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Limit cycles for piecewise linear differential systems via Poincaré-Miranda theorem

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Abstract: In [**2**] we develop an effective procedure to prove the existence, determine the number, and locate periodic orbits of dynamical systems of both discrete and continuous nature. It is based on the use of the Poincaré-Miranda theorem. This note presents one of the results obtained in that paper: a new example of piecewise linear differential system with three limit cycles.

1 Introduction and main result

The study of the number of limit cycles for planar differential systems is a classical topic in the theory of dynamical systems. In the last years, many attention has been devoted to the study of nested limit cycles of piecewise linear systems, steered by the applicability of these systems in the modelling of biological and mechanical applications. In 2012, S.M. Huan and X.S. Yang gave numerical evidences of a piecewise linear system with two zones and a discontinuity straight line, having three nested limit cycles ([**3**]). A proof based on the Newton–Kantorovich theorem of the existence of these limit cycles for this example and a nearby one, was given by J. Llibre and E. Ponce ([**5**]). A different proof, from a bifurcation viewpoint, was presented by E. Freire, E. Ponce and F. Torres in [**1**]. Until now, as far as we know, three is the maximum observed number of limit cycles in a piecewise linear systems with two zones and a discontinuity straight line, but it is not known if this is the maximum number that such type of systems can have.

In this work we present a new example, again with 3 limit cycles, inspired on the ones given in [**3, 5**]. The main contribution is that our proof relies on the so called Poincaré-Miranda theorem and it is very simple. This theorem is essentially the extension of the intermediate value theorem (or more precisely the Bolzano's theorem) to higher dimensions. It was stated by H. Poincaré in 1883 and 1884, and proved by himself in 1886 ([**7, 8**]). In 1940, C. Miranda re-obtained the result as an equivalent formulation of Brouwer fixed point theorem ([**6**]). Recent proofs are presented in [**4, 10**]. For completeness, we recall it. As usual, $\overline{S}$ and ∂S denote, respectively, the closure and the boundary of a set $S \subset \mathbb{R}^n$.

Theorem 1 (Poincaré-Miranda) *Set* $\mathcal{B} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : L_i < x_i < U_i, 1 \leq i \leq n\}$. *Suppose that* $f = (f_1, f_2, \ldots, f_n) : \overline{\mathcal{B}} \to R^n$ *is continuous,* $f(\mathbf{x}) \neq \mathbf{0}$ *for all* $\mathbf{x} \in \partial\mathcal{B}$, *and for* $1 \leq i \leq n$,

$$f_i(x_1, \ldots, x_{i-1}, L_i, x_{i+1}, \ldots, x_n) \leq 0 \quad \textit{and} \quad f_i(x_1, \ldots, x_{i-1}, U_i, x_{i+1}, \ldots, x_n) \geq 0,$$

Then, there exists $\mathbf{s} \in \mathcal{B}$ *such that* $f(\mathbf{s}) = \mathbf{0}$.

We prove:

Theorem 2 *The two-zones piecewise linear differential system*

$$\dot{\mathbf{x}} = \begin{cases} A^{+}\mathbf{x} & \text{if } x \geq 1, \\ A^{-}\mathbf{x} & \text{if } x \leq 1, \end{cases} \tag{1}$$

where $\mathbf{x} = (x, y)^t$,

$$A^{-} := \begin{pmatrix} \frac{67}{50} & -\frac{833}{125} \\ \frac{1}{2} & -\frac{87}{50} \end{pmatrix} \text{ and } A^{+} := \begin{pmatrix} \frac{3}{8} & -1 \\ 1 & \frac{3}{8} \end{pmatrix},$$

has at least three nested hyperbolic limit cycles surrounding the origin.

2 Proof of Theorem 2

Let $\varphi^{\pm}(t;p) = (x^{\pm}(t;p), x^{\pm}(t;p))$ denote the flow associated to the linear systems $\dot{\mathbf{x}} = A^{\pm}\mathbf{x}$. Observe that if there exists a limit cycle then it must lie on both sides of the line $x = 1$, so let $t^{-} > 0$ be the smaller time such that $x^{-}(t^{-};(1,y)) = 1$ for a point $(1, y)$ with $y > 0$, and let $t^{+} > 0$ be the first positive time such that $x^{+}(-t^{+};(1,y)) = 1$. Then any limit cycle must satisfy both conditions and $y^{+}(-t^{+};(1,y)) - y^{-}(t^{-};(1,y)) = 0$, or equivalently,

$$\mathrm{e}^{-\frac{3}{8}u}\left(\cos(u) + y\sin(u)\right) - 1 = 0, \tag{2}$$

$$\left(35\cos\left(\frac{49}{50}v\right) + (-238y + 55)\sin\left(\frac{49}{50}v\right)\right)\frac{\mathrm{e}^{-\frac{1}{5}v}}{35} - 1 = 0, \tag{3}$$

$$\left(-49\cos\left(\frac{49}{50}v\right)y + (77y - 25)\sin\left(\frac{49}{50}v\right)\right)\frac{\mathrm{e}^{-\frac{v}{5}}}{49} + \mathrm{e}^{-\frac{3}{8}u}\left(\cos(u)\,y - \sin(u)\right) = 0, \tag{4}$$

where $u = t^{+} > 0$ and $v = t^{-} > 0$. By solving equation (2) we get $y = (\mathrm{e}^{-3u/8} - \cos(u))/\sin(u)$. By substituting this expression in equations (3) and (4), we obtain

$$\begin{aligned} g_1(u,v) :=&\ a(v)\cos(u) + b(v)\sin(u) - a(v)\mathrm{e}^{\frac{3}{8}u} = 0, \\ g_2(u,v) :=&\ c(v)\cos(u) + d(v)\sin(u) + e(v)\mathrm{e}^{\frac{3}{8}u} + f(v)\mathrm{e}^{-\frac{3}{8}u} = 0, \end{aligned} \tag{5}$$

where

$$a(v) = 238\,\mathrm{e}^{-\frac{v}{5}}\sin\left(\frac{49}{50}\,v\right), \quad b(v) = 55\,\mathrm{e}^{-\frac{v}{5}}\sin\left(\frac{49}{50}\,v\right) + 35\,\mathrm{e}^{-\frac{v}{5}}\cos\left(\frac{49}{50}\,v\right) - 35$$

$$c(v) = 49\,\mathrm{e}^{-\frac{v}{5}}\cos\left(\frac{49}{50}\,v\right) - 77\,\mathrm{e}^{-\frac{v}{5}}\sin\left(\frac{49}{50}\,v\right) + 49, \quad d(v) = -25\,\mathrm{e}^{-\frac{v}{5}}\sin\left(\frac{49}{50}\,v\right),$$

$$e(v) = 77\,\mathrm{e}^{-\frac{v}{5}}\sin\left(\frac{49}{50}\,v\right) - 49\,\mathrm{e}^{-\frac{v}{5}}\cos\left(\frac{49}{50}\,v\right), \quad f(v) = -49.$$

Numerically it is easy to guess that there are 3 different solutions of system (5), see the figure. Their approximate values in (u, v) variables are $(0.441441, 4.554696)$, $(0.639391, 4.105752)$ and $(1.686596, 3.458345)$. Once we prove that near these values there are actual solutions of system (5), each one of them will correspond to a solution of the system of equations (2)–(4) and, consequently, all them will give rise to 3 limit cycles of (1), see again the figure.

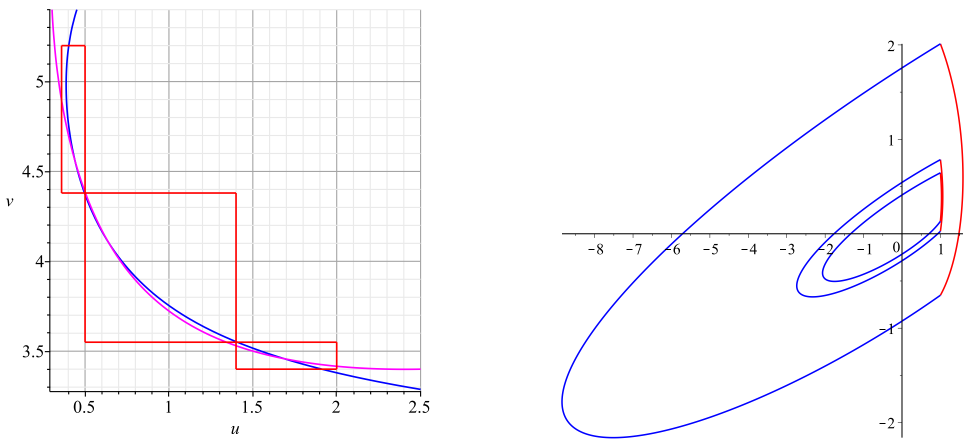

FIGURE 1. Left part: Intersection points between $g_1(u,v)=0$ (in blue) and $g_2(u,v)=0$ (in magenta) and some boxes containing them. Right part: the 3 limit cycles of system (1).

To prove the existence of 3 solutions of system (5), we consider the 3 boxes:

$$\mathcal{B}_1 := \left[\frac{9}{25},\frac{1}{2}\right]\times\left[\frac{219}{50},\frac{26}{5}\right],\quad \mathcal{B}_2 := \left[\frac{1}{2},\frac{7}{5}\right]\times\left[\frac{71}{20},\frac{219}{50}\right]\ \text{ and } \mathcal{B}_3 := \left[\frac{7}{5},2\right]\times\left[\frac{17}{5},\frac{71}{20}\right]$$

which are also shown in the figure and we apply the Poincaré-Miranda theorem to each of them. For short we only give some details for $\mathcal{B}_1$. We write $[\underline{u},\overline{u}] := [9/25, 1/2]$ and $[\underline{v},\overline{v}] := [219/50, 26/5]$.

The existence of a solution in $\mathcal{B}_1$ will follow by applying the Poincaré-Miranda theorem to this box if we prove the following two claims:

(i) It holds that $g_2(u,\underline{v}) > 0$ and $g_2(u,\overline{v}) < 0$ for all $u \in [\underline{u},\overline{u}]$.
(ii) It holds that $g_1(\underline{u},v) < 0$ and $g_1(\overline{u},v) > 0$ for all $v \in [\underline{v},\overline{v}]$.

To control the sign of g_j on the sides of each box we use next lemma:

Lemma 3 *Set* $h(x) = A\cos(\alpha x)+B\sin(\alpha x)+C\mathrm{e}^{\beta x}+D\mathrm{e}^{-\beta x}$, *with* $A,B,C,D\in\mathbb{R}$, $\alpha\neq 0$, $\beta>0$ *and* $x\in[\underline{x},\overline{x}]\subset\mathbb{R}^+$. *Then for each* $n\geq 0$ *we have* $h(x)=\sum_{j=0}^{n} a_j x^j + m_n(x)x^{n+1}$, *where*

$$a_j = \frac{1}{j!}\left(\alpha^j\left[A\cos\left(j\frac{\pi}{2}\right)+B\sin\left(j\frac{\pi}{2})\right)\right]+\beta^j\left[C+(-1)^jD\right]\right), \tag{6}$$

$$|m_n(x)| \leq \overline{m}_n = \frac{|\alpha|^{n+1}\left(|A|+|B|\right)+|\beta|^{n+1}\left(|C|\mathrm{e}^{\beta\overline{x}}+|D|\mathrm{e}^{-\beta\underline{x}}\right)}{(n+1)!}. \tag{7}$$

In fact, we only give the details to prove in item (i) that $g_2(u,\underline{v})>0$ for all $u\in[\underline{u},\overline{u}]$. All the other sides of the box and the study of the other two boxes can be done by adapting the same procedure. We have that

$$g_2(u,\underline{v}) = c\left(\frac{219}{50}\right)\cos(u)+d\left(\frac{219}{50}\right)\sin(u)+e\left(\frac{219}{50}\right)\mathrm{e}^{\frac{3}{8}u}+f\left(\frac{219}{50}\right)\mathrm{e}^{-\frac{3}{8}u},\quad \text{with}$$

$$
\begin{aligned}
&A = c\left(\tfrac{219}{50}\right) = 49\,\mathrm{e}^{-\frac{219}{250}} \cos\left(\tfrac{10731}{2500}\right) - 77\,\mathrm{e}^{-\frac{219}{250}} \sin\left(\tfrac{10731}{2500}\right) + 49,\\
&B = d\left(\tfrac{219}{50}\right) = -25\,\mathrm{e}^{-\frac{219}{250}} \sin\left(\tfrac{10731}{2500}\right),\\
&C = e\left(\tfrac{219}{50}\right) = \left(-49\cos\left(\tfrac{10731}{2500}\right) + 77\sin\left(\tfrac{10731}{2500}\right)\right)\mathrm{e}^{-\frac{219}{250}}, \quad D = f\left(\tfrac{219}{50}\right) = -49.
\end{aligned}
$$

By applying Lemma 3 with $n = 4$, $\alpha = 1$ and $\beta = 3/8$, we have that $g_2(u, \underline{v}) = \sum_{j=0}^{4} a_j u^j + m_4(u)u^5$, with a_j given in (6) and $|m_4(u)| < \overline{m}_4 \simeq 0.66642 < 0.7 = M$, see (7). Taking $a_j^- := \mathrm{Trunc}(a_j \cdot 10^k) \cdot 10^{-k} - 10^{-k}$ with $k = 3$, for each $j = 0, \ldots, 4$ we obtain that $\sum_{j=0}^{4} a_j u^j > \sum_{j=0}^{4} a_j^- u^j$ in $[\underline{u}, \overline{u}]$, where

$$
\sum_{j=0}^{4} a_j^- u^j = -\frac{1}{1000} + \frac{1001}{50}\,u - \frac{39899}{1000}\,u^2 - \frac{669}{500}\,u^3 + \frac{357}{125}\,u^4.
$$

Putting all the inequalities together we get that in $[\underline{u}, \overline{u}]$,

$$
g_2(u, \underline{v}) = \sum_{j=0}^{4} a_j u^j + m(u)u^5 > \sum_{j=0}^{4} a_j^- u^j - \frac{7}{10}u^5 := Q_5(u).
$$

Finally, Q_5 is a polynomial with rational coefficients. Computing its Sturm sequence ([**9**]) we get that it has no zeroes $[\underline{u}, \overline{u}]$ and it is positive, as we wanted to prove.

On the other two sides of $\mathcal{B}_1$ or on the boundaries of the other 2 boxes we can use the same approach. The only changes are that n and k vary from one to another, the corresponding upper bound M must be computed and sometimes instead of g_j is is convenient to consider $\mathrm{e}^{v/5} g_j$, see [**2**] for more details.

To prove the hyperbolicity of the limit cycles we can follow the same ideas that in [**5**].

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