The role of teachers’ knowledge in the use of learning opportunities triggered by mathematical connections

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In the context of non-standard measurement in second grade of primary school, connections related to the mathematical foundations of length measurement are perceived as an important focus of attention when seeking to gain a better understanding of such a context. Learning opportunities stemming from connections are described and analysed with relation to teachers’ and students’ interventions in the classroom. This paper discusses the relationship between extra- and intra-mathematical connections and the role of a teacher’s knowledge in the use of learning opportunities emerging from such connections. Our results suggest that extra-mathematical connections are strongly based on intra-mathematical connections and confirm that different types of knowledge can help teachers to make the most of the learning opportunities arising from connections in a classroom context.

1. Introduction

The making of connections is a linchpin of Mathematics Education, as it is related to the development of a broader and deeper knowledge of mathematics (Skemp, 1978; Triantafillou & Potari, 2010). Mathematics, and in particular mathematics problem solving, is characterized by the interconnectivity between different content areas (e.g. Algebra and Geometry), different representations, or different procedures, and between mathematics and outside-mathematics situations. Therefore, connections play a major role at all educational levels, especially in primary school, where traditionally formal mathematical education begins.

Following a research trend in recent years, several perspectives on connections have been developed: both regarding the ways connections occur in mathematics (e.g. Zazkis & Mamolo, 2011) and the way connections are established in the classroom (e.g. De Gamboa & Figueiras, 2014; Montes, Ribeiro, Carrillo, & Kilpatrick, 2016). We focus on the latter perspective and on the categorizations that emerge which describe connections as a complex system of relationships, in which outside-mathematics situations, systems of representation and/or heuristics are linked.

Making connections in the classroom can help students to identify new applications of mathematics to real-world problems and to give meaning to such problems in school contexts. It may also make it easier to use different representations when solving problems, as well as to reinterpret and rebuild connections between concepts, properties and/or procedures. Consequently, promoting the emergence of connections in the classroom has the potential to trigger a wide array of learning opportunities. The usefulness of these opportunities depends on teachers' knowledge and on their ability to identify, interpret and promote learning opportunities stemming from connections, and to make decisions during the classroom activity that help students to build up a broader and deeper mathematical knowledge.

The presence of connections when conceptualizing teachers' knowledge – assuming a practice-based perspective – reveals a relationship between teachers' knowledge and the way connections are established and used in the classroom. For instance, Rowland, Turner, Thwaites and Huckstep (2009) consider connections as a domain of teachers' knowledge that refers to teachers' ability to anticipate complexity, make decisions about sequencing, make connections between procedures, and make connections between concepts. As regards Ball, Thames, and Phelps (2008), connections are related to teachers' awareness of how mathematical concepts are connected throughout school years.

In managing the development of students' understanding, teachers need to mobilize their knowledge (both mathematical knowledge and pedagogical content knowledge) in a very specialized way. In order to capture the nuances of that knowledge and to characterize the specialized features of mathematics teachers' knowledge in terms of what is mobilised when promoting and exploring mathematics learning opportunities arising from mathematical connections, we consider such specialization in the sense of the framework of the Mathematics Teachers' Specialized Knowledge – MTSK (Carrillo, Montes, Contreras, & Climent 2018).

With a view to deepening our understanding of connections and their role in practice (potentialities and constraints), and to conceptualize ways of improving the effectiveness and utility of such connections in terms of teacher knowledge, the concretization of the aforementioned categorizations of connections need to be studied and expanded in relation to several mathematical topics. Amongst the diversity of topics in school mathematics, measurement is perceived as a crucial element in pupils' development of mathematical understanding and knowledge (e.g. Sarama, Clements, Barret, Van Dine, & McDonel, 2011). It is also a rich environment in relation to the emergence of mathematical connections, as it represents a natural linkage between numbers and operations, geometry and real-world problems. In particular, the introduction of the measurement of length – its different dimensions (Clements & Sarama, 2007) and properties – can enact several aspects of mathematical connections related to natural and rational numbers, different representations of numbers, the procedures related to the measurement of length, and different units of measurement (e.g. Szilágyi, Clements, & Sarama, 2013).

With the aim of gaining a better understanding of what features of teachers' knowledge can help teachers to foster the making of connections in the classroom and to make the most of these connections in terms of exploiting the learning opportunities stemming from them, in this chapter we present the case of Carla, a prospective teacher developing an introduction to standard length units in the second grade of primary school. We start by characterizing mathematical connections that emerge during Carla's lesson and we identify and discuss the learning opportunities stemming from those connections. Then, we analyse what features of her knowledge are related to the use of connections and what other features of teacher knowledge would have helped her to effectively use connections to build a deeper and broader knowledge of length measurement.

2. Length measurement teaching and learning

The main ultimate aim in performing a measurement is to assign a numerical value to an object's attribute. Before measuring, we need to identify which of the object's properties are measurable and differentiate it from those that are not (Campbell, 1928). When assigning numerical values to measurable properties we establish a relation between the property that is being measured and the mathematical properties of the numbers that are used for measuring. Measurement fosters the establishment of relationships between geometrical concepts and numbers, by analysing shape, position, symmetries, rotations and translations using the numerical properties of measurable magnitudes such as angles, distances, lengths and areas. This relation between geometrical properties and numerical properties also allows us to explore the relationship between measurable magnitudes such as perimeter and area when shape or position is changed.

The richness of measurement in terms of the relations between geometry and numbers is related to a high level of complexity. With a view to gaining a better understanding of how connections can be established when working with length measurement in the classroom, an exhaustive analysis of the mathematical elements involved can be useful. Like in many mathematical topics (e.g. adding or dividing fractions) understanding the steps involved may be more complex than the process itself as it involves several stages including many key concepts that are articulated through the connections between problems, representations, definitions, properties and procedures.

Stephan and Clements (2003) posited that six key concepts must be mastered to develop a full understanding of measurement and the skills required for it: (i) equal partitioning – the mental process of dividing an object into equal parts, requiring the acknowledgement of the divisibility of the object; (ii) unit iteration – the skill of exhaustively repeating the unit successively to cover the object; (iii) transitivity – recognition of the mathematical property of measure that ensures that if the measure of A is bigger than the measure of B and the measure of B is bigger than the measure of C, then the measure of A is bigger than the measure of C; (iv) conservation of the measure through rigid transformations that do not change the amount of magnitude; (v) addition and accumulation of distance – recognition that the measurement process outcome is the measure of the object (how many units have to be repeated to equal the measurement of the object); and (vi) relationship between number and measure, implying acceptance that a variation of the unit of measure would generate a change in the measurement outcome (total amount of units).

Length measurement can be divided up into several mathematical packs that include conceptual and procedural elements, as proposed by Ma (1999). In the context of classroom activity, the comprehension of length measurement is associated with that of a system of practices that allow problems related to measure to be solved (Rondero & Font, 2015). In the case of length measurement in primary school, we use an epistemic configuration of length measurement to analyse its complexity (Rondero & Font, 2015) and how the six previous concepts are supposed to be mastered by the pupils. Specifically, we apply the six levels of complexity proposed by Rondero and Font (2015) to length measurement in the early years of primary school which become useful to structure the discussion around the teaching and learning of measurement, as will be shown later in this chapter.

The first level refers to problematic situations related to the need to establish a universal method for the measurement of length in a quotidian context. The second level of complexity concerns the different representations associated with the measurement of length. The third level of complexity consists of the definitions and concepts related to the measurement of length. The fourth level refers to propositions and properties of the concepts related to the measurement of length. The fifth level is formed by the procedures used when performing a length measurement. Finally, the sixth level is related to the arguments that can be used when interpreting the results of length measurements and the decision-making stemming from those results.

The specificities of this framework serve to interpret teachers' and students' interventions in terms of the construction of knowledge of length measurement in a more explicit way than through the six key concepts proposed by Clements and Sarama (2007). These six levels are used when discussing the nature of the connections employed by teachers and those emerging from the answers and/or comments students make in response to different problems of length measurement. Moreover, the different elements in each level allow us to identify learning opportunities stemming from connections in the classroom.

Problematic situations	Representations
PS1: How to compare lengths indirectly? PS2: How to establish numerical comparisons between lengths? PS3: How to communicate quantities of length in a universal way? PS4: What is the most exact way of communicating lengths? PS5: What is the best way of subdividing a unit of length? PS6: How to establish equivalencies between units of measurement? PS7: How to compare measures made with different units of measurement?	R1: Verbal representation using adjectives like “long” or “short” and adverbs like “more” or “exactly”. R2: Symbolic representations using natural, rational and irrational numbers. R3: Graphic representations on the number line. R4: Enactive representations using parts of the body.
Definitions and concepts	**Properties and propositions**
D1: Cardinal and ordinal numbers. D2: Natural and positive rational numbers. D3: Approximation and estimation. D4: Occupied space and comprised space. D5: Distance. D6: Units and subunits of measurement. D7: Standard and non-standard units of measurement. D8: Perimeter of plane shapes.	P1: Transitivity of measure. P2: Transitivity of numbers. P3: Conservation of length. P4: Accumulation and additivity: recognition that the outcome of the measurement process is the measure of the object. P5: Inverse relation between unit’s length and the numerical outcome. P6: Equivalence between procedures if results differ less than a given percentage.
Procedures	**Arguments**
PR1: Addition of positive quantities. PR2: Multiplication of positive quantities. PR3: Representation of quantities on the	A1: When a length of measurement and its fractions are defined numerical comparisons can be established. A2: Units of length have to be well

number line. PR4: Estimation of lengths. PR5: Strategies of mental calculation. PR6: Comparison of lengths. PR7: Discretization of rational quantities. PR8: Use of instruments of measurement. PR9: Definition of an algorithm of the process of measurement (choose a unit of measurement, extend the unit repeatedly and exhaustively along the object, count the number of iterations needed to cover the length of the object, approximate the results at a certain level of exactness, assess the plausibility of the result obtained).	known (e.g. hand span, arm or foot) so the communication of length measurement can be effective. A3: The most exact way to communicate lengths is through the use of consistent units (so the length of the unit does not depend on the particular instrument used). A4: The suitability of the unit's partition depends on the context in which the unit is used. The more exactness is needed, the smaller the subunits have to be. A5: In order to establish equivalencies between units, they should be consistent. The exactness of such equivalencies depends on the possibility of measuring one unit using the other one and obtaining a natural, rational or irrational result. A6: There is an inverse relationship between the length of the unit and the numerical outcome.

Table 1: Mathematical complexity of the measurement of length in early stages.

In the analysis we focus on the mathematical complexity of the measurement of length in early stages and its relationships with the different subdomains of the MTSK conceptualization.

3. Connections

The mathematical connections that are present in the classroom activity are bonds between different mathematical ideas. In particular, we can define mathematical connections as a network of links that coordinate definitions, properties, procedures and/or representations by means of coherent and logical relations (De Gamboa & Figueiras, 2014). The establishment of such connections implies, in most cases, the construction of complex structures between the different links (De Gamboa, 2015). In this way, coordination between the links that form a connection entails the assessment of previous connections, or the creation of new ones.

In a classroom context, mathematical connections occur –among other classroom situations– when students and teacher interact, generating linear and non-linear chains of links, considering the varying nature of the interactions that happen in a classroom. The complexity associated with the construction of the connections can sometimes generate misinterpretations or incompleteness of the links produced, which is related to common mistakes in school mathematics (De Gamboa, 2015). Therefore, the emergence of connections in a class generates learning opportunities related to the possibility of reorganizing conceptual structures and reorienting the misinterpretations based on establishing new links and restructuring the existing ones.

When thinking about the kinds of connections, two major dimensions can be perceived: intra-mathematical and extra-mathematical. Intra-mathematical connections are produced in a mathematical context where only mathematical representations, properties, procedures and arguments are used. Extra-mathematical connections, on the other hand, are connections that link mathematical concepts – from a broader perspective, including definitions, properties, procedures, and representations – and problematic situations in a non-mathematical context.

Intra-mathematical connections can be (a) *conceptual connections* (implying treatment or implying conversions), or, (b) *transversal-process* connections. The former consists of the relationships that are established between representations, procedures, or techniques associated with a single concept or different concepts, while the latter are the relationships between a mathematical concept and a mathematical process that is transversal to different mathematical concepts. Specifically, we consider the connections with arguments, proof, and heuristics in problem solving.

The connections related to transversal processes are associated with the transition between a mathematical activity focused on algorithmic activities, and a mathematical activity characterized by a deeper degree of abstraction, based on the identification of patterns, the justification and proof of results, and the rigorous communication of mathematical information. For example, while observing during a classroom activity that there is an inverse relationship between the measurement unit used and the numerical outcome in the measurement process, a connection related to processes would establish a relation between the concrete relationship identified and the process of justification of that property, with emphasis on its generalization.

Conceptual connections can be divided into two kinds: (i) connections that imply treatment, and (ii) connections that imply conversions, in the sense of Duval (2006). Concerning (i), no changes of register occur, hence connections of this kind are the most common in the mathematics classroom, because they are the connections associated with algorithmic procedures, such as the equivalence between measurement units. In the second case, changes of register do occur, implying the coordination of meanings between different registers of representation, such as the calculation of a perimeter starting from the picture of the geometrical shape, and the representation of this calculation with symbols and letters. This coordination between different registers of representation of the same context is one of the main elements for the understanding of a concept, because the transit between them allows us to distinguish the elements that are mathematically relevant from those that are not (Duval, 2006).

Extra-mathematical connections are characterized by connecting mathematical concepts with situations that: (a) have clearly different objectives to those of the mathematical activity in the classroom; (b) use a different kind of discourse from that used in mathematics classrooms, and; (c) require a set of symbols and a language that clearly differ from those used in mathematics (Walkerdine, 1998). Therefore, when *extra-mathematical connections* are established, the validity and coherence of the links are determined not only by mathematical rules, which imply a complex coordination between the usual concepts and procedures of validation used in mathematics, but also by the procedures of validation that are used in other disciplines.

In the mathematics classroom, connections happen mainly in two situations. Firstly, when the teacher plans the sequence of activities with the aim of making explicit connections, and secondly, in contingency situations in which a student’s comment or the classroom discussion triggers the establishment of connections. In both cases, the ability of the teacher to manage and use the learning opportunities that the connections can provide is crucial. In order to explore and gain a better understanding of how connections can be used in the classroom as a learning opportunity, it is important to analyse teachers’ knowledge.

4. Teachers’ knowledge

Teachers’ knowledge has been a central concern of mathematics education research over the past 30 years (e.g. Shulman, 1986). In particular, teachers’ knowledge of measurement has been a focus of research (Olivero & Robutti, 2001; Lanciano, 2003; Steele, 2013), with the aim of exploring and characterizing it, as well as analysing the challenges related to measurement teaching and learning. Some of this research has addressed, in particular, prospective teachers’ knowledge to teach measurement (Policastro, Almeida, & Ribeiro, 2017; Subramanian, 2014). In this paper we use the MTSK model (Carrillo et al., 2013; Carrillo, Climent, et al., 2018), which proposes a specialized perspective on teachers' knowledge, to help us to understand teachers' knowledge, here in particular in the scope of measurement.

This model follows the seminal reflection by Shulman (1986) that considers three domains – mathematical knowledge, pedagogical content knowledge and includes the affective domain (although this last subdomain will not be under scrutiny in this chapter) – as elements that, while possibly used by the teacher in a complex and integrated way, can be analysed separately in order to gain a deeper understanding of professional knowledge.

Concerning teachers' knowledge of Measurement, with the focus only on mathematical knowledge, this model identifies three subdomains. Knowledge of Topics (KoT) which, in the context of measurement would correspond to Knowledge of Measurement (KoM). This includes teachers' knowledge of the different mathematical elements that make up the topic of measurement: definitions, properties, procedures, registers of representations, phenomenology and applications. Here we can find research into, for example, teachers’ knowledge of measurement of perimeter and area, as well as the difference between both (Steele, 2013). In this subdomain we can also find teachers’ knowledge of how the different properties and representations of the measurement are related. Knowledge of the Structure of the Mathematical foundations (KSM) refers to the knowledge of the kind of connections in and to relate the topic under discussion with other(s), or with the same topic with different levels of complexity. In the case of measurement, it comprises the knowledge of the different relationships of measurement with other mathematical topics, such as Arithmetic (Meissner, 2011), Numbers (Rafiepour & Karimianzade, 2017), and Geometry (Fonseca & Cunha, 2011), or the relationship between the content in each topic worked on in the classroom and their connection to other topics at a higher mathematical level. Knowledge of the Practice of Mathematics (KPM), covering the knowledge of the processes of justification, proof, and refutation that can appear when dealing with measurement, together with heuristic strategies of problem solving that can be used concerning measurement. For instance, the relationship of the idea of proof with the precision of the measurement is of huge importance (Mariotti, 2011).

As regards pedagogical content knowledge, in the particular case of measurement, we can find three subdomains. Knowledge of the Mathematics Teaching (KMT), including knowledge of tasks to manage the learning of measurement, and methodological resources, whether physical, such as puzzles (Sensevy, 2009), or technological (Kortenkamp & Rolka, 2009). Here we can also find theories of teaching, both formal and personal, that help teachers to make sense of the approach adopted. One other subdomain of teachers’ PCK concerns the Knowledge of the Features of Learning Mathematics (KFLM), encompassing teachers’ knowledge of, among others, the elements into which we can unpack measurement learning, the usual ways in which students interact with measurement, usual mistakes and obstacles in the learning of the concept, and theories of learning measurement. Knowledge of Measurement Learning Standards (KMLS), namely the knowledge of the degree of competence and performance that students are expected to reach in each grade, as indicated in curricula, or by professional associations (e.g. NCTM).

4. Methods and context

Data was collected as part of the 4th year of the initial teachers' training programme at the *Universitat Autònoma de Barcelona*. During the previous three years, prospective teachers had taken mathematics and mathematics education. In particular, one of those courses addressed teaching and learning magnitudes and measurement, dealing specifically with teaching and learning length and its measurement, focusing on the importance of making connections in the classroom. In the 4th year, prospective teachers have an internship of 240 hours in a primary school. Prospective teachers are assigned to a particular classroom where they design, implement and assess a teaching unit on a mathematical topic. One of the activities they are required to develop during the internship – as part of a course at university – is the analysis of a video-episode they select from their own practice. Prospective teachers are required to record some of their teaching moments – classes (a total of 10 hours) during the field practice. Then they have to choose one of these classes and identify what they consider to be a significant episode in terms of the mathematical content approach and students' understanding. To select the episodes, they should use Sherin, Linsenmeier and van Es's (2009) criteria: window, clarity and depth. These criteria have been discussed previously as part of a course aimed at discussing the use of such criteria in order to obtain powerful information about and for improving teachers' practices.

This interpretative study is framed in the qualitative research paradigm. Considering the nature of our research question we use a case study (Bryman, 2004) to analyse connections, learning opportunities and teachers' knowledge in a context that has not been designed ad hoc for the purposes of this research. Amongst the several case studies that have been developed as part of a broader research project – aimed at identifying and deepening the understanding of the content of teachers' knowledge and how it intertwines with teachers' actions and beliefs when they choose and analyse videos from their own practice– five prospective teachers focused their intervention on the topic of length measurement. We focus on one of those prospective teachers, Carla, who has been selected, according to two criteria: (a) she made significant use of Sherin et al. (2009) criteria, which allowed her to conduct a detailed and profound analysis of her own practice and (b) her design of the teaching unit was focused on the making of connections between several aspects of length measurement.

We have selected episode fragments from Carla's recorded sessions, with 7-8 year-old pupils, in which the three dimensions have been identified at a high level. Window dimension is related to explicit evidences of students' different levels of comprehension of length measurement. Depth dimension concerns interactions in which students take part in the decision making about the units, the instruments and the procedures that can be used. Finally, evidences of the clarity dimension have been identified when students transparently express mathematical arguments about their comprehension of length measurement. The richness of the evidences obtained in the light of the three previous dimensions, along with the analysis of the mathematical complexity of length measurement in the context of non-standard units, are an ideal source of data for the analysis of how teachers' knowledge determines the use of connections in the classroom.

We focus on an episode involving three tasks exploring the use of non-standard units for measuring length in a "real life" (classroom) context – foot length, hand span and width of a finger to measure classroom length, the height of a book cover, and the height of a glue pot, respectively – as it helps to reveal Carla's knowledge of connections. The episode is first analysed focusing on identifying mathematical connections. Afterwards we analysed the links that form the connections using the epistemic configuration (presented in Table 1). When analysing Carla's work, focusing on connections, we adopt the categorization proposed by De Gamboa and Figueiras (2014) to identify how emerging intra-mathematical connections sustain the extra-mathematical connection, in terms of their nature and their relationship with the task performed.

The characterization of connections using the epistemic configuration of length measurement allows us to interpret such connection in light of specific problems, representations, definitions, properties, procedures and arguments, which provides a framework to identify learning opportunities in the context of length measurement. It is important to note that, although the analysis tends to focus on identifying missed instruction opportunities, such situations are perceived as learning opportunities, and as a powerful tool for use in education. Thus, their inclusion enhances the discussion presented in the subsequent sections.

Finally, we analyse how teachers' knowledge can help/constrain students to make the most of the learning opportunities that emerge within the classroom. To analyse teachers' knowledge, we use the MTSK conceptualization which serves to focus, in-depth, on the measurement-related knowledge Carla employed. This model allows us to identify concrete aspects of Carla’s knowledge used in the episode, enhancing our understanding of the specialized knowledge supporting the management of connections as well as some specificities of her knowledge that could have helped her to use some missed learning opportunities. The episode we analyse here concerns revisiting three activities about the use of non-standard measures. Analysing the episode enables us to identify three blocks of connections, as we show in the following section.

5. Analysis and Discussion

The three different tasks involved measuring lengths in a “real life” context, establishing an extra-mathematical connection related to a problem that needs to be tackled from a mathematical perspective (how can we compare and represent lengths?). This connection relates length to the numerical value representing it, and it is based on comparing the particular length to be measured with the length of an object that can be used as a reference (hands, feet, etc.), referred to as the unit of measurement.

The analysis showed that eight other particular connections were established in relation to the general extra-mathematical connection. These connections can be grouped in three different blocks defined by the elements of the mathematical complexity of length measurement that is highlighted. To begin with, a block of three connections involving the inverse relation between the length of the instrument and the numerical outcome was identified. Secondly, three connections related to alternative procedures for length measurement form another block. Finally, two connections between standard and non-standard units form the third block. The details of each of the connections in each block regarding the mathematical complexity of measurement and the teachers’ knowledge related to the use of the learning opportunities triggered by connections are discussed below.

In the first block, an initial connection is identified when reviewing the results of the measurement of the height of a book cover using hand span. The results obtained by the students are similar but not the same (different hand spans). The connection occurs when the teacher asks “What happened when we measured?”, to which a student answers that they are all wrong, in response to whom the teacher remarks that the results are not wrong. A link between the set of results and their mathematical validity is established, based on the need to have a criterion to decide whether two different results of the same measurement are equivalent or not. This link is related to problematic situations PS2, PS3 and PS4 which refer to comparison, communication and exactness in the measurement of length.

The difference between the results (the numerical answer) reported by the students can be tied to two characteristics of measurement. On the one hand, the approximate character of measurement, along with the possibility of expressing the numerical result using natural or rational numbers, or even imprecise expressions such as “a little bit more”, means a margin of error needs to be created to consider two results as equivalent. On the other hand, the difference in the results may be due to the inconsistency of non-standard units, with different results obtained depending on the instrument being used.

When the teacher asks the students why the results are different, two students (Daniel and Hugo) answer by formulating a hypothesis regarding the inverse relation between the length of the instrument and the numerical outcome. Therefore, a link is made between the validity of the results and the formulation of a hypothesis regarding the dependence of the result on the length of the instrument. The formulation of this hypothesis is related to the arguments referring to the exactness of measurement (A2: units of length have to be well known so the communication of length measurement can be effective; and A3: the most exact way of communicating lengths is through the use of consistent units (so the length of the unit does not depend on the particular instrument used).

Finally, the teacher justifies the validity of the hypothesis by an enactive verification (R4: enactive representations using parts of the body) comparing her own hand span with a student’s hand.

> Teacher: Can anyone tell me why we have obtained different results? Let’s see, Daniel…
>
> Daniel: Because one of us has a big hand, a small hand… […]
>
> Hugo: Each person has a different sized hand, and the bigger the hand, the fewer hands fit. And the smaller the hands, the more hands fit.

The inverse relationship between the length of the instrument and the numerical outcome (property P5) appears again when reviewing the results of measuring the length of the class using their feet and the height of a pot of glue using the width of their fingers. In both cases a second and a third connection are identified when Daniel emphasizes the relationship between the length of the feet and the numerical outcome, and the relationship between the width of the fingers and the numerical outcome. In both cases the teacher makes enactive verifications, showing the difference between the space occupied by her feet and fingers in comparison with those of the students. Finally, at the end of the third connection, the teacher formulates a question related to the reliability of the measures being reviewed, emphasizing the connection between the inverse relationship (P5) and the initial question related to the validity of a set of different results when measuring the same object using the same unit of measurement. However, the latter intervention goes beyond the validity of the results, drawing attention to the lack of reliability of the results obtained when using non-standard units, and therefore to the usefulness of standard units.

The analysis of these three connections shows how elements from several levels of the complexity of measurement at early stages (Table 1) appear in the classroom activity. The teacher's management of classroom activities may foster the use of learning opportunities stemming from the previous three connections. The content of the Knowledge of the Practice of Mathematics (KPM) can contribute towards helping the teacher to make remarks regarding the validation and verification of results and the formulation of hypotheses, as the mathematical rules to justify whether or not two results are the same or equivalent in a measurement context are different to those used in arithmetic. In this sense the enactive verification of this relation carried out by the teacher should lead to more elaborate justifications. Knowledge of the Structure of the Mathematical foundations of Measurement serves to make recursive remarks in the approximate nature of measurement since the use of decimal expressions and fractions may be needed in order to represent real length measurements. In relation to the problems generated by the inconsistency of non-standard units, the teachers' Knowledge of Topics (KoT) – in the particular scope of Measurement – along with Knowledge of the Features of Learning Measurement can help the teacher to address usual mistakes and obstacles related to the use of nonstandard units by emphasising that the numerical result of the measurement process depends on the instrument used.

The three connections are therefore intra-mathematical connections with conversion, as there are changes from a numerical and symbolic register to an enactive and physical register, formed by different links. The fact that all three connections rely on a change of register of representation underlines the importance of the teacher knowing different registers of representation (KoT), how the physical and numerical representations are related (KSM), what resources she can use to make this correspondence of registers clear (KMT) and how this relation is also important in other school topics such as area and volume (KMLS).

During the review of the measurement of the height of the book cover using the hand span and the height of a pot of glue using fingers, three connections related to procedures were identified. In the case of the height of the book cover, Isaac asks if it is also correct to do the measurement by opening the hand partially.

Isaac: Can we do it like this as well? (Opening his hand completely.)

Teacher: Of course, another thing is how we place our hands. If some of you place them like this (partially opened) and some of you place them as

> Isaac has (completely open), Isaac will get less . . . . But it doesn't mean that it is wrong; it simply indicates that we have different hands and we have measured differently.

Isaac's intervention and the teacher's answer establish a fourth connection between the correct procedure for the measurement of length and an alternative procedure. The teacher's explanation remains only at a superficial level, thus not allowing students to deepen their knowledge (and conception) of length. In fact, by saying "…it doesn't mean that it is wrong…." she is likely confusing the students, who consequently may not appreciate the importance of following the same procedure when measuring length. The main relation at the core of this intra-mathematical connection with treatment is between the standard procedure and an alternative one. While the standard procedure (completely extended hand) can be replicable if it is performed by the same person, in the case of the alternative procedure replicability is much more difficult, and therefore they cannot be equivalent procedures.

This connection is related to procedure PR9 in the epistemic configuration of length measurement, concretely, to the step "extend the unit repeatedly and exhaustively along the object" that is being measured. Therefore, the connection triggers an opportunity for discussing the importance of using well defined procedures in measurement in order to establish reliable comparisons between the results of measurements, related to the problematic situation on how to establish numerical comparisons between lengths (PS2).

The teacher's confusion is a lost opportunity for clarifying two possible reasons for different results. In the first block of connections, the difference was due to the inverse relation between the length of the instrument and the numerical outcome, while in this case the different results are generated by a lack of accuracy when performing the procedure of measurement. The teacher's incorrect answer can create a false link between the inverse relationship discussed in the first block of connections and the lack of accuracy in the performance of the procedure.

During the discussion of the results of measuring the length of the classroom using their feet, another student (Miguel) asks if separating their feet while performing the measurement would produce a different result.

Miguel: If we separate our feet we are not doing (measuring) anything.

Teacher: Of course, we have to put them next to each other.

Moreover, Miguel does an enactive representation of his question, not only separating his feet while measuring but also changing the direction of his feet. Thus, another intra-mathematical connection with treatment between the standard procedure and an alternative one is established. However, the teacher fails to take this opportunity to continue exploring the sub-connection by emphasizing that the procedure must be performed exhaustively (intuitive introduction of the notion of algorithm), in order to obtain the same answer when using the same measurement unit and representational system.

The third intra-mathematical conceptual connection with treatment between procedures is detected during the discussion of the way students measured the height of a pot of glue using their fingers (Ribeiro, Badillo, Sánchez-Matamoros & Artès, 2016). While most students obtained numbers close to 10, Miguel's answer was one. When Carla asked him to explain how he obtained that result, Miguel elucidated his reasoning by placing his finger perpendicularly to the base of the pot, while the rest of the class had placed their fingers parallel to the base:

Teacher: No? How many fingers did you get?

Miguel (placing his finger vertically next to the glue pot): One!

Teacher: One? Like this? (The teacher repeats the measurement process using the index finger horizontally) […]

Miguel: Ah, four, four . . .

Teacher: No! It can't be . . . you should get eight; you are doing it wrong.

This situation provides an excellent opportunity to discuss the importance of establishing a common procedure when conducting measurements. In this case, Miguel's answers show the use of non-standard units in a non-standard way. However, Carla's arguments and exemplification indicate her sole focus on the standard measurement process. In this sense, she seems to be taking for granted the underlying procedures, thus disregarding an answer that differs from her own (Jakobsen, Ribeiro, & Mellone, 2014). Finally, she also fails to grasp (or at least discuss with her students) the relationship between the number obtained and the measurement method used.

These three latter connections are closely related to the sequence of steps that have to be followed to perform a measurement correctly (PR9) from the epistemic configuration of length measurement. They are also related to the "universal" communication of lengths and property (P3) as well as to accumulation and additivity in the measurement of length (PR4). The opportunities stemming from the latter three connections can be addressed by, on the one hand, conducting discussions in the classroom about the importance of unifying procedural criteria and using students' interventions to emphasize important elements in the comprehension of measurement using a deep knowledge of the procedures involved in length measurement and their characteristics (KoT). Although in the first three connections the teacher shows a sound knowledge of the inverse relation between the length of the instrument and the numerical outcome, in the fourth connection the teacher shows a lack of knowledge of the importance of following the steps in the measurement procedure thoroughly (PR9), which could have caused confusion in the students, reinforcing common misconceptions in primary school (Sisman & Asku, 2016). On the other hand, the identification and interpretation of common mistakes in students' answers and comments based on knowledge of common mistakes and obstacles in learning length measurement (KFLM) may help the teacher to find the suitable moments to emphasize key concepts in length measurement. Knowledge of the Features of the Learning Mathematics, in the particular case of Measurement (KFLM) also appears to be linked to knowing, identifying and anticipating students' difficulties and misconceptions. Making use of students' interventions can be a good opportunity to emphasize mathematical contents that can be problematic to them.

Finally, two connections between the concepts of standard and non-standard units are identified. The first is established during the review of the measurement of the height of a book cover, when one of the students describes his measurement as two hand spans and one centimetre. This conceptual intra-mathematical connection with treatment between the concepts of hand span and centimetre is related to the problematic situations related to the communication of lengths, its exactness, the ways of subdividing units of measurement and the establishment of equivalencies between units (P3, P4, P5 and P6 respectively). Moreover, the establishment of this connection is related to the numerical and enactive representations of length, the definitions of standard and non-standard units, the property of accumulation and additivity, and the standard procedure of measurement.

The relationship between the use of standard and non-standard units appears again when reviewing the measured height of the pot of glue, but from a broader perspective. At the end of the review the teacher reflects upon the usefulness of the non-standard units.

> Teacher: […] Did I really ask you to measure with your feet, hands and fingers? Would we obtain the same results? […] We need to find another way, another system so we can measure and obtain the same results, don't we?

An intra-mathematical connection with treatment between standard and non-standard units is established. This connection is related to the problem of using non-standard units that started the first connection in the first block, and sums up the implications of using non-consistent units, since the inverse relation between the length of the instrument and the numerical outcome implies that the results may differ considerably if different people perform the measurement. Therefore, this connection draws attention to the extra-mathematical connection that encompasses the whole episode and the other seven connections identified.

This connection triggers an opportunity to show the importance of establishing consistent units that can be divided into subunits, and of defining equivalencies between subunits. Moreover, the episode points to the familiarity of some students with standard units, which can be important information for planning teaching and learning activities on the topic of length. However, the teacher does not use the student's reference to centimetres to compare the use of standard units and non-standard units. The use of these learning opportunities can be fostered by a deep knowledge of the foundations of length measurement (KoT), such as problematic situations, the comprehension of definitions (standard and non-standard units), certain properties (accumulation and additivity) or procedures (steps in the standard procedure), and some arguments related to the length of measurement. Likewise, Knowledge of the Features of Learning Mathematics (KFLM) related to the children's use of standard units, the common obstacles in the learning of length measurement and theories of learning length measurement would contribute towards helping the teacher to design her teacher interventions and to use children's references to standard for emphasising the importance of a consistent unit.

6. Final remarks

The analysis of the episode shows a clear-cut case of a classroom situation in which several intra-mathematical connections sustain a broader extra-mathematical connection. The coordination of the eight connections analysed, and therefore the use of the learning opportunities triggered by them, relies on different kinds of teachers' knowledge being used. This example of an extra-mathematical connection provides insight into the structure of extra-mathematical connections and how they can be established in the classroom.

Understanding extra-mathematical connections as something more than the use of extra-mathematical contexts for the practice of mathematical concepts and skills means considering more specific connections between representations, concepts, properties and procedures that form the complex network of links of the extra-mathematical connection. In this case, the three blocks of connections are related to the foundations of length measurement that are at the core of the activity. Conceptual connections with conversion allow measures made with non-standard units to be translated into numerical values, seeking to show the inverse relationship between the length of the instrument and the numerical outcome. Conceptual connections with treatment are related to the need to move the instrument repeatedly along the object without leaving empty spaces or changing direction, and also to the properties of standard and non-standard units. In addition, the seven connections are related to all the levels in the mathematical complexity of measurement (Table 1) as well as to different subdomains of MTSK. As a result, the usefulness of extra-mathematical connection for the construction of mathematical knowledge depends, at least partially, on the use of intra-mathematical connections for understanding the mathematical foundations of length.

The analysis of these connections shows again how the coordination between the analysis of mathematical complexity and the analysis of the connections in the classroom practice highlights moments in the teaching practice that reveal the need for teachers to be able to identify and interpret the classroom activity so as to make decisions aimed at consolidating key concepts of length measurement at early stages. However, when those intra-mathematical connections are not enhanced or when they are misused, students' learning opportunities will be underutilized. In three of the intra-mathematical connections presented in the analysis, lost opportunities can be identified. Therefore, the extra-mathematical connection was misused in relation to those learning opportunities, as students' attention was primarily drawn to units and instruments, while some core aspects of measurement, such as equal partition, unit iteration, or the relation between non-standard units of measurement and the centimetre, were not addressed.

Although there are several reasons for the teacher to misuse those learning opportunities, e.g. classroom management or her choice of emphasizing the inverse relationship between the length of the instrument and the numerical outcome, knowledge analysis reveals that there are some types of knowledge that would help the teacher to take greater advantage of the learning opportunities arising from the intra-mathematical connections, especially those related to correctness, exactness and reliability.

This knowledge is related to a deep understanding of the mathematical foundations of measurement (KoT). Even if the teacher's main goal was to show the students the need to use standard units, some important issues related to the procedure of length measurement need not have been ignored. Knowledge of the Structure of Mathematics (KSM) is also present when natural and rational numbers are used to represent quantities of length, which is related to the approximate nature of measurements of continuous magnitudes. It is also important for teachers to know the way concepts such as measurement evolve during the school years, as many properties of length measurement are also properties of area or volume measurement (KMLS).

However, a deep understanding of the foundations of length is not sufficient to take advantage of learning opportunities. Knowledge of Features of the Learning of

Mathematics (KFLM) in the context of Measurement is related also to the identification and interpretation of common mistakes such as those related to equal partition and unit iteration. Besides having a deep knowledge of the topics and recognising some of the potential difficulties students can face when performing measurements of length, it is fundamental to know when is the best moment to use students' ideas and how these ideas can be used and framed to make some important points in the classroom, such as the importance of performing measurement procedures correctly.

It is important to emphasize that the content of all the aforementioned subdomains of knowledge are intertwined, as the MK subdomains (KoT and KSM) are the pillars on which KFLM and KMT are based. Thus, coordination of different kinds of knowledge allows teachers' knowledge to acquire its specialized dimension. The above analysis shows that extra-mathematical connections are based on the coordinated establishment of intra-mathematical connections, triggering learning opportunities that need several types of teacher knowledge to be used.

The above analysis leads to a reflection on certain didactic questions related to the use of non-standard units in early years. In the particular case of length measurement, some studies (Clements, 1999) note that the early introduction of non-standard units of measure to show the need to use standard units can be premature. Our results may reinforce this idea, as some students proposed the use of standard units to perform some of the tasks.

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