is given as the fundamental matrix of the differential equation $u' = \partial F_0(t, \varphi(t, z))u$. So the matrix $M(t)$ is invertible for each $t \in [0, T]$. From here, the proof follows immediately from the derivative of $\psi(t, \xi, \varepsilon)$ with respect to t. □

Proof of Theorem 2.1.6. Let $x(\cdot, z, \varepsilon)$ be a solution of (2.3) such that $x(0, z, \varepsilon) = z$. For each $z \in \overline{V}$, there exists $\varepsilon_1 > 0$ such that if $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ then $x(\cdot, z, \varepsilon)$ is defined in $[0, T]$. Indeed, from Lemma 2.2.2, $x(t, z, \varepsilon) = \varphi\left(t, w(t, z, \varepsilon)\right)$ for each $z \in \overline{V}$, where $w(\cdot, z, \varepsilon)$ is the solution of (2.28). Moreover for $|\varepsilon_1| > 0$ sufficiently small, $w(t, z, \varepsilon) \in W$ for each $(t, z, \varepsilon) \in [0, T] \times \overline{V} \times [-\varepsilon_1, \varepsilon_1]$. Repeating the argument of the proof of Theorem 2.1.5 we can show that $\check{t}_z = T$ for every $z \in \overline{V}$. Since $\varphi(\cdot, z)$ is defined in [0,T] for every $z \in W$, it follows that $\tilde{t}_z = T$, i.e. $x(\cdot, z, \varepsilon)$ is also defined in $[0, T]$.

Now, denoting

$$f(z,\varepsilon) = x(T,z,\varepsilon) - z,$$

the proof follows similarly of Theorem 2.1.5. □

2.3 Computing formulae

In this section we illustrate how to compute the formulae of Theorems 2.1.5 and 2.1.6 for some $k \in \mathbb{N}$. In 3.1 we compute the formulae when $F_0 = 0$ for Theorem 2.1.5 up to $k = 5$. In 3.2 we compute the formulae when $F_0 \neq 0$ for Theorem 2.1.6 up to $k = 4$.

First of all from (2.7) we should determine the sets S_l for $l = 1, 2, 3, 4, 5$.

$$\begin{aligned}
&S_1 = \{1\},\\
&S_2 = \{(0,1),(2,0)\},\\
&S_3 = \{(0,0,1),(1,1,0),(3,0,0)\},\\
&S_4 = \{(0,0,0,1),(1,0,1,0),(2,1,0,0),(0,2,0,0),(4,0,0,0)\}.
\end{aligned}$$

To compute S_l is conveniently to exhibit a table of possibilities with the value b_i in the column i. We starts it from the last column.

Clearly the last column can be only filled by 0 and 1, because $5b_5 > 5$ for $b_5 > 1$. The same happens with the fourth and the third column, because $3b_3$, $4b_4 > 5$, for b_3, $b_4 > 1$. Taking $b_5 = 1$, the unique possibility is $b_1 = b_2 = b_3 = b_4 = 0$, thus any other solution satisfies $b_5 = 0$. Taking $b_5 = 0$ and $b_4 = 1$, the unique possibility is $b_1 = 1$ and $b_2 = b_3 = 0$, thus any other solution must have $b_4 = b_5 = 0$. Finally, taking $b_5 = b_4 = 0$ and $b_3 = 1$, we have two possibilities either $b_1 = 2$ and $b_2 = 0$, or $b_1 = 0$ and $b_2 = 1$. Thus any other solution satisfies $b_3 = b_4 = b_5 = 0$.

Now we observe that the second column can be only filled by 0, 1 or 2, since $2b_2 > 5$ for $b_2 > 2$; and taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 1$ the unique possibility is $b_1 = 3$. Taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 2$ the unique possibility is $b_1 = 1$, thus any other solution satisfies $b_2 = b_3 = b_4 = b_5 = 0$. Finally, taking $b_2 = b_3 = b_4 = b_5 = 0$ the unique possibility is $b_1 = 5$. Therefore the complete table of solutions is

$S_5 =$

b_1	b_2	b_3	b_4	b_5
0	0	0	0	1
1	0	0	1	0
0	1	1	0	0
2	0	1	0	0
3	1	0	0	0
1	2	0	0	0
5	0	0	0	0

Now we can use the (2.7) and (2.6) to compute the expressions of y_i and f_i.

2.4 Fifth order averaging of Theorem 2.1.5

We assume that $F_0 \equiv 0$. From (2.7) we obtain the functions $y_i(t,z)$ for $k = 1,2,3,4,5$.

$$
\begin{aligned}
y_1(t,z) &= \int_0^t F_1(s,z)ds,\\
y_2(t,z) &= \int_0^t \left(2F_2(s,z) + 2\frac{\partial F_1}{\partial x}(s,z)y_1(s,z)\right) ds,\\
y_3(t,z) &= \int_0^t \Bigg(6F_3(s,z) + 6\frac{\partial F_2}{\partial x}(s,z)y_1(t,z)\\
&\quad + 3\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z)^2 + 3\frac{\partial F_1}{\partial x}(s,z)\, y_2(s,z)\Bigg) ds,\\
y_4(t,z) &= \int_0^t \Bigg(24F_4(s,z) + 24\frac{\partial F_3}{\partial x}(s,z)y_1(s,z)\\
&\quad + 12\frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z)^2 + 12\frac{\partial F_2}{\partial x}(s,z)y_2(s,z)\\
&\quad + 12\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z) \odot y_2(s,z)\\
&\quad + 4\frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^3 + 4\frac{\partial F_1}{\partial x}(s,z)y_3(s,z)\Bigg) ds,\\
y_5(t,z) &= \int_0^t \Bigg(120F_5(s,z) + 120\frac{\partial F_4}{\partial x}(s,z)y_1(s,z\\
&\quad + 60\frac{\partial^2 F_3}{\partial x^2}(s,z)y_1(s,z)^2\\
&\quad + 60\frac{\partial F_3}{\partial x}(s,z)y_2(s,z) + 60\frac{\partial^2 F_2}{\partial x^2}(s,z)y_1(s,z) \odot y_2(s,z)\\
&\quad + 20\frac{\partial^3 F_2}{\partial x^3}(s,z)y_1(s,z)^3 + 20\frac{\partial F_2}{\partial x}(s,z)y_3(s,z)\\
&\quad + 20\frac{\partial^2 F_1}{\partial x^2}(s,z)y_1(s,z) \odot y_3(s,z)\\
&\quad + 15\frac{\partial^2 F_1}{\partial x^2}(s,z)y_2(s,z)^2 + 30\frac{\partial^3 F_1}{\partial x^3}(s,z)y_1(s,z)^2 \odot y_2(s,z)\\
&\quad + 5\frac{\partial^4 F_1}{\partial x^4}(s,z)y_1(s,z)^4 + 5\frac{\partial F_1}{\partial x}(s,z)y_4(s,z)\Bigg) ds.
\end{aligned}
$$

So from (2.6) we have that

$$
\begin{aligned}
f_0(z) =&0,\\
f_1(z) =&\int_0^T F_1(t,z)dt,\\
f_2(z) =&\int_0^T \left(F_2(t,z)ds + \frac{\partial F_1}{\partial x}(t,z)y_1(t,z)\right)dt,\\
f_3(z) =&\int_0^T \left(F_3(t,z) + \frac{\partial F_2}{\partial x}(t,z)y_1(t,z)\right.\\
&\left.+\frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)^2 + \frac{1}{2}\frac{\partial F_1}{\partial x}(t,z)y_2(t,z)\right)dt,\\
f_4(z) =&\int_0^T \left(F_4(t,z) + \frac{\partial F_3}{\partial x}(t,z)y_1(t,z)\right.\\
&+\frac{1}{2}\frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)^2 + \frac{1}{2}\frac{\partial F_2}{\partial x}(t,z)y_2(t,z)\\
&+\frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)dt\\
&\left.+\frac{1}{6}\frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^3 + \frac{1}{6}\frac{\partial F_1}{\partial x}(t,z)y_3(t,z)\right)dt,\\
f_5(z) =&\int_0^T \left(F_5(t,z) + \frac{\partial F_4}{\partial x}(t,z)y_1(t,z)\right.\\
&+\frac{1}{2}\frac{\partial^2 F_3}{\partial x^2}(t,z)y_1(t,z)^2 + \frac{1}{2}\frac{\partial F_3}{\partial x}(t,z)y_2(t,z)\\
&+\frac{1}{2}\frac{\partial^2 F_2}{\partial x^2}(t,z)y_1(t,z)\odot y_2(t,z)\\
&+\frac{1}{6}\frac{\partial^3 F_2}{\partial x^3}(t,z)y_1(t,z)^3 + \frac{1}{6}\frac{\partial F_2}{\partial x}(t,z)y_3(t,z)\\
&+\frac{1}{6}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_1(t,z)\odot y_3(t,z)\\
&+\frac{1}{8}\frac{\partial^2 F_1}{\partial x^2}(t,z)y_2(t,z)^2 + \frac{1}{4}\frac{\partial^3 F_1}{\partial x^3}(t,z)y_1(t,z)^2\odot y_2(t,z)\\
&\left.+\frac{1}{24}\frac{\partial^4 F_1}{\partial x^4}(t,z)y_1(t,z)^4 + \frac{1}{24}\frac{\partial F_1}{\partial x}(t,z)y_4(t,z)\right)dt.
\end{aligned}
$$

2.5 Fourth order averaging of Theorem 2.1.6

Now we assume that $F_0 \not\equiv 0$. First a Cauchy problem, or equivalently an integral equation (see Remark 2.1.4), must be solved to compute the expressions $y_i(t,z)$ for $i = 1, 2, \ldots, k$. We give the integral equations and its solutions for $k = 1, 2, 3, 4$.

Let $\eta(t,z)$ be the function defined in 2.10 and let $M(z) = \eta(T,z)$. Hence, from (2.7) and (2.6) we obtain the functions $y_1(t,z)$ and $f_1(z)$:

$$y_1(t,z) = \int_0^t \left(F_1(s,\varphi(s,z)) + \frac{\partial F_0}{\partial x}(s,\varphi(s,z))y_1(s,z) \right) ds,$$

so

$$y_1(t,z) = e^{\eta(t,z)} \int_0^t e^{-\eta(s,z)} F_1(s,\varphi(s,z))ds,$$

and

$$f_1(z) = M(z) \int_0^T e^{-\eta(t,z)} F_1(t,\varphi(t,z))dt.$$

Similarly, the functions $y_2(t,z)$ and $f_2(z)$ are given by:

$$\begin{aligned} y_2(t,z) = \int_0^t & \left(2F_2(s,\varphi(s,z)) + 2\frac{\partial F_1}{\partial x}(s,\varphi(s,z))y_1(s,z) \right. \\ & \left. + \frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_1(s,z)^2 + \frac{\partial F_0}{\partial x}(s,\varphi(s,z))\, y_2(s,z) \right) dt, \end{aligned}$$

so

$$\begin{aligned} y_2(t,z) =& e^{\eta(t,z)} \int_0^t e^{-\eta(s,z)} \left(2F_2(s,\varphi(s,z)) + 2\frac{\partial F_1}{\partial x}(s,\varphi(s,z))y_1(s,z) \right. \\ & \left. \frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_1(s,z)^2 \right) ds, \end{aligned}$$

and

$$\begin{aligned} f_2(z) =& M(z) \int_0^T e^{-\eta(t,z)} \left(F_2(t,\varphi(t,z)) + \frac{\partial F_1}{\partial x}(t,\varphi(t,z))y_1(t,z) \right. \\ & \left. \frac{1}{2}\frac{\partial^2 F_0}{\partial x^2}(t,\varphi(t,z))y_1(t,z)^2 \right) dt, \end{aligned}$$

The functions $y_3(t,z)$ and $f_3(z)$ are given by

$$\begin{aligned} y_3(t,z) = \int_0^t & \left(6F_3(s,\varphi(s,z)) + 6\frac{\partial F_2}{\partial x}(s,\varphi(s,z))y_1(s,z) \right. \\ & + 3\frac{\partial^2 F_1}{\partial x^2}(s,\varphi(s,z))y_1(s,z)^2 + 3\frac{\partial F_1}{\partial x}(s,\varphi(s,z))\, y_2(s,z) \\ & + 3\frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_1(s,z) \odot y_2(s,z) \\ & \left. + \frac{\partial^3 F_0}{\partial x^3}(s,\varphi(s,z))y_1(s,z)^3 + \frac{\partial F_0}{\partial x}(s,\varphi(s,z))y_3(s,z) \right) ds, \end{aligned}$$

so

$$
\begin{aligned}
y_3(t,z) =& e^{\eta(t,z)} \int_0^t e^{-\eta(s,z)} \bigg(6F_3(s,\varphi(s,z)) + 6\frac{\partial F_2}{\partial x}(s,\varphi(s,z))y_1(s,z) \\
&+ 3\frac{\partial^2 F_1}{\partial x^2}(s,\varphi(s,z))y_1(s,z)^2 + 3\frac{\partial F_1}{\partial x}(s,\varphi(s,z))\, y_2(s,z) \\
&+ 3\frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_1(s,z) \odot y_2(s,z) \\
&+ \frac{\partial^3 F_0}{\partial x^3}(s,\varphi(s,z))y_1(s,z)^3 \bigg)\, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
f_3(z) =& M(z) \int_0^T e^{-\eta(t,z)} \bigg(F_3(t,\varphi(t,z)) + \frac{\partial F_2}{\partial x}(t,\varphi(t,z))y_1(t,z) \\
&+ \frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,\varphi(t,z))y_1(t,z)^2 + \frac{1}{2}\frac{\partial F_1}{\partial x}(t,\varphi(t,z))\, y_2(t,z) \\
&+ \frac{1}{2}\frac{\partial^2 F_0}{\partial x^2}(t,\varphi(t,z))y_1(t,z) \odot y_2(t,z) \\
&+ \frac{1}{6}\frac{\partial^3 F_0}{\partial x^3}(t,\varphi(t,z))y_1(t,z)^3 \bigg)\, ds,
\end{aligned}
$$

Finally, the functions $y_4(t,z)$ and $f_4(z)$ are given by

$$
\begin{aligned}
y_4(t,z) = \int_0^t \bigg(& 24F_4(s,\varphi(s,z)) + 24\frac{\partial F_3}{\partial x}(s,\varphi(s,z))y_1(s,z) \\
&+ 12\frac{\partial^2 F_2}{\partial x^2}(s,\varphi(s,z))y_1(s,z)^2 + 12\frac{\partial F_2}{\partial x}(s,\varphi(s,z))y_2(s,z) \\
&+ 12\frac{\partial^2 F_1}{\partial x^2}(s,\varphi(s,z))y_1(s,z) \odot y_2(s,z) \\
&+ 4\frac{\partial^3 F_1}{\partial x^3}(s,\varphi(s,z))y_1(s,z)^3 + 4\frac{\partial F_1}{\partial x}(s,\varphi(s,z))y_3(s,z) \\
&+ 4\frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_1(s,z) \odot y_3(s,z) \\
&+ 3\frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_2(s,z)^2 ds + 6\frac{\partial^3 F_0}{\partial x^3}(s,\varphi(s,z))y_1(s,z)^2 \odot y_2(s,z) \\
&+ \frac{\partial^4 F_0}{\partial x^4}(s,\varphi(s,z))y_1(s,z)^4 + \frac{\partial F_0}{\partial x}(s,\varphi(s,z))y_4(s,z) \bigg)\, ds.
\end{aligned}
$$

so

$$
\begin{aligned}
y_4(t,z) =& e^{\eta(t,z)}\int_0^t e^{-\eta(s,z)}\bigg(24F_4(s,\varphi(s,z)) + 24\frac{\partial F_3}{\partial x}(s,\varphi(s,z))y_1(s,z) \\
&+ 12\frac{\partial^2 F_2}{\partial x^2}(s,\varphi(s,z))y_1(s,z)^2 + 12\frac{\partial F_2}{\partial x}(s,\varphi(s,z))y_2(s,z) \\
&+ 12\frac{\partial^2 F_1}{\partial x^2}(s,\varphi(s,z))y_1(s,z)\odot y_2(s,z) \\
&+ 4\frac{\partial^3 F_1}{\partial x^3}(s,\varphi(s,z))y_1(s,z)^3 + 4\frac{\partial F_1}{\partial x}(s,\varphi(s,z))y_3(s,z) \\
&+ 4\frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_1(s,z)\odot y_3(s,z) \\
&+ 3\frac{\partial^2 F_0}{\partial x^2}(s,\varphi(s,z))y_2(s,z)^2 ds + 6\frac{\partial^3 F_0}{\partial x^3}(s,\varphi(s,z))y_1(s,z)^2\odot y_2(s,z) \\
&+ \frac{\partial^4 F_0}{\partial x^4}(s,\varphi(s,z))y_1(s,z)^4\bigg)\, ds.
\end{aligned}
$$

and

$$
\begin{aligned}
f_4(z) =& M(z)\int_0^T e^{-\eta(t,z)}\bigg(F_4(t,\varphi(t,z)) + \frac{\partial F_3}{\partial x}(t,\varphi(t,z))y_1(t,z) \\
&+ \frac{1}{2}\frac{\partial^2 F_2}{\partial x^2}(t,\varphi(t,z))y_1(t,z)^2 + \frac{1}{2}\frac{\partial F_2}{\partial x}(t,\varphi(t,z))y_2(t,z) \\
&+ \frac{1}{2}\frac{\partial^2 F_1}{\partial x^2}(t,\varphi(t,z))y_1(t,z)\odot y_2(t,z) \\
&+ \frac{1}{6}\frac{\partial^3 F_1}{\partial x^3}(t,\varphi(t,z))y_1(t,z)^3 + \frac{1}{6}\frac{\partial F_1}{\partial x}(t,\varphi(t,z))y_3(t,z) \\
&+ \frac{1}{6}\frac{\partial^2 F_0}{\partial x^2}(t,\varphi(t,z))y_1(t,z)\odot y_3(t,z) \\
&+ \frac{1}{8}\frac{\partial^2 F_0}{\partial x^2}(t,\varphi(t,z))y_2(t,z)^2 ds + \frac{1}{4}\frac{\partial^3 F_0}{\partial x^3}(t,\varphi(t,z))y_1(t,z)^2\odot y_2(t,z) \\
&+ \frac{1}{24}\frac{\partial^4 F_0}{\partial x^4}(t,\varphi(t,z))y_1(t,z)^4\bigg)\, ds.
\end{aligned}
$$

2.6 Appendix: Basic results on the Brouwer degree

In this appendix we present the existence and uniqueness result from the degree theory in finite dimensional spaces. We follow the Browder's paper [16], where are formalized the properties of the classical Brouwer degree. We also present some results that we shall need for proving the main results of this paper.

Theorem 2.6.1. *Let $X = \mathbb{R}^n = Y$ for a given positive integer n. For bounded open subsets V of X, consider continuous mappings $f\colon \overline{V} \to Y$, and points y_0 in Y such that y_0 does not lie in $f(\partial V)$ (as usual ∂V denotes the boundary of V). Then to each such triple (f, V, y_0), there corresponds an integer $d_B(f, V, y_0)$ having the following three properties.*

(i) *If $d_b(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of X onto Y, then for every bounded open set V and $y_0 \in V$, we have*

$$d\left(f_0\big|_V, V, y_0\right) = \pm 1.$$

(ii) (*Additivity*) *If $f\colon \overline{V} \to Y$ is a continuous map with V a bounded open set in X, and V_1 and V_2 are a pair of disjoint open subsets of V such that*

$$y_0 \notin f(\overline{V}\backslash(V_1 \cup V_2)),$$

then,

$$d\left(f_0, V, y_0\right) = d\left(f_0, V_1, y_0\right) + d\left(f_0, V_1, y_0\right).$$

(iii) (*Invariance under homotopy*) *Let V be a bounded open set in X, and consider a continuous homotopy $\{f_t : 0 \leq t \leq 1\}$ of maps of $\overline{V}$ in to Y. Let $\{y_t : 0 \leq t \leq 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$. Then $d_B(f_t, V, y_t)$ is constant in t on $[0, 1]$.*

Theorem 2.6.2. *The degree function $d_B(f, V, y_0)$ is uniquely determined by the conditions of Theorem 2.6.1.*

For the proofs of Theorems 2.6.1 and 2.6.2 see [16].

Lemma 2.6.3. *We consider the continuous functions $f_i\colon \overline{V} \to \mathbb{R}^n$, for $i=0,1,\cdots,k$, and $f, g, r\colon \overline{V} \times [\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$, given by*

$$g(\cdot, \varepsilon) = f_1(\cdot) + \varepsilon f_2(\cdot) + \varepsilon^2 f_3(\cdot) + \cdots + \varepsilon^{k-1} f_k(\cdot),$$

$$f(\cdot, \varepsilon) = g(\cdot, \varepsilon) + \varepsilon^k r(\cdot, \varepsilon).$$

Assume that $g(z, \varepsilon) \neq 0$ for all $z \in \partial V$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. If for $|\varepsilon| > 0$ sufficiently small $d_B\left(f(\cdot, \varepsilon), V, y_0\right)$ is well defined, then

$$d_B\left(f(\cdot, \varepsilon), V, y_0\right) = d_B\left(g(\cdot, \varepsilon), V, y_0\right).$$

For a proof of Proposition 2.6.3 see Lemma 2.1 in [19].

Chapter 3

Three applications of Theorem 2.1.5

The first application studies the periodic solutions of the Hénon–Heiles Hamiltonian using the averaging theory of second order. The other two examples analyze the limit cycles of some classes of polynomial differential systems in the plane. These last two applications use the averaging theory of third orden. More precisely these three applications are based in Theorem 2.1.5.

In the next section we summarize the results of Theorem 2.1.5 up to third order, which are the ones that we shall use in the applications here considered.

3.1 The averaging theory of first, second and third order

As far as we know the averaging theory of third order for studying specifically periodic orbits was developed by first time in [19]. Now we summarize it here from Theorem 2.1.5 which is given at any order.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{3.1}$$

where $F_1, F_2, F_3 \colon \mathbb{R} \times D \to \mathbb{R}$, $R\colon \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t,\cdot) \in C^2(D)$, $F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2 F_1$, $D_x F_2$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0}\colon D \to \mathbb{R}$ for $k = 1, 2, 3$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_z F_1(s,z)\cdot y_1(s,z) + F_2(s,z)\right] ds,$$

$$\begin{aligned} F_{30}(z) =& \frac{1}{T}\int_0^T \Big[\frac{1}{2} y_1(s,z)^T \frac{\partial^2 F_1}{\partial z^2}(s,z) y_1(s,z) + \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z) y_2(s,z) \\ &+ \frac{\partial F_2}{\partial z}(s,z)(y_1(s,z)) + F_3(s,z)\Big] ds, \end{aligned}$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)dt,$$

$$y_2(s,z) = \int_0^s \left[\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)dr + F_2(t,z)\right] dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\varphi(\cdot,\varepsilon)$ of the system such that $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}\colon V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If F_{10} and F_{20} are identically zero and F_{30} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}$ are mainly the zeros of F_{30} for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

3.2 The Hénon–Heiles Hamiltonian

The results presented in this section have been proved by Jiménez and Llibre in [69].

The classical Hénon–Heiles potential consist of a two dimensional harmonic potential plus two cubic terms. It was introduced in 1964, as a model for studying the existence of a third integral of motion of a star in an rotating meridian plane of a galaxy in the neighborhood of a circular orbit [52]. The classical *Hénon–Heiles* potential has been generalized by introducing two parameters to each cubic term

$$\frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + Bxy^2 + \frac{1}{3}Ax^3. \tag{3.2}$$

such that $B \neq 0$, with $x, y, p_x, p_y \in \mathbb{R}$. Then the classical Hénon–Heiles Hamiltonian system corresponds to $A = -1$, $B = 1$. The Hamiltonian system is given by

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{p}_x &= -x - (Ax^2 + By^2), \\ \dot{y} &= p_y, \\ \dot{p}_y &= -y - 2Bxy. \end{aligned} \tag{3.3}$$

As usual the dot denotes derivative with respect to the independent variable $t \in \mathbb{R}$, the time. We name (3.3) the *Hénon–Heiles Hamiltonian systems with two parameters*, or simply the *Hénon–Heiles systems*.

The periodic orbits in the Hénon–Heiles potential have been numerically studied and classified by Churchil *et. al.* [28], Davies *et. al.* [34] and others [15, 42, 100]. Maciejewski *et. al.* [94] did an analytical study of a more general Hénon–Heiles Hamiltonians including a third cubic term of the form $C\,x^2y$, which can be removed by a proper rotation, and two more parameters associated with the quadratic part of the potential. They proved the existence of connected branches of non–stationary periodic orbits in the neighborhood of a given degenerate stationary point.

Theorem 3.2.1. *At every positive energy level the Hénon–Heiles Hamiltonian system* (3.3) *has at least*

(a) *one periodic orbit if* $(2B - 5A)(2B - A) < 0$ *(see Figure 3.1),*

(b) *two periodic orbits if* $A + B = 0$ *and* $A \neq 0$ *(this case contains the classical Hénon–Heiles system), and*

(c) *three periodic orbits if* $B(2B - 5A) > 0$ *and* $A + B \neq 0$ *(see Figure 3.2).*

Proof. For proving this theorem we shall apply Theorem 2.1.5 to the Hamiltonian system (3.3). Generically the periodic orbits of a Hamiltonian system with more

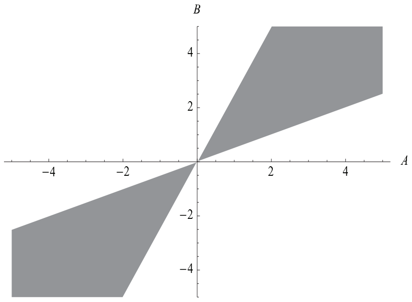

Figure 3.1: Open region $(2B-5A)(2B-A)<0$ in the parameter space (A,B) where there is at least one periodic orbit with multipliers different from 1.

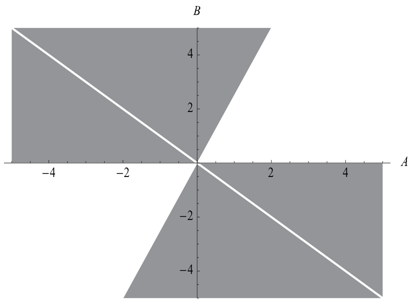

Figure 3.2: Open region $B(2B-5A)>0$ and $A+B\neq 0$ in the parameter space (A,B) where there are at least three periodic orbits with multipliers different from 1. When $A+B=0$, there are at least two periodic orbits with multipliers different from 1.

than one degree of freedom are on cylinders fulfilled of periodic orbits. Therefore we cannot apply directly Theorem 2.1.5 to a Hamiltonian system, since the Jacobian of the function f at the fixed point a will be always zero. Then we must apply Theorem 2.1.5 to every Hamiltonian fixed level where the periodic orbits generically are isolated.

On the other hand in order to apply Theorem 2.1.5 we need a small parameter ε. So in the Hamiltonian system (3.3) we change the variables (x,y,p_x,p_y) to (X,Y,p_X,p_Y) where $x=\varepsilon X$, $y=\varepsilon Y$, $p_x=\varepsilon p_X$ and $p_y=\varepsilon p_Y$. In the new variables, system (3.3) becomes

$$\begin{aligned}\dot{X} &= p_X,\\ \dot{p}_X &= -X-\varepsilon(AX^2+BY^2),\\ \dot{Y} &= p_Y,\\ \dot{p}_Y &= -Y-2\varepsilon BXY.\end{aligned} \tag{3.4}$$

This system again is Hamiltonian with Hamiltonian

$$\frac{1}{2}(p_X^2 + p_Y^2 + X^2 + Y^2) + \varepsilon \left(BXY^2 + \frac{1}{3}AX^3 \right). \tag{3.5}$$

As the change of variables is only a scale transformation, for all ε different from zero, the original and the transformed systems (3.3) and (3.4) have essentially the same phase portrait, and additionally system (3.4) for ε sufficiently small is close to an integrable one

First we change the Hamiltonian (3.5) and the equations of motion (3.4) to polar coordinates for $\varepsilon = 0$, which is a harmonic oscillator. Thus we have

$$X = r\cos\theta, \quad p_X = r\sin\theta, \quad Y = \rho\cos(\theta+\alpha), \quad p_Y = \rho\sin(\theta+\alpha).$$

Recall that this is a change of variables when $r > 0$ and $\rho > 0$. Moreover doing this change of variables appear in the system the angular variables θ and α. Later on the variable θ will be used for obtaining the periodicity necessary for applying the averaging theory.

The fixed value of the energy in polar coordinates is

$$h = \frac{1}{2}(r^2 + \rho^2) + \varepsilon \left(\frac{1}{3}Ar^3\cos^3\theta + Br\rho^2\cos\theta\cos^2(\theta+\alpha) \right), \tag{3.6}$$

and the equations of motion are given by

$$\begin{aligned}
\dot r &= -\varepsilon\sin\theta\left(A\,r^2\cos^2\theta + B\,\rho^2\cos^2(\theta+\alpha)\right), \\
\dot\theta &= -1 - \varepsilon\cos\theta\left(A\,r\cos^2\theta + \frac{\rho^2}{r}B\,\cos^2(\theta+\alpha)\right), \\
\dot\rho &= -\varepsilon B\,r\rho\cos\theta\sin(2(\theta+\alpha)), \\
\dot\alpha &= \varepsilon\frac{\cos\theta}{r}\left(A\,r^2\cos^2\theta + B(\rho^2 - 2r^2)\cos^2(\theta+\alpha)\right).
\end{aligned} \tag{3.7}$$

However the derivatives of the left hand side of these equations are with respect to the time variable t, which is not periodic. We change to the θ variable as the independent one, and we denote by a prime the derivative with respect to θ. The angular variable α cannot be used as the independent variable since the new differential system would not have the form (2.1) for applying Theorem 2.1.5. The system (3.7) goes over to

$$\begin{aligned}
r' &= \frac{\varepsilon\,r\sin\theta\left(A\,r^2\cos^2\theta + B\rho^2\cos^2(\theta+\alpha)\right)}{r + \varepsilon(A\,r^2\cos^3\theta + B\rho^2\cos\theta\cos^2(\theta+\alpha))}, \\
\rho' &= \frac{\varepsilon\,Br^2\rho\cos\theta\sin(2(\theta+\alpha))}{r + \varepsilon(A\,r^2\cos^3\theta + B\rho^2\cos\theta\cos^2(\theta+\alpha))}, \\
\alpha' &= -\frac{\varepsilon\cos\theta\left(B\left(\rho^2 - 2r^2\right)\cos^2(\theta+\alpha) + A\,r^2\cos^2\theta\right)}{r + \varepsilon(B\rho^2\cos\theta\cos^2(\theta+\alpha) + A\,r^2\cos^3\theta)}.
\end{aligned}$$

Of course this system has now only three equations because we do not need the θ equation. If we write the previous system as a Taylor series in powers of ε, we have

$$
\begin{aligned}
r' &= \varepsilon \sin\theta(A\,r^2\cos^2\theta + B\rho^2\cos^2(\theta+\alpha)) \\
&\quad - \varepsilon^2 \frac{\sin 2\theta}{8r}\,\big(A\,r^2(1+\cos(2\theta)) + B\rho^2(1+\cos(2(\theta+\alpha))\big)^2 + O(\varepsilon^3), \\
\rho' &= \varepsilon B r\rho\cos\theta\sin(2(\theta+\alpha)) \\
&\quad - \varepsilon^2 B\rho\cos^2\theta\sin(2(\theta+\alpha))(Ar^2\cos^2\theta + B\rho^2\cos^2(2(\theta+\alpha))) + O(\varepsilon^3), \quad (3.8)\\
\alpha' &= -\varepsilon\frac{\cos\theta}{r}(A\,r^2\cos^2\theta + B(\rho^2-2r^2)\cos^2(\theta+\alpha)) \\
&\quad + \varepsilon^2\,\frac{\cos^2\theta}{r^2}\,(Ar^2\cos^2\theta + B\rho^2\cos^2(\theta+\alpha)) \\
&\qquad (A\,r^2\cos^2\theta + B(\rho^2-2r^2)\cos^2(\theta+\alpha)) + O(\varepsilon^3).
\end{aligned}
$$

Now system (3.8) is 2π-periodic in the variable θ. In order to apply Theorem 2.1.5 we must fix the value of the first integral at $h>0$, and by solving equation (3.6) for ρ we obtain

$$
\rho = \sqrt{\frac{h - r^2/2 - \varepsilon A\,r^3\cos^3\theta/3}{1/2 + \varepsilon\,B\,r\cos\theta\cos^2(\theta+\alpha)}}. \tag{3.9}
$$

Then substituting ρ in equations (3.8), we obtain the two differential equations

$$
\begin{aligned}
r' &= \varepsilon\sin\theta(A\,r^2\cos^2\theta + B(2h-r^2)\cos^2(\theta+\alpha)) \\
&\quad - \varepsilon^2\Big(\frac{\sin 2\theta}{8r}\big(A\,r^2(1+\cos(2\theta)) + B\left(2h-r^2\right)(1+\cos(2(\theta+\alpha)))\;\big)^2 \\
&\quad + \frac{2}{3}AB\,r^3\sin\theta\cos^3\theta\cos^2(\theta+\alpha) \\
&\quad + 2B^2hr\sin(2\theta)\cos^4(\theta+\alpha) - B^2r^3\sin(2\theta)\cos^4(\theta+\alpha)\,\Big) + O(\varepsilon^3), \\
\alpha' &= \varepsilon\left(\frac{B}{r}(3r^2-2h)\cos\theta\cos^2(\theta+\alpha) - Ar\cos^3\theta\right) \\
&\quad + \varepsilon^2(A^2r^2\cos^6\theta + \frac{2}{3}AB(6h-5r^2)\cos^4\theta\cos^2(\theta+\alpha) \\
&\quad + \frac{B^2}{r^2}(r^2-2h)^2\cos^2\theta\cos^4(\theta+\alpha)\;) + O(\varepsilon^3).
\end{aligned} \tag{3.10}
$$

Clearly system (3.10) satisfies the assumptions of Theorem 2.1.5, and it has the form (2.1) with $F_1 = (F_{11}, F_{12})$ and $F_2 = (F_{21}, F_{22})$, where

$$
\begin{aligned}
F_{11} &= \sin\theta\big(A\,r^2\cos^2\theta + B(2h-r^2)\cos^2(\theta+\alpha)\big), \\
F_{12} &= \frac{B}{r}(3r^2-2h)\cos\theta\cos^2(\theta+\alpha) - Ar\cos^3\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
F_{21} &= -\frac{\sin 2\theta}{8r}\left(A\,r^2(1+\cos(2\theta)) + B\left(2h-r^2\right)(1+\cos(2(\theta+\alpha)))\right)^2 \\
&\quad -\frac{2}{3}AB\,r^3\sin\theta\cos^3\theta\cos^2(\theta+\alpha) - 2B^2hr\sin(2\theta)\cos^4(\theta+\alpha) \\
&\quad + B^2r^3\sin(2\theta)\cos^4(\theta+\alpha), \\
F_{22} &= A^2r^2\cos^6\theta + \frac{2}{3}AB(6h-5r^2)\cos^4\theta\cos^2(\theta+\alpha) \\
&\quad + \frac{B^2}{r^2}(r^2-2h)^2\cos^2\theta\cos^4(\theta+\alpha).
\end{aligned}
$$

As $r \neq 0$ the functions F_1 and F_2 are analytical. Furthermore they are 2π-periodic in the variable θ, the independent variable of system (3.10). However the averaging theory of first order does not apply because the average functions of F_1 and F_2 in the period vanish

$$
f_1(r,\alpha) = \int_0^{2\pi} (F_{11}, F_{12})\, d\theta = (0,0)\,.
$$

As the function f_1 of Theorem 2.1.5 is zero, we procede to calculate the function f_2 by applying the second order averaging theory. We have that f_2 is defined by

$$
f_2(r,\alpha) = \int_0^{2\pi} \left[D_{r\alpha}F_1(\theta,r,\alpha).y_1(\theta,r,\alpha) + F_2(\theta,r,\alpha)\right] d\theta, \tag{3.11}
$$

where

$$
y_1(\theta,r,\alpha) = \int_0^{\theta} F_1(t,r,\alpha)\, dt\,.
$$

The two components of the vector y_1 are

$$
\begin{aligned}
y_{11} &= \int_0^{\theta} F_{11}(t,r,\alpha)\, dt \\
&= \frac{1}{3}\left(B(2h-r^2)\sin^2(\theta/2)\left(\cos(2(\theta+\alpha))+2\cos(2\alpha+\theta)+3\right) - Ar^2(\cos^3\theta - 1)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
y_{12} &= \int_0^{\theta} F_{12}(t,r,\alpha)\, dt \\
&= -\frac{Ar}{12}(9\sin\theta + \sin 3\theta) - \frac{Bh}{6r}(3\sin(2\alpha+\theta) + \sin(2\alpha+3\theta) - 4\sin 2\alpha + 6\sin\theta) \\
&\quad + \frac{Br}{4}(3\sin(2\alpha+\theta) + \sin(2\alpha+3\theta) - 4\sin(2\alpha) + 6\sin\theta).
\end{aligned}
$$

For the Jacobian matrix

$$D_{r\alpha}F_1(\theta,r,\alpha)=\begin{pmatrix}\dfrac{\partial F_{11}}{\partial r} & \dfrac{\partial F_{11}}{\partial \alpha}\\ \dfrac{\partial F_{12}}{\partial r} & \dfrac{\partial F_{12}}{\partial \alpha}\end{pmatrix},$$

we obtain

$$\begin{pmatrix}\big(2Ar\cos^2\theta-2Br\cos^2(\theta+\alpha)\big)\sin\theta & -2B(2h-r^2)\cos(\theta+\alpha)\sin\theta\sin(\theta+\alpha)\\ \\ -A\cos^3\theta+6B\cos^2(\theta+\alpha)\cos\theta & -\dfrac{2B}{r}(3r^2-2h)\cos\theta\cos(\theta+\alpha)\sin(\theta+\alpha)\\ -\dfrac{B}{r^2}\left(3r^2-2h\right)\cos^2(\theta+\alpha)\cos\theta & \end{pmatrix}.$$

We can now calculate from Theorem 2.1.5 the function (3.11) and we obtain

$$f_2=\Big(-\frac{Br}{12}(6B-A)(r^2-2h)\sin 2\alpha,$$
$$\frac{1}{12}\big(r^2(5A^2-12AB-3B^2)-2B(A-6B)(h-r^2)\cos(2\alpha)+2Bh(6A-B)\big)\Big).$$

We have to find the zeros (r^*,α^*) of $f_2(r,\alpha)$, and to check that the Jacobian determinant

$$|D_{r,\alpha}f_2(r^*,\alpha^*)|\neq 0. \tag{3.12}$$

Solving the equation $f_2(r,\alpha)=0$ we obtain five solutions (r^*,α^*) with $r^*>0$, namely

$$\left(\sqrt{2}h,\pm\operatorname{arcsec}\frac{B(A-6B)}{4B^2+6AB-5A^2}\right),\ \left(\sqrt{\frac{2Bh}{3B-A}},0\right),\ \left(\sqrt{\frac{14Bh}{9B-5A}},\pm\pi/2\right). \tag{3.13}$$

The first two solutions are not good, because for them we get from (3.9) that $\rho=0$ when $\varepsilon=0$, and ρ must be positive. The third solution exists if $B(3B-A)>0$. The last two solutions exist if $B(9B-5A)>0$. The Jacobian (3.12) of the third solution is

$$-\frac{5B^2h^2(A-6B)(A-2B)(A+B)}{9(A-3B)}, \tag{3.14}$$

and for the last two solutions the Jacobian coincides and is equal to

$$\frac{7B^2h^2(A-6B)(5A-2B)(A-B)}{9(5A-9B)}. \tag{3.15}$$

Summarizing, from Theorem 2.1.5 the third solution of $f_2(r,\alpha)=0$ provides a periodic orbit of system (3.10) (and consequently of the Hamiltonian system (3.4) on the Hamiltonian level $h>0$) if $B(3B-A)>0$, $(A-6B)(A-2B)(A+B)\neq 0$, and from (3.9) we get $\rho=\sqrt{2(A-2B)h/(A-3B)}$, we also need $(2B-A)(3B-A)>0$. The conditions $B(3B-A)>0$ and $(2B-A)(3B-A)>0$ can be reduced to $B(2B-A)>0$, where $(A-6B)(A-2B)\neq 0$ is included, but $A+B\neq 0$ is not. Then the third solution provides a periodic orbit when $B(2B-A)>0$ and $A+B\neq 0$.

In a similar way the last two solutions of $f_2(r,\alpha)=0$ provide two periodic orbits of system (3.10) if $B(9B-5A)>0$, $(A-6B)(5A-2B)(A-B)\neq 0$, and from (3.9) we get $\rho=\sqrt{2(5A-2B)h/(5A-9B)}$, we also need $(2B-5A)(9B-5A)>0$. The conditions $B(9B-5A)>0$ and $(2B-5A)(9B-5A)>0$ can be reduced to $B(2B-5A)>0$, where the condition $(A-6B)(5A-2B)(A-B)\neq 0$ is included. Then the fourth and fifth solutions provide two periodic orbits whenever $B(2B-5A)>0$.

There is one periodic orbit if the third solution exists, and the last two solutions do not. There are two periodic orbits if the two last solutions exist, and not the third one, *i.e.* when $A+B=0$. Finally there are three periodic orbits if the third, fourth and fifth solutions exist. Now the statements of Theorem 2.1.5 follow easily.

The regions in the parameter space where periodic orbits exist are summarized in Figures 3.1 and 3.2. □

3.3 Limit cycles of polynomial differential systems

The results presented in this section come from Llibre and Swirszcz [82].

After the definition of limit cycle due to Poincaré [104], the statement of the 16–th Hilbert's problem [53], the discover that the limit cycles are important in the nature by Liénard [70],... the study of the limit cycles of the planar differential systems has been one of the main problems of the qualitative theory of the differential equations.

One of the best ways of producing limit cycles is by perturbing the periodic orbits of a center. This has been studied intensively perturbing the periodic orbits of the centers of the quadratic polynomial differential systems see the book of Christopher and Li [25], and the references quoted there.

It is well known that if a quadratic polynomial differential system has a limit cycles this must surround a focus. Up to know the maximum number of known limit cycles surrounding a focus of a quadratic polynomial differential system is 3, which coincides with the maximum number of small limit cycles which can bifurcate by Hopf from a singular point of a quadratic polynomial differential system, see Bautin [7]. But as far as we know up to now there are few quadratic centers for which it is proved that the perturbation of their periodic orbits inside the class of all quadratic polynomial differential systems can produce 3 limit cycles. These are the center whose exterior boundary is formed by three invariant straight lines (see Żołądek [126]), three different families of reversible quadratic centers (see Świrszcz [117]), and the center $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$ (see Buică, Gasull and Yang [3]). The study of the perturbation of this last center has been made through the Melnikov function of third order computed using the algorithm developed by Françoise [46] and Iliev [55]. Here we can provide a new and shorter proof of this second result by using the averaging theory, see Theorem 3.3.1.

We study the limit cycles of the following two differential systems: the *quadratic systems*

$$\begin{aligned} \dot{x} &= -y(1+x) + \varepsilon(\lambda x + \bar{A}x^2 + \bar{B}xy + \bar{C}y^2), \\ \dot{y} &= \quad x(1+x) + \varepsilon(\lambda y + \bar{D}x^2 + \bar{E}xy + \bar{F}y^2), \end{aligned} \tag{3.16}$$

such that for $\varepsilon = 0$ have a straight line consisting of singular points, and the *cubic systems* of the form

$$\begin{aligned} \dot{x} &= -y(1-x^2-y^2) + \varepsilon^3\lambda x + \sum_{s=1}^{3} \varepsilon^s \sum_{i=0}^{3} a_{i,s} x^i y^{3-i}, \\ \dot{y} &= \quad x(1-x^2-y^2) + \varepsilon^3\lambda y + \sum_{s=1}^{3} \varepsilon^s \sum_{i=0}^{3} b_{i,s} x^i y^{3-i}, \end{aligned} \tag{3.17}$$

such that for $\varepsilon = 0$ have a unit circle consisting of singular points. Note that the perturbation of this cubic systems is inside the class of all polynomial differential system with linear and cubic homogeneous nonlinearities.

We study for $\varepsilon \neq 0$ sufficiently small the number of limit cycles of systems (3.16) and (3.17) bifurcating from the periodic orbits of the centres of (3.16) and (3.17) for $\varepsilon = 0$, respectively. Our main results are the following.

Theorem 3.3.1. *For convenient* λ, $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$, $\bar{F}$ *system* (3.16) *has* 3 *limit cycles bifurcating from the periodic orbits of the center for* $\varepsilon = 0$.

Theorem 3.3.2. *The following statements hold for system* (3.17).

(a) *Using the averaging theory of third order for* $\varepsilon \neq 0$ *sufficiently small we can obtain at most* 5 *limit cycles of system* (3.17) *bifurcating from the periodic orbits of the center located at the origin of system* (3.17) *with* $\varepsilon = 0$.

(b) *For convenient* λ, $a_{i,s}$, $b_{i,s}$, $i = 0, 1, 2, 3$, $s = 1, 2, 3$ *system* (3.17) *has* 0, 1, 2, 3, 4 *or* 5 *limit cycles bifurcating from the periodic orbits of the center for* $\varepsilon = 0$.

It is known that systems of the form $\dot{x} = -y + P_3(x, y)$, $\dot{y} = x + Q_3(x, y)$, with P_3 and Q_3 homogeneous polynomials of degree 3 can have 5 small limit cycles bifurcating by Hopf from the origin, see [112, 88].

We are going to use the following result due to Cherkas [23].

Lemma 3.3.3. *The differential equation*

$$\frac{dr}{d\varphi} = \frac{\lambda r + a(\varphi) r^k}{1 + b(\varphi) r^{k-1}}$$

after the change of variable

$$\rho(\varphi) = \frac{r(\varphi)^{k-1}}{1 + b(\varphi) r(\varphi)^{k-1}}$$

becomes the Abel equation

$$\begin{aligned}\frac{d\rho}{d\varphi} &= (k-1)b(\varphi)(\lambda b(\varphi) - a(\varphi))\rho^3 \\ &\quad + \left[(k-1)(a(\varphi) - 2\lambda b(\varphi)) - b'(\varphi)\right]\rho^2 + (k-1)\lambda\rho,\end{aligned}$$

Combining Lemma 3.3.3 with polar coordinates transformation we immediately get the next result.

Corollary 3.3.4. *Let $P(x,y)$ and $Q(x,y)$ be homogenous polynomials of degree n. Then the differential system*

$$\begin{aligned}\dot{x} &= -y + \lambda x + P_n(x,y) \\ \dot{y} &= x + \lambda y + Q_n(x,y)\end{aligned} \tag{3.18}$$

can be transformed into the Abel equation

$$\begin{aligned}\frac{d\rho}{d\varphi} &= (k-1)B(\varphi)(\lambda B(\varphi) - A(\varphi))\rho^3 \\ &\quad + \left[(k-1)(A(\varphi) - 2\lambda B(\varphi)) - B'(\varphi)\right]\rho^2 + (k-1)\lambda\rho.\end{aligned}$$

where

$$A(\varphi) = \cos\varphi P_n(\cos\varphi, \sin\varphi) + \sin\varphi Q_n(\sin\varphi, \cos\varphi)$$

and

$$B(\varphi) = \cos\varphi Q_n(\cos\varphi, \sin\varphi) - \sin\varphi P_n(\sin\varphi, \cos\varphi).$$

Proof. System (3.18) expressed in polar coordinates becomes

$$\begin{aligned}\dot{r} &= \lambda r + A(\varphi) r^n, \\ \dot{y} &= 1 + B(\varphi) r^n.\end{aligned}$$

Dividing $\dot{r}$ by $\dot{\varphi}$ and using Lemma 3.3.3 the proof of the corollary follows. □

Proof of Theorem 3.3.1. From Corollary 3.3.4 applied to system (3.16) it follows that finding limit cycles of (3.16) is equivalent to finding periodic solutions of

$$\begin{aligned}\frac{d\rho}{d\varphi} = (\sin\varphi)\rho^2 + \epsilon\Big[&-\frac{1}{4}\cos\varphi((3\bar{A}+\bar{C}+\bar{E}-4\lambda)\cos\varphi \\ &+ (\bar{A}-\bar{C}-\bar{E})\cos 3\varphi \\ &+ 2(\bar{B}+\bar{D}+\bar{F}+(\bar{B}+\bar{D}-\bar{F})\cos 2\varphi)\sin\varphi)\rho^3 \\ &+ ((\bar{A}+\bar{C}-2\lambda)\cos\varphi + (\bar{A}-\bar{C}-\bar{E})\cos 3\varphi \\ &+ (\bar{D}+\bar{F})\sin\varphi + (\bar{B}+\bar{D}-\bar{F})\sin 3\varphi)\rho^2 + \lambda\rho\Big].\end{aligned} \tag{3.19}$$

We are going to apply Theorem 2.1.5 to system (3.19). We first solve the differential equation

$$\frac{d\rho}{d\varphi} = (\sin\varphi)\rho^2,$$

with initial condition $\rho(0) = R/(1+R)$ and we get $\rho(\varphi, R) = R/(1+R\cos\varphi)$. Thus $M_R(\varphi)$ in (1.35) will be a solution of a differential equation $M_R'(\varphi) = (2R\sin\varphi)/(1+R\cos\varphi)$, namely, $M_R(\varphi) = 1 + 2\ln(1+R) - 2\ln(1+r\cos\varphi)$. Thus formula (1.35) yields

$$\begin{aligned}\mathcal{F}(R) = \int_0^{2\pi} \Big(&\lambda\frac{R}{\Xi(\varphi,R)} \\ &+ \bar{A}\frac{\cos\varphi(R\cos\varphi + 8\cos(2\varphi) + 3R\cos(3\varphi))R^2}{4\Xi(\varphi,R)} \\ &+ \bar{B}\frac{(2R\sin 2\varphi + 8\sin 3\varphi + 3R\sin 4\varphi)R^2}{8\Xi(\varphi,R)} \\ &- \bar{C}\frac{\cos\varphi(3R\cos\varphi+4)\sin^2\varphi R^2}{\Xi(\varphi,R)} \\ &+ \bar{D}\frac{\cos^2\varphi(3R\cos\varphi+4)\sin\varphi R^2}{\Xi(\varphi,R)} \\ &- \bar{E}\frac{\cos\varphi(R\cos\varphi + 8\cos 2\varphi + 3R\cos 3\varphi - 4)R^2}{4\Xi(\varphi,R)} \\ &+ \bar{F}\frac{(5R\cos\varphi + 8\cos 2\varphi + 3R\cos 3\varphi)\sin\varphi R^2}{4\Xi(\varphi,R)}\Big)\, d\varphi,\end{aligned} \tag{3.20}$$

where $\Xi(\varphi, R) = (R\cos\varphi+1)^3(2\log(R+1) - 2\log(R\cos\varphi+1)+1)$. Now observe that the terms in front of $\bar{B}$, $\bar{D}$ and $\bar{F}$ are odd π-periodic functions of φ, thus their integrals from 0 to 2π are equal to zero. Therefore

$$\begin{aligned}\mathcal{F}(R) = \int_0^{2\pi} \Bigg(& \lambda \frac{R}{\Xi(\varphi, R)} \\ & + \bar{A} \frac{\cos\varphi(R\cos\varphi + 8\cos(2\varphi) + 3R\cos(3\varphi))R^2}{4\Xi(\varphi, R)} \\ & + \bar{C} \frac{\cos\varphi(3R\cos\varphi + 4)\sin^2\varphi R^2}{\Xi(\varphi, R)} \\ & + \bar{E} \frac{\cos\varphi(R\cos\varphi + 8\cos 2\varphi + 3R\cos 3\varphi - 4)R^2}{4\Xi(\varphi, R)} \Bigg) d\varphi \\ = {} & \lambda f_1(R) + \bar{A} f_2(R) + \bar{C} f_3(R) - \bar{E} f_4(R).\end{aligned} \tag{3.21}$$

We claim that the four functions f_1, f_2, f_3 and f_4 are linearly independent. Now we prove the claim. By straightforward calculation we obtain the following Taylor expansions:

$$\begin{aligned} f_1(R) &= \frac{1}{24}\pi R \left(2615R^4 - 800R^3 + 312R^2 - 96R + 48\right) + \mathcal{O}(R^6), \\ f_2(R) &= \frac{1}{24}\pi R^3 \left(313R^2 - 60, R - 18\right) + \mathcal{O}(R^6), \\ f_3(R) &= \frac{1}{24}\pi R^3 \left(401R^2 - 84R - 6\right) + \mathcal{O}(R^6), \\ f_4(R) &= -\frac{1}{24}\pi R^3 \left(43R^2 - 12R + 6\right) + \mathcal{O}(R^6). \end{aligned}$$

The determinant of the coefficient matrix of terms $R^2, \ldots, R^5$ is $\pi^4/3$ and the claim follows.

A well-known classical result states that if a family n functions is linearly independent, then there exists a linear combination of them with at least $n-1$ zeroes. Thus Theorem 3.3.1 follows. □

Proof of Theorem 3.3.2. First we prove statement (b). We shall use third order averaging to show that the system

$$\begin{aligned}\dot{x} = {} & -y(1 - x^2 - y^2) + \varepsilon^3 \lambda x \\ & - \frac{1}{1200}(75\mathcal{B}\varepsilon + 108\mathcal{E} + 19840)\varepsilon x^3 + (j + 24)\varepsilon x^2 y \\ & + \left(4\varepsilon^3(\mathcal{A} - 4\lambda) + \varepsilon^2 \left(\frac{27\mathcal{B}}{128} - \mathcal{C}\right) + \frac{(81\mathcal{E} + 16480)\varepsilon}{300}\right) xy^2 \\ & + \frac{1}{2}\varepsilon(2j + \mathcal{D}\varepsilon)y^3,\end{aligned} \tag{3.22}$$

$$\begin{aligned}\dot{y} = {}& x(1-x^2-y^2)+\varepsilon^3\lambda y \\ &+\frac{1}{2}(\mathcal{D}\varepsilon-2j)\varepsilon x^3+\left(\varepsilon^2\left(\mathcal{C}-\frac{3\mathcal{B}}{128}\right)+\frac{(81\mathcal{E}+18080)\varepsilon}{300}\right)x^2y \\ &-(j+40)\varepsilon xy^2-\frac{1}{300}(27\mathcal{E}+6560)\varepsilon y^3,\end{aligned}$$

can have 0, 1, 2, 3, 4 or 5 limit cycles for an appropriate choice of the parameters λ, $\mathcal{A}$, $\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$ and $\mathcal{E}$. System (3.22) is clearly a special case of system (3.17), thus once we show it, statement (b) will be proved.

Using Cherkas Transformation (Lemma 3.3.3) we transform system (3.22) into the Abel equation

$$\frac{d\rho}{d\varphi}=\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3, \tag{3.23}$$

where

$$\begin{aligned}F_1 ={}& \rho^3\left(\frac{3}{50}(3\mathcal{E}+640)\cos(4\varphi)+8(\sin(2\varphi)-2\sin(4\varphi))-\frac{16}{3}\cos(2\varphi)\right) \\ &+\rho^2\left(-\frac{9}{50}(3\mathcal{E}+640)\cos(4\varphi)-8\sin(2\varphi)+48\sin(4\varphi)+\frac{16}{3}\cos(2\varphi)\right), \\ F_2 ={}& \frac{\rho^3}{30000}\Big[25(6400j+75\mathcal{B}+432\mathcal{E}+117760)\cos(2\varphi) \\ &-75\cos(4\varphi)(72(j+8)\mathcal{E}+15360(j+8)-25\mathcal{B}) \\ &-600\sin(2\varphi)(400j+25\mathcal{D}+12\mathcal{E}+7360) \\ &+480000(j+8)\sin(4\varphi)-7200(\mathcal{E}+80)\sin(6\varphi) \\ &+3(9\mathcal{E}+1120)(9\mathcal{E}+2720)\sin(8\varphi) \\ &-400(27\mathcal{E}+7360)\cos(6\varphi)+14400(3\mathcal{E}+640)\cos(8\varphi)\Big] \\ &+\rho^2\left(\left(\frac{3\mathcal{B}}{128}-\mathcal{C}\right)\cos(2\varphi)-\frac{3}{16}\mathcal{B}\cos(4\varphi)+3\mathcal{D}\sin(\varphi)\cos(\varphi)\right), \\ F_3 ={}& -2\lambda\rho \\ &+\rho^2\Big((\mathcal{A}-4\lambda)(2\cos(2\varphi)-3\cos(4\varphi))+\mathcal{A}\Big) \\ &+\rho^3\Big\{\mathcal{A}\cos 4\varphi-\mathcal{A}-\frac{11\mathcal{B}}{64}+2\mathcal{C}-\frac{4\mathcal{D}}{3}+2\lambda \\ &+\frac{1}{76800}\Big[\sin(2\varphi)(384(100(j+4)\mathcal{D}-3\mathcal{C}(3\mathcal{E}+640))+\mathcal{B}(513\mathcal{E}+103040)) \\ &-96\cos(2\varphi)(25(2j-7)\mathcal{B}+3200\mathcal{C}-6\mathcal{D}(3\mathcal{E}+640)) \\ &-400\cos(4\varphi)(3(4j+21)\mathcal{B}+128(3\mathcal{C}+2\mathcal{D}+6\lambda)) \\ &+\sin(6\varphi)(1152(3\mathcal{C}\mathcal{E}+640\mathcal{C}-400\mathcal{D})-\mathcal{B}(81\mathcal{E}+23680)) \\ &-96\cos(6\varphi)(175\mathcal{B}-640(5\mathcal{C}+18\mathcal{D})-54\mathcal{D}\mathcal{E})\end{aligned}$$

$$+ 800\sin(4\varphi)(11\mathcal{B} + 64(3\mathcal{D} - 2\mathcal{C})) + 144\mathcal{B}(3\mathcal{E} + 640)\sin(8\varphi) + 38400\mathcal{B}\cos(8\varphi)\Big]\Big\}.$$

By straightforward calculation we verify that $F_{10} = 0$,

$$y_1(\rho,\varphi) = \frac{\rho^3}{300}\sin\varphi((27\mathcal{E} + 4160)\cos\varphi + 3(3(3\mathcal{E} + 640)\cos 3\varphi - 800\sin 3\varphi)) - \frac{\rho^2}{600}\left(2\sin(2\varphi)(27(3\mathcal{E} + 640)\cos 2\varphi - 800(9\sin 2\varphi + 1)) + 4800\sin^2\varphi\right),$$

and $F_{20} = 0$. Next

$$\begin{aligned} y_2(\rho,\varphi) = {} & \frac{1}{128}\rho^2(9\mathcal{B}\cos\varphi + 12\mathcal{B}\cos(3\varphi) + 128\mathcal{C}\cos\varphi - 192\mathcal{D}\sin\varphi)\sin\varphi \\ & + \rho^3\Bigg[\left(\frac{8j}{3} + \frac{\mathcal{B}}{32} - \frac{9\mathcal{E}}{25} + \frac{128}{15}\right)\sin(2\varphi) \\ & - \frac{1}{50}(400j + 25\mathcal{D} - 24\mathcal{E} + 1280)\sin^2\varphi \\ & - \frac{9}{200}j\mathcal{E}\sin(4\varphi) + \frac{8}{9}(9j + 494)\sin^2(2\varphi) - \frac{48}{5}j\sin(4\varphi) \\ & + \frac{1}{64}\mathcal{B}\sin(4\varphi) + \frac{81\mathcal{E}^2\sin^2(4\varphi)}{4000} - \frac{4}{5}\mathcal{E}\sin^2(3\varphi) + \frac{216}{25}\mathcal{E}\sin^2(4\varphi) \\ & - \frac{63}{25}\mathcal{E}\sin(4\varphi) - \frac{3}{5}\mathcal{E}\sin(6\varphi) + \frac{9}{5}\mathcal{E}\sin(8\varphi) - 64\sin^2(3\varphi) \\ & + \frac{3808}{5}\sin^2(4\varphi) - \frac{7904}{15}\sin(4\varphi) - \frac{1472}{9}\sin(6\varphi) + 384\sin(8\varphi)\Bigg] \\ & + \rho^4\Bigg[-\frac{243\mathcal{E}^2\sin^2(4\varphi)}{16000} - \frac{1}{25}(21\mathcal{E} + 2480)\sin^2\varphi + \frac{29}{25}\mathcal{E}\sin^2(3\varphi) \\ & - \frac{162}{25}\mathcal{E}\sin^2(4\varphi) + \frac{1}{300}(189\mathcal{E} + 9920)\sin(2\varphi) + \frac{27}{25}\mathcal{E}\sin(4\varphi) \\ & + \frac{87}{100}\mathcal{E}\sin(6\varphi) - \frac{27}{20}\mathcal{E}\sin(8\varphi) - \frac{1528}{9}\sin^2(2\varphi) + \frac{464}{5}\sin^2(3\varphi) \\ & - \frac{2856}{5}\sin^2(4\varphi) + \frac{3056}{15}\sin(4\varphi) + \frac{10672}{45}\sin(6\varphi) - 288\sin(8\varphi)\Bigg] \\ & + \rho^5\frac{((27\mathcal{E} + 4160)\cos\varphi + 3(3(3\mathcal{E} + 640)\cos(3\varphi) - 800\sin(3\varphi)))^2\sin^2\varphi}{60000} \end{aligned}$$

and

$$\begin{aligned} F_{30}(\rho) = {} & -2\lambda\rho + \mathcal{A}\rho^2 - \left(\mathcal{A} - \mathcal{B} - \frac{2\mathcal{D}}{3} - 2\lambda\right)\rho^3 \\ & - \left(\frac{91\mathcal{B}}{128} - \mathcal{C} + \frac{7\mathcal{D}}{3} - \frac{4\mathcal{E}}{5}\right)\rho^4 + \left(\mathcal{D} - \frac{9\mathcal{E}}{5}\right)\rho^5 + \mathcal{E}\rho^6. \end{aligned}$$

The coefficients of F_{30} are linearly independent (linear) functions of λ, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$. Therefore for any $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \in \mathbb{R}$ there exist λ, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ such that $F_{30}(\rho_i) = 0$ for $i = 1, 2, 3, 4, 5$. This ends the proof of statement (b).

Now we sketch the proof of statement (a). If instead of doing the computations of the proof of statement (b) for system (3.22) we did them for the general system (3.17) we would obtain a function $F_{30}(\rho)$ which again is a polynomial of degree 6 in ρ without independent term. Thus the averaging theory of third order can only produce for $\varepsilon \neq 0$ sufficiently small at most 5 limit cycles of system (3.17) bifurcating from the periodic orbits at the origin of system (3.17) with $\varepsilon = 0$. □

3.4 The generalized polynomial differential Liénard equation

The results of this section have been prove by Llibre, Mereu and Teixeira in [75].

The second part of the Hilbert's problem is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. The *generalized polynomial Liénard differential equations*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3.24}$$

was introduced in [72]. Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively. For this subclass of polynomial vector fields we have a simplified version of Hilbert's problem, see [71] and [110].

In 1977 Lins, de Melo and Pugh [71] studied the classical polynomial Liénard differential equations (3.24) obtained when $g(x) = x$ and stated the following conjecture: *if $f(x)$ has degree $n \geq 1$ and $g(x) = x$, then* (3.24) *has at most* $[n/2]$ *limit cycles.* They also proved the conjecture for $n = 1, 2$. The conjecture for $n \in \{3, 4, 5\}$ is still open. For $n \geq 5$ this conjecture is not true as it has been proved recently by Dumortier, Panazzolo and Roussarie in [39], and De Maesschalck and F. Dumortier [33]. Recently the conjecture has been proved for $n = 3$, see Chengzhi and Llibre [87]. So at this moment only remains to know if the conjecture holds or not for $n = 4$.

We note that a classical polynomial Liénard differential equation has a unique singular point. However it is possible for generalized polynomial Liénard differential equations to have more than one singular point.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point, that are so called *small amplitud limit cycles*, see [86]. We denote by $\hat{H}(m, n)$ the maximum number of small amplitude limit cycles for systems of the form (3.24). The values of $\hat{H}(m, n)$ give a lower bound for the maximum number $H(m, n)$ (i.e. the *Hilbert number*) of limit cycles that the differential equation

(3.24) with m and n fixed can have. It is unknown the finitude of $H(m,n)$ for every positive integers m and n. For more information about the Hilbert's 16th problem and related topics see [59] and [69].

Now we shall describe briefly the main results about the limit cicles on Liénard differential systems.

(i) In 1928 Liénard [72] proved if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (3.24) has a unique limit cycle.

(ii) In 1973 Rychkov [106] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is an odd polynomial of degree five, then equation (3.24) has at most two limit cycles.

(iii) In 1977 Lins, de Melo and Pugh [71] proved that $H(1,1) = 0$ and $H(1,2) = 1$.

(iv) In 1998 Coppel [32] proved that $H(2,1) = 1$.

(v) Dumortier, Li and Rousseau in [40] and [37] proved that $H(3,1) = 1$.

(vi) In 1997 Dumortier and Chengzhi [38] proved that $H(2,2) = 1$.

Up to now and as far as we know only for these four cases ((iii)-(vi)) marked with asterisks in Table 3.1 the Hilbert numbers $H(m,n)$ are determined.

Blows, Lloyd and Lynch, [10], [87] and [90] have used inductive arguments in order to prove the following results.

(I) If g is odd then $\hat{H}(m,n) = [n/2]$.

(II) If f is even then $\hat{H}(m,n) = n$, whatever g is.

(III) If f is odd then $\hat{H}(m, 2n+1) = [(m-2)/2] + n$.

(IV) If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$.

Christopher and Lynch [27], [91], [92], [93] have developed a new algebraic method for determining the Liapunov quantities of system (3.24) and proved the following:

(V) $\hat{H}(m,2) = [(2m+1)/3]$.

(VI) $\hat{H}(2,n) = [(2n+1)/3]$.

(VII) $\hat{H}(m,3) = 2[(3m+2)/8]$ for all $1 < m \leq 50$.

(VIII) $\hat{H}(3,n) = 2[(3n+2)/8]$ for all $1 < m \leq 50$.

(IX) The values of Table 3.1 for $\hat{H}(4,k) = \hat{H}(k,4)$, $k = 6,7,8,9$ and $\hat{H}(5,6) = \hat{H}(6,5)$.

In 1998 Gasull and Torregrosa [47] obtained upper bounds for $\hat{H}(7,6)$, $\hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$.

In 2006 the values of Table 3.1 for $\hat{H}(m,n) = \hat{H}(n,m)$, for $n = 4$, $m = 10,11,12,13$; $n = 5$, $m = 6,7,8,9$; $n = 6$, $m = 5,6$ were given by Yu and Han in [124].

Table 3.1: The values of $H(m,n)$ or $\hat{H}(m,n)$ for the Liénard systems in function of the degrees m and n.

		n																
		1	2	3	4	5	6	7	8	9	10	11	12	13	...	48	49	50
	1	0	1*	1	2	2	3	3	4	4	5	5	6	6	...	24	24	→
	2	1*	1*	2	3	3	4	5	5	6	7	7	8	9	⋯	32	33	→
	3	1*	2	2	4	4	6	6	6	8	8	8	10	10	...	36	38	38
	4	2	3	4	4	6	7	8	9	9	10	11	12	13				
	5	2	3	4	6	6	8	9	10	11								
	6	3	4	6	7	8	8	9										
	7	3	5	6	8	9	9	9										
m	8	4	5	6	9	10												
	9	4	6	8	9	11												
	10	5	7	8	10													
	11	5	7	8	11													
	12	6	8	10	12													
	13	6	9	10	13													
	⋮	⋮	⋮	⋮														
	20	10	13	14	17													
	⋮	⋮	⋮	⋮														
	48	24	32	36														
	49	24	33	38														
	50	↓	↓	38														

By using the averaging theory we shall study in this work the maximum number of limit cycles $\tilde{H}(m,n)$ which can bifurcate from the periodic orbits of a linear center perturbed inside the class of all generalized polynomial Liénard differential equations of degrees m and n as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \sum_{k\geq 1} \varepsilon^k (f_n^k(x)y + g_m^k(x)), \end{aligned} \tag{3.25}$$

where for every k the polynomials $g_m^k(x)$ and $f_n^k(x)$ have degree m and n respectively, and ε is a small parameter, i.e. the maximal number of *medium amplitude limit cycles* which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, perturbed as in (3.25).

In fact we mainly shall compute lower estimations of $\tilde{H}(m,n)$. More precisely we compute the maximum number of limit cycles $\tilde{H}_k(m,n)$ which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of order k, for $k = 1,2,3$. Of course $\tilde{H}_k(m,n) \leq \tilde{H}(m,n) \leq H(m,n)$. Note that up to now there were no lowers estimations for $H(m,n)$ when

(a) $m = 4$ and $n > 13$, or $m > 20$ and $n = 4$,

(b) $m = 5$ and $n > 9$, or $m > 9$ and $n = 5$,

(c) $m = 6$ and $n > 7$, or $m > 7$ and $n = 6$,

(d) $m, n > 7$.

After our results we will have lowers estimations of $H(m,n)$ for all $m, n \geq 1$. From these estimations we obtain that $\tilde{H}_k(m,n) \leq \hat{H}(m,n)$ for $k = 1,2,3$ for the values which $\hat{H}(m,n)$ is known.

Theorem 3.4.1. *If for every $k = 1,2,3$, the polynomials $f_n^k(x)$ and $g_m^k(x)$ have degree n and m respectively, with $m,n \geq 1$, then for $|\varepsilon|$ sufficiently small, the maximum number of medium limit cycles of the polynomial Liénard differential systems* (3.25) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory*

(a) *of first order is* $\tilde{H}_1(m,n) = \left[\dfrac{n}{2}\right]$;

(b) *of second order is* $\tilde{H}_2(m,n) = \max\left\{\left[\dfrac{n-1}{2}\right] + \left[\dfrac{m}{2}\right], \left[\dfrac{n}{2}\right]\right\}$; *and*

(c) *of third order is* $\tilde{H}_3(m,n) = \left[\dfrac{n+m-1}{2}\right]$.

From Theorem 3.4.1 follows immediately Table 2.

Table 3.2: Values of $\tilde{H}_3(m,n)$. The numbers written in the style 6 coincide with the ones of Table 1. The numbers written in the style *6* are smaller than the corresponding of Table 1. The numbers written in the style **6** are unknown in Table 1.

		n																
		1	2	3	4	5	6	7	8	9	10	11	12	13	…	48	49	50
	1	0	1	1	2	2	3	3	4	4	5	5	6	6	…	24	24	→
	2	1	1	2	*2*	*3*	*3*	*4*	*4*	*5*	*5*	*6*	*6*	*7*	⋯	*24*	*25*	→
	3	1	2	2	*3*	*3*	*4*	*4*	*5*	*5*	*6*	*6*	*7*	*7*	⋯	*25*	*25*	→
	4	2	*2*	*3*	*3*	*4*	*4*	*5*	*5*	*6*	*6*	*7*	*7*	*8*	⋯	**25**	**26**	→
	5	2	*3*	*3*	*4*	*4*	*5*	*5*	*6*	*6*	**7**	**7**	**8**	**8**	⋯	**26**	**26**	→
	6	3	*3*	*4*	*4*	*5*	*5*	*6*	**6**	**7**	**7**	**8**	**8**	**9**	⋯	**26**	**27**	→
	7	3	*4*	*4*	*5*	*5*	*6*	*6*	**7**	**7**	**8**	**8**	**9**	**9**	⋯	**27**	**27**	→
m	8	4	*4*	*5*	*5*	*6*	*6*	**7**	**7**	**8**	**8**	**9**	**9**	**10**	⋯	**27**	**28**	→
	9	4	*5*	*5*	*6*	*6*	*7*	**7**	**8**	**8**	**9**	**9**	**10**	**10**	⋯	**28**	**28**	→
	10	5	*5*	*6*	*6*	*7*	**7**	**8**	**8**	**9**	**9**	**10**	**10**	**11**	⋯	**28**	**29**	→
	11	5	*6*	*6*	*7*	*7*	**8**	**8**	**9**	**9**	**10**	**10**	**11**	**11**	⋯	**29**	**29**	→
	12	6	*6*	*7*	*7*	*8*	**8**	**9**	**9**	**10**	**10**	**11**	**11**	**12**	⋯	**29**	**30**	→
	13	6	*7*	*7*	*8*	*8*	**9**	**9**	**10**	**10**	**11**	**11**	**12**	**12**	⋯	**30**	**30**	→
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	20	10	*10*	*11*	*11*	**12**	**12**	**13**	**13**	**14**	**14**	**15**	**15**	**16**	⋯	**33**	**34**	→
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	48	24	*24*	*25*	*25*	**26**	**26**	**27**	**27**	**28**	**28**	**29**	**29**	**30**	⋯	**47**	**48**	→
	49	24	*25*	*25*	*26*	**26**	**27**	**27**	**28**	**28**	**29**	**29**	**30**	**30**	⋯	**48**	**48**	→
	50	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	

It seems that the numbers $\hat{H}(m,n)$ can be symmetric with respect m and n. Some studies is this direction are made in [89]. We remark that in general $\tilde{H}_k(m,n) \neq \tilde{H}_k(n,m)$ for $k = 1, 2$, but $\tilde{H}_3(m,n) = \tilde{H}_3(n,m)$.

Proof of statement (*a*) *of Theorem 3.4.1.* We shall need the first order averaging theory to prove statement (a) of Theorem 3.4.1.

In order to apply the first order averaging method we write system (3.25) with $k = 1$, in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta, r > 0$. In this way system (3.25) is written in the standard form for applying the averaging theory. If we write $f_1(x) = \sum_{i=0}^{n} a_i x^i$ and $g_1(x) = \sum_{i=0}^{m} b_i x^i$ with $b_0 = 0$, then system (3.25) becomes

$$\begin{aligned}
\dot{r} &= -\varepsilon\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_i r^i\cos^i\theta\sin\theta\right),\\
\dot{\theta} &= -1 - \frac{\varepsilon}{r}\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{m} b_i r^i\cos^{i+1}\theta\right).
\end{aligned} \tag{3.26}$$

Now taking θ as the new independent variable system, (3.26) becomes

$$\frac{dr}{d\theta} = \varepsilon\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_i r^i\cos^i\theta\sin\theta\right) + O(\varepsilon^2),$$

and

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_i r^i\cos^i\theta\sin\theta\right)d\theta.$$

In order to calculate the exact expression of F_{10} we use the following formulas

$$\int_0^{2\pi}\cos^{2k+1}\theta\sin^2\theta d\theta = 0, \qquad k = 0, 1, \ldots$$

$$\int_0^{2\pi}\cos^{2k}\theta\sin^2\theta d\theta = \alpha_{2k} \neq 0, \qquad k = 0, 1, \ldots$$

$$\int_0^{2\pi}\cos^{k}\theta\sin\theta d\theta = 0, \qquad k = 0, 1, \ldots$$

Hence

$$F_{10}(r) = \frac{1}{2} \sum_{\substack{i=0 \\ i \text{ even}}}^{n} a_i \alpha_i r^{i+1}. \tag{3.27}$$

Then the polynomial $F_{10}(r)$ has at most $[n/2]$ positive roots, and we can choose the coefficients a_i with i even in such a way that $F_{10}(r)$ has exactly $[n/2]$ simple positive roots. Hence statement (a) of Theorem 3.4.1 is proved. □

Proof of statement (*b*) *of Theorem 3.4.1.* For proving statement (b) of Theorem 3.4.1 we shall use the second order averaging theory.

If we write $f_1(x) = \sum_{i=0}^{n} a_i x^i$, $f_2(x) = \sum_{i=0}^{n} c_i x^i$, $g_1(x) = \sum_{i=0}^{m} b_i x^i$ and $g_2(x) = \sum_{i=0}^{m} d_i x^i$ with $b_0 = d_0 = 0$, then system (3.25) with $k = 2$ in polar coordinates (r, θ), $r > 0$ becomes

$$\begin{aligned} \dot{r} =& -\varepsilon \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_i r^i \cos^i\theta \sin\theta \right) - \\ & -\varepsilon^2 \left(\sum_{i=0}^{n} c_i r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} d_i r^i \cos^i\theta \sin\theta \right), \\ \dot{\theta} =& -1 - \frac{\varepsilon}{r} \left(\sum_{i=0}^{n} a_i r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{m} b_i r^i \cos^{i+1}\theta \right) - \\ & -\frac{\varepsilon^2}{r} \left(\sum_{i=0}^{n} c_i r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{m} d_i r^i \cos^{i+1}\theta \right). \end{aligned} \tag{3.28}$$

Taking θ as the new independent variable system, (3.28) writes

$$\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + O(\varepsilon^3),$$

where

$$\begin{aligned} F_1(\theta, r) =& \sum_{i=0}^{n} a_i r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_i r^i \cos^i\theta \sin\theta, \\ F_2(\theta, r) =& \left(\sum_{i=0}^{n} c_i r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} d_i r^i \cos^i\theta \sin\theta \right) \\ & - r \sin\theta \cos\theta \left(\sum_{i=0}^{n} a_i r^i \cos^i\theta \sin\theta + \sum_{i=0}^{m} b_i r^{i-1} \cos^i\theta \right)^2. \end{aligned}$$

Now we determine the corresponding function F_{20}. For this we compute

$$\frac{d}{dr}F_1(\theta,r)=\sum_{i=0}^{n}(i+1)a_ir^i\cos^i\theta\sin^2\theta+\sum_{i=1}^{m}ib_ir^{i-1}\cos^i\theta\sin\theta,$$

and $\displaystyle\int_0^\theta F_1(\phi,r)d\phi$ which is equal to

$$\begin{aligned}&a_1r^2\left(\alpha_{11}\sin\theta+\alpha_{21}\sin(3\theta)\right)+\ldots\\&\quad+a_lr^{l+1}\Big(\alpha_{1l}\sin\theta+\alpha_{2l}\sin(3\theta)+\ldots.+\alpha_{(\frac{l+3}{2})l}\sin((l+2)\theta\Big)\\&\quad+a_0r\left(\alpha_{10}\theta+\alpha_{20}\sin(2\theta)\right)+\ldots\\&\quad+a_br^{b+1}\Big(\alpha_{1b}\theta+\alpha_{2b}\sin(2\theta)+\ldots+\alpha_{(\frac{b+4}{2})b}\sin(b+2)\theta\Big)\\&b_0(1-\cos\theta)+\ldots+b_mr^m\left(\frac{1}{m+1}(1-\cos^{m+1}\theta)\right),\end{aligned}\tag{3.29}$$

where l is the greatest odd number less than or equal to n, b is the greatest even number less than or equal to n, and α_{ij} are real constants exhibited during the computation of $\int_0^\theta\cos^i\phi\sin^2\phi\;d\phi$ for all i. We know from (3.27) that F_{10} is identically zero if and only if $a_i=0$ for all i even. Moreover

$$\int_0^{2\pi}\cos^i\theta\sin^3\theta d\theta=0,\qquad i=0,1,\ldots$$

$$\int_0^{2\pi}\cos^i\theta\sin^2\theta\sin((2k+1)\theta)d\theta=0,\qquad i,k=0,1,\ldots$$

$$\int_0^{2\pi}\cos^{2i+1}\theta\sin^2\theta d\theta=0,\qquad i=0,1,\ldots$$

$$\int_0^{2\pi}\cos^{2i}\theta\sin^2\theta d\theta=A_{2i}\neq0,\qquad i=0,1,\ldots$$

$$\int_0^{2\pi}\cos^i\theta\sin\theta d\theta=0,\qquad i=0,1,\ldots$$

$$\int_0^{2\pi}\cos^{2i}\theta\sin\theta\sin((2k+1)\theta)d\theta=B_{2i}^{2k+1}\neq0,\qquad i,k=0,1,\ldots$$

$$\int_0^{2\pi}\cos^{2i+1}\theta\sin\theta\sin((2k+1)\theta)d\theta=0,\qquad i,k=0,1,\ldots$$

So

$$\int_0^{2\pi}\frac{d}{dr}F_1(\theta,r)y_1(\theta,r)d\theta=$$

$$\sum_{\substack{j=2\\ j \text{ even}}}^{k} \sum_{\substack{i=1\\ i \text{ odd}}}^{l} -\frac{i+1}{j+1} a_i b_j r^{i+j} \int_0^{2\pi} \cos^{i+j+1}\theta \sin^2\theta d\theta +$$

$$\sum_{\substack{j=2\\ j \text{ even}}}^{k} \sum_{\substack{i=1\\ i \text{ odd}}}^{l} j a_i b_j r^{i+j} \int_0^{2\pi} \cos^j\theta \sin\theta \left(\alpha_{1i}\sin\theta + \ldots + \alpha_{\frac{i+3}{2}i}\sin((i+2)\theta)\right) d\theta =$$

$$r\left(\tilde{\alpha}_{10}a_1b_0 + (\tilde{\alpha}_{12}a_1b_2 + \tilde{\alpha}_{30}a_3b_0)r^2 + \ldots\ldots + \sum_{i+j=l+k} \tilde{\alpha}_{ij}a_ib_jr^{l+k-1}\right),$$

where $\tilde{\alpha}_{ij} = -\dfrac{1+i}{j+i}A_{i+j+1} + j\left(\alpha_{1i}B_j^1 + \alpha_{2i}B_j^2 + \ldots + \alpha_{\frac{i+3}{2}i}B_j^{i+2}\right)$, for all i, j and k being the greatest even number less than or equal to m.

Moreover

$$\begin{aligned}\int_0^{2\pi} F_2(\theta, r) d\theta = & \sum_{\substack{i=0\\ i \text{ even}}}^{b} c_i r^{i+1} \int_0^{2\pi} \cos^i\theta \sin^2\theta d\theta \\ & + \sum_{\substack{j=0\\ j \text{ even}}}^{k} \sum_{\substack{i=1\\ i \text{ odd}}}^{l} 2r^{i+j} a_i b_j \int_0^{2\pi} \cos^{i+j+1}\theta \sin^2\theta d\theta \\ = & A_0c_0r + \ldots + A_bc_br^{b+1} \\ & + 2\left(A_2a_1b_0r + A_4(a_3b_0 + a_1b_2)r^3 + \ldots + A_{l+k+1}r^{l+k} \sum_{i+j=l+k} a_ib_j\right).\end{aligned}$$

Then $F_{20}(r)$ is the polynomial

$$\begin{aligned}& r\Big(\rho_{10}a_1b_0 + (\rho_{12}a_1b_2 + \rho_{30}a_3b_0)r^2 + (\rho_{14}a_1b_4 + \rho_{32}a_3b_2 + \rho_{50}a_5b_0)r^4 + \\ & \ldots + \rho_{lk}a_lb_kr^{l+k-1} + A_0c_0 + A_2c_2r^2 + \ldots + A_bc_br^b\Big),\end{aligned} \tag{3.30}$$

where $\rho_{ij} = \tilde{\alpha}_{ij} + 2A_{i+j+1}$ for all i, j. Note that in order to find the positive roots of F_{20} we must find the zeros of a polynomial in r^2 of degree equal to the $\max\left\{\dfrac{l+k-1}{2}, \dfrac{b}{2}\right\}$. We have that $\dfrac{b}{2} = \left[\dfrac{n}{2}\right]$ and $\dfrac{l+k-1}{2} = \left[\dfrac{n-1}{2}\right] + \left[\dfrac{m}{2}\right]$. See Table 3.3.

We conclude that F_{20} has at most $\max\{[(n-1)/2] + [m/2], [n/2]\}$ positive roots. Moreover we can choose the coefficients a_i, b_j, c_k in such a way that (3.30) has exactly $\max\{[(n-1)/2] + [m/2], [n/2]\}$ simple positive roots. Hence the statement (b) of Theorem 3.4.1 follows. □

Table 3.3: Values of $(l+k-1)/2$ written using the integer part function.

n	m	l	k	$(l+k-1)/2$	$[(n-1)/2]+[m/2]$
odd	even	n	m	$(n+m-1)/2$	$(n-1)/2+m/2$
even	even	n-1	m	$(n-1+m-1)/2$	$((n-1)-1)/2+m/2$
odd	odd	n	m-1	$(n+m-1-1)/2$	$(n-1)/2+(m-1)/2$
even	odd	n-1	m-1	$(n-1+m-1-1)/2$	$((n-1)-1)/2+(m-1)/2$

Proof of statement (*c*) *of Theorem 3.4.1.* The proof of statement (c) of Theorem 3.4.1 is based in the third order averaging theory.

If we write $f_1(x) = \sum_{i=0}^{n} a_i x^i$, $f_2(x) = \sum_{i=0}^{n} c_i x^i$, $f_3(x) = \sum_{i=0}^{n} p_i x^i$, $g_1(x) = \sum_{i=0}^{m} b_i x^i$, $g_2(x) = \sum_{i=0}^{m} d_i x^i$ and $g_3(x) = \sum_{i=0}^{m} q_i x^i$ with $b_0 = d_0 = q_0 = 0$, then an equivalent system to (3.25) with $k = 3$ will be found by considering polar coordinates (r, θ). So

$$\begin{aligned} \dot{r} &= -\sin\theta\left(\varepsilon A + \varepsilon^2 B + \varepsilon^3 C\right), \\ \dot{\theta} &= -1 - \frac{\cos\theta}{r}\left(\varepsilon A + \varepsilon^2 B + \varepsilon^3 C\right), \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} A &= \sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin\theta + \sum_{i=0}^{m} b_i r^i\cos^i\theta, \\ B &= \sum_{i=0}^{n} c_i r^{i+1}\cos^i\theta\sin\theta + \sum_{i=0}^{m} d_i r^i\cos^i\theta, \\ C &= \sum_{i=0}^{n} p_i r^{i+1}\cos^i\theta\sin\theta + \sum_{i=0}^{m} q_i r^i\cos^i\theta. \end{aligned}$$

Taking θ as the new independent variable system (3.31) becomes

$$\begin{aligned} \frac{dr}{d\theta} =& \varepsilon A\sin\theta + \varepsilon^2\left(B\sin\theta - \frac{A^2\cos\theta\sin\theta}{r}\right) \\ &+ \varepsilon^3\left(\frac{A^3\cos^2\theta\sin\theta}{r^2} - \frac{2AB\cos\theta\sin\theta}{r} + C\sin\theta\right). \end{aligned} \tag{3.32}$$

We know by (3.27) that F_{10} is identically zero if and only if $a_i = 0$ for all i even, and by (3.30) we obtain that F_{20} is identically zero if and only if the coefficients a_i, b_j and c_k satisfy

$$c_\mu = \frac{1}{A_\mu} \sum_{\substack{i+j=\mu+1 \\ i \text{ odd, } j \text{ even}}} \rho_{i,j}\ a_i\ b_j \tag{3.33}$$

where μ is even, A_μ and $\rho_{i,j}$ are given in section 2.2.

In order to apply the third order averaging method we need to compute the corresponding function F_{30}. So the proof of statement (c) of Theorem 3.4.1 will be direct consequence of the next auxiliary lemmas.

The proof of the next lemma is straightforward and follows from some tedious computations. It will be omitted.

Lemma 3.4.2. *The corresponding functions $y_1(\theta, r)$ and $y_2(\theta, r)$ of third order averaging method are expressed by* (3.29) *and*

$$y_2(\theta, r) = \quad C_0 + C_1 r + C_2 r^2 + \ldots + C_\lambda r^\lambda,$$

respectively, where $\lambda = \max\{2n+1, 2m-1\}$ *and*

$$\begin{aligned}
C_{2k+1} = & \sum_{i+j+=2k} c_{ij}^0 a_i a_j + \sum_{i+j=2k+2} d_{ij}^0 b_i b_j + \sum_{i+j=2k+1} e_{ij}^0 a_i b_j \theta \\
& + \sum_{i+j=2k} f_{ij}^0 a_i a_j \theta^2 + d_{2k+1} + c_{2k}\theta + \sum_{i+j=2k+2} b_i b_j \left(\sum_{i=0}^{k+1} a_{2i+1}^0 \cos(2i+1)\theta \right) \\
& + \left(\sum_{i+j+=2k} a_i a_j + \sum_{i+j=2k+2} b_i b_j + \sum_{i+j=2k+1} a_i b_j \theta + d_{2k+1} \right) \left(\sum_{i=0}^{k+1} a_{2i+2}^0 \cos(2i+2)\theta \right) \\
& + \sum_{i+j=2k+1} a_i b_j \left(\sum_{i=0}^{k+1} a_{2i+1}^1 \sin(2i+1)\theta \right) \\
& + \left(\sum_{i+j+=2k+1} a_i b_j + \sum_{i+j=2k} a_i a_j \theta + c_{2k} \right) \left(\sum_{i=0}^{k+1} a_{2i+2}^1 \sin(2i+2)\theta) \right),
\end{aligned}$$

$$C_{2k} = \sum_{i+j+=2k-1} c^1_{ij} a_i a_j + \sum_{i+j=2k+1} d^1_{ij} b_i b_j + \sum_{i+j=2k} e^1_{ij} a_i b_j \theta$$
$$+ \left(\sum_{i+j=2k-1} a_i a_j + \sum_{i+j=2k+1} b_i b_j + \sum_{i+j=2k} a_i b_j \theta \right) \left(\sum_{i=0}^{k+1} b^0_{2i+1} \cos(2i+1)\theta \right)$$
$$+ \left(\sum_{i+j+=2k+1} b_i b_j \right) \left(\sum_{i=0}^{k+1} b^0_{2i+2} \cos(2i+2)\theta \right)$$
$$+ \left(\sum_{i+j=2k} a_i b_j + c_{2k-1} + \sum_{i+j=2k} a_i b_j \theta \right) \left(\sum_{i=0}^{k+1} b^1_{2i+1} \sin(2i+1)\theta \right)$$
$$+ \left(\sum_{i+j+=2k} a_i b_j \right) \left(\sum_{i=0}^{k+1} b^1_{2i+2} \sin(2i+2)\theta) \right),$$

where a^l_{2i+1}, a^l_{2i+2}, b^l_{2i+1}, a^l_{2i+2}, c^l_{ij}, d^l_{ij}, e^l_{ij}, f^l_{ij} *are real constants for* $l = 1, 2$ *and* $k = 0, 1, \ldots, \dfrac{\lambda}{2}$.

Lemma 3.4.3. *The integral* $\displaystyle\int_0^{2\pi} \frac{1}{2} \frac{\partial^2 F_1}{\partial r^2}(s, r)(y_1(s, r))^2 ds$ *is the polynomial*

$$\pi(D_0 + D_1 r + D_2 r^2 + \ldots + D_\kappa r^\kappa) \tag{3.34}$$

where $\kappa = \begin{cases} n + 2m - 1 & \text{if } m > n+1 \text{ and } m \text{ or } n \text{ even}, \\ n + 2m - 2 & \text{if } m > n+1 \text{ and } m \text{ and } n \text{ odd}, \\ 3n + 1 & \text{if } m \le n+1 \text{ and } n \text{ even}, \\ 3n & \text{if } m \le n+1 \text{ and } n \text{ odd}, \end{cases}$

and

$$D_\chi = \sum_{i+j+k=\chi-1} \beta^1_{ijk} a_i a_j a_k + \sum_{i+j+k=\chi+1} \gamma^1_{ijk} a_i b_j b_k + \sum_{i+j+k=\chi} \delta^1_{ijk} a_i a_j b_k,$$

for $\chi = 0, 1, \ldots, \kappa$ *where* β^1_{ijk}, γ^1_{ijk}, δ^1_{ijk} *are real constants.*

Proof. We will denote

$$\frac{\partial^2 F_1}{\partial r^2}(s, r) = h_1(r) + h_2(r),$$

where

$$h_1(r) = \sum_{i=1}^{n} i(i+1) a_i r^{i-1} \cos^i \theta \sin^2 \theta,$$
$$h_2(r) = \sum_{i=2}^{m} i(i-2) b_i r^{i-2} \cos^i \theta \sin \theta,$$

and

$$(y_1(s,r))^2 = g_1^2(r) + 2g_1(r)g_2(r) + g_2^2(r),$$

with

$$g_1(r) = s_1(r) + s_2(r),$$

where

$$\begin{aligned} s_1(r) =& a_1r^2\left(\alpha_{11}\sin\theta + \alpha_{21}\sin(3\theta)\right) + \ldots \\ &+ a_l r^{l+1}\left(\alpha_{1l}\sin\theta + \alpha_{2l}\sin(3\theta) + \ldots + \alpha_{(\frac{l+3}{2})l}\sin((l+2)\theta\right), \\ s_2(r) =& a_0 r\left(\alpha_{10}\theta + \alpha_{20}\sin(2\theta)\right) + \ldots \\ &+ a_b r^{b+1}\left(\alpha_{1b}\theta + \alpha_{2b}\sin(2\theta) + \ldots + \alpha_{(\frac{b+4}{2})b}\sin(b+2)\theta\right), \end{aligned}$$

and

$$g_2(r) = \quad b_0(1-\cos\theta) + \ldots + b_m r^m\left(\frac{1}{m+1}(1-\cos^{m+1}\theta)\right).$$

Then

$$\begin{aligned} \frac{\partial^2 F_1}{\partial r^2}(s,r)\,(y_1(s,r))^2 =& h_1(r)\left(g_1^2(r) + 2g_1(r)g_2(r) + g_2^2(r)\right) \\ &+ h_2(r)\left(g_1^2(r) + 2g_1(r)g_2(r) + g_2^2(r)\right). \end{aligned}$$

From

$$\int_0^{2\pi}\cos^{2i}\theta\sin^2\theta\sin(\rho_1\theta)\sin(\rho_2\theta)d\theta = M_1(2i,\rho_1,\rho_2) \neq 0, \qquad \rho_1,\rho_2 \ \textit{odd},$$

$$\int_0^{2\pi}\cos^{2i+1}\theta\sin^2\theta\sin(\rho_1\theta)\sin(\rho_2\theta)d\theta = 0, \qquad \rho_1,\rho_2 \ \textit{odd},$$

for $i = 1, 2, \ldots$ we have that

$$\int_0^{2\pi} h_1(r)s_1(r)^2 d\theta = \sum_{\substack{k=1 \\ k \text{ odd}}}^{l}\ \sum_{\substack{j=1 \\ j \text{ odd}}}^{l}\ \sum_{\substack{i=2 \\ i \text{ even}}}^{b} \zeta^1_{ijk} a_i a_j a_k r^{i-1} r^{j+1} r^{k+1}$$

where $\zeta^1_{ijk} = \sum\limits_{\substack{\rho_1=1 \\ \rho \text{ odd}}}^{k+2}\ \sum\limits_{\substack{\rho'=1 \\ \rho_1 \text{ odd}}}^{j+2} \delta^{jk}_{\rho_1\rho_2} i(i+1)\alpha_{\frac{\rho_1+1}{2}j}\alpha_{\frac{\rho_2+1}{2}k} M_1(i,\rho_1,\rho_2)$, with

$$\delta^{jk}_{\rho_1\rho_2} = \begin{cases} 1 & \text{if } \rho_1 = \rho_2 \text{ and } j = k, \\ 2 & \text{if } \rho_1 \neq \rho_2 \text{ or } j \neq k. \end{cases}$$

Thus $H_1(r) = \int_0^{2\pi} h_1(r)s_1(r)^2 d\theta$ is a polynomial in r of degree $3n-1$ if n even, and $3n$ if n odd.

Knowing that

$$\int_0^{2\pi} \cos^i\theta \sin^2\theta \sin(\rho_1\theta)\theta d\theta = M_2(i,\rho_1,0) \neq 0, \qquad \rho_1 \text{ odd},$$

$$\int_0^{2\pi} \cos^{2i}\theta \sin^2\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta = 0, \qquad \rho_1 \text{ odd}, \rho_2 \text{ even},$$

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta = M_3(2i,\rho_1,\rho_2) \neq 0, \quad \rho_1 \text{ odd}, \rho_2 \text{ even},$$

for $i = 1, 2, \ldots$ we have that

$$\begin{aligned}\int_0^{2\pi} 2h_1(r)s_1(r)s_2(r)d\theta = & \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=1\\ j \text{ odd}}}^{l} \sum_{i=1}^{n} \zeta_{ijk}^2 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1} \\ & + \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=1\\ j \text{ odd}}}^{l} \sum_{\substack{i=1\\ i \text{ odd}}}^{l} \zeta_{ijk}^3 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1},\end{aligned}$$

where $\zeta_{ijk}^{\lambda} = \sum_{\substack{\rho_1=1\\ \rho_1 \text{ odd}}}^{k+2} \sum_{\substack{\rho_2=0\\ \rho_2 \text{ even}}}^{j+2} 2i(i+1)\alpha_{\frac{\rho_1+1}{2}j}\alpha_{\frac{\rho_2+2}{2}k} M_{\lambda}(i,\rho_1,\rho_2),\ \lambda = 2, 3.$

Thus the degree of the polynomial $H_2(r) = \int_0^{2\pi} 2h_1(r)s_1(r)s_2(r)d\theta$ in r is $3n$.

From

$$\int_0^{2\pi} \cos^i\theta (\sin^2\theta)\theta^2 d\theta = M_4(i,0,0) \neq 0,$$

$$\int_0^{2\pi} \cos^{2i}\theta \sin^2\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta = M_5(2i,\rho_1,\rho_2) \neq 0, \quad \rho_1,\ \rho_2 \text{ even},$$

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta = 0, \qquad \rho_1,\ \rho_2 \text{ even},$$

$$\int_0^{2\pi} \cos^i\theta \sin^2\theta \sin(\rho_1\theta)\theta d\theta = M_6(i,\rho_1,0) \neq 0, \qquad \rho_1 \text{ even},$$

for $i = 1, 2, \ldots$ we have that

$$\begin{aligned}\int_0^{2\pi} h_1(r)s_2^2(r)d\theta = & \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{i=1}^{n} \zeta_{ijk}^4 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1} \\ & + \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=1\\ j \text{ even}}}^{b} \sum_{\substack{i=2\\ i \text{ even}}}^{n} \zeta_{ijk}^5 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1}\end{aligned}$$

$$+\sum_{\substack{k=0\\k \text{ even}}}^{b}\sum_{\substack{j=0\\j \text{ even}}}^{b}\sum_{i=1}^{n}\zeta_{ijk}^{6}a_ia_ja_kr^{i-1}r^{j+1}r^{k+1},$$

where $\zeta_{ijk}^{\lambda} = \sum_{\substack{\rho_1=0\\\rho_1 \text{ even}}}^{k+2}\sum_{\substack{\rho_2=0\\\rho_2 \text{ even}}}^{j+2}\delta_{\rho_1\rho_2}^{jk}i(i+1)\alpha_{\frac{\rho_1+2}{2}j}\alpha_{\frac{\rho_2+2}{2}k}M_\lambda(i,\rho_1,\rho_2)$, $\lambda = 4,5,6$ with $\delta_{\rho_1\rho_2}^{jk}$ as above. Thus $H_3(r) = \int_0^{2\pi} h_1(r)s_2^2(r)d\theta$ is a polynomial in r of degree $3n+1$ if n even, and $3n-1$ if n odd.

Knowing that

$$\int_0^{2\pi}\cos^i\theta\sin^2\theta\sin(\rho_1\theta)d\theta = 0, \qquad \rho_1 = 1,2,\ldots$$
$$\int_0^{2\pi}\cos^{2i}\theta(\sin^2\theta)\theta d\theta = M_7(i,0,0) \neq 0,$$
$$\int_0^{2\pi}\cos^{2i+1}\theta(\sin^2\theta)\theta d\theta = 0,$$

for $i = 1,2,\ldots$ we have that

$$\int_0^{2\pi}h_1(r)(s_1(r)+s_2(r))g_2(r)d\theta = \sum_{k=0}^{m}\sum_{\substack{j=0\\j \text{ even}}}^{b}\sum_{i=1}^{n}\zeta_{ijk}^{7}a_ia_jb_kr^{i-1}r^{j+1}r^k,$$

where $k+i$ is odd, and $\zeta_{ijk}^{7} = i(i+1)\alpha_{1j}M_7(i,0,0)$. Thus $H_4(r) = \int_0^{2\pi} h_1(r)(s_1(r) + s_2(r))g_2(r)d\theta$ is a polynomial in r of degree $2n+m-1$ if m even, $2n+m$ if n even, m odd, and $2n+m-2$ if n, m odd.

The equalities

$$\int_0^{2\pi}\cos^{2i}\theta\sin^2\theta d\theta = M_8(i,0,0) \neq 0,$$
$$\int_0^{2\pi}\cos^{2i+1}\theta\sin^2\theta d\theta = 0,$$

for $i = 1,2,\ldots$ imply

$$\int_0^{2\pi}h_1(r)g_2^2(r)d\theta = \sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{i=1}^{n}\zeta_{ijk}^{8}a_ib_jb_kr^{i-1}r^jr^k,$$

where $\zeta_{ijk}^{8} = \delta_{jk}i(i+1)M_8(i,0,0)$ with $\delta_{jk} = \begin{cases}1 & \text{if } j = k,\\ 2 & \text{if } j \neq k.\end{cases}$

Thus $H_5(r) = \displaystyle\int_0^{2\pi} h_1(r) g_2^2(r) d\theta$ is a polynomial in r of degree $2m+n-1$ if n or m even, and $2m+n-2$ if n and m odd.

From

$$\int_0^{2\pi} \cos^i\theta \sin\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta = 0, \qquad \rho_1,\ \rho_2 \text{ odd}$$

for $i = 1, 2, \ldots$ we have that

$$H_6(r) = \int_0^{2\pi} h_2(r) s_1^2(r) d\theta = 0.$$

From the values of the integrals

$$\begin{aligned}
\int_0^{2\pi} \cos^{2i}\theta (\sin\theta)\theta \sin(\rho_1\theta) d\theta &= M_9(i, \rho_1, 0) \neq 0, && \rho_1 \text{ odd},\\
\int_0^{2\pi} \cos^{2i+1}\theta (\sin\theta)\theta \sin(\rho_1\theta) d\theta &= 0, && \rho_1 \text{ odd},\\
\int_0^{2\pi} \cos^{i}\theta \sin\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta &= 0, && \rho_1 \text{ even},\ \rho_2 \text{ odd},
\end{aligned}$$

for $i = 1, 2, \ldots$ we have that

$$\int_0^{2\pi} h_2(r) s_1(r) s_2(r) d\theta = \sum_{\substack{k=1\\ k \text{ odd}}}^{l} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{\substack{i=2\\ i \text{ even}}}^{m} \zeta_{ijk}^9 b_i a_j a_k r^{i-2} r^{j+1} r^{k+1},$$

where $\zeta_{ijk}^9 = \displaystyle\sum_{\substack{\rho_1=1\\ \rho_1 \text{ odd}}}^{l+2} i(i-1)\alpha_{1j}\alpha_{\frac{\rho_1+1}{2}k} M_9(i, \rho_1, 0)$.

Thus $H_7(r) = \displaystyle\int_0^{2\pi} h_2(r) s_1(r) s_2(r) d\theta$ is a polynomial in r of degree $2n+m-1$ if m even and $2m+n-2$ if m odd.

The formulas

$$\begin{aligned}
\int_0^{2\pi} \cos^{i}\theta (\sin\theta)\theta^2 d\theta &= M_{10}(i, 0, 0) \neq 0, &&\\
\int_0^{2\pi} \cos^{2i}\theta (\sin\theta)\theta \sin(\rho_1\theta) d\theta &= 0, && \rho_1 \text{ even},\\
\int_0^{2\pi} \cos^{2i+1}\theta (\sin\theta)\theta \sin(\rho_1\theta) d\theta &= M_{11}(i, \rho_1, 0) \neq 0, && \rho_1 \text{ even},\\
\int_0^{2\pi} \cos^{i}\theta \sin\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta &= 0, && \rho_1,\ \rho_2 \text{ odd},
\end{aligned}$$

for $i=1,2,\dots$ imply

$$\int_0^{2\pi} h_2(r)s_2^2(r)d\theta = \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{i=1}^{m} \zeta_{ijk}^{10} b_i a_j a_k r^{i-2} r^{j+1} r^{k+1} + \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{\substack{i=1\\ i \text{ odd}}}^{m} \zeta_{ijk}^{11} b_i a_j a_k r^{i-2} r^{j+1} r^{k+1},$$

where

$$\zeta_{ijk}^{10} = \delta_{jk}^1 i(i-1)\alpha_{1j}\alpha_{1k} M_{10}(i,\rho_1,0),$$
$$\zeta_{ijk}^{11} = \sum_{\substack{\rho_1=1\\ \rho_1 \text{ even}}}^{b+2} \delta_{jk\rho_1}^2 i(i-1)\alpha_{1j}\alpha_{\frac{\rho_1+2}{2}k} M_{11}(i,\rho_1,0),$$

with $\delta_{jk}^1 = \begin{cases} 1 & \text{if } j=k, \\ 2 & \text{if } j\neq k, \end{cases} \quad \delta_{jk\rho_1}^2 = \begin{cases} 1 & \text{if } j=k,\ \rho_1=0, \\ 2 & \text{if } j\neq k,\ \rho_1\neq 0. \end{cases}$.
Thus $H_8(r) = \displaystyle\int_0^{2\pi} h_2(r)s_2^2(r)d\theta$ is a polynomial in r of degree $m+2n$ if n even, and $m+2n-2$ if n odd.

From

$$\int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin(\rho_1\theta)d\theta = M_{12}(i,\rho_1,0) \neq 0, \qquad \rho_1 \text{ odd},$$
$$\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin(\rho_1\theta)d\theta = 0, \qquad \rho_1 \text{ odd},$$
$$\int_0^{2\pi} \cos^{i}\theta (\sin\theta)\theta d\theta = M_{13}(i,0,0) \neq 0,$$
$$\int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin(\rho_1\theta)d\theta = M_{14}(i,\rho_1,0) \neq 0, \qquad \rho_1 \text{ even},$$
$$\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin(\rho_1\theta)d\theta = 0, \qquad \rho_1 \text{ even},$$

for $i=1,2,\dots$ we have that

$$\int_0^{2\pi} h_2(r)(s_1(r)+s_2(r))g_2(r)d\theta = \sum_{k=0}^{m} \sum_{\substack{j=1\\ j \text{ odd}}}^{l} \sum_{i=1}^{m} \zeta_{ijk}^{12} b_i a_j b_k r^{i-2} r^{j+1} r^{k}$$

$$+\sum_{k=0}^{m}\sum_{\substack{j=0\\ j \text{ even}}}^{b}\sum_{i=1}^{m}\zeta_{ijk}^{13} b_i a_j b_k r^{i-2} r^{j+1} r^{k}$$
$$+\sum_{k=0}^{m}\sum_{\substack{j=1\\ j \text{ even}}}^{l}\sum_{i=1}^{m}\zeta_{ijk}^{14} b_i a_j b_k r^{i-2} r^{j+1} r^{k},$$

where

$$\zeta_{ijk}^{12} = \begin{cases} \displaystyle\sum_{\substack{\rho_1=1\\ \rho_1 \text{ odd}}}^{j+2} \frac{i(i-1)}{k+2}\alpha_{\frac{\rho+1}{2}j} M_{12}(i,\rho_1,0) & \text{for } k+i \text{ even},\\ 0 & \text{for } k+i \text{ odd},\end{cases}$$

$$\zeta_{ijk}^{13} = \frac{i(i-1)}{k+1}\alpha_{1j} M_{13}(i,0,0),$$

$$\zeta_{ijk}^{14} = \begin{cases} \displaystyle\sum_{\substack{\rho_1=0\\ \rho_1 \text{ even}}}^{j+2} \frac{i(i-1)}{k+2}\alpha_{\frac{\rho_1+2}{2}j} M_{14}(i,\rho_1,0) & \text{for } k+i \text{ even},\\ 0 & \text{for } k+i \text{ odd}.\end{cases}$$

Thus $H_9(r) = \int_0^{2\pi} h_2(r)(s_1(r) + s_2(r))g_2(r)d\theta$ is a polynomial in r of degree $2m+n-1$ if n even, and $2m+n-2$ if n odd.

From the value of the integral

$$\int_0^{2\pi} \cos^i\theta \sin\theta d\theta = 0,$$

for $i = 1, 2, \ldots$ we have that

$$H_{10}(r) = \int_0^{2\pi} h_2(r) g_2^2(r) d\theta = 0.$$

We conclude that $\int_0^{2\pi} \frac{1}{2}\frac{\partial^2 F_1}{\partial r^2}(s,r)(y_1(s,r))^2 ds = \sum_{i=1}^{10} H_i$ whose degree is the greatest of the degrees of H_i. Hence the proof of the lemma follows. □

The proofs of the next three lemmas follow in a similar way to the previous one. They will be omitted.

Lemma 3.4.4. *The integral* $\displaystyle\int_0^{2\pi} \frac{1}{2}\frac{\partial F_1}{\partial r}(s,r)(y_2(s,r))ds$ *is the polynomial*

$$\frac{\pi}{r}(E_0 + E_1 r + E_2 r^2 + \ldots + E_\vartheta r^\vartheta), \tag{3.35}$$

where $\vartheta = \begin{cases} n+2m & \text{if } m > n+1 \text{ and } n \text{ even},\\ n+2m-1 & \text{if } m > n+1 \text{ and } n \text{ odd},\\ 3n+2 & \text{if } m \le n+1 \text{ and } n \text{ even},\\ 3n+1 & \text{if } m \le n+1 \text{ and } n \text{ odd}, \end{cases}$

and

$$\begin{aligned} E_{2l+1} = & \sum_{i+j+k=2l-1} \beta_{ijk}^2 a_i a_j a_k + \sum_{i+j+k=2l+1} \gamma_{ijk}^2 a_i b_j b_k + \sum_{i+j=2l} \delta_{ij}^2 b_i c_j \\ & + \sum_{i+j=2l} \eta_{ij}^2 a_i d_j + \sum_{\substack{i+j+k=2l \\ i \text{ even}}} \upsilon_{ijk}^2 a_i a_j b_k \pi, \end{aligned}$$

$$\begin{aligned} E_{2l} = & \sum_{i+j+k=2l-2} \beta_{ijk}^2 a_i a_j a_k + \sum_{i+j+k=2l} \gamma_{ijk}^2 a_i b_j b_k + \sum_{i+j=2l-1} \delta_{ij}^2 b_i c_j \\ & + \sum_{i+j=2l-1} \eta_{ij}^2 a_i d_j + \sum_{\substack{i+j+k=2l-1 \\ i \text{ even}}} \upsilon_{ijk}^2 a_i a_j b_k \pi + \sum_{\substack{i+j=2l-2 \\ i \text{ even}}} \varsigma_{ij}^2 a_i c_j \pi, \end{aligned}$$

for $l = 0, 1, \ldots, \dfrac{\vartheta}{2}$, *where* β_{ijk}^2, γ_{ijk}^2, δ_{ij}^2, η_{ij}^2, υ_{ijk}^2, ς_{ij}^2 *are real constants.*

Lemma 3.4.5. *The integral* $\displaystyle\int_0^{2\pi} \frac{1}{2}\frac{\partial F_2}{\partial r}(s,r)(y_1(s,r))ds$ *is the polynomial*

$$\frac{\pi}{r}(F_0 + F_1 r + F_2 r^2 + \ldots + F_\nu r^\nu), \tag{3.36}$$

where $\nu = \begin{cases} n+2m & \text{if } m > n+1 \text{ and } n \text{ even},\\ n+2m-1 & \text{if } m > n+1 \text{ and } n \text{ odd},\\ 3n+2 & \text{if } m \le n+1 \text{ and } n \text{ even},\\ 3n+1 & \text{if } m \le n+1 \text{ and } n \text{ odd}, \end{cases}$

and

$$F_{2l+1} = \sum_{i+j+k=2l-1} \beta^3_{ijk} a_i a_j a_k + \sum_{i+j+k=2l+1} \gamma^3_{ijk} a_i b_j b_k + \sum_{i+j=2l} \delta^3_{ij} b_i c_j + \sum_{i+j=2l} \eta^3_{ij} a_i d_j,$$

$$F_{2l} = \sum_{i+j+k=2l-2} \beta^3_{ijk} a_i a_j a_k + \sum_{i+j+k=2l} \gamma^3_{ijk} a_i b_j b_k + \sum_{i+j=2l-1} \delta^3_{ij} b_i c_j + \sum_{i+j=2l-1} \eta^3_{ij} a_i d_j + \sum_{\substack{i+j+k=2l-1 \\ i\ even}} \upsilon^3_{ijk} a_i a_j b_k \pi + \sum_{\substack{i+j+=2l-2 \\ i\ even}} \varsigma^3_{ij} a_i c_j \pi,$$

for $l = 0, 1, \ldots, \dfrac{\nu}{2}$, *where* β^3_{ijk}, γ^3_{ijk}, δ^3_{ij}, η^3_{ij}, υ^3_{ijk}, ς^3_{ij} *are real constants.*

Lemma 3.4.6. *The integral* $\displaystyle\int_0^{2\pi} F_3(s,r)ds$ *is the polynomial*

$$\frac{\pi}{r}(G_0 + G_2 r^2 + \ldots + G_\psi r^\psi), \tag{3.37}$$

where $\psi = \begin{cases} n+2m & \text{if } m > n+1 \text{ and } n \text{ even}, \\ n+2m-1 & \text{if } m > n+1 \text{ and } n \text{ odd}, \\ 3n+2 & \text{if } m \le n+1 \text{ and } n \text{ even}, \\ 3n+1 & \text{if } m \le n+1 \text{ and } n \text{ odd}, \end{cases}$

and

$$G_{2l} = \sum_{i+j+k=2l-2} \beta^4_{ijk} a_i a_j a_k + \sum_{i+j+k=2l} \gamma^4_{ijk} a_i b_j b_k + \sum_{i+j=2l-1} \delta^4_{ij} b_i c_j + \sum_{i+j=2l-1} \eta^4_{ij} a_i d_j + p_{2l-2},$$

for $l = 0, 1, \ldots, \dfrac{\psi}{2}$, *where* β^4_{ijk}, γ^4_{ijk}, δ^4_{ij}, η^2_{ij}, υ^4_{ijk} *are real constants.*

By Lemmas 3.4.3, 3.4.4, 3.4.5 and 3.4.6 we obtain

$$F_{30}(r) = \frac{\alpha}{r}\left(M_0 + M_1 r + M_2 r^2 + M_3 r^3 + M_4 r^4 + \ldots + M_{\varrho-1} r^{\varrho-1} + M_\varrho r^\varrho\right),$$

where

$$M_{2l+1} = \sum_{i+j+k=2l-1} \beta_{ijk} a_i a_j a_k + \sum_{i+j+k=2l+1} \gamma_{ijk} a_i b_j b_k + \sum_{i+j=2l} \delta_{ij} b_i c_j$$

$$+ \sum_{i+j=2l} \eta_{ij} a_i d_j + \sum_{\substack{i+j=2l \\ i \text{ even}}} \nu_{ij} a_i a_j b_k \pi,$$

$$M_{2l} = \sum_{i+j+k=2l} \beta_{ijk} a_i b_j b_k + \sum_{i+j+k=2l-2} \gamma_{ijk} a_i a_j a_k + \sum_{i+j=2l-1} \delta_{ij} b_i c_j$$
$$+ \sum_{i+j=2l-1} \eta_{ij} a_i d_j + \sum_{i+j+k=2l-2} \mu_{ijk} a_i a_j a_k + \varpi_{2l-2} p_{2l-2}$$
$$+ \left(\sum_{\substack{i+j+k=2l-1 \\ i \text{ even}}} \nu_{ijk} a_i a_j b_k + \sum_{\substack{i+j=2l-2 \\ i \text{ even}}} \rho_{ijk} a_i c_j \right) \pi$$
$$+ \sum_{\substack{i+j+k=2l-2 \\ i \text{ even}}} \tau_{ijk} a_i a_j a_k \pi^2,$$

for $l = 0, 1, 2, \ldots \dfrac{\varrho}{2}$ and

$$\varrho = \begin{cases} n + 2m & \text{if } m > n+1 \text{ and n even}, \\ n + 2m - 1 & \text{if } m > n+1 \text{ and n odd}, \\ 3n + 2 & \text{if } m \leq n+1 \text{ and n even}, \\ 3n + 1 & \text{if } m \leq n+1 \text{ and n odd}. \end{cases}$$

Applying the equalities $a_i = 0$, for all i even and (3.33), we obtain that $M_0 = 0$ and $M_\kappa = 0$ for κ odd. Moreover from (3.33) we obtain $c_k = \sum_{\substack{i+j=k+1 \\ i \text{ odd} \\ j \text{ even}}} a_i b_j = 0$ for $k > b$. Then $M_k = 0$ for k greater than

$$\lambda = \begin{cases} n + m - 2 & \text{if n, m odd}, \\ n + m - 1 & \text{if n odd, m even}, \\ n + m - 2 & \text{if n, m even}, \\ n + m - 1 & \text{if n even, m odd}. \end{cases}$$

Thus

$$F_{30}(r) = \alpha r \left(M_2 + M_4 r^2 + M_6 r^4 + \ldots + M_{\lambda-4} r^{\lambda-2} + M_{\lambda-2} r^{\lambda} \right)$$

where

$$M_\omega = \sum_{\substack{i+j+k=\omega \\ i \text{ odd} \\ j \text{ even} \\ k \text{ odd}}} \beta'_{ijk} a_i b_j b_k + \sum_{\substack{i+j=\omega-1 \\ i \text{ even} \\ j \text{ odd}}} \delta'_{ij} b_i c_j + \sum_{\substack{i+j=\omega-1 \\ i \text{ odd} \\ j \text{ even}}} \eta'_{ij} a_i d_j \;+\; \varpi_\omega p_{\omega-2}.$$

Consequently $F_3(z)$ is a polynomial of degree λ in the variable r^2. Then $F_3(z)$ has at most $\left[\dfrac{n+m-1}{2}\right]$ positive roots, and from the third order averaging method we conclude that this is the maximum number of limit cycles of the polynomial Liénard differential systems (3.25) with $k = 3$ bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$. This completes the proof of statement (c) of Theorem 3.4.1. □

Chapter 4

On the 16–th Hilbert problem

In this chapter we present a brief survey on the 16–th Hilbert problem . The notes of this chapter are based in the paper [74].

We consider differential equations in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{4.1}$$

where P and Q are polynomials of degree at most d. We recall that a *limit cycle* of the differential equation (4.1) is a periodic orbit of this equation isolated in the set of all periodic orbits of equation (4.1).

The notion of limit cycle appears in the years 1891 and 1897 in the works of Poincaré [103]. Moreover, he proved that a polynomial differential equation (4.1) without saddle connections has finitely many limit cycles, see [103].

Hilbert [53] at the Second International Congress of Mathematicians, celebrated in Paris in 1900, proposed a list of 23 relevant problems for being solved during the XX century. The 16–th problem of the list is:

16. *Problem of the topology of algebraic curves and surfaces*

The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. As to curves of the 6th order, I have satisfied myself–by a complicated process, it is true–that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the *maximum number of sheets which a surface of the 4th order in three dimensional space can really have.*

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X and Y are rational integral functions of the nth degree in x and y. Written homogeneously, this is

$$X\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + Y\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + Z\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0,$$

where X, Y, and Z are rational integral homogeneous functions of the nth degree in x, y, z, and the latter are to be determined as functions of the parameter t.

It is clear that the 16–th Hilbert problem is formulated in two parts. The first part is about the mutual disposition of the maximal number of separate branches of an algebraic curve, and its extension to nonsingular real algebraic varieties. In the second part he asked for the maximal number and relative position of the limit cycles of the differential system (4.1). Usually the first part of the 16–th Hilbert problem is studied by researchers in real algebraic geometry, while the second part is considered by mathematicians working in dynamical systems or differential equations. Hilbert also pointed out that there exist possible connections between these two parts. Some of these connections are described in the survey about the 16–th Hilbert problem written by Jibin Li, see [72].

In what follows when we talk about the 16–th Hilbert problem we always are talking on the second part of the 16–th Hilbert problem.

In 1988 Noel Lloyd [85] observed with respect the 16–th Hilbert problem that *the striking aspect is that the hypothesis is algebraic, while the conclusion is topological.*

Arnold in 1977 and 1983 (see [1] and [2], respectively) stated the *weakened, infinitesimal or tangential 16–th Hilbert problem* which we do not consider here, but there are excellent surveys for this modified problem, see for instance the survey of Ilyashenko [59] on the 16–th Hilbert problem, the already mentioned survey of Jibin Li, or the book of Colin Christopher and Chengzhi Li [25], or the survey due to Kaloshin [64], or the one of Yakovenko [122], or more recently the work of Binyamini, Novikov and Yakovenko [9], ...

According with Smale [113] except for the Riemann hypothesis, the second part of the 16–th Hilbert problem seems to be the most elusive of the Hilbert's problems. Smale states the following modern version of the second half of 16–th Hilbert problem:

Consider the polynomial differential equation (4.1) *in* $\mathbb{R}^2$*. Is there a bound* K *on the number of limit cycles of the form* $K \leq d^q$ *where* d *is the maximum of the degrees of* P *and* Q*, and* q *is a universal constant*?

The possible distribution or topological configurations of limit cycles mentioned as position for Hilbert has also interested to many authors. Coleman in his work [30] on the 16–Hilbert problem said: *For* $d > 2$ *the maximal number of eyes is not known, nor is it known just which complex patterns of eyes within eyes, or eyes enclosing more than a single critical point, can exist.* Here "eye" means a nest of limit cycles. We shall see later on that some of the questions on the possible topological configurations of limit cycles realized by polynomial differential equations can be solved easily.

Another problem very related with the 16–th Hilbert problem is the study of the possible bifurcations of limit cycles. Again this problem will not be considered here, see good information about it in the survey of Jibin Li, or the books of Christopher and Chengzhi Li, Yankian Ye [123], Zhifen Zang et al. [125], ...

Our approach to the 16–Hilbert problem is done through the following seven problems:

Problem 1: *Is it true that a polynomial differential equation* (4.1) *has a finite number of limit cycles* ?

Problem 2: *Is it true that the number of limit cycles of a polynomial differential equation* (4.1) *is bounded by a constant depending only on the degree of the polynomials* ?

If the problem 2 has a positive answer then its bound is denoted by $H(d)$, and called the *Hilbert number* for the polynomial differential equations (4.1) of degree d.

Problem 3: *If the problem 2 has a positive answer, provide an upper bound for* $H(d)$*.*

Smale [113] in 1998 said that the 16–Hilbert problem looks very difficult, and that first we must consider a special class of simpler polynomial differential equations, and he propose to study the 16–Hilbert problem restricted to the *Liénard polynomial differential equations*, i.e. to the polynomial differential equations of the form

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{4.2}$$

where $F(x)$ is a polynomial in the variable x of degree d.

Problem 4: *What about the problems* 2 *and* 3 *if we restrict the study to the Liénard polynomial differential equations* (4.2) ?

For the Liénard polynomial differential equations we do not talk about the problem 1 because as we shall see the problem 1 has been solved in positively for all polynomial differential equations (4.1).

Problem 5: *What are the possible topological configurations of limit cycles for the polynomial differential equations* (4.1) *?*

An *algebraic limit cycle* is an oval of an algebraic curve which is a limit cycle of a polynomial differential equation (4.1).

Problem 6: *Is it true that the number of algebraic limit cycles of a polynomial differential equation* (4.1) *is bounded by a constant depending only on the degree of the polynomials* ?

If the problem 6 has a positive answer then its bound is denoted by $H^a(d)$, and we called it the *algebraic Hilbert number* for the polynomial differential equation (4.1) of degree d.

Problem 7: *If the problem 6 has a positive answer, provide an upper bound for* $H^a(d)$.

The first four problems have considered by several authors, see for instance the surveys of Ilyashenko and of Jibin Li. Here, we pass fast for these first four problems, and we shall dedicate more space to the last three problems which as far as we know there has not been considered for the moment in any other survey.

4.1 Problem 1

Dulac [36] in 1923 claimed that any polynomial differential equation (4.1) always has finitely many limit cycles. Ilyashenko [57] in 1985 found an error in Dulac's paper. Later on, two long works have appeared, independently, providing proofs of Dulac's assertion, one due to Écalle [41] in 1992 and the other to Ilyashenko [58] in 1991. As Smale mentioned in [113] these two papers have yet to be thoroughly digested by the mathematical community.

Bamon [6] in 1986 proved that any polynomial differential equation of degree 2 has finitely many limit cycles. His result uses previous results of Ilyashenko.

From the work of Dulac [36] it follows that if a polynomial differential equation (4.1) has all its saddle connections forming a simple homoclinic or heteroclinic loop, then also the equation has finitely many limit cycles, see for more details the nice work of Sotomayor [114]. Here a *homoclinic* or *heteroclinic loop* is formed by $k = 1$ or $k > 1$ saddles (eventually some saddles can be repeated) and k different separatrices connecting these saddles and forming a loop (eventually some points of this loop can be identified in a repeated saddle) in such a way that at least in one of the two sides of the loop is defined a Poincaré return map. Let $\mu_i < 0 < \lambda_i$ the eigenvalues of these saddles, if

$$\prod_{i=1}^{k} \frac{\lambda_i}{\mu_i} \neq 1,$$

then the loop is called *simple*.

4.2 Problem 2

Since the polynomial differential equations of degree 1 or linear differential equations have no limit cycles, it follows that the Hilbert number $H(1) = 0$.

Unfortunately we do not know if an uniform upper bound for the maximum number of limit cycles exists for all polynomial vector fields of degree d if $d \geq 2$.

4.3 Problem 3

Since problem 2 remains open for degree $d \geq 2$, we do not know if the Hilbert number $H(d)$ exist for $d \geq 2$.

In 1957 Petrovskii and Landis [101] claimed that the polynomial differential equations of degree $d = 2$ has at most 3 limit cycles, i.e. that the Hilbert number $H(2) = 3$. Soon (in 1959) a gap was found in the arguments of Petrovskii and Landis see [102]. Later on Lan Sun Chen and Ming Shu Wang [20] in 1979 and Songling Shi [111] in 1982 provided the first polynomial differential equations of degree 2 having 4 limit cycles, and consequently showing that $H(2) \geq 4$.

Some lower bounds for $H(d)$ have been given, mainly by Christopher and Lloyd [26] and Jibin Li, see the survey of this last author who analyze these lower bounds.

4.4 Problem 4

The study of Liénard differential equations (not necessarily polynomial) has a long history and a lot of results were obtained on them, see for example the book [125].

If $F(x) = x^3 - x$ then the Liénard differential equation (4.2) is the famous van der Pol's equation which has at most one limit cycle.

Van der Pol in 1926, Liénard in 1928 and Andronov in 1929 proved that the periodic solution of a self–sustained oscillation in a vacuum tube was a limit cycle in the sense defined by Poincaré. After this observation of the existence of a limit cycle in the nature, the existence, non–existence, uniqueness and other properties of the limit cycles have been intensively studied not only by the mathematicians, which were already motivated by the works of Poincaré and Hilbert, also by the physiciens, and later on by the chemists, biologists, economists, and many others. The limit cycles started to be important in the sciences.

For the Liénard polynomial differential equations (4.2) of degree d the existence of a uniform bound for the maximum number of limit cycles also remains unproved. But when the degree d of these systems is odd Ilyashenko and Panov in [60] obtained an uniform upper bound for the number of limit cycles in a subclass of systems such that the polynomial $F(x)$ is monic and its coefficients satisfy some estimations.

In 1977 Lins, de Melo and Pugh conjectured in [71] that the Liénard polynomial differential equation (4.2) of degree $d \geq 3$ has at most $[(d-1)/2]$ limit cycles, where $[(d-1)/2]$ means the largest integer less than or equal to $(d-1)/2$. Moreover, they provide Liénard polynomial differential equations (4.2) for any degree $d \geq 3$ having at least $[(d-1)/2]$ limit cycles. They also proved that the conjecture is true for $d = 3$. It is not difficult to show that their conjecture also holds for the degrees $d = 1, 2$.

In 2007 Dumortier, Panazzolo and Roussarie [39] gave a counterexample to this conjecture for $d = 7$ and mentioned that it can be extended to $d \geq 7$ odd. Recently, de Maesschalck and Dumortier proved in [33] that the Liénard polynomial differential equation (4.2) of degree $d \geq 6$ can have $[(d-1)/2]+2$ limit cycles. In the last two papers the results are proved using singular perturbation theory, and the authors work with relaxation oscillation solutions to study the number of limit cycles.

Chengzhi Li and Llibre [87] shows in 2012 that the Lins–de Melo–Pugh's conjecture is true for the Liénard polynomial differential equations of degree $d = 4$. So at this moment only remains open the conjecture for degree $d = 5$.

4.5 Problem 5

A *topological configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Given a topological configuration of limit cycles $C = \{C_1, \ldots, C_n\}$ the curve C_i is *primary* if there is no curve C_j of C contained into the bounded region limited by C_i.

Two topological configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are (*topologically*) *equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^n C_i\right) = \left(\cup_{i=1}^m C'_i\right)$. Of course, for equivalent configurations of limit cycles C and C' we have that $n = m$.

We say that a polynomial differential equation (4.1) *realizes* the configuration of limit cycles C if the set of all limit cycles of (4.1) is equivalent to C.

In 2004 Llibre and Rodríguez [81] proved the following result.

Theorem 4.5.1. *Let $C = \{C_1, \ldots, C_n\}$ be a topological configuration of limit cycles, and let r be its number of primary curves. Then the following statements hold.*

(a) The configuration C is realizable by some polynomial differential equation.

(b) The configuration C is realizable as algebraic limit cycles by a polynomial differential equation of degree $\leq 2(n+r)-1$. Moreover, such a polynomial differential equation has a first integral of Darboux type.

Of course, statement (a) of Theorem 4.5.1 follows immediately from statement (b).

Statement (a) of Theorem 4.5.1 was solved by first time by Schecter and Singer [109] and Sverdlove [116], but they do not provide an explicit polynomial vector field satisfying the given configuration of limit cycles, as it was provided in the proof of statement (b) of Theorem 4.5.1.

If $f = f(x, y)$ is a polynomial we denote its partial derivatives with respect to the variables x and y as f_x and f_y, respectively. Christopher [24] in 2001 proved the following result.

Theorem 4.5.2. *Let $f = 0$ be a non–singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the straight line $D = 0$ lies outside all bounded components of $f = 0$. Choose the constants α and β so that $\alpha D_x + \beta D_y \neq 0$, then the polynomial differential equation of degree m,*

$$\dot{x} = \alpha f - D f_y, \qquad \dot{y} = \beta f + D f_x,$$

has all the bounded components of $f = 0$ as hyperbolic limit cycles. Furthermore, the differential equation has no other limit cycles.

Theorem 4.5.2 improves a similar result due to Winkel [121], but the polynomial differential equation obtained by Winkel has degree $2m - 1$.

Given a topological configuration of n limit cycles we can consider an equivalent topological configuration formed by circles. Then, consider the algebraic curve $f = 0$ formed by the product of all the circles. Applying Theorem 4.5.2 to the curve $f = 0$, we obtain a polynomial differential equation of degree $2n$ which realizes the given topological configuration of n limit cycles with algebraic limit cycles. A difference between the polynomial differential equations of Theorems 4.5.1 and 4.5.2, is that the first always has a first integral, and the second, in general, has no first integrals.

In short, both theorems show that any topological configuration of limit cycles is realizable with algebraic limit cycles for some polynomial differential equation, and provide the degree of such polynomial differential equations. But there are many questions which remains open, as for instance: *what are the possible topological configurations of limit cycles realizable for the polynomial differential equations of a given degree*? Of course this question is strongly more difficult than the question to provide a uniform upper bound for the maximum number of limit cycles that the polynomial differential equations of a given degree can have.

4.6 Problems 6 and 7

Associated to the polynomial differential equation (4.1) there is the *polynomial vector field*

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{4.3}$$

The algebraic curve $f(x,y)=0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the polynomial vector field $\mathcal{X}$ or of the polynomial differential equation (4.1) if for some polynomial $K \in \mathbb{R}[x,y]$ we have

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{4.4}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f=0$.

Since on the points of an algebraic curve $f=0$ the gradient $(\partial f/\partial x, \partial f/\partial y)$ of the curve is orthogonal to the vector field $\mathcal{X}$ (see (4.4)), the vector field $\mathcal{X}$ is tangent to the curve $f=0$. Hence the curve $f=0$ is formed by orbits of the vector field $\mathcal{X}$. This justifies the name of invariant algebraic curve given to the algebraic curve $f=0$ satisfying (4.4) for some polynomial K, because it is *invariant* under the flow defined by the vector field $\mathcal{X}$.

The next well known result tell us that we can restrict our attention to the irreducible invariant algebraic curves, for a proof see for instance [73]. Here, as it is usual, $\mathbb{R}[x,y]$ denotes the ring of all polynomials in the variables x and y and coefficients in $\mathbb{R}$.

Proposition 4.6.1. *We suppose that $f \in \mathbb{R}[x,y]$ and let $f = f_1^{n_1}\cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{R}[x,y]$. Then for a polynomial vector field $\mathcal{X}$, $f=0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i=0$ is an invariant algebraic curve for each $i=1,\ldots,r$ with cofactor K_{f_i}. Moreover $K_f = n_1K_{f_1}+\ldots+n_rK_{f_r}$.*

Consider the space Σ' of all real polynomial vector fields (4.4) of degree d having real irreducible invariant algebraic curves. A simpler version of the second part of the 16th Hilbert's problem is: *Is there an uniform upper bound for the maximal number of algebraic limit cycles of any polynomial vector field of* Σ'? Now we cannot provide an answer to this question for general real algebraic curves, but we give the answer for the following class of algebraic curves.

We say that a set $f_j=0$, for $j=1,\ldots,k$, of irreducible algebraic curves is *generic* if it satisfies the following five conditions:

(i) There are no points at which $f_j=0$ and its first derivatives all vanish (i.e. $f_j=0$ is a non–singular algebraic curve).

(ii) The highest order homogeneous terms of f_j have no repeated factors.

(iii) If two curves intersect at a point in the affine plane, they are transversal at this point.

(iv) There are no more than two curves $f_j=0$ meeting at any point in the affine plane.

(v) There are no two curves having a common factor in the highest order homogeneous terms.

The next result was proved by Llibre, Ramírez and Sadovskaia [78] in 2010.

Theorem 4.6.2. *For a polynomial vector field $\mathcal{X}$ of degree $d \geq 2$ having all its irreducible invariant algebraic curves generic, the maximum number of algebraic limit cycles is at most $1 + (d-1)(d-2)/2$ if d is even, and $(d-1)(d-2)/2$ if d is odd. Moreover these upper bounds are reached.*

For cubic polynomial vector fields having all their irreducible invariant algebraic curves generic Theorem 4.6.2 says that one is the maximum number of algebraic limit cycles, but there are examples of cubic polynomial vector fields having two algebraic limit cycles, of course such vector fields have non–generic invariant algebraic curves. Thus the polynomial differential equation of degree 3

$$\dot{x} = 2y(10 + xy), \quad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3,$$

has two algebraic limit cycles contained into the invariant algebraic curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$, see Proposition 19 of [84].

Up to now all the polynomial vector fields having non–generic invariant algebraic curves and more algebraic limit cycles than the upper bounds given in Theorem 4.6.2 for the generic case have degree odd, and at most one limit cycle than the upper bound of Theorem 4.6.2. So, in [78] we did the following conjecture.

Conjecture 1. *The algebraic Hilbert number is*

$$H^a(d) = 1 + (d-1)(d-2)/2.$$

The easiest version of this conjecture is it restriction to the polynomial vector fields of degree 2.

Conjecture 2. $H^a(2) = 1$.

Note that both conjectures are true when d is even and we restrict the algebraic limit cycles to generic invariant algebraic curves.

An interesting result on the limit cycles of a C^1 differential equation in the plane is the following one due to Giacomini, Llibre and Viano [50], see an easier proof in [81]. This result has been used in the proofs of Theorems 4.5.1 and 4.6.2. First we need a definition.

Let U be an open subset of $\mathbb{R}^2$. A function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of a C^1 vector field $\mathcal{X}$ defined on U if V verifies the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V$$

in U.

Theorem 4.6.3. *Let X be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let $V : U \to \mathbb{R}$ be an inverse integrating factor of X. If γ is a limit cycle of X, then γ is contained in $\{(x, y) \in U : V(x, y) = 0\}$.*

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