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PERIODS OF HOMEOMORPHISMS ON CLOSED SURFACES

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Abstract. The goal of this paper is to show what information on the set of periodic points of a homeomorphism on a closed surface can be obtained using the action of this homeomorphism on the homological groups of the closed surface.

1. Introduction

Here a *closed surface* means a connected compact surface with or without boundary, orientable or not. More precisely, *an orientable connected compact surface without boundary of genus* $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. *An orientable connected compact surface with boundary of genus* $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A non–orientable connected compact surface without boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. *A non–orientable connected compact surface with boundary of genus* $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \to \mathbb{X}$ be a homeomorphism on a closed surface $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f. The aim of the present paper is to provide some information on $\mathrm{Per}(f)$. The statement of our first result is.

Theorem 1. *Let* $\mathbb{X}$ *be a closed surface and let* f *be a self–homeomorphism into* $\mathbb{X}$. *If* $\mathbb{X} = \mathbb{M}_{g,b}$, *then the following statement hold.*

(a) *If* $(g, b) \in \{(1, 0), (0, 2)\}$ (*i.e. the torus and the closed annulus respectively*), *then there is no information on the set* $\mathrm{Per}(f)$.
(b) *If* $(g, b) = (0, 0)$ (*i.e. the* 2*–dimensional sphere*), *then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.
(c) *If* $(g, b) = (0, 1)$ (*i.e. the* 2*–dimensional disc*), *then* $1 \in \mathrm{Per}(f)$.
(d) *If* $g > 1$ *and* $b = 0$, *then* $\mathrm{Per}(f) \cap \{1, 2, ..., 2g\} \neq \emptyset$.

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(e) *If* $g > 1$ *and* $b > 0$*, then* $\mathrm{Per}(f) \cap \{1, 2, ..., 2g + b - 1\} \neq \emptyset$.

If $\mathbb{X} = \mathbb{N}_{g,b}$*, then the following statements hold.*

(f) *If* $(g, b) \in \{(2, 0), (1, 1)\}$ *(i.e. the Klein bottle and the Möebius band respectively), there is no information on the set* Per*(f).*

(g) *If* $(g, b) = (1, 0)$ *(i.e. the projective plane), then* $1 \in \mathrm{Per}(f)$.

(h) *If* (g, b) *does not satisfy the assumptions of statements* (f) *or* (g)*, then* $\mathrm{Per}(f) \cap \{1, 2, ..., g + b - 1\} \neq \emptyset$.

The proof of Theorem 1 is done in section 2. The main tool for proving it is a result due to Fuller [4].

The results of Theorem 1 restricted to the orientable closed surfaces without boundary, i.e. for the closed surfaces $\mathbb{M}_{g,0}$, where already obtained by Franks and Llibre in [3].

The objective of the rest of the paper is to improve the information provided in Theorem 1 using as a main tool the Lefschetz fixed point theory. We shall follow the ideas of Franks and Llibre in [3] when they improve the results of Theorem 1 for the homeomorphisms of the closed surfaces $\mathbb{M}_{g,0}$, see Theorems 5 and 6.

Let A be an $n \times n$ complex matrix. A $k \times k$ *principal submatrix* of A is a submatrix lying in the same set of k rows and columns, and a $k \times k$ *principal minor* is the determinant of such a principal submatrix. There are $\binom{n}{k}$ different $k \times k$ principal minors of A, and the sum of these is denoted by $E_k(A)$. In particular, $E_1(A)$ is the trace of A, and $E_n(A)$ is the determinant of A, denoted by $\det(A)$.

It is well known that the characteristic polynomial of A is given by

$$\det(tI - A) = t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \ldots + (-1)^n E_n(A).$$

Our main result is state in the following theorem.

Theorem 2. *Let* $f : \mathbb{X} \to \mathbb{X}$ *be a homeomorphism and let* A *be the integral matrix of the isomorphism* $f_{*1} : H_1(\mathbb{X}, \mathbb{Q}) \to H_1(\mathbb{X}, \mathbb{Q})$ *induced by* f *on the first homology group of* $\mathbb{X}$*. If* $\mathbb{X}$ *is either* $\mathbb{M}_{g,b}$ *with* $b > 0$*, or* $\mathbb{N}_{g,b}$ *with* $b \geq 0$*, then the following statements hold.*

(a) *If* $E_1(A) \neq 1$*, then* $1 \in \mathrm{Per}(f)$.

(b) *If* $E_1(A) = 1$ *and* $E_2(A) \neq 0$*, then* $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

If $\mathbb{X} = \mathbb{M}_{g,b}$ *with* $b > 0$*, then the following statement hold.*

(c) *If* $2g + b - 1 \geq 3$*,* $E_1(A) = 1$*,* $E_2(A) = 0$ *and* k *is the smallest integer of the set* $\{3, 4, ..., 2g + b - 1\}$ *such that* $E_k(A) \neq 0$*, then* $\mathrm{Per}(f)$ *has a periodic point of period a divisor of* k.

If $\mathbb{X} = \mathbb{N}_{g,b}$ *with* $b \geq 0$*, then the following statement hold.*

(d) *If* $g + b - 1 \geq 3$*,* $E_1(A) = 1$*,* $E_2(A) = 0$ *and* k *is the smallest integer of the set* $\{3, 4, ..., g + b - 1\}$ *such that* $E_k(A) \neq 0$*, then* $\mathrm{Per}(f)$ *has a periodic point of period a divisor of* k.

Theorem 2 is proven in section 3.

2. Proof of Theorem 1

Let f be continuous self–map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface, the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by f on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$, $f_{*2} = (d)$ where d is the *degree* of the map f if $b = 0$, $f_{*2} = 0$ if $b > 0$, and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see for additional details [6, 7]).

We recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see again for additional details [6, 7]).

The proof of Theorem 1 is a consequence of a general result from polyhedron homeomorphisms proved in [4], see also Halpern [5] and Brown [1] for more details on it.

Theorem 3 (Fuller's Theorem)**.** *Let f be a homeomorphism of a compact polyhedron X into itself. If the Euler characteristic of X is not zero, then f has a periodic point with period not greater that the maximum of $\sum_{k\, odd} B_k(X)$ and $\sum_{k\, even} B_k(X)$, where $B_k(X)$ denotes the k–th Betti number of X.*

Proof of Theorem 1. Assume $\mathbb{X} = \mathbb{M}_{g,b}$. Since for a closed surface $\mathbb{M}_{g,b}$ its homological groups with rational coefficients are $H_0(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{2g}{\cdots} \oplus \mathbb{Q}$ and $H_2(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q}$ if $b = 0$, and $H_0(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{2g+b-1}{\cdots} \oplus \mathbb{Q}$ and $H_2(\mathbb{M}_{g,b}, \mathbb{Q}) = 0$ if $b > 0$, then its Euler characteristic $\mathcal{X}(\mathbb{M}_{g,b}) = B_0(\mathbb{M}_{g,b}) - B_1(\mathbb{M}_{g,b}) + B_2(\mathbb{M}_{g,b})$ is equal to $2 - 2g$ if $b = 0$, and $2 - 2g - b$ if $b > 0$, where $B_k(\mathbb{M}_{g,b}) = \dim_{\mathbb{Q}}(H_k(\mathbb{M}_{g,b}, \mathbb{Q}))$.

Since

$$\sum_{k\, even} B_k(\mathbb{M}_{g,b}) = 2 \neq 0 \text{ and } \sum_{k\, odd} B_k(\mathbb{M}_{g,b}) = 2g \quad \text{if } b = 0,$$

$$\sum_{k\, even} B_k(\mathbb{M}_{g,b}) = 1 \neq 0 \text{ and } \sum_{k\, odd} B_k(\mathbb{M}_{g,b}) = 2g + b - 1 \quad \text{if } b > 0,$$

the orientable closed surfaces for which the Fuller's Theorem does not provide any information on the set of periods $\mathrm{Per}(f)$ are the ones having zero Euler characteristic, i.e. when $g = 1$ and $b = 0$, and $g = 0$ and $b = 2$. Therefore statement (a) is proved.

If $(g, b) = (0, 0)$ then $\max\{\sum_{k\, even} B_k(\mathbb{M}_{g,b}) = 2, \sum_{k\, odd} B_k(\mathbb{M}_{g,b}) = 0\} = 2$, then by Theorem 3 it follows that $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. Hence statement (b) follows.

If $(g,b)=(0,1)$ then $\max\{\sum_{k\,even} B_k(\mathbb{M}_{g,b})=1, \sum_{k\,odd} B_k(\mathbb{M}_{g,b})=0\}=1$, then by Theorem 3 it follows that $1\in \mathrm{Per}(f)$. So statement (c) follows.

If $g>1$ and $b=0$ then $\max\{\sum_{k\,even} B_k(\mathbb{M}_{g,b})=2, \sum_{k\,odd} B_k(\mathbb{M}_{g,b})=2g\}=2g$, then by Theorem 3 it follows that $\mathrm{Per}(f)\cap\{1,2,...,2g\}\neq\emptyset$. Hence statement (d) is proved.

If $g>1$ and $b>0$ then $\max\{\sum_{k\,even} B_k(\mathbb{M}_{g,b})=1, \sum_{k\,odd} B_k(\mathbb{M}_{g,b})=2g\}=2g+b-1$, then by Theorem 3 it follows that $\mathrm{Per}(f)\cap\{1,2,...,2g+b-1\}\neq\emptyset$. Therefore statement (e) follows.

Assume $\mathbb{X}=\mathbb{N}_{g,b}$. Since for a closed surface $\mathbb{N}_{g,b}$ its homological groups with rational coefficients are $H_0(\mathbb{M}_{g,b},\mathbb{Q})=\mathbb{Q}$, $H_1(\mathbb{M}_{g,b},\mathbb{Q})=\mathbb{Q}\oplus \overset{g+b-1}{\cdots} \oplus\mathbb{Q}$ and $H_2(\mathbb{M}_{g,b},\mathbb{Q})=0$, then its Euler characteristic $\mathcal{X}(\mathbb{N}_{g,b})=B_0(\mathbb{N}_{g,b})-B_1(\mathbb{N}_{g,b})+B_2(\mathbb{N}_{g,b})=2-g-b$, where $B_k(\mathbb{N}_{g,b})=\dim_{\mathbb{Q}}(H_k(\mathbb{N}_{g,b},\mathbb{Q}))$.

Since

$$\sum_{k\,even} B_k(\mathbb{M}_{g,b})=1\neq 0 \text{ and } \sum_{k\,odd} B_k(\mathbb{M}_{g,b})=g+b-1,$$

the non–orientable closed surfaces for which the Fuller's Theorem does not provide any information on the set of periods $\mathrm{Per}(f)$ are the ones having zero Euler characteristic, i.e. when $g=2$ and $b=0$, and $g=1$ and $b=1$. Therefore statement (f) is proved.

If $(g,b)=(1,0)$ then $\max\{\sum_{k\,even} B_k(\mathbb{M}_{g,b})=1, \sum_{k\,odd} B_k(\mathbb{M}_{g,b})=0\}=1$, then by Theorem 3 it follows that $1\in \mathrm{Per}(f)$. So statement (g) follows.

If (g,b) does not satisfy the assumptions of statements (f) or (g), then $\max\{\sum_{k\,even} B_k(\mathbb{M}_{g,b})=1, \sum_{k\,odd} B_k(\mathbb{M}_{g,b})=g+b-1\}=g+b-1$, then by Theorem 3 it follows that $\mathrm{Per}(f)\cap\{1,2,...,g+b-1\}\neq\emptyset$. Hence statement (h) is proved. □

3. Proof of Theorems 2

Let $f:\mathbb{X}\to\mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of f is defined by

$$L(f)=\mathrm{trace}(f_{*0})-\mathrm{trace}(f_{*1})+\mathrm{trace}(f_{*2}).$$

For continuous maps and in particular for homeomorphisms f defined on $\mathbb{X}$ the Lefschetz fixed point theorem states (see for instance [1]).

Theorem 4. *If $L(f)\neq 0$ then f has a fixed point.*

With the aim of studying the periodic points of f we shall use the Lefschetz numbers of the iterates of f, i.e. $L(f^n)$. Note that if $L(f^n)\neq 0$ then f^n has a fixed point, and consequently f has a periodic point of period a divisor of n. In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$ it is defined the formal *Lefschetz zeta function* of f as

$$Z_f(t)=\exp\left(\sum_{n=1}^{\infty}\frac{L(f^n)}{n}t^n\right). \tag{1}$$

The Lefschetz zeta function is in fact a generating function for the sequence of Lefschetz numbers n. In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$.

For a continuous self–map of a closed surface the Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})},$$

see for more details Franks [2]. Then, for an orientation–preserving homeomorphism $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{(1-t)^2} & \text{if } b = 0. \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0, \end{cases} \tag{2}$$

where $f_{*1} = A$. Note that $f_{*2} = (1)$ if $b = 0$, and $f_{*2} = (0)$ if $b > 0$.

For an orientation–reversing homeomorphism $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{1-t^2} & \text{if } b = 0. \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0. \end{cases} \tag{3}$$

Note that $f_{*2} = (-1)$ if $b = 0$, and $f_{*2} = (0)$ if $b > 0$.

Finally, for a homeomorphism $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ we have

$$Z_f(t) = \frac{\det(I - tA)}{1-t}. \tag{4}$$

Using the Lefschetz zeta function the orientation preserving and reversing homeomorphisms on $\mathbb{M}_{g,0}$ were studied in [3]. The results there obtained are the following two theorems. Here we shall study the homeomorphisms on $\mathbb{M}_{g,b}$ with $b > 0$ and on $\mathbb{N}_{g,b}$ with $b \geq 0$.

Theorem 5 (Theorem 4 of [3]). *Let $f : \mathbb{M}_{g,0} \to \mathbb{M}_{g,0}$ be an orientation–preserving homeomorphism and let A be the $2g \times 2g$ integral matrix of the isomorphism $f_{*1} : H_1(\mathbb{M}_{g,0}, \mathbb{Q}) \to H_1(\mathbb{M}_{g,0}, \mathbb{Q})$ induced by f on the first homology group of $\mathbb{M}_{g,0}$. Then the following statements hold.*

(a) *If $g = 0$, then $1 \in \text{Per}(f)$.*
(b) *If $g > 0$ and $E_1(A) \neq 2$, then $1 \in \text{Per}(f)$.*
(c) *If $g > 0$, $E_1(A) = 2$ and $E_2(A) \neq 1$, then $\text{Per}(f) \cap \{1, 2\} \neq \emptyset$.*
(d) *If $g = 1$, $E_1(A) = 2$ and $E_2(A) = 1$, then there is no information on $\text{Per}(f)$.*
(e) *If $g > 1$, $E_1(A) = 2$, $E_2(A) = 1$ and k is the smallest integer of the set $\{3, 4, \ldots, 2g\}$ such that $E_k(A) \neq 0$, then f has a periodic point of period a divisor of k.*

Theorem 6 (Theorem 3 of [3]). *Let $f : \mathbb{M}_{g,0} \to \mathbb{M}_{g,0}$ be an orientation–reversing homeomorphism and let A be the $2g \times 2g$ integral matrix of the isomorphism $f_{*1} : H_1(\mathbb{M}_{g,0}, \mathbb{Q}) \to H_1(\mathbb{M}_{g,0}, \mathbb{Q})$ induced by f on the first homology group of $\mathbb{M}_{g,0}$. Then the following statements hold.*

(a) *If $g = 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*
(b) *If $g > 0$ and $E_1(A) \neq 0$, then $1 \in \mathrm{Per}(f)$.*
(c) *If $g > 0$, $E_1(A) = 0$ and $E_2(A) \neq -1$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*
(d) *If $g = 1$, $E_1(A) = 0$ and $E_2(A) = -1$, then there is no information on $\mathrm{Per}(f)$.*
(e) *If $g > 1$, $E_1(A) = 0$, $E_2(A) = 1$ and k is the smallest integer of the set $\{3, 4, \dots, 2g\}$ such that $E_k(A) \neq 0$, then f has a periodic point of period a divisor of k.*

Unfortunately we cannot distinguish using the Lefschetz zeta function the orientation–preserving homeomorphisms from the orientation–reversing on $\mathbb{M}_{g,b}$ when $b > 0$, see (2) and (3). They can be distinguished if $b = 0$, because then they have different Lefschetz zeta functions, see again (2) and (3).

Proof of Theorem 2. Assume $b > 0$. Let $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a homeomorphism, and let A be the $(2g + b - 1) \times (2g + b - 1)$ integral matrix of the isomorphism $f_{*1} : H_1(\mathbb{X}, \mathbb{Q}) \to H_1(\mathbb{X}, \mathbb{Q})$ induced by f on the first homology group of $\mathbb{X}$. Then, combining the expressions (1), (2) with $b > 0$, (3) with $b > 0$ and (4) with $b \geq 0$ we obtain the following equalities

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{1 - t}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \dots + (-1)^m E_m(A)t^m}{1 - t}\right) \\
&= \log(1 - E_1(A)t + E_2(A)t^2 - \dots) - \log(1 - t) \\
&= \left(-E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right) t^2 - \dots\right) - \left(-t - \frac{t^2}{2} - \dots\right) \\
&= (1 - E_1(A))t + \left(\frac{1}{2} - \frac{E_1(A)^2}{2} + E_2(A)\right) t^2 + O(t^3).
\end{aligned}
$$

Here $m = 2g + b - 1$ if $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, or $m = g + b - 1$ if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$. Therefore we have

$$
L(f) = 1 - E_1(A), \quad \text{and} \quad L(f^2) = 1 - E_1(A)^2 + 2E_2(A).
$$

Hence, if $E_1(A) \neq 1$ then $L(f) \neq 0$, and by Theorem 4 statement (a) follows.

If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, and again by Theorem 4 we get that $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. So statement (b) is proved.

Assume now that $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, $2g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., 2g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log\left(\frac{1 - t + (-1)^k E_k(A)t^k + ... + (-1)^{b-1} E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right) \\ &= \log\left(1 + \frac{(-1)^k E_k(A)t^k + ... + (-1)^{b-1} E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right) \\ &= (-1)^k E_k(A)t^k + O(t^{k+1}).\end{aligned}$$

Hence, $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. So, from Theorem 4, it follows the statement (c).

Suppose that $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, $g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log\left(\frac{1 - t + (-1)^k E_k(A)t^k + ... + (-1)^{g+b-1} E_{g+b-1}(A)t^{g+b-1}}{1-t}\right) \\ &= \log\left(1 + \frac{(-1)^k E_k(A)t^k + ... + (-1)^{g+b-1} E_{g+b-1}(A)t^{g+b-1}}{1-t}\right) \\ &= (-1)^k E_k(A)t^k + O(t^{k+1}).\end{aligned}$$

Again $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. Therefore, from Theorem 4, it follows the statement (d). □

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